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HÖLDER STABLE RECOVERY OF TIME-DEPENDENT ELECTROMAGNETIC POTENTIALS APPEARING IN A DYNAMICAL ANISOTROPIC SCHRÖDINGER EQUATION

YAVAR KIAN AND ALEXANDER TETLOW

Abstract. We consider the inverse problem of Höldder-stably determining the time- and space-dependent coefficients of the Schrödinger equation on a simple Riemannian manifold with boundary of dimension $n \geq 2$ from knowledge of the Dirichlet-to-Neumann map. Assuming the divergence of the magnetic potential is known, we show that the electric and magnetic potentials can be Hölder-stably recovered from these data. Here we also remove the smallness assumption for the solenoidal part of the magnetic potential present in previous results.

1. Introduction

1.1. **Statement of the Problem.** Let $T > 0$, let $(\mathcal{M}, g)$ be a compact, connected, smooth Riemannian manifold of dimension $n \geq 2$, and denote by $\partial\mathcal{M}$ its boundary. Further assume that $(\mathcal{M}, g)$ is simple (see definition 1). Let $A \in W^{2,\infty}((0,T) \times \mathcal{M}; T^*\mathcal{M})$ be given by $A = \sum_{j=1}^n a_j dx^j$, and consider the magnetic Laplacian given by

$$\Delta_{g,A(t)} u = \sum_{j,k=1}^n |g|^{-\frac{1}{2}} \left(\partial_{x^j} + i a_j(t,x)\right)\Big(|g|^{\frac{1}{2}} g^{jk} \left(\partial_{x^k} + i a_k(t,x)\right) u\Big),$$

where $g^{-1} = g^{ij}$ and $|g| = \det(g)$. If $A = 0$, this is just the usual Laplace-Beltrami operator Δ_g. For $T > 0$ and $q \in W^{1,\infty}((0,T) \times \mathcal{M})$ we consider the initial boundary value problem (IBVP)

$$\begin{aligned} i\partial_t u(t,x) + \Delta_{g,A(t)} u(t,x) + q(t,x) u(t,x) &= 0 \text{ in } (0,T) \times \mathcal{M}, \\ u(t,x) &= f \text{ on } (0,T) \times \partial\mathcal{M}, \\ u(0,x) &= 0 \text{ in } \mathcal{M}, \end{aligned} \tag{1.1}$$

with inhomogeneous Dirichlet data f. For all $r, s \in (0,\infty)$ and $X = \mathcal{M}$ or $X = \partial\mathcal{M}$ define the spaces $H^{r,s}((0,T) \times X) = H^r(0,T; L^2(X)) \cap L^2(0,T; H^s(X))$ with the associated norm

$$\|u\|^2_{H^{r,s}((0,T)\times X)} = \|u\|^2_{H^r(0,T;L^2(X))} + \|u\|^2_{L^2(0,T;H^s(X))}.$$

We further define the space

$$H_0^{r,s}((0,T) \times \partial\mathcal{M}) = \Big\{ f \in H^{r,s}\big((0,T) \times \partial\mathcal{M}\big) : \text{ for all } k \in \big(-1, s - \frac{1}{2}\big) \cap \mathbb{N},\ \partial_t^k f|_{t=0} = 0 \Big\}.$$

The problem (1.1) admits a unique solution $u \in H^{1,2}((0,T) \times \mathcal{M})$ for $f \in H^{\frac{9}{4},\frac{3}{2}}((0,T) \times \partial\mathcal{M})$ (see [10, Proposition 2.1]). Further, the Dirichlet-to-Neumann (D-to-N map in short) map

$$\Lambda_{A,q}(f) = (\partial_\nu + iA\nu)u, \text{ for } f \in H^{\frac{9}{4},\frac{3}{2}}((0,T) \times \partial\mathcal{M}), \tag{1.2}$$

where $\nu = \nu(x)$ denotes the unit outward normal to $\partial\mathcal{M}$ with respect to the metric g, is a bounded operator from $H_0^{\frac{9}{4},\frac{3}{2}}((0,T) \times \partial\mathcal{M})$ to $L^2((0,T) \times \partial\mathcal{M})$. For $j = 1,2$, let $A_j \in W^{2,\infty}((0,T) \times \mathcal{M}; T * \mathcal{M})$, and $q_j \in W^{1,\infty}((0,T) \times \mathcal{M})$. We call (A_1, q_1) and (A_2, q_2) gauge equivalent if there exists $\phi \in W^{3,\infty}((0,T) \times \mathcal{M})$ such that $\phi|_{(0,T)\times\partial\mathcal{M}} = 0$, $A_2 = A_1 + d\phi$ and $q_2 = q_1 - \partial_t\phi$ and let u_j be the solution of (1.1) with potentials $A = A_j$ and $q = q_j$. If ϕ is as above, we recall that the D-to-N map is invariant under this gauge transformation. More precisely, we have

$$(i\partial_t + \Delta_{g,A_1(t)} + q_1)e^{i\phi} u_2(x,t) = e^{i\phi}(i\partial_t + \Delta_{g,A_2(t)} + q_2)u_2(x,t) = 0,$$

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and we deduce that $e^{i\phi}u_2 = u_1$ and

$$(\partial_\nu + iA_1\nu)u_1 = (\partial_\nu + i(A_1 + d\phi)\nu)u_2 = (\partial_\nu + iA_2\nu)u_2,$$

which then implies that $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$. This obstruction to uniqueness notwithstanding, the aim of this paper is to prove Hölder-stable recovery of the time-dependent electric and magnetic potentials (A, q) from knowledge of the D-to-N map $\Lambda_{A,q}$.

1.2. **History of the Problem.** In the case of the dynamic Schrödinger equation with time-independent potentials, Hölder-stable recovery of the magnetic field from knowledge of the Dirichlet-to-Neumann map was shown in [3], and stable recovery of the electric potential of the Schrödinger equation on a Riemannian manifold was proved in [4]. This latter result is extended to stable determination of the electromagnetic potentials on a Riemannian manifold from the D-to-N map in [2]. We mention also the recent work of [5], where such results have been extended to unbounded cylindrical domain.

Literature dealing with the inverse problem of recovering time-dependent potentials of the Schrödinger equation is rather sparse. To the best of the authors knowledge, the only results establishing recovery of time-dependent potentials of the Schrödinger equation from the D-to-N map deal with Euclidean domains. In particular, it was proved in [8] that the time-dependent electric and magnetic potentials are uniquely determined by the D-to-N map. Logarithmic-stable determination was shown for the electric potential in [7]. This result was extended to the full electromagnetic potential in [6], provided that the time-independent part of the magnetic potential is sufficiently small. Indeed, it was only recently shown in [10] that the electromagnetic potential in a Euclidean domain can be Hölder-stably recovered from knowledge of the D-to-N map.

In the current work, we show that it is possible to Hölder-stably recover the time-and-space-dependent coefficients of the dynamic Schrödinger equation on a simple Riemannian manifold.

1.3. **Main Results.** Here and in the rest of this paper we write $\|\cdot\|$ for the norm of an operator in $\mathcal{B}\big(H_0^{\frac{9}{4},\frac{3}{2}}((0,T)\times\partial\mathcal{M}), L^2((0,T)\times\partial\mathcal{M})\big)$. In this paper we aim to prove the following:

Theorem 1. *(Uniqueness):For $j = 1, 2$, let $A_j \in W^{6,\infty}((0,T)\times\mathcal{M}; T^*\mathcal{M})$ and $q_j \in W^{4,\infty}((0,T)\times\mathcal{M})$. Assume also that*

$$\partial_x^\alpha A_1(t,x) = \partial_x^\alpha A_2(t,x), \quad (t,x) \in (0,T)\times\partial\mathcal{M}, \quad \alpha \in \mathbb{N}^n, |\alpha| \le 5. \tag{1.3}$$

Then the condition $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$ implies that (A_1, q_1) and (A_2, q_2) are gauge equivalent.

Theorem 2. *(Stable Recovery of the Magnetic Potential): Let the condition of Theorem 1 be fulfilled and, for $j = 1, 2$, let $A_j \in W^{6,\infty}((0,T)\times\mathcal{M}; T^*\mathcal{M}) \cap H^{3n+4}((0,T)\times\mathcal{M}; T^*\mathcal{M})$ be such that*

$$\partial_x^\alpha A_1(t,x) = \partial_x^\alpha A_2(t,x), \quad (t,x) \in (0,T)\times\partial\mathcal{M}, \quad \alpha \in \mathbb{N}^n, |\alpha| \le 3n+3. \tag{1.4}$$

Assume also that there exists a constant B such that

$$\sum_{j=1,2} \|q_j\|_{W^{5,\infty}((0,T)\times\mathcal{M};T^*\mathcal{M})} + \|A_j\|_{W^{5,\infty}((0,T)\times\mathcal{M};T^*\mathcal{M})} + \|A_j\|_{H^{3n+4}((0,T)\times\mathcal{M};T^*\mathcal{M})} \le B. \tag{1.5}$$

Then we have

$$\left\|A_1^{sol} - A_2^{sol}\right\| \le C\left\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\right\|^{s_1},$$

where $s_1 > 0$ is a general constant, $C > 0$ a constant depending only on B, T, $\mathcal{M}$ and A_j^{sol} is the solenoidal part of the Hodge decomposition of A_j, given in Lemma 1.

Theorem 3. *(Stable Recovery of the Electric Potential): Let the condition of Theorem 2 be fulfilled with*

$$\delta A_1 = \delta A_2. \tag{1.6}$$

Fix also $q_j \in W^{4,\infty}((0,T)\times\mathcal{M}) \cap H^5((0,T)\times\mathcal{M})$ and assume that the condition

$$\partial_x^\alpha q_1(t,x) = \partial_x^\alpha q_2(t,x), \quad (t,x) \in (0,T)\times\partial\mathcal{M}, \quad \alpha \in \mathbb{N}^n, \ |\alpha| \le 4, \tag{1.7}$$

is fulfilled. We also assume that there exists a constant $B_1 > 0$ such that

$$\sum_{j=1,2} \left(\|q_j\|_{W^{4,\infty}((0,T)\times\mathcal{M})} + \|q_j\|_{H^5((0,T)\times\mathcal{M})} \right) \le B_1. \tag{1.8}$$

Then we have

$$\|q_1 - q_2\|_{L^2((0,T)\times\mathcal{M})} \leq C \left\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\right\|^{s_2}, \tag{1.9}$$

where C depends only on B, B_1 T, and $\mathcal{M}$, and s_2 is a general constant.

As far as the authors are aware, the following work is the first dealing with recovery of time-dependent potentials appearing in a Schrödinger equation with variable coefficients of order two. In fact, the above estimates are the first showing Hölder-stable recovery of a coefficient dependent on all variables of a second order partial differential equation with variable coefficients of order two. The only other work where similar results have been obtained is [10], where the authors consider the case of a bounded subset of $\mathbb{R}^n$ with the Euclidean metric.

Furthermore, stable recovery of the magnetic potential appearing in a Schrödinger equation on a manifold with non-Euclidean metric has, thus far, relied upon the a priori assumption that the magnetic potential is small in some appropriate norm, even in the time-independent case (see, for example, [2]). This smallness assumption is also utilized when recovering the magnetic potential of the wave equation (as seen in [12]). In fact, it happens that this assumption is not necessary when dealing with the Schrödinger equation, even when the magnetic potential is allowed to depend on time, as we shall demonstrate herein.

In Section 2, we introduce the geodesic ray-transforms for 1-forms and for functions. In Section 3 we construct geometric optics solutions to the equation (1.1). We devote Section 4 to the proof of Theorem 1, using the geometric optics solutions as the main tool. The estimate of Theorem 2 is proved in Section 5, whereas the estimate of Theorem 3 is proved in Section 6.

2. Notations

In this section, we list some notation used in the rest of the paper. We denote by $\langle\cdot,\cdot\rangle_g$ the inner product with respect to g on $T\mathcal{M}$, that is for $x \in \mathcal{M}$ and $Y, Z \in T_x\mathcal{M}$ given by $Y = \sum_{j=1}^n y_j\partial_{x^j}$, $Z = \sum_{j=1}^n z_j\partial_{x^j}$ we have

$$\langle Y, Z\rangle_{g(x)} = \sum_{j,k=1}^n g_{jk}(x) y_j z_k.$$

Similarly, we denote by $\langle\cdot,\cdot\rangle_g$ the inner product with respect to g on $T^*\mathcal{M}$, that is for $U, V \in T_x^*\mathcal{M}$ given by $U = \sum_{j=1}^n u_j dx^j$, $V = \sum_{j=1}^n v_j dx^j$ we have

$$\langle U, V\rangle_g(x) = \sum_{j,k=1}^n g^{jk}(x) u_j v_k.$$

We denote by dV_g the Riemannian volume on $\mathcal{M}$, which is given in local coordinates by $dV_g = |g|^{\frac{1}{2}}\, dx^1 \wedge \cdots \wedge dx^n$. We further define on $\partial\mathcal{M}$ the surface measure σ_g such that for $X \in H^1(\mathcal{M}; T\mathcal{M})$ we have

$$\int_{\mathcal{M}} \mathrm{div}_g(X) dV_g = \int_{\partial\mathcal{M}} \langle X, \nu\rangle_g \, d\sigma_g,$$

where $\mathrm{div}_g(X) = \sum_{j=1}^n |g|^{-\frac{1}{2}} \partial_{x_j}\big(|g|^{\frac{1}{2}} X^j\big)$. Additionally, we recall the Riemannian gradient operator given by $\nabla_g f = \big(g^{j1}\partial_{x_j} f, \cdots, g^{jn}\partial_{x_j} f\big)$.

We recall the coderivative operator δ is the operator sending the 1-form $\omega = \sum_{i=1}^n \omega_i dx^i \in W^{1,\infty}(\mathcal{M}; T^*\mathcal{M})$ to the function $\delta\omega$ given in local coordinates by

$$\delta\omega = |g|^{-\frac{1}{2}} \sum_{j,k=1}^n \partial_{x^j}\big(|g|^{\frac{1}{2}} g^{jk}\omega_k\big). \tag{2.1}$$

We recall also the definition of a simple manifold. Let D be the Levi-Civita connection on $(\mathcal{M}, g)$. For $x \in \partial\mathcal{M}$ we consider the second quadratic form of the boundary

$$\Pi(\theta,\theta) = \langle D_\theta \nu, \theta\rangle_{g(x)}, \ \theta \in T_x\partial\mathcal{M}.$$

We say that $\partial\mathcal{M}$ is strictly convex if the form Π is positive-definite for every $x \in \partial\mathcal{M}$.

Definition 1. *We say that $(\mathcal{M}, g)$ is simple if $\partial\mathcal{M}$ is strictly convex, $\mathcal{M}$ is simply connected, and for any $x \in \mathcal{M}$ the exponential map $\exp_x : \exp_x^{-1}(\mathcal{M}) \to \mathcal{M}$ is a diffeomorphism.*

We write $\gamma_{x,\theta}$ for the unique geodesic in $\mathcal{M}$ with initial point $x \in \mathcal{M}$ and initial direction $\theta \in T_x\mathcal{M}$. We define the sphere bundle of $\mathcal{M}$ by

$$S\mathcal{M} = \{(x,\theta) \in T\mathcal{M} : |\theta|_g = 1\},$$

and likewise the submanifold of inner vectors $\partial_+ S\mathcal{M}$ by

$$\partial_+ S\mathcal{M} = \{(x,\theta) \in S\mathcal{M},\ x \in \partial\mathcal{M},\ \langle \theta, \nu(x)\rangle_g (x) < 0\}.$$

Given that $\mathcal{M}$ is assumed to be simple, we can also define $\tau_+(x,\theta)$ to be the maximal time of existence in M of the geodesic $\gamma_{x,\theta}$ for $x \in \partial\mathcal{M}$, that is

$$\tau_+(x,\theta) = \min\{s > 0 : \gamma_{x,\theta}(s) \in \partial\mathcal{M}\} \text{ for } (x,\theta \in \partial_+ S\mathcal{M}).$$

We also introduce here the geodesic ray transforms on a simple Riemannian manifold $\mathcal{M}$.

Definition 2. *The geodesic ray transform for* 1*-forms is the linear operator* $I_1 : \mathcal{C}^\infty(\mathcal{M}; T^*\mathcal{M}) \to \mathcal{C}^\infty(\partial_+ S\mathcal{M})$ *which is defined by*

$$I_1\omega(x,\theta) = \int_0^{\tau_+(x,\theta)} \omega(\gamma_{x,\theta}(s))\gamma'_{x,\theta}(s)ds, \quad (x,\theta) \in \partial_+ S\mathcal{M},\ \omega \in \mathcal{C}^\infty(\mathcal{M}; T^*\mathcal{M}). \tag{2.2}$$

Definition 3. *The geodesic ray transform for functions is the linear operator* $I_0 : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\partial_+ S\mathcal{M})$ *which is given by*

$$I_0 f(x,\theta) = \int_0^{\tau_+(x,\theta)} f(\gamma_{x,\theta}(s))ds, \quad (x,\theta) \in \partial_+ S\mathcal{M},\ f \in \mathcal{C}^\infty(\mathcal{M}). \tag{2.3}$$

3. Geometric Optics Solutions

We now seek to construct GO solutions of the magnetic Schrödinger equation in $(0,T) \times \mathcal{M}$. We fix $A_j \in W^{6,\infty}((0,T) \times \mathcal{M}; T^*\mathcal{M})$, $q_j \in W^{4,\infty}((0,T) \times \mathcal{M})$ and assume that

$$\partial_x^\alpha A_1(t,x) = \partial_x^\alpha A_2(t,x), \quad (t,x) \in (0,T) \times \partial\mathcal{M}, \quad \alpha \in \mathbb{N}^n,\ |\alpha| \le 5. \tag{3.1}$$

We consider the equations

$$\begin{aligned} i\partial_t u_j + \Delta_{g,A_j(t)} u_j + q_j u_j &= 0 \text{ in } (0,T) \times \mathcal{M}, \\ u_1(0,\cdot) = u_2(T,\cdot) &= 0 \text{ in } \mathcal{M}. \end{aligned} \tag{3.2}$$

We seek to find, for $\lambda > 1$, $j = 1,2$, solutions $u_j \in H^{1,2}((0,T) \times \mathcal{M})$ of (3.2) of the form

$$u_j(t,x) = \Big(a_j(t,x) + \frac{b_j(t,x)}{\lambda}\Big) e^{i\lambda(\psi(x) - \lambda t)} + R_{j,\lambda}(t,x). \tag{3.3}$$

In (3.3) above, ψ, a_j, b_j satisfy the following eikonal and transport equations:

$$|\nabla_g \psi|_g^2 = 1, \tag{3.4}$$

$$2i\langle \nabla_g\psi, \nabla_g a_j\rangle_g + i(\Delta_g\psi)a_j - 2(A_j\nabla_g\psi)a_j = 0, \tag{3.5}$$

$$2i\langle \nabla_g\psi, \nabla_g a_j\rangle_g + i(\Delta_g\psi)b_j - 2(A_j\nabla_g\psi)b_j = -(i\partial_t + \Delta_{g,A_j(t)} + q_j)a_j. \tag{3.6}$$

Taken together, equations (3.4) - (3.6) yield

$$(i\partial_t + \Delta_{g,A(t)} + q_j)\Big[e^{i\lambda(\psi(x)-\lambda t)}\Big(a_j(t,x) + \frac{b_j(t,x)}{\lambda}\Big)\Big] = e^{i\lambda(\psi(x)-\lambda t)}\frac{(i\partial_t + \Delta_{g,A(t)} + q_j)b_j(t,x)}{\lambda}.$$

We also assume that there exists $\tau \in \big(0, \frac{T}{4}\big)$ such that a_j, b_j are supported in $[\tau, T-\tau] \times \mathcal{M}$ and further assume that $a_j, b_j \in H^3((0,T) \times \mathcal{M})$, whence $(i\partial_t + \Delta_{g,A_j(t)} + q_j)b_j \in H^1(0,T; L^2(\mathcal{M}))$. Thus we can choose $R_{j,\lambda}$ solving

$$\begin{aligned} (i\partial_t + \Delta_{g,A_j(t)} + q_j)R_{j,\lambda} &= -e^{i\lambda(\psi(x)-\lambda t)}\frac{(i\partial_t + \Delta_{g,A_j(t)} + q_j)b_j}{\lambda} && \text{in } (0,T) \times \mathcal{M}, \\ R_{1,\lambda}(0,\cdot) = R_{2,\lambda}(T,\cdot) &= 0 && \text{in } \mathcal{M}, \\ R_{j,\lambda}(t,x) &= 0 && \text{on } (0,T) \times \partial\mathcal{M}. \end{aligned} \tag{3.7}$$

Since $(\mathcal{M}, g)$ is simple, the eikonal equation (3.4) can be solved globally on $\mathcal{M}$. To see this, we first extend the simple manifold $(\mathcal{M}, g)$ to a simple, compact manifold $(\mathcal{M}_1, g)$ with $\mathcal{M}$ contained in the interior of $\mathcal{M}_1$. We pick $y \in \partial\mathcal{M}_1$ and consider polar normal coordinates (r, θ) on $\mathcal{M}_1$ given by $x = \exp_y(r\theta)$ for $r > 0$ and $\theta \in S_y\mathcal{M}_1 = \{v \in T_y\mathcal{M}_1 : |v|_{g(y)} = 1\}$. Letting $\nu(y)$ denote the outward unit normal to $\partial\mathcal{M}_1$ with respect to the metric g, we define $\partial_+ S_y\mathcal{M}_1 = \{\theta \in S_y\mathcal{M}_1 : \langle\theta, \nu(y)\rangle_{g(y)} < 0\}$. According to the Gauss Lemma (see e.g. [15, Chapter 9, Lemma 15]), in these coordinates the metric takes the form $g(r, \theta) = dr^2 + g_0(r, \theta)$ with $g_0(r, \theta)$ a metric on $\{\theta \in S_y\mathcal{M}_1 : \langle\nu(y), \theta\rangle_{g(y)} \leq 0\}$ depending smoothly on r. In polar normal coordinates $dV_g = \mu(r, \theta)^{\frac{1}{2}} dr d\theta$, where $\mu = \det g_0$ and $d\theta$ is the usual spherical volume form on $\partial_+ S_y\mathcal{M}$. For a function $f \in L^1(\mathcal{M})$ extended by zero to $\mathcal{M}_1$, we can extend dV_g to a volume form on $T_y(\mathcal{M}_1)$ and get

$$\int_{\mathcal{M}} f(x) dV_g(x) = \int_0^\infty \int_{\partial_+ S_y\mathcal{M}_1} f(r, \theta)\mu(r, \theta)^{\frac{1}{2}} dr d\theta.$$

We choose

$$\psi(x) = dist_g(y, x) \tag{3.8}$$

where $dist_g$ denotes the Riemannian distance function. Since $\psi(r, \theta) = r$, we can easily check that ψ solves the eikonal equation (3.4).

We now look towards solving the transport equations (3.5)-(3.6). First, note that

$$\nabla_g \psi(r, \theta) = \partial_r = \gamma'_{y,\theta}(r) = \theta. \tag{3.9}$$

Therefore, we rewrite the transport equations (3.5)-(3.6) in polar normal coordinates based at $y \in \partial\mathcal{M}_1$ to obtain

$$\partial_r a_j + \Big(\frac{\partial_r \mu}{4\mu}\Big) a_j + i\big(A_j\theta\big) a_j = 0, \tag{3.10}$$

$$\partial_r b_j + \Big(\frac{\partial_r \mu}{4\mu}\Big) b_j + i(A_j\theta) b_j = \beta_j(t, r, \theta), \tag{3.11}$$

where $A_j\theta$ denotes $A_j(t, r, \theta)\theta$ and β_j denotes $(i\partial_t + \Delta_{g,A_j(t)} + q_j)a_j/2$.

Applying [17, Section 3, Theorem 5], we find $\tilde{A}_1 \in W^{6,\infty}((0, T) \times \mathcal{M}_1; T^*\mathcal{M}_1)$ such that for $t \in (0, T)$ the support of $\tilde{A}_1(t, \cdot)$ is contained in the interior of $\mathcal{M}_1$, and we have $\tilde{A}_1 = A_1$ on $(0, T) \times \mathcal{M}$ and $\|\tilde{A}_1\|_{W^{6,\infty}((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \leq C \|A_1\|_{W^{6,\infty}((0,T)\times\mathcal{M};T^*\mathcal{M})}$, where C depends only on $\mathcal{M}$. Then for all $t \in (0, T)$ we put:

$$\tilde{A}_2(t, x) = \begin{cases} A_2(t, x), & \text{if } x \in \mathcal{M}, \\ \tilde{A}_1(t, x), & \text{if } x \in \mathcal{M}_1 \setminus \mathcal{M}. \end{cases}$$

Then according to (3.1), $\tilde{A}_2 \in W^{6,\infty}((0, T) \times \mathcal{M}_1; T^*\mathcal{M}_1)$ and

$$\max_{j=1,2} \left\|\tilde{A}_j\right\|_{W^{6,\infty}((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \leq C \max_{j=1,2} \|A_j\|_{W^{6,\infty}((0,T)\times\mathcal{M};T^*\mathcal{M})}.$$

Similarly, for $j = 1, 2$, we consider $\tilde{q}_j \in W^{4,\infty}((0, T) \times \mathcal{M}_1)$ such that for $t \in (0, T)$ the support of $\tilde{q}_j(t, \cdot)$ is contained in the interior of $\mathcal{M}_1$, and we have $\tilde{q}_j = q_j$ on $(0, T) \times \mathcal{M}$ and $\|\tilde{q}_j\|_{W^{4,\infty}((0,T)\times\mathcal{M}_1)} \leq C \|q_j\|_{W^{4,\infty}((0,T)\times\mathcal{M})}$. Note that here we do not impose that $\tilde{q}_1$ and $\tilde{q}_2$ should coincide on $(0, T) \times (\mathcal{M}_1 \setminus \mathcal{M})$.

For any $h \in H^5((0, T) \times \partial_+ S_y\mathcal{M}_1)$, the functions

$$a_1(t, r, \theta) = \chi(t)h(t, \theta)\mu(r, \theta)^{-\frac{1}{4}} \exp\Big(i \int_0^{+\infty} \tilde{A}_1(t, r + s, \theta)\theta ds\Big), \tag{3.12}$$

$$a_2(t, r, \theta) = \chi(t)\mu(r, \theta)^{-\frac{1}{4}} \exp\Big(i \int_0^{+\infty} \tilde{A}_2(t, r + s, \theta)\theta ds\Big), \tag{3.13}$$

are solutions to the transport equations (3.10). In the same way, for $\tilde{\beta}_j = (i\partial_t + \Delta_{g,\tilde{A}_j(t)} + \tilde{q}_j)a_j/2$, we fix

$$b_j(t, r, \theta) = \mu(r, \theta)^{-\frac{1}{4}} \int_0^r \Big[\exp\Big(-i \int_{s_2}^r \tilde{A}_j(t, s_1, \theta)\theta ds_1\Big)\beta_j(t, s_2, \theta)\mu^{\frac{1}{4}}(s_2, \theta)\Big] ds_2 \tag{3.14}$$

which is a solution of (3.11). Here we fix $\chi \in C_0^\infty((\tau, T-\tau))$ satisfying $\chi = 1$ on $[2\tau, T-2\tau]$, $0 \le \chi \le 1$ and $\|\chi\|_{W^{k,\infty}(\mathbb{R})} \le C_k \tau^{-k}$ with C_k independent of τ.

Let us now consider the remainder terms $R_{j,\lambda}$, $j = 1, 2$. In view of (3.12)-(3.14), we deduce the following bounds:

$$\|a_1\|_{H^3((0,T)\times\mathcal{M})} \le C \|h\|_{H^3((0,T)\times\partial_+ S_y\mathcal{M}_1)} \tau^{-3}, \quad \|b_1\|_{H^3((0,T)\times\mathcal{M})} \le C \|h\|_{H^5((0,T)\times\partial_+ S_y\mathcal{M}_1)} \tau^{-4}, \tag{3.15}$$

$$\left\|(i\partial_t + \Delta_{g,A_1(t)} + q_1)b_1\right\|_{L^2((0,T)\times\mathcal{M})} \le C \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M}_1)} \tau^{-2}, \tag{3.16}$$

$$\|a_2\|_{H^3((0,T)\times\mathcal{M})} \le C\tau^{-3}, \quad \|b_2\|_{H^3((0,T)\times\mathcal{M})} \le C\tau^{-4}, \quad \left\|(i\partial_t + \Delta_{g,A_2(t)} + q_2)b_2\right\|_{L^2((0,T)\times\mathcal{M})} \le C\tau^{-2}, \tag{3.17}$$

where C depends only on $\mathcal{M}$, T and $\|A_1\|_{W^{5,\infty}((0,T)\times\mathcal{M}} + \|A_2\|_{W^{5,\infty}((0,T)\times\mathcal{M}}$. Then applying [10, Lemma 2.1], we see that problem (3.7) admits unique solutions $R_{j,\lambda}$ for $j = 1, 2$ with $R_{j,\lambda} \in C([0,T]; H_0^1(\mathcal{M}) \cap H^2(\mathcal{M})) \cap \mathcal{C}^1([0,T]; L^2(\mathcal{M}))$. On the other hand, from the a priori estimate [11, (10.10), page 324], we deduce that

$$\|R_{1,\lambda}\|_{L^2((0,T)\times\mathcal{M})} \le C \frac{\left\|(i\partial_t + \Delta_{A_1(t)} + q_1)b_1\right\|_{L^2((0,T)\times\mathcal{M})}}{\lambda} \le C \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M}_1)} \tau^{-2}\lambda^{-1}. \tag{3.18}$$

Moreover, applying [10, Lemma 2.1] we find that

$$\|R_{1,\lambda}\|_{L^2(0,T;H^2(\mathcal{M}))} \le C \frac{\left\|e^{i\lambda(\psi(x)-\lambda t)}(i\partial_t + \Delta_{A_1(t)} + q_1)b_1\right\|_{H^1(0,T;L^2(\mathcal{M}))}}{\lambda} \le C \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M}_1)} \tau^{-3}\lambda,$$

and by interpolation between this estimate and (3.18) we deduce

$$\|R_{1,\lambda}\|_{L^2(0,T;H^1(\mathcal{M}))} \le C \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M}_1)} \tau^{-3}.$$

Combining this with (3.18) we obtain

$$\|R_{1,\lambda}\|_{L^2(0,T;H^1(\mathcal{M}))} + \lambda \|R_{1,\lambda}\|_{L^2((0,T)\times\mathcal{M})} \le C \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M}_1)} \tau^{-3}. \tag{3.19}$$

In a similar manner, we derive the estimate

$$\|R_{2,\lambda}\|_{L^2(0,T;H^1(\mathcal{M}))} + \lambda \|R_{2,\lambda}\|_{L^2((0,T)\times\mathcal{M})} \le C\tau^{-3}. \tag{3.20}$$

This completes our construction of the geometric optics solutions of (3.2).

4. Unique Determination of the Potentials Modulo Gauge Invariance

We recall that any 1-form $\omega \in W^{1,p}(\mathcal{M}; T^*\mathcal{M})$, with $p \in [2, \infty)$ admits a Hodge decomposition via $\omega = \omega^{sol} + d\phi$, where $\omega^{sol} \in W^{1,p}(\mathcal{M}; T^*\mathcal{M})$ is the solenoidal part of ω which satisfies $\delta\omega^{sol} = 0$ (see (2.1) for the definition of coderivative operator δ) and $\phi \in W^{2,p}(\mathcal{M}) \cap H_0^1(\mathcal{M})$. Let us first prove an extension of this Hodge decomposition for the 1-form $A \in W^{6,\infty}((0,T) \times \mathcal{M}; T^*\mathcal{M})$ given by the following:

Lemma 1. *Let $A \in W^{6,\infty}((0,T) \times \mathcal{M}; T^*\mathcal{M})$. Then we can decompose A into*

$$A = A^{sol} + d\phi, \tag{4.1}$$

where, for any $p \in (2, \infty)$, $A^{sol} \in W^{5,\infty}((0,T)\times\mathcal{M}; T^*\mathcal{M})$, and $\phi \in L^\infty(0,T; W^{7,p}(\mathcal{M})) \cap W^{5,\infty}(0,T; L^\infty(\mathcal{M}))$, we have $\phi|_{(0,T)\times\partial\mathcal{M}} = 0$ and $\delta A^{sol} = 0$.

Proof. We fix ϕ to be the solution for all $t \in [0,T]$ of the boundary value problem

$$\begin{aligned} -\Delta_g \phi(t,\cdot) &= -\delta A(t,\cdot) \quad &\text{in } \mathcal{M}, \\ \phi(t,\cdot) &= 0 &\text{on } \partial\mathcal{M}. \end{aligned}$$

Since $\delta A(t,\cdot) \in W^{5,\infty}(\mathcal{M})$, according to [9, Theorem 2.5.1.1], this problem admits a unique solution $\phi(t,\cdot) \in \bigcap_{p\in[2,\infty)} W^{7,p}(\mathcal{M})$. Moreover, since $\delta A \in L^\infty(0,T; W^{5,\infty}(\mathcal{M}))$, we also deduce that $\phi \in \bigcap_{p\in[2,\infty)} L^\infty(0,T; W^{7,p}(\mathcal{M}))$. In the same way, using the fact that $\delta A \in W^{5,\infty}(0,T; L^\infty(\mathcal{M}))$, we prove that $\phi \in \bigcap_{p\in[2,\infty)} W^{5,\infty}(0,T; W^{2,p}(\mathcal{M}))$. We then use the Sobolev embedding theorem to deduce that $\phi \in W^{5,\infty}(0,T; L^\infty(\mathcal{M}))$. We fix $A^{sol} = A - d\phi$

and by the Sobolev embedding theorem, deduce that $A^{sol} \in W^{5,\infty}((0,T)\times\mathcal{M};T^*\mathcal{M})$. Moreover, we see that

$$\delta A^{sol} = \delta A - \delta d\phi = \delta A - \Delta_g \phi = 0.$$

Thus (4.1) is the Hodge decomposition of A and the proof of the lemma is complete. □

We start by considering the implication

$$\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2} \Rightarrow A^{sol} = 0,$$

where A^{sol} is the solenoidal part of the Hodge decomposition (4.1) of A. For this purpose, we establish the following intermediate result.

Lemma 2. *Let $A_1, A_2 \in W^{6,\infty}((0,T)\times\mathcal{M};T^*\mathcal{M})$ satisfy the matching condition (1.3), and fix $A = A_1 - A_2$ extended by 0 on $(0,T)\times(\mathcal{M}_1\setminus\mathcal{M})$. In particular, for $\tilde{A}_j$ the extension of A_j to $(0,T)\times\mathcal{M}_1$ introduced in the previous section, we have $A = \tilde{A}_1 - \tilde{A}_2$. Assuming these conditions are fulfilled, we find that*

$$\begin{aligned}\left|\int_0^T\int_0^\infty\int_{\partial_+S_y\mathcal{M}_1} i(A(r,\theta)\theta)\chi^2(t)h(t,\theta)\exp\Big(i\int_0^\infty A(t,r+s,\theta)\theta ds\Big)d\theta drdt\right| \\ \leq C\big[\,\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\,\lambda^5\tau^{-8}\,\|h\|_{H^5((0,T)\times\partial_+S_y\mathcal{M}_1)} + \|h\|_{H^4((0,T)\times\partial_+S_y\mathcal{M}_1)}\,\tau^{-6}\lambda^{-1}\big].\end{aligned} \tag{4.2}$$

Proof. We fix u_j, $j=1,2$ the solutions for $j=1,2$ respectively of (3.2) taking the form (3.3). We write also $\psi_{j,\lambda} = u_j - R_{j,\lambda}$. We consider $v \in H^{1,2}((0,T)\times\mathcal{M})$ solving

$$\begin{aligned} &i\partial_t v + \Delta_{g,A_2(t)}v + q_2 v = 0 && \text{in } (0,T)\times\mathcal{M},\\ &v(0,\cdot) = 0 && \text{in } \mathcal{M},\\ &v = \psi_{1,\lambda} && \text{on } (0,T)\times\partial\mathcal{M},\end{aligned}$$

and consider $w = v - u_1$ which solves

$$\begin{aligned} &i\partial_t w + \Delta_{g,A_2(t)}w + q_2 w = 2iA\nabla_g u_1 + Vu_1 && \text{in } (0,T)\times\mathcal{M},\\ &w(0,\cdot) = 0 && \text{in } \mathcal{M},\\ &w = 0 && \text{on } (0,T)\times\partial\mathcal{M},\end{aligned}$$

where $V = i\delta A + |A_2|_g^2 - |A_1|_g^2 + q_1 - q_2$. Multiplying this equation by $\overline{u_2}$ and integrating by parts yields

$$\int_0^T\int_{\mathcal{M}}(2iA\nabla_g u_1 + Vu_1)\overline{u_2}dV_g(x)dt = \int_0^T\int_{\partial\mathcal{M}}\partial_\nu w\overline{u_2}d\sigma_g dt. \tag{4.3}$$

Moreover,

$$\left|\int_\Sigma \partial_\nu w\overline{u_2}d\sigma_g dt\right| \leq \|(\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2})\psi_{1,\lambda}\|_{L^2((0,T)\times\partial\mathcal{M})}\,\|\psi_{2,\lambda}\|_{L^2((0,T)\times\partial\mathcal{M})},$$

and (3.15)-(3.16) imply

$$\begin{aligned}\left|\int_{(0,T)\times\partial\mathcal{M}}\partial_\nu w\overline{u_2}d\sigma_g dt\right| &\leq C\,\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\,\|\psi_{1,\lambda}\|_{H^{\frac{9}{4},\frac{3}{2}}((0,T)\times\mathcal{M})}\,\|\psi_{2,\lambda}\|_{L^2((0,T)\times\mathcal{M})}\\ &\leq C\,\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\,\lambda^6\,\|h\|_{H^5((0,T)\times\partial_+S_y\mathcal{M}_1)}\,\tau^{-8}.\end{aligned} \tag{4.4}$$

Here C is a generic constant which depends only on $\mathcal{M}$, T and $\|A_1\|_{W^{5,\infty}((0,T)\times\mathcal{M})}+\|A_2\|_{W^{5,\infty}((0,T)\times\mathcal{M})}$. On the other hand, we have that

$$\begin{aligned}
&\int_0^T\int_{\mathcal{M}}(2iA\nabla_g u_1+Vu_1)\overline{u_2}dV_g(x)dt=\\
&=\lambda\int_{(0,T)\times\mathcal{M}}2i(A\nabla_g\psi)a_1\overline{a_2}dV_g(x)dt+\lambda\int_{(0,T)\times\mathcal{M}}2i(A\nabla_g\psi)a_1\Big(\frac{\overline{b_2}}{\lambda}+e^{i\lambda(\psi(x)-\lambda t)}\overline{R_{2,\lambda}}\Big)dV_g(x)dt\\
&\quad+\lambda\int_{(0,T)\times\mathcal{M}}2i(A\nabla_g\psi)\Big(\frac{b_1}{\lambda}+e^{-i\lambda(\psi(x)-\lambda t)}R_{1,\lambda}\Big)\overline{a_2}dV_g(x)dt\\
&\quad+\lambda\int_{(0,T)\times\mathcal{M}}2i(A\nabla_g\psi)\Big(\frac{b_1}{\lambda}+e^{-i\lambda(\psi(x)-\lambda t)}R_{1,\lambda}\Big)\Big(\frac{\overline{b_2}}{\lambda}+e^{i\lambda(\psi(x)-\lambda t)}\overline{R_{2,\lambda}}\Big)dV_g(x)dt\\
&\quad+\int_{(0,T)\times\mathcal{M}}\Big(2ie^{i\lambda(\psi(x)-\lambda t)}A\Big(\nabla_g a_1+\frac{\nabla_g b_1}{\lambda}+\nabla_g R_{1,\lambda}\Big)+Vu_1\Big)\overline{u_2}dV_g(x)dt.
\end{aligned}\tag{4.5}$$

We then divide (4.5) by λ and apply (3.19)-(3.20) to obtain

$$\begin{aligned}
&\left|\int_{(0,T)\times\mathcal{M}}i(A\nabla_g\psi)a_1\overline{a_2}dV_g(x)dt\right|\\
&\le\lambda^{-1}\left|\int_{(0,T)\times\mathcal{M}}(2iA\nabla_g u_1+Vu_1)\overline{u_2}dV_g(x)dt\right|+C\,\|h\|_{H^4((0,T)\times\partial_+S_y\mathcal{M}_1)}\,\tau^{-6}\lambda^{-1}.
\end{aligned}$$

Using polar normal coordinates in the left hand side of the above gives us

$$\begin{aligned}
&\left|\int_0^T\int_0^\infty\int_{\partial_+S_y\mathcal{M}_1}i(A(t,r,\theta)\theta)\chi^2(t)h(t,\theta)\mu(r,\theta)^{-\frac12}\exp\Big(i\int_0^\infty A(t,r+s,\theta)\theta ds\Big)dV_g(r,\theta)dt\right|\\
&\le\lambda^{-1}\left|\int_{(0,T)\times\mathcal{M}}(2iA\nabla_g u_1+Vu_1)\overline{u_2}dV_g(x)dt\right|+C\,\|h\|_{H^4((0,T)\times\partial_+S_y\mathcal{M}_1}\,\tau^{-6}\lambda^{-1}.
\end{aligned}$$

Using now the fact that $\mu(r,\theta)^{-\frac12}dV_g(r,\theta)=drd\theta$, we conclude that

$$\begin{aligned}
&\left|\int_0^T\int_0^\infty\int_{\partial_+S_y\mathcal{M}_1}i(A(t,r,\theta)\theta)\chi^2(t)h(t,\theta)\exp\Big(i\int_0^\infty A(t,r+s,\theta)\theta ds\Big)d\theta drdt\right|\\
&\le\lambda^{-1}\left|\int_{(0,T)\times\mathcal{M}}(2iA\nabla_g u_1+Vu_1)\overline{u_2}\right|dV_g(x)dt+C\,\|h\|_{H^4((0,T)\times\partial_+S_y\mathcal{M}_1)}\,\tau^{-6}\lambda^{-1}.
\end{aligned}$$

We use this last estimate together with (4.3) and (4.4) to obtain (4.2). □

Armed with the above, we are now in a position to complete the proof of the uniqueness result.

Proof of Theorem 1. Let us assume that $\Lambda_{A_1,q_1}=\Lambda_{A_2,q_2}$, and begin by proving that this condition implies that $A^{sol}=0$. We recall also Definition 2 of I_1, the geodesic ray transform for 1-forms given by (2.2). According to s-injectivity of the transform I_1 (consult e.g. [1] or [16, Theorem 4]), it is enough to show that $I_1A(t,\cdot)=0$. Then, sending $\lambda\to\infty$ in (4.2) we obtain

$$\int_0^T\int_0^\infty\int_{\partial_+S_y\mathcal{M}_1}i(A(t,r,\theta)\theta)\chi^2(t)h(t,\theta)\exp\Big(i\int_0^\infty A(t,r+s,\theta)\theta ds\Big)d\theta drdt=0.\tag{4.6}$$

On the other hand, notice that, due to (3.9), for $A=\sum_{j=1}^n a_jdx^j$ we have

$$\begin{aligned}
\int_0^\infty A(t,r,\theta)\theta dr&=\int_0^{\tau_+(y,\theta)}A(t,r,\theta)\theta dr\\
&=\int_0^{\tau_+(y,\theta)}A(t,\gamma_{y,\theta}(s))\gamma'_{y,\theta}(s)ds=I_1[A(t,\cdot)](y,\theta).
\end{aligned}$$

Thus we deduce that

$$\begin{aligned}\int_0^\infty i(A(t,r,\theta)\theta)\exp\Big(i\int_0^\infty A(t,r+s,\theta)\theta ds\Big)dr &= \int_0^\infty i(A(t,r,\theta)\theta)\exp\Big(i\int_r^\infty A(t,s,\theta)\theta ds\Big)dr\\ &= -\int_0^\infty \partial_r \exp\Big(i\int_r^\infty A(t,s,\theta)\theta ds\Big)dr\\ &= \exp\Big(i\int_0^\infty A(t,s,\theta)\theta ds\Big)-1 = e^{iI_1[A(t,\cdot)](y,\theta)}-1.\end{aligned}$$

Using this identity in (4.6) and applying Fubini's theorem, we get

$$\int_0^T\int_{\partial_+S_y\mathcal{M}_1}\chi^2(t)\big[e^{iI_1[A(t,\cdot)](y,\theta)}-1\big]h(t,\theta)d\theta dt = 0.$$

Since $h\in C_0^\infty((0,T)\times\partial_+S\mathcal{M}_1)$ is arbitrary, we deduce that

$$\chi^2(t)\big[e^{iI_1[A(t,\cdot)](y,\theta)}-1\big]=0,\quad t\in(0,T),\ (y,\theta)\in\partial_+S\mathcal{M}_1.$$

But since $\tau\in(0,\frac{T}{4})$ is arbitrary and $\chi(t)=1$ for $t\in[2\tau,T-2\tau]$, we see that

$$e^{iI_1[A(t,\cdot)](y,\theta)}=1,\quad t\in[0,T],\ (y,\theta)\in\partial_+S\mathcal{M}_1,$$

and hence deduce that for all $t\in[0,T]$, $I_1[A(t,\cdot)](y,\theta)\in 2\pi\mathbb{Z}$. Since $A\in W^{6,\infty}((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)$ one can check that $I_1A\in C([0,T]\times\partial_+S\mathcal{M}_1)$. Then since for all $y\in\partial\mathcal{M}_1$ it holds that $\partial_+S_y\mathcal{M}_1$ is connected, we conclude that the map $[0,T]\times\partial_+S_y\mathcal{M}_1\ni(t,\theta)\mapsto I_1[A(t,\cdot)](y,\theta)$ is constant. On the other hand, note that $A=0$ on $\mathcal{M}_1\setminus\mathcal{M}$, so that for any $y\in\partial\mathcal{M}_1$ there exists $\theta\in\partial_+S_y\mathcal{M}_1$ such that for all $t\in[0,T]$ we have $I_1[A(t,\cdot)](y,\theta)=0$. Therefore we conclude that $A^{sol}=0$.

We can then use the Hodge decomposition (4.1), to deduce the existence of $\phi\in W^{5,\infty}((0,T)\times\mathcal{M})$ satisfying $\phi|_{(0,T)\times\partial\mathcal{M}}=0$ such that $A_2=A_1+d\phi$. Thus the proof will be completed if we show that $q_2=q_1-\partial_t\phi$. Since $A_2=A_1+d\phi$ we can put $q_3=q_1-\partial_t\phi$ and by gauge invariance we have $\Lambda_{A_1,q_1}=\Lambda_{A_2,q_3}$. Thus, by assumption it follows that

$$\Lambda_{A_2,q_3}=\Lambda_{A_1,q_1}=\Lambda_{A_2,q_2}. \tag{4.7}$$

Therefore, the proof will be complete if we prove that condition (4.7) implies that $q_3=q_2$. For this purpose, we let $y\in\partial\mathcal{M}_1$, $h\in C_0^\infty((0,T)\times\partial_+S_y\mathcal{M}_1)$. We consider u_2 the solution of (3.2) for $j=2$ taking the form (3.3), and u_1 the solution of (3.2) but with A_j replaced by A_2 and q_j replaced by q_3, again taking the form (3.2). Note that $q_3=q_1-\partial_t\phi\in W^{4,\infty}((0,T)\times\mathcal{M})$, so this construction is still valid. In particular, taking $A_1=A_2$ in (4.3) we obtain

$$\int_0^T\int_{\mathcal{M}}(q_3-q_2)u_1\overline{u_2}dV_g(x)dt=\int_0^T\int_{\partial\mathcal{M}}\big[(\Lambda_{A_2,q_3}-\Lambda_{A_2,q_2})\psi_{1,\lambda}\big]\overline{u_2}d\sigma_g dt=0.$$

Fixing $q=q_3-q_2$ extended by 0 on $(0,T)\times(\mathcal{M}_1\setminus\mathcal{M})$, we get

$$\begin{aligned}\int_0^T\int_{\mathcal{M}}qu_1\overline{u_2}dV_g(x)dt &= \int_{(0,T)\times\mathcal{M}}qa_1\overline{a_2}dV_g(x)dt+\int_{(0,T)\times\mathcal{M}}qa_1\Big(\frac{\overline{b_2}}{\lambda}+e^{i\lambda(\psi(x)-\lambda t)}\overline{R_{2,\lambda}}\Big)dV_g(x)dt\\ &+\int_{(0,T)\times\mathcal{M}}q\Big(\frac{b_1}{\lambda}+e^{-i\lambda(\psi(x)-\lambda t)}R_{1,\lambda}\Big)\overline{a_2}dV_g(x)dt\\ &+\int_{(0,T)\times\mathcal{M}}q\Big(\frac{b_1}{\lambda}+e^{-i\lambda(\psi(x)-\lambda t)}R_{1,\lambda}\Big)\Big(\frac{\overline{b_2}}{\lambda}+e^{i\lambda(\psi(x)-\lambda t)}\overline{R_{2,\lambda}}\Big)dV_g(x)dt.\end{aligned}$$

Then, we argue similarly to the proof of Lemma 2. Using polar normal coordinates and (3.19)-(3.20) we get

$$\left|\int_0^T\int_0^\infty\int_{\partial_+S_y\mathcal{M}_1}\chi^2(t)q(t,r,\theta)\overline{h(t,\theta)}d\theta drdt\right|\le C\,\|h\|_{H^4((0,T)\times\partial_+S_y\mathcal{M}_1)}\,\tau^{-6}\lambda^{-1}.$$

And we send $\lambda\to\infty$ to obtain

$$\int_0^T\int_0^\infty\int_{\partial_+S_y\mathcal{M}_1}\chi^2(t)q(t,r,\theta)\overline{h(t,\theta)}d\theta drdt=0. \tag{4.8}$$

Let us recall the definition of the geodesic ray transform I_0 acting on functions, given by (2.3). In light of (4.8), we allow $y \in \partial\mathcal{M}$ and $h \in C_0^\infty((0,T)\times\partial_+S_y\mathcal{M}_1)$ to be arbitrary, whence we deduce that

$$\chi^2(t)I_0[q(t,\cdot)](y,\theta) = \int_0^{\tau_+(y,\theta)} \chi^2(t)q(t,r,\theta)dr = 0, \quad t\in(0,T),\ (y,\theta)\in\partial_+S\mathcal{M}_1.$$

Now, since $\tau \in (0,\frac{T}{4})$ is arbitrary and $\chi = 1$ on $[2\tau, T-2\tau]$, we conclude that $I_0[q(t,\cdot)] = 0$ for all $t\in(0,T)$. Then by injectivity of I_0 on $L^2(\mathcal{M})$ (e.g. [16, Theorem 3]) implies that $q = 0$, whence $q_2 = q_3 = q_1 - \partial_t\phi$. This completes the proof of Theorem 1. □

5. Stable Determination of the Magnetic Potential

In this section we establish the stability estimate in the recovery of the magnetic potential stated in Theorem 2. For $j = 1,2$, we assume that $A_j \in W^{6,\infty}((0,T)\times\mathcal{M};T^*\mathcal{M}) \cap H^{3n+4}((0,T)\times\mathcal{M};T^*\mathcal{M})$ fulfill (1.4). Then, for $A = A_1 - A_2$ extended by 0 on $(0,T)\times(\mathcal{M}_1\setminus\mathcal{M})$ we have $A \in W^{6,\infty}((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)\cap H^{3n+4}((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)$. We will also assume for the moment that for some small $\varepsilon > 0$ it holds that

$$\left\|A^{sol}\right\|_{L^2((0,T)\times\mathcal{M}_1)} \le \varepsilon. \tag{5.1}$$

Before proving Theorem 2, let us recall some facts about the geodesic ray transform I_1.
Firstly, according to [14, Theorem 4.2.1], the ray transform for 1-forms extends to a bounded linear operator $I_1 : H^k(\mathcal{M}_1;T^*\mathcal{M}_1) \to H^k(\partial_+S\mathcal{M}_1)$. Fixing $w(x,\theta) = \left|\langle\theta,\nu(x)\rangle_g\right|$, we can also extend I_1 to a bounded linear operator $I_1 : L^2(\mathcal{M}_1;T^*\mathcal{M}_1) \to L^2_w(\partial_+S\mathcal{M}_1)$, where $L^2_w(\partial_+S\mathcal{M}_1)$ is the L^2 space with respect to the weighted measure $w(y,\theta)d\theta d\sigma_g(y)$, and thus define $I_1^* : L^2_w(\partial_+S\mathcal{M}_1) \to L^2(\mathcal{M}_1;T^*\mathcal{M}_1)$ as the adjoint of I_1. By condition (1.3) we have $A \in H^5((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)$ with supp $A(t,\cdot)\subset\mathcal{M}$ for $t\in(0,T)$. Moreover, according to [16, Section 8], the operator $I_1^*I_1$, is an elliptic pseudodifferential operator of order -1. Together with condition (1.5), we have for $0\le k\le 5$

$$\|I_1^*I_1A\|_{H^k((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \le C\|A\|_{H^k((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \le CB. \tag{5.2}$$

Also according to [16, Section 8], we can find constants $C_1, C_2 > 0$ such that for $0\le k\le 5$

$$C_1\left\|A^{sol}\right\|_{L^2(0,T;H^k(\mathcal{M}_1))} \le \|I_1^*I_1A\|_{L^2(0,T;H^{k+1}(\mathcal{M}_1))} \le C_2\left\|A^{sol}\right\|_{L^2(0,T;H^k(\mathcal{M}_1))}. \tag{5.3}$$

.

Proof of Theorem 2 subject to (5.1). Following the work of the previous section, we allow $h(t,\theta)$ to depend on $y\in\partial\mathcal{M}_1$. We can rewrite inequality (4.2) in the form

$$\begin{aligned}&\left|\int_0^T\int_{\partial_+S_y\mathcal{M}_1}\left(e^{iI_1[A(t,\cdot)](y,\theta)}-1\right)\chi^2(t)h(t,y,\theta)d\theta dt\right| \\ &\le C\Big(\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\,\lambda^5\tau^{-8}\|h(y,\cdot)\|_{H^5((0,T)\times\partial_+S_y\mathcal{M}_1)} + \lambda^{-1}\tau^{-6}\|h(y,\cdot)\|_{H^4((0,T)\times\partial_+S_y\mathcal{M}_1)}\Big).\end{aligned} \tag{5.4}$$

We can use the Taylor expansion $e^t = 1+t+t^2\int_0^1 e^{st}(1-s)ds$ to see that

$$e^{iI_1[A(t,\cdot)](y,\theta)} - 1 = iI_1[A(t,\cdot)](y,\theta) - I_1[A(t,\cdot)]^2(y,\theta)\int_0^1 e^{isI_1[A(t,\cdot)](y,\theta)}(1-s)ds,$$

and using this identity in (5.4) yields

$$\begin{aligned}&\left|\int_0^T\chi^2(t)\int_{\partial_+S_y\mathcal{M}_1}I_1[A(t,\cdot)](y,\theta)h(t,y,\theta)d\theta dt\right| \le C\Big(\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\,\lambda^5\tau^{-8}\|h(y,\cdot)\|_{H^5((0,T)\times\partial_+S_y\mathcal{M}_1)} \\ &+\lambda^{-1}\tau^{-6}\|h(y,\cdot)\|_{H^4((0,T)\times\partial_+S_y\mathcal{M}_1)} + \|h(y,\cdot)\|_{L^2((0,T)\times\partial_+S_y\mathcal{M}_1)}\|I_1A\|^2_{\mathcal{C}^0([0,T]\times\partial_+S\mathcal{M})}\Big).\end{aligned}$$

Combining this with the fact that

$$I_1A = I_1d\phi + I_1A_{sol} = I_1A_{sol}$$

and the definition of I_1, we deduce that

$$\|I_1A\|_{\mathcal{C}^0([0,T]\times\partial_+S\mathcal{M})} \le C\left\|A^{sol}\right\|_{\mathcal{C}^0([0,T]\times\mathcal{M};T^*\mathcal{M})}.$$

This implies that

(5.5)
$$\left|\int_0^T \chi^2(t)\int_{\partial_+ S_y\mathcal{M}_1} I_1[A(t,\cdot)](y,\theta)h(t,y,\theta)d\theta dt\right| \le C\Big(\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\,\lambda^5\tau^{-8}\,\|h(y,\cdot)\|_{H^5((0,T)\times\partial_+S_y\mathcal{M}_1)} + \lambda^{-1}\tau^{-6}\,\|h(y,\cdot)\|_{H^4((0,T)\times\partial_+S_y\mathcal{M}_1)} + \|h(y,\cdot)\|_{L^2((0,T)\times\partial_+S_y\mathcal{M}_1)}\,\|A^{sol}\|^2_{\mathcal{C}^0([0,T]\times\mathcal{M};T^*\mathcal{M})}\Big).$$

Since I_1 extends to a bounded linear operator $I_1 : H^k(\mathcal{M}_1;T^*\mathcal{M}_1) \to H^k(\partial_+S\mathcal{M}_1)$, we can choose $h(t,y,\theta) = I_1I_1^*I_1[A(t,\cdot)](y,\theta)$ and then integrate (5.5) with respect to the volume form $d\sigma_g$ of $\partial\mathcal{M}_1$. Using the compactness of $\mathcal{M}_1$ we deduce that

(5.6)
$$\int_0^T \chi^2(t)\int_{\mathcal{M}_1} |I_1^*I_1[A(t,\cdot)](x)|^2\,dV_g(x)dt = \int_0^T \chi^2(t)\int_{\partial_+S\mathcal{M}_1} I_1[A(t,\cdot)](y,\theta)h(t,\theta)\left|\langle\theta,\nu(y)\rangle_g\right| d\theta d\sigma_g(y)dt$$
$$\le C\Big(\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\,\lambda^5\tau^{-8}\,\|I_1^*I_1A\|_{H^5((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} + \lambda^{-1}\tau^{-6}\,\|I_1^*I_1A\|_{H^4((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} + \|I_1^*I_1A^{sol}\|_{L^2((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)}\,\|A^{sol}\|^2_{\mathcal{C}^0([0,T]\times\mathcal{M};T^*\mathcal{M})}\Big).$$

Moreover, using (5.2) we can further simplify (5.6) in order to obtain

(5.7)
$$\int_0^T \chi^2(t)\int_{\mathcal{M}_1} |I_1^*I_1[A(t,\cdot)](x)|^2\,dV_g(x)dt \le C\Big(\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\,\lambda^5\tau^{-8} + \lambda^{-1}\tau^{-6} + \|A^{sol}\|_{L^2((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)}\,\|A^{sol}\|^2_{\mathcal{C}^0([0,T]\times\mathcal{M};T^*\mathcal{M})}\Big).$$

Since we also have

(5.8)
$$\left|\int_0^T \chi^2(t)\int_{\mathcal{M}_1} |I_1^*I_1[A(t,\cdot)](x)|^2\,dV_g(x)dt - \int_0^T\int_{\mathcal{M}_1} |I_1^*I_1[A(t,\cdot)](x)|^2\,dV_g(x)dt\right| \le C\Big[\int_0^\tau (1-\chi^2(t))dt + \int_{T-\tau}^T (1-\chi^2(t))dt\Big] \le C\tau,$$

we obtain the estimate

(5.9)
$$\int_0^T\int_{\mathcal{M}_1} |I_1^*I_1[A(t,\cdot)](x)|^2\,dV_g(x)dt \le C\Big(\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\,\lambda^5\tau^{-8} + \lambda^{-1}\tau^{-6} + \tau + \|A^{sol}\|_{L^2((0,T)\times\mathcal{M}_1)}\,\|A^{sol}\|^2_{\mathcal{C}^0([0,T]\times\mathcal{M}_1)}\Big).$$

We now set $\gamma_* = \min\big((\frac{T}{4})^{44},1\big)$. Let $\gamma = \|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|$. For $\gamma<\gamma_*$, we can choose $\tau=\gamma^{\frac{1}{44}}$, $\lambda=\tau^{-7}$, and deduce that

(5.10)
$$\|I_1^*I_1A\|^2_{L^2((0,T)\times\mathcal{M}_1)} \le C\Big(\gamma^{\frac{1}{44}} + \|A^{sol}\|_{L^2((0,T)\times\mathcal{M}_1)}\,\|A^{sol}\|^2_{\mathcal{C}^0([0,T]\times\mathcal{M}_1)}\Big).$$

By the Sobolev embedding theorem, interpolation, and condition (1.5), we observe that

(5.11)
$$\|A^{sol}\|_{\mathcal{C}^0([0,T]\times\mathcal{M}_1)} \le C\,\|A^{sol}\|_{H^{\frac{n+1}{2}+\frac{1}{6}}((0,T)\times\mathcal{M}_1)} \le C\,\|A^{sol}\|^{\frac{5}{6}}_{L^2((0,T)\times\mathcal{M}_1)}\,\|A^{sol}\|^{\frac{1}{6}}_{H^{3n+4}((0,T)\times\mathcal{M}_1)} \le C\,\|A^{sol}\|^{\frac{5}{6}}_{L^2((0,T)\times\mathcal{M}_1)}.$$

Then, using (5.3) and condition (1.5), interpolation also yields the estimate

(5.12)
$$\|A^{sol}\|^2_{L^2((0,T)\times\mathcal{M}_1)} \le C\,\|I_1^*I_1A\|^2_{L^2(0,T;H^1(\mathcal{M}_1))} \le C\,\|I_1^*I_1A\|^{\frac{10}{6}}_{L^2((0,T)\times\mathcal{M}_1)}\,\|I_1^*I_1A\|^{\frac{1}{3}}_{L^2(0,T;H^6(\mathcal{M}_1))} \le C\,\|I_1^*I_1A\|^{\frac{10}{6}}_{L^2((0,T)\times\mathcal{M}_1)}\,\|A^{sol}\|^{\frac{1}{3}}_{L^2(0,T;H^5(\mathcal{M}_1))} \le C\,\|I_1^*I_1A\|^{\frac{10}{6}}_{L^2((0,T)\times\mathcal{M}_1)}.$$

Finally we combine (5.10), (5.11) and (5.12) to obtain

$$\|A^{sol}\|^2_{L^2((0,T)\times\mathcal{M}_1)} \le C\,\|I_1^*I_1A\|^{\frac{10}{6}}_{L^2} \le C\gamma^{\frac{5}{264}} + C\,\|A^{sol}\|^{\frac{80}{36}}_{L^2} \le C\gamma^{\frac{5}{264}} + C\varepsilon^{\frac{8}{36}}\,\|A^{sol}\|^2_{L^2((0,T)\times\mathcal{M}_1)}.$$

Thus for small ε we deduce that

$$\left\|A^{sol}\right\|_{L^2((0,T)\times\mathcal{M})} \leq C\gamma^{\frac{5}{528}}.$$

Similarly for $\gamma \geq \gamma_*$, we have

$$\left\|A^{sol}\right\|_{L^2((0,T)\times\mathcal{M})} \leq \frac{\left\|A^{sol}\right\|_{L^2((0,T)\times\mathcal{M})}\gamma^{\frac{5}{528}}}{\gamma_*^{\frac{5}{528}}} \leq C\gamma^{\frac{5}{528}}. \tag{5.13}$$

Thus the proof of Theorem 2 is complete, subject to the smallness assumption (5.1). □

We will now show that the assumption that (5.1) holds a priori is unnecessary. Define $\eta \in \mathcal{C}^\infty(\mathbb{R}^n)$ by

$$\eta(x) = \begin{cases} C\exp(\frac{1}{|x|^2-1}) & \text{if } |x|<1, \\ 0 & \text{if } |x|\geq 1, \end{cases}$$

where $C>0$ is chosen so that $\int_{\mathbb{R}^n}\eta(x)dx = 1$. We further define the function

$$\eta_\rho(x) = \frac{1}{\rho^n}\eta\Big(\frac{x}{\rho}\Big).$$

Note that η_ρ approximates the Dirac delta distribution on $\mathbb{R}^n$ as $\rho\to 0$. Arguing as we did in (5.8), we use the estimate (5.4) to deduce that

$$\begin{aligned} &\left|\int_0^T\int_{\partial_+S_y\mathcal{M}_1}\big(e^{iI_1[A(t,\cdot)](y,\theta)}-1\big)h(t,y,\theta)d\theta dt\right| \\ &\leq C\Big(\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\,\lambda^5\tau^{-8}\,\|h(y,\cdot)\|_{H^5((0,T)\times\partial_+S_y\mathcal{M}_1)} + \lambda^{-1}\tau^{-6}\,\|h(y,\cdot)\|_{H^4((0,T)\times\partial_+S_y\mathcal{M}_1)} + \tau\Big). \end{aligned} \tag{5.14}$$

Since A is extended by 0 to $(0,T)\times(\mathcal{M}_1\setminus\mathcal{M})$, it follows that $e^{iI_1[A(t,\cdot)](y,\theta)}-1$ is compactly supported in $[0,T]\times\partial_+S_y\mathcal{M}_1$. We can find a finite open cover $\{U_i\}_{i=1}^N$ of $\partial\mathcal{M}_1$ so that for all $y\in U_i$ we can choose the same spherical coordinates $\theta := \mathbb{R}^{n-1}\ni\alpha\mapsto\theta(\alpha)$ on $S_y\mathcal{M}_1$ in such a way that $\theta(\alpha)$ gives coordinates in a neighborhood of $\text{supp}(e^{iI_1[A(t,\cdot)](y,\theta)}-1)\subset\partial_+S_y\mathcal{M}_1$.

We can then fix $y\in\partial\mathcal{M}_1$, $\theta_0\in\partial_+S_y\mathcal{M}_1$, $t_0\in(0,T)$. Let $\alpha_0=\alpha(\theta_0)$, $\gamma=\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|$. We define the function $f(\alpha,t)=e^{iI_1[A(t,\cdot)](y,\theta(\alpha))}-1$ and let $h(t,y,\theta)$ approximate the cylindrical Dirac delta distribution, that is

$$h(t,y,\theta(\alpha)) = \frac{1}{sin^{n-2}(\alpha_1)sin^{n-3}(\alpha_2)\cdots sin(\alpha_{n-2})}\eta_\rho\big((\alpha_0,t_0)-(\alpha,t)\big).$$

It is well known (see for instance [13, Lemma 2.1]) that

$$\|h\|_{H^k((0,T)\times\partial_+S_y\mathcal{M}_1)} \leq \rho^{-(n+k)}, \quad k\in\mathbb{N}.$$

In addition, we fix

$$f^\rho(\alpha_0,t_0) = \int_{\mathbb{R}^n} f(\alpha,t)h\big[\big(t_0,y,\theta(\alpha_0)\big)-\big(t,y,\theta(\alpha)\big)\big]dtd\alpha.$$

We use (5.14) to deduce that

$$\left|\int_{\mathbb{R}^n} f(\alpha,t)\eta_\rho\big((\alpha_0,t_0)-(\alpha,t)\big)dtd\alpha\right| \leq C\Big(\gamma\lambda^5\tau^{-8}\rho^{-n-5}+\lambda^{-1}\tau^{-6}\rho^{-n-4}+\tau\Big). \tag{5.15}$$

In particular, C is a positive constant depending only on $\mathcal{M}$, T and B, and independent of y. In order to deal with the left hand side above, we need the following Lemma:

Lemma 3. *Let $f:\mathbb{R}^n\mapsto\mathbb{R}$ be $\mathcal{C}^1$, and let $f^\rho(x_0)=\int_{B(x_0,\rho)}f(x)\eta_\rho(x_0-x)dx$. Then for any $x_0\in\mathbb{R}^n$ we have that*

$$|f^\rho(x_0)-f(x_0)| \leq C\,\|f\|_{\mathcal{C}^1}\,\rho.$$

Proof.

$$|f^\rho(x_0)-f(x_0)| = \left|\int_{B(x_0,\rho)} \eta_\rho(x_0-x)[f(x)-f(x_0)]dx\right| \le \int_{B(x_0,\rho)} \eta_\rho(x_0-x)\,|f(x)-f(x_0)|\,dx$$
$$\le \int_{B(x_0,\rho)} \eta_\rho(x_0-x)\,\|f\|_{\mathcal{C}^1}\,\rho dx \le C\,\|f\|_{\mathcal{C}^1}\cdot\rho.$$

□

Since $I_1 : \mathcal{C}^k(\mathcal{M}_1;T^*\mathcal{M}_1) \mapsto \mathcal{C}^k(\partial S\mathcal{M}_1)$ is bounded, $\|A\|_{W^{5,\infty}((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \le B$, then we must have $\|f\|_{\mathcal{C}^1} \le CB$ when $f(\alpha,t) = e^{iI_1[A(t,\cdot)](y,\theta(\alpha))}-1$. Thus, Lemma (3) together with (5.15) tells us that

$$\left|e^{iI_1[A(t_0,\cdot)](y,\theta_0)}-1\right| \le C\Big(\gamma\lambda^5\tau^{-8}\rho^{-n-5}+\lambda^{-1}\tau^{-6}\rho^{-n-4}+\tau+\rho\Big).$$

For $\gamma \le \min\big((\frac{T}{4})^{6n+69},1\big)$ we can choose $\tau=\gamma^{\frac{1}{6n+69}}$, $\lambda=\tau^{-n-11}$, $\rho=\tau$ to deduce that

$$\left|e^{iI_1[A(t_0,\cdot)](y,\theta_0)}-1\right| \le C\gamma^{\frac{1}{6n+69}},$$

with C independent of y. We now choose γ_0 small enough so the right hand side is near 0 when $\gamma<\gamma_0$. But this implies that $I_1[A(t_0,\cdot)](y,\theta_0)$ remains close to integer multiples of 2π whenever $\gamma<\gamma_0$. Recall that A is extended to $(0,T)\times\mathcal{M}_1\setminus\mathcal{M}$ by zero. Thus, for choices of y,θ_0 corresponding to short geodesics remaining close to the boundary of $\mathcal{M}_1$, we have $I_1[A(t_0,\cdot)](y,\theta_0)=0$. Then, the continuity of $I_1[A(t_0,\cdot)]$ in y,θ_0, together with the previous argument implies $I_1[A(t_0,\cdot)](y,\theta_0)$ is close to zero when $\gamma<\gamma_0$. But $\|I_1A\|_{\mathcal{C}^0([0,T]\times\partial_+S\mathcal{M}_1)} \le \varepsilon^2$ implies $\|I_1A\|_{L^2(0,T)\times\partial_+S\mathcal{M}_1} \le C\varepsilon^2$, and in turn $\|I_1^*I_1A\|_{L^2((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \le C\varepsilon^2$.
Then interpolation gives

$$\left\|A^{sol}\right\|_{L^2((0,T)\times\mathcal{M}_1)} \le C\,\|I_1^*I_1A\|_{L^2(0,T;H^1(\mathcal{M}_1))} \le C\,\|I_1^*I_1A\|^{\frac12}_{L^2((0,T)\times\mathcal{M}_1)}\,\|I_1^*I_1A\|^{\frac12}_{L^2(0,T;H^2(\mathcal{M}_1))}$$
$$\le C\,\|I_1^*I_1A\|^{\frac12}_{L^2((0,T)\times\mathcal{M}_1)}\,\left\|A^{sol}\right\|^{\frac12}_{L^2(0,T;H^1(\mathcal{M}_1))} \le C\,\|I_1^*I_1A\|^{\frac12}_{L^2((0,T)\times\mathcal{M}_1)} \le C\varepsilon.$$

Thus, for $\gamma<\gamma_0$ we conclude that the smallness assumption $\left\|A^{sol}\right\|_{L^2((0,T)\times\mathcal{M}_1)} \le \varepsilon$ holds. Therefore, we can rerun the argument of the previous section with γ_* replaced by γ_0, and reach the same conclusion without the need to assume smallness a priori. On the other hand, if $\gamma\ge\gamma_0$, we proceed as in (5.13). With this, the proof of Theorem 2 is now complete.

6. Stable Recovery of the Electric Potential

This section is devoted to proving the stability estimate in the recovery of the electric potential stated in Theorem 3. Henceforth, for $j=1,2$ we assume that $A_j \in W^{5,\infty}((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)$ with $\delta A_1=\delta A_2$ (so that $A=A^{sol}$), $q_j\in W^{4,\infty}((0,T)\times\mathcal{M}_1)$ and that conditions (1.7) and (1.8) are fulfilled. Additionally, we continue to assume that condition (1.5) holds true for the magnetic potential. In light of (3.15)-(3.20), we can use (4.3)-(4.4) to deduce that

$$\left|\int_0^T\int_{\mathcal{M}} Vu_1\overline{u_2}dV_g(x)dt\right| \le C\Big(\lambda\tau^{-6}\,\|A\|_{L^\infty((0,T)\times\mathcal{M};T^*\mathcal{M})}\,\|h\|_{H^4((0,T)\times\partial_+S_y\mathcal{M})} + \gamma\tau^{-8}\lambda^6\,\|h\|_{H^5((0,T)\times\partial_+S_y\mathcal{M}_1)}\Big), \tag{6.1}$$

where again γ denotes $\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|$. Using the fact that

$$\int_0^T\int_{\mathcal{M}} Vu_1\overline{u_2}\,dV_g(x)dt = \int_0^T\int_{\mathcal{M}} Va_1\overline{a_2}\,dV_g(x)dt + \int_0^T\int_{\mathcal{M}} Va_1\Big(\frac{\overline{b_2}}{\lambda}+e^{i\lambda(\psi-\lambda t)}\overline{R_{2,\lambda}}\Big)\,dV_g(x)dt$$
$$+\int_0^T\int_{\mathcal{M}} V\Big(\frac{b_1}{\lambda}+e^{-i\lambda(\psi-\lambda t)}R_{1,\lambda}\Big)\overline{a_2}\,dV_g(x)dt,$$
$$+\int_0^T\int_{\mathcal{M}} V\Big(\frac{b_1}{\lambda}+e^{-i\lambda(\psi-\lambda t)}R_{1,\lambda}\Big)\Big(\frac{\overline{b_2}}{\lambda}+e^{i\lambda(\psi-\lambda t)}\overline{R_{2,\lambda}}\Big)\,dV_g(x)dt$$

together with (6.1) and (3.15)-(3.20), we obtain

$$\int_0^T \int_{\mathcal{M}} V a_1 \overline{a_2}\, dV_g(x)dt \leq C\Big(\lambda\tau^{-6} \|A\|_{L^\infty((0,T)\times\mathcal{M};T^*\mathcal{M})} \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M})} + \gamma\tau^{-8}\lambda^6 \|h\|_{H^5((0,T)\times\partial_+ S_y\mathcal{M}_1)} + \lambda^{-1}\tau^{-4} \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M}_1)}\Big). \tag{6.2}$$

Then, by the definition of V together with Stokes' theorem, we deduce

$$\int_0^T \int_{\mathcal{M}} V a_1 \overline{a_2}\, dV_g(x)dt = \int_0^T \int_{\mathcal{M}} q a_1 \overline{a_2}\, dV_g(x)dt - i\int_0^T \int_{\mathcal{M}} A\nabla_g(a_1\overline{a_2})\, dV_g(x)dt - \int_0^T \int_M \langle A, A_1 + A_2\rangle_g\, a_1\overline{a_2}\, dV_g(x)dt,$$

whence we have

$$\int_0^T \int_{\mathcal{M}} q a_1 \overline{a_2}\, dV_g(x)dt \leq C\Big(\lambda\tau^{-6} \|A\|_{L^\infty((0,T)\times\mathcal{M};T^*\mathcal{M})} \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M})} + \gamma\tau^{-8}\lambda^6 \|h\|_{H^5((0,T)\times\partial_+ S_y\mathcal{M}_1)} + \lambda^{-1}\tau^{-4} \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M}_1)}\Big). \tag{6.3}$$

Since it holds that

$$\int_0^T \int_{\mathcal{M}} q a_1 \overline{a_2}\, dV_g(x)dt = \int_0^T \int_{\partial_+ S_y\mathcal{M}_1} \int_0^\infty q(t,r,\theta)\chi^2(t)h(t,\theta)\exp\Big(i\int_0^\infty A(t,r+s,\theta)\theta ds\Big)drd\theta dt$$

we deduce

$$\left|\int_0^T \int_{\partial_+ S_y\mathcal{M}_1} \int_0^\infty \chi^2(t)q(t,r,\theta)h(t,\theta)drd\theta dt\right| \leq \left|\int_0^T \int_{\mathcal{M}} q a_1\overline{a_2} dV_g(x)dt\right| + \left|\int_0^T \int_{\partial_+ S_y\mathcal{M}_1} \int_0^\infty \chi^2(t)q(t,r,\theta)h(t,\theta)\Big[\exp\Big(i\int_0^\infty A(t,r+s,\theta)\theta ds\Big) - 1\Big]drd\theta dt\right|.$$

Applying the mean value theorem to the second term on the right, we find that

$$\left|\int_0^T \int_{\mathcal{M}} I_0[q(t,\cdot)](y,\theta)\chi^2(t)h(t,y,\theta)d\theta dt\right| \leq \left|\int_0^T \int_{\mathcal{M}} q a_1\overline{a_2} dV_g(x)dt\right| + C\|A\|_{L^\infty((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)},$$

and, by combining the above with (6.3), we deduce that

$$\left|\int_0^T \int_{\partial_+ S_y\mathcal{M}_1} I_0[q(t,\cdot)](y,\theta)\chi^2(t)h(t,y,\theta)\right| d\theta dt \leq C\Big(\lambda\tau^{-6} \|A\|_{L^\infty((0,T)\times\mathcal{M};T^*\mathcal{M})} \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M})} + \gamma\tau^{-8}\lambda^6 \|h\|_{H^5((0,T)\times\partial_+ S_y\mathcal{M}_1)} + \lambda^{-1}\tau^{-4} \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M}_1)}\Big). \tag{6.4}$$

By the Sobolev interpolation theorem, we can choose $p \in (n+1,\infty)$ such that $\|A\|_{L^\infty((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \leq C\|A\|_{W^{1,p}((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)}$, and by interpolation together with condition (1.5) we deduce that

$$\begin{aligned}\|A\|_{L^\infty((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} &\leq C\|A\|^{\frac{1}{2}}_{W^{2,p}((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \|A\|^{\frac{1}{2}}_{L^p((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \\ &\leq \|A\|^{\frac{1}{2}}_{L^p((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \leq C\|A\|^{\frac{1}{p}}_{L^2((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)}.\end{aligned}$$

By combining this estimate with the result Theorem 2, we conclude that

$$\|A\|_{L^\infty((0,T)\times\mathcal{M}_1;T^*\mathcal{M}_1)} \leq C\gamma^{\frac{s_1}{p}}.$$

Thus, we can rewrite (6.4) as

$$\left|\int_0^T \int_{\partial_+ S_y\mathcal{M}_1} I_0[q(t,\cdot)](y,\theta)\chi^2(t)h(t,y,\theta)\right| d\theta dt \leq C\Big(\lambda\tau^{-6}\gamma^{\frac{s_1}{p}} \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M})} + \gamma\tau^{-8}\lambda^6 \|h\|_{H^5((0,T)\times\partial_+ S_y\mathcal{M}_1)} + \lambda^{-1}\tau^{-4} \|h\|_{H^4((0,T)\times\partial_+ S_y\mathcal{M}_1)}\Big). \tag{6.5}$$

Proof of Theorem 3. In order to prove (1.9) we will use the estimate (6.5) together with a suitable choice of h. First, note that according to condition (1.7) we have $q \in H^5((0,T)\times\mathcal{M}_1)$ with supp $q(t,\cdot)\subset\mathcal{M}$ when $t\in(0,T)$. Recall, according to [16, Section 7], that $I_0^*I_0$ with I_0^* the adjoint of I_0 (see for instance [2, Subsection 2.2] for details) is an elliptic pseudodifferential operator of order -1 for $\xi\in T^*\mathcal{M}$. Therefore, for all $t\in(0,T)$, we have $\|I_0^*I_0[q(t,\cdot)]\|\in H^5((0,T)\times\mathcal{M}_1)$ and condition (1.8) implies

$$\|I_0^*I_0q\|_{H^5((0,T)\times\mathcal{M}_1)}\le C\|q\|_{H^5((0,T)\times\mathcal{M}_1)}\le CB_1. \tag{6.6}$$

Moreover, according to [14, Theorem 4.2.1], for all $k\in\mathbb{N}$, the operator $I_0: H^k(\mathcal{M}_1)\to H^k(\partial_+S\mathcal{M}_1)$ is bounded. Thus, we can choose $h(t,\cdot)=I_0I_0^*I_0[q(t,\cdot)]\in H^5((0,T)\times\partial_+S\mathcal{M}_1)$. Integrating the left hand side of (6.5) with respect to $y\in\partial\mathcal{M}_1$ and applying Fubini's theorem yields

$$\int_0^T\chi^2(t)\int_{\partial_+S\mathcal{M}_1}I_0[q(t,\cdot)](y,\theta)h(t,y,\theta)\left|\langle\theta,\nu(y)\rangle_g\right|d\theta d\sigma_g(y)dt=\int_0^T\chi^2(t)\int_{\mathcal{M}_1}|I_0^*I_0[q(t,\cdot)](x)|^2\,dV_g(x)dt.$$

Combining this with (6.5) and (6.6), and using the fact that $\mathcal{M}_1$ is compact, we get

$$\int_0^T\chi^2(t)\int_{\mathcal{M}_1}|I_0^*I_0[q(t,\cdot)](x)|^2\,dV_g(x)dt\le C\Big(\gamma^{\frac{s_1}{p}}\lambda\tau^{-6}+\gamma\tau^{-8}\lambda^{-6}+\tau^{-4}\lambda^{-1}\Big), \tag{6.7}$$

with C depending only on $\mathcal{M}_1$, T and B_1. Further, by the same argument as in (5.8), the estimate (6.7) can be rewritten as

$$\int_0^T\int_{M_1}|I_0^*I_0[q(t,\cdot)](x)|^2\,dV_g(x)\le C\Big[\gamma^{\frac{s_1}{p}}\lambda\tau^{-6}+\gamma\tau^{-8}\lambda^{6}+\tau^{-4}\lambda^{-1}+\tau\Big]. \tag{6.8}$$

Note that for all $t\in(0,T)$ we have supp $q(t,\cdot)\subset\mathcal{M}$. Thus, according to [16, Theorem 3], we have

$$\int_{\mathcal{M}_1}|q(t,x)|^2\,dV_g(x)\le C\|I_0^*I_0[q(t,\cdot)]\|^2_{H^1(\mathcal{M}_1)},\quad t\in(0,T).$$

Integrating with respect to $t\in(0,T)$ yields

$$\int_0^T\int_{\mathcal{M}_1}|q(t,x)|^2\,dV_g(x)\le C\|I_0^*I_0[q(t,\cdot)]\|^2_{L^2(0,T;H^1(\mathcal{M}_1))}.$$

Then, by interpolation we obtain

$$\begin{aligned}\int_0^T\int_{\mathcal{M}_1}|q(t,x)|^2\,dV_g(x)\le&C\|I_0^*I_0[q(t,\cdot)]\|_{L^2((0,T)\times\mathcal{M}_1)}\|I_0^*I_0[q(t,\cdot)]\|_{L^2(0,T;H^2(\mathcal{M}_1))}\\ \le&C\|I_0^*I_0[q(t,\cdot)]\|_{L^2((0,T)\times\mathcal{M}_1)},\end{aligned}$$

where C depends on $\mathcal{M}$, T and B_1. Combining this with estimate (6.8), we find that

$$\int_0^T\int_{\mathcal{M}_1}|q(t,x)|^2\,dV_g(x)\le C\Big[\gamma^{\frac{1}{p}\frac{5}{528}}\lambda\tau^{-6}+\gamma\tau^{-8}\lambda^{6}+\tau^{-4}\lambda^{-1}+\tau\Big] \tag{6.9}$$

and (1.9) follows from (6.9) by a similar argument to the one used to prove Theorem 2 from (5.9). □

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