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Open and closed factors in Arnoux-Rauzy words *

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In Memory of the Late Professor Aldo de Luca

Abstract: Given a finite non-empty set \mathbb{A} , let \mathbb{A}^+ denote the free semigroup generated by \mathbb{A} consisting of all finite words $u_1u_2\cdots u_n$ with $u_i \in \mathbb{A}$. A word $u \in \mathbb{A}^+$ is said to be closed if either $u \in \mathbb{A}$ or if u is a complete first return to some factor $v \in \mathbb{A}^+$, meaning u contains precisely two occurrences of v , one as a prefix and one as a suffix. We study the function $f_x^c : \mathbb{N} \rightarrow \mathbb{N}$ which counts the number of closed factors of each length in an infinite word x . We derive an explicit formula for f_x^c in case x is an Arnoux-Rauzy word. As a consequence we prove that $\liminf_{n \rightarrow \infty} f_x^c(n) = +\infty$.

Keywords: Arnoux-Rauzy word, first return, complexity, return word, Sturmian word, closed word

1 Introduction

Throughout this paper we let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\omega = \{0, 1, 2, 3, \dots\}$ be the smallest transfinite ordinal. Given a finite non-empty set \mathbb{A} , we let \mathbb{A}^+ denote the

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free semigroup generated by \mathbb{A} consisting of all words $u_1u_2 \cdots u_n$ with $u_i \in \mathbb{A}$, and $\mathbb{A}^{\mathbb{N}}$ denote the set of (right) infinite words $x = x_1x_2x_3 \cdots$ with $x_i \in \mathbb{A}$. For each infinite word $x = x_1x_2x_3 \cdots \in \mathbb{A}^{\mathbb{N}}$, the factor complexity $p_x(n)$ counts the number of distinct blocks (or factors) $x_i x_{i+1} \cdots x_{i+n-1}$ of length n occurring in x . First introduced by Hedlund and Morse in their seminal 1938 paper [19] under the name of *block growth*, the factor complexity provides a useful measure of the extent of randomness of x . Periodic words have bounded factor complexity while digit expansions of normal numbers have maximal complexity. A celebrated theorem of Morse and Hedlund in [19] states that every aperiodic (meaning not ultimately periodic) word contains at least $n + 1$ distinct factors of each length n . Sturmian words are those aperiodic words of minimal factor complexity: $p_x(n) = n + 1$ for each $n \geq 1$.

Several notions of complexity have been successfully used in the study of infinite words and their combinatorial properties. They include Abelian complexity [1, 7, 8, 9, 20, 21], palindrome complexity [1], cyclic complexity [7], privileged complexity [20], group complexity [8] and maximal pattern complexity [17] to name just a few. In this paper we introduce and study two new complexity functions based on the notions of open and closed words [13]. A word $u \in \mathbb{A}^+$ is said to be *closed* if either $u \in \mathbb{A}$ or if u is a complete first return to some proper factor $v \in \mathbb{A}^+$, meaning u has precisely two occurrences of v , one as a prefix and one as a suffix. If u is not closed then u is said to be *open*. Thus a word $u \in \mathbb{A}^+ \setminus \mathbb{A}$ is closed if and only if it is bordered and its longest border only occurs in u as a prefix and as a suffix. The longest border of a closed word is called *frontier*. For example, *aabaaabaa* is closed and its frontier is equal to *aabaa*. In contrast, *ab* is open as it is unbordered while *abaabbababbaaba* is open since its frontier *aba* occurs internally in u . It is easily seen that all privileged words [20] are closed and hence so are all palindromic factors of rich words [14]. The terminology open and closed was first introduced by the authors in [5] although the notion of a closed word had already been introduced earlier by A. Carpi and A. de Luca in [6]. For a nice overview of open and closed words we refer the reader to the recent survey article by G. Fici [13].

For each infinite word $x \in \mathbb{A}^{\mathbb{N}}$ we consider the functions $f_x^c, f_x^o : \mathbb{N} \rightarrow \mathbb{N}$ which count the number of closed and open factors of x of each length $n \in \mathbb{N}$. In this note we investigate the function f_x^c where x is an Arnoux-Rauzy word. Arnoux-Rauzy words were first introduced in [2] in the special case of a 3-letter alphabet. They are a natural generalization of Sturmian words to alphabets of cardinality greater than two. If $x \in \mathbb{A}^{\mathbb{N}}$ is an Arnoux-Rauzy word, then $p_x(n) = (|\mathbb{A}| - 1)n + 1$ for each $n \in \mathbb{N}$. Moreover each factor u of x has precisely $|\mathbb{A}|$ distinct complete first returns in x .

Our main result in Theorem 1 below provides an explicit formula for the closed complexity function $f_x^c(n)$ for an Arnoux-Rauzy word x on a t -letter alphabet \mathbb{A} . Since for any word $x \in \mathbb{A}^{\mathbb{N}}$ we have that $f_x^c(n) + f_x^o(n) = p_x(n)$, a formula for $f_x^c(n)$ also yields a formula for $f_x^o(n)$. Our formula is expressed in terms of two related sequences associated to x . The first is the sequence $(b_k)_{k \geq 0}$ of the lengths of the

bispecial factors $\varepsilon = B_0, B_1, B_2, \dots$ of x , ordered according to increasing length. The second is the sequence $(p_a^{(k)})_{a \in \mathbb{A}}^{k \in \omega}$ where for each $k \in \omega$, the t coordinates of $(p_a^{(k)})_{a \in \mathbb{A}}$ are the lengths of the t first returns to B_k in x . Both sequences have been extensively studied in the literature. For each $k \in \omega$, the coordinates of $(p_a^{(k)})_{a \in \mathbb{A}}$ are coprime and each is a period of the word B_k . Moreover, each B_k is an extremal Fine and Wilf word, i.e., any word u having periods $(p_a^{(k)})_{a \in \mathbb{A}}$ and of length greater than b_k is a constant word, i.e., $u = a^n$ for some n (see [24]).

Theorem 1. *Let $x \in \mathbb{A}^{\mathbb{N}}$ be an Arnoux-Rauzy word. For each $k \in \omega$ and $a \in \mathbb{A}$ set $I_{k,a} = [b_k - 2p_k + p_a^{(k)} + 2, b_k + p_a^{(k)}]$ where $p_k = \min_{b \in \mathbb{A}} \{p_b^{(k)}\}$. Let*

$$F(a, n) = \sum_{\substack{k \in \omega \\ n \in I_{k,a}}} (d(n, I_{k,a}) + 1), \quad (1)$$

where for $n \in I_{k,a}$, the quantity $d(n, I_{k,a})$ denotes the minimal distance from n to the endpoints of the interval $I_{k,a}$. Then the number of closed factors of x for each length n is $f_x^c(n) = \sum_{a \in \mathbb{A}} F(a, n)$.

For each fixed $n \in \mathbb{N}$ and $a \in \mathbb{A}$, the sum in (1) is finite since it only involves those k for which $n \in I_{k,a}$.

As a corollary of Theorem 1, we show that if x is an Arnoux-Rauzy word, then $\liminf f_x^c(n) = +\infty$. In contrast, it follows from [23] that if x is the regular paperfolding word, then $\liminf f_x^c(n) = 0$, in other words, for infinitely many n , all factors of x of length n are open.

We end this section by recalling a few basic notions in combinatorics on words relevant to the paper. Throughout this text \mathbb{A} will denote a finite non-empty set (the alphabet). For $n \in \mathbb{N}$, let \mathbb{A}^n denote the set of all words $a_1 a_2 \cdots a_n$ with $a_i \in \mathbb{A}$. For $u = a_1 a_2 \cdots a_n \in \mathbb{A}^n$, we let $\bar{u} \in \mathbb{A}^n$ denote the reversal of u , i.e., $\bar{u} = a_n a_{n-1} \cdots a_1$. Let $\mathbb{A}^+ = \bigcup_{n \in \mathbb{N}} \mathbb{A}^n$ denote the free semigroup generated by \mathbb{A} . For $u = a_1 a_2 \cdots a_n \in \mathbb{A}^+$ the quantity n is called the length of u and denoted $|u|$. We set $\mathbb{A}^* = \mathbb{A}^+ \cup \{\varepsilon\}$ where ε is the empty word (of length equal to 0). We let $\mathbb{A}^{\mathbb{N}}$ denote the set of all infinite words $a_1 a_2 a_3 \cdots$ with $a_i \in \mathbb{A}$. For $x \in \mathbb{A}^+ \cup \mathbb{A}^{\mathbb{N}}$ and $v \in \mathbb{A}^*$ we say that v is a factor of x if $x = uv y$ for some $u \in \mathbb{A}^*$ and $y \in \mathbb{A}^* \cup \mathbb{A}^{\mathbb{N}}$. We let $\text{Fac}(x)$ denote the set of all factors of x . A factor v of x is called right (resp. left) special if va and vb (resp. av and bv) are each factors of x for some choice of distinct $a, b \in \mathbb{A}$. A factor which is both right and left special is said to be bispecial. Given factors u and v of x , we say that u is a first return to v in x if uv is a factor of x having precisely two first occurrences of v , one as a prefix and one as a suffix. In which case the word uv is called a complete first return to v .

2 Counting closed factors in Arnoux-Rauzy words

Throughout this section we let \mathbb{A} denote a finite set of cardinality $t \geq 2$. A recurrent word $x \in \mathbb{A}^{\mathbb{N}}$ is called an Arnoux-Rauzy word if x contains, for each $n \geq 0$, precisely one right special factor R_n of length n and one left special factor L_n of length n . Furthermore, R_n is a prefix of t -many distinct factors of x of length $n + 1$ while L_n is a suffix of t -many distinct factors of x of length $n + 1$. In particular one has $p_x(n) = (t - 1)n + 1$ and each factor u of x has precisely t distinct complete first returns. In the special case of a binary alphabet, we see that x is a Sturmian word. Arnoux-Rauzy words constitute a special class of episturmian words (see [3, 12, 16]) and hence each factor u of an Arnoux-Rauzy word is (palindromically) rich, i.e., u contains exactly $|u| + 1$ many distinct palindromic factors (including the empty word ε). We will make use of the following alternative characterisation of rich words given in [12].

Lemma 2.1. *[Proposition 3 in [12]] A word $u \in \mathbb{A}^+$ is rich if and only if for every prefix v of u , the longest palindromic suffix of v is uni-occurrent in v .*

Let us now fix an Arnoux-Rauzy word $x \in \mathbb{A}^{\mathbb{N}}$. Recall that for each length $n \in \omega$ an Arnoux-Rauzy word contains either zero or one bispecial factor of length n . Let $\varepsilon = B_0, B_1, B_2, \dots$ be the sequence of bispecial factors of x ordered according to increasing length. Put $b_k = |B_k|$ so that $0 = b_0 < b_1 < b_2 < \dots$. We recall the following characterization of the bispecial factors B_k of x in terms of palindromic closures (see [10, 12]). For each $k \in \mathbb{N}$ there exists a unique $a_k \in \mathbb{A}$ such that $B_{k-1}a_k$ is a left special factor of x . The sequence $(a_k)_{k \in \mathbb{N}}$ is called the directive sequence of x . It follows that $B_{k-1}a_k$ is a prefix of B_k but in fact B_k is the palindromic closure of $B_{k-1}a_k$, i.e., the shortest palindrome beginning in $B_{k-1}a_k$. More precisely, if we let S_k denote the longest palindromic suffix of $B_{k-1}a_k$ and write $B_{k-1}a_k = x_k S_k$ with $x_k \in \mathbb{A}^*$, then $B_k = x_k S_k \overline{x_k}$ (see for instance Lemma 5 in [10] in case the alphabet \mathbb{A} is binary).

Lemma 2.2. *For each $k \in \mathbb{N}$ we have that S_k is a uni-occurrent factor of B_k . In particular S_k is not a factor of B_{k-1} .*

Proof. Clearly S_k is a factor of B_k . To see that it is uni-occurrent, suppose that S_k occurs more than once in B_k . Since S_k and B_k are each palindromes and $B_k = x_k S_k \overline{x_k}$, it follows that S_k occurs at least twice in $x_k S_k = B_{k-1}a_k$. But this contradicts Lemma 2.1 since S_k was defined as the longest palindromic suffix of $B_{k-1}a_k$. \square

Define $\varphi : \text{Fac}(x) \rightarrow \omega$ by $\varphi(v)$ is the least $k \in \omega$ such that v is a factor of B_k . In particular $\varphi(v) = 0 \Leftrightarrow v = \varepsilon$.

Lemma 2.3. *Let $k \in \mathbb{N}$ and $v \in \mathbb{A}^+$. Then $v \in \varphi^{-1}(k)$ if and only if v is a factor of B_k containing S_k as a factor. In particular each $v \in \varphi^{-1}(k)$ is uni-occurrent in B_k .*

Proof. Suppose v is a factor of B_k containing S_k as a factor. Then by Lemma 2.2, v is not a factor of B_{k-1} and hence not a factor of any B_j with $j < k$. Hence $\varphi(v) = k$. Conversely suppose that $\varphi(v) = k$. Then v is a factor of B_k but not of B_{k-1} . Since

$$B_k = x_k S_k \bar{x}_k = B_{k-1} a_k \bar{x}_k = x_k a_k B_{k-1},$$

it follows that v must contain S_k as a factor. Having established that each $v \in \varphi^{-1}(k)$ contains S_k , it follows by Lemma 2.2 that v is uni-occurrent in B_k . \square

For each $k \in \omega$ and $a \in \mathbb{A}$, let $R_a^{(k)}$ denote the complete first return to B_k in x beginning in $B_k a$ and put $p_a^{(k)} = |R_a^{(k)}| - b_k$. In other words $p_a^{(k)}$ is the length of the first return to B_k determined by $R_a^{(k)}$. We note that $R_a^{(0)} = a$ for each $a \in \mathbb{A}$. The sequence $(p_a^{(k)})_{a \in \mathbb{A}}^{k \in \omega}$ is computed recursively as follows : $p_a^{(0)} = 1$ for each $a \in \mathbb{A}$. For $k \geq 1$, we have $p_{a_k}^{(k)} = p_{a_k}^{(k-1)}$, and $p_b^{(k)} = p_b^{(k-1)} + p_{a_k}^{(k-1)}$ for $b \in \mathbb{A} \setminus \{a_k\}$. It is easily verified by induction that

$$b_k = \frac{\sum_{a \in \mathbb{A}} p_a^{(k)} - t}{t - 1}.$$

For each $k \in \mathbb{N}$, we set $p_k = p_{a_k}^{(k)}$. Since B_k is a complete first return to B_{k-1} beginning in $B_{k-1} a_k$ i.e., $R_{a_k}^{(k-1)} = B_k$, it follows that

$$p_k = p_{a_k}^{(k)} = p_{a_k}^{(k-1)} = |R_{a_k}^{(k-1)}| - b_{k-1} = b_k - b_{k-1}. \quad (2)$$

It follows immediately from our recursive definition of the $p_a^{(k)}$ that

$$p_k = \min\{p_a^{(k)} \mid a \in \mathbb{A}\}.$$

Lemma 2.4. *Let $k \in \mathbb{N}$ and let J_k denote the interval $[b_k - 2p_k + 2, b_k]$. If $v \in \varphi^{-1}(k)$ then $|v| \in J_k$ and, for each $m \in J_k$, the set $\varphi^{-1}(k)$ contains precisely $d(m, J_k) + 1$ distinct words of length m , where $d(m, J_k)$ is the minimal distance between m and the two boundary points of the interval J_k . In particular $|\varphi^{-1}(k)| = p_k^2$.*

Proof. In view of Lemma 2.3 we have that $v \in \varphi^{-1}(k)$ if and only if v is a factor of B_k which contains S_k as a subfactor. It follows that $|S_k| \leq |v| \leq |B_k|$. Also, since $B_k = B_{k-1} a_k \bar{x}_k$, by (2) we deduce that $p_k = b_k - b_{k-1} = |x_k| + 1$. Furthermore, as $B_k = x_k S_k \bar{x}_k$ we have $|S_k| = |B_k| - 2|x_k| = b_k - 2(p_k - 1) = b_k - 2p_k + 2$. Hence $b_k - 2p_k + 2 \leq |v| \leq b_k$. Now suppose $m \in J_k$. To see that $\varphi^{-1}(k)$ contains $d(m, J_k) + 1$ distinct words of length m we simply use the fact that each $v \in \varphi^{-1}(k)$ contains S_k and is uni-occurrent in B_k (see Lemma 2.3). Finally,

$$|\varphi^{-1}(k)| = 1 + 2 + \cdots + (p_k - 1) + p_k + (p_k - 1) + \cdots + 2 + 1 = 2 \left(\frac{p_k(p_k - 1)}{2} \right) + p_k = p_k^2.$$

\square

Let $k \in \mathbb{N}$ and $v \in \varphi^{-1}(k)$. As a consequence of Lemma 2.3, there exists a unique decomposition $B_k = u_1 v u_2$ with $u_1, u_2 \in \mathbb{A}^*$. In particular, $v u_2$ is right special in x and $u_1 v$ is left special in x . Now suppose u is a closed factor of x with frontier v . In particular u begins and ends in v . Since x is recurrent and aperiodic, it follows that $v u_2$ is a prefix of u and $u_1 v$ is a suffix of u , whence $u_1 u u_2$ is a complete first return to B_k . In fact, $u_1 u u_2$ begins and ends in B_k and does not admit other occurrences of B_k for otherwise v would occur in u internally (meaning not as a prefix or as a suffix). Thus $u_1 u u_2 = R_a^{(k)}$ for some $a \in \mathbb{A}$.

Definition 2.5. Let u be a closed factor of x and $a \in \mathbb{A}$. We say u is of *type* a if and only if either $u = a$ or, if u is closed with frontier $v \in \mathbb{A}^+$, then $u_1 u u_2 = R_a^{(k)}$ where $k = \varphi(v)$ and $B_k = u_1 v u_2$.

If u is a closed factor of x of type $a \in \mathbb{A}$ and frontier $v \in \mathbb{A}^+$, then

$$|u| - |v| = |R_a^{(k)}| - (|u_1| + |u_2| + |v|) = |R_a^{(k)}| - |B_k| = p_a^{(k)}, \quad (3)$$

where $k = \varphi(v)$. We observe that the equality $|u| - |v| = p_a^{(k)}$ in (3) also holds in case $u \in \mathbb{A}$ taking $v = \varepsilon$ and $k = 0$.

Let $C(x)$ denote the set of all closed factors of x and for each $u \in C(x)$ let $\text{fr}(u) \in \mathbb{A}^*$ denote its frontier. By convention we define $\text{fr}(a) = \varepsilon$ for each $a \in \mathbb{A}$. For each $k \in \omega$ and $a \in \mathbb{A}$ we let $C_{k,a}(x)$ denote the set of all closed factors u of x of type a whose frontier $\text{fr}(u)$ belongs to $\varphi^{-1}(k)$.

Lemma 2.6. *The sets $\{C_{k,a}(x) : k \in \omega, a \in \mathbb{A}\}$ define a partition of $C(x)$ and $\text{fr} : C_{k,a}(x) \rightarrow \varphi^{-1}(k)$ is a bijection.*

Proof. Each closed factor $u \in C(x)$ has a unique type and its frontier $\text{fr}(u)$ belongs to $\varphi^{-1}(k)$ for a unique value of $k \in \omega$. Whence each closed factor u of x belongs to a unique $C_{k,a}(x)$. By definition, if $u \in C_{k,a}(x)$ then $\text{fr}(u) \in \varphi^{-1}(k)$. Moreover u is uniquely determined by its frontier $\text{fr}(u)$ and its type. In fact, if $u \in C_{k,a}(x)$ then $u_1 u u_2 = R_a^{(k)}$ where u_1, u_2 are determined by the (unique) factorization $B_k = u_1 \text{fr}(u) u_2$. This proves fr is injective. To see that fr is also surjective, let $v \in \varphi^{-1}(k)$. Then we can write $B_k = u_1 v u_2$ for some $u_1, u_2 \in \mathbb{A}^*$. Hence $R_a^{(k)}$ begins in u_1 and ends in u_2 . It follows that $u = u_1^{-1} R_a^{(k)} u_2^{-1}$ is a closed factor of x of type a and $\text{fr}(u) = v$. \square

Proof of Theorem 1. Fix $n \in \mathbb{N}$. By Lemma 2.6 we have

$$f_x^c(n) = |C(x) \cap \mathbb{A}^n| = \sum_{\substack{k \in \omega \\ a \in \mathbb{A}}} |C_{k,a}(x) \cap \mathbb{A}^n|.$$

Now assume $u \in C_{k,a} \cap \mathbb{A}^n$ and put $v = \text{fr}(u) \in \varphi^{-1}(k)$. Then by (3) we have that $n = |u| = |v| + p_a^{(k)}$. By Lemma 2.4, $|v| = n - p_a^{(k)} \in J_k = [b_k - 2p_k + 2, b_k]$. By Lemma 2.6 the number of words $u \in C_{k,a}(x) \cap \mathbb{A}^n$ is equal to the number of words $v \in \varphi^{-1}(k)$ of length $n - p_a^{(k)}$ which by Lemma 2.4 is equal to $d(n - p_a^{(k)}, J_k) + 1 = d(n, I_{k,a}) + 1$ where $I_{k,a} = [b_k - 2p_k + p_a^{(k)} + 2, b_k + p_a^{(k)}]$. This completes the proof of Theorem 1. \square

In case $|\mathbb{A}| = 2$, i.e., x is Sturmian, each bispecial factor B_k has precisely two first returns, the shortest one is of length p_k , and we let q_k denote the length of the other first return. So for fixed $a \in \mathbb{A}$ and $k \in \mathbb{N}$ we have

$$p_a^{(k)} = \begin{cases} p_k, & \text{if } a = a_k; \\ q_k, & \text{otherwise.} \end{cases}$$

If $a = a_k$ then $I_{k,a} = [q_k, q_k + 2p_k - 2]$ and if $a \neq a_k$ then $I_{k,a} = [2q_k - p_k, 2q_k + p_k - 2]$. Putting $P_k = [q_k, q_k + 2p_k - 2]$ and $Q_k = [2q_k - p_k, 2q_k + p_k - 2]$, we obtain that for a Sturmian word x the number of closed factors of x of each length n is given by

$$f_x^c(n) = \sum_{\substack{k \in \omega \\ n \in P_k}} (d(n, P_k) + 1) + \sum_{\substack{k \in \omega \\ n \in Q_k}} (d(n, Q_k) + 1). \quad (4)$$

Example 2.7. Consider the Fibonacci word

$$x_{fib} = abaababaabaababaa \dots$$

fixed by the morphism $a \mapsto ab$, $b \mapsto a$. Then $p_k = F_k$ and $q_k = F_{k+1}$ where the sequence $(F_k)_{k \in \omega}$ is the Fibonacci sequence given by $F_0 = F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 1$.

Table 1 shows the number of closed factors of length $n \leq 15$ in the Fibonacci word computed using (4).

Table 1: The number of closed factors in the Fibonacci word.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f_{x_{fib}}^c(n)$	2	1	2	3	4	3	4	5	6	5	6	7	8	9	10

For example, for $n = 11$ we must determine those k for which either $11 \in P_k$ or $11 \in Q_k$. It is easily checked that 11 only belongs to $P_4 = [8, 16]$, $Q_3 = [7, 11]$ and $Q_4 = [11, 19]$. So

$$f_{x_{fib}}^c(11) = d(11, P_4) + 1 + d(11, Q_3) + 1 + d(11, Q_4) + 1 = 4 + 1 + 1 = 6.$$

The graph of the function $f_{x_{fib}}^c$ is shown in Figure 1. The function is clearly not monotone.

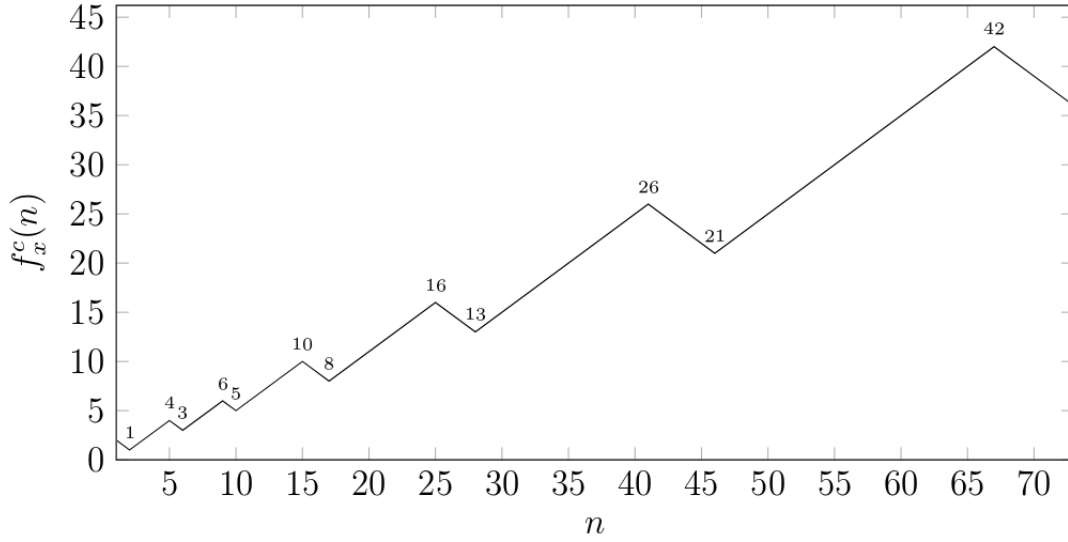


Figure 1: The number of closed factors in the Fibonacci word.

Figure 2 illustrates the behavior of the number of closed factors of the Tribonacci word $x_{trib} \in \{a, b, c\}^{\mathbb{N}}$ defined as the fixed point of the morphism $a \mapsto ab$, $b \mapsto ac$, $c \mapsto a$.

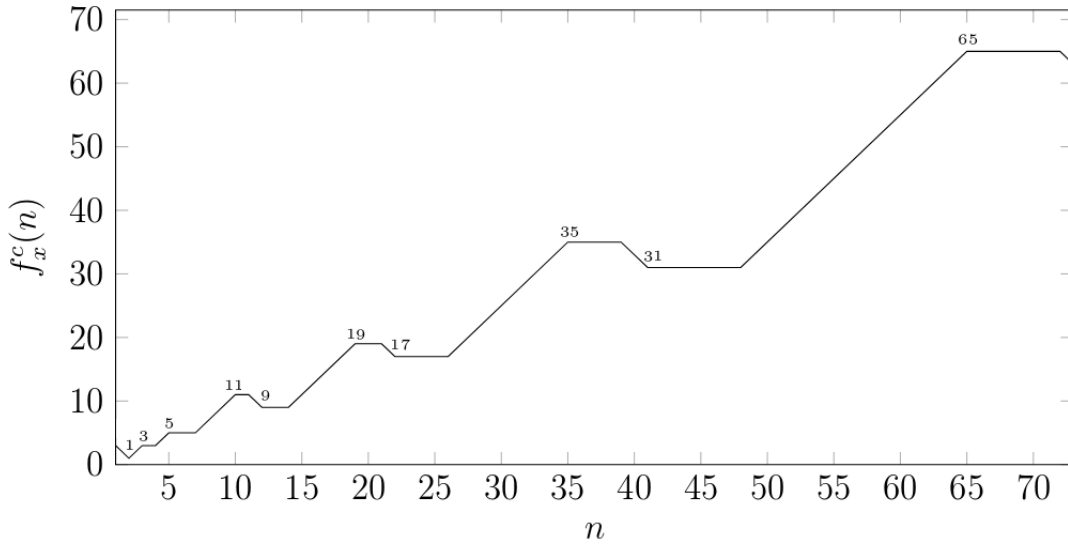


Figure 2: The number of closed factors in the Tribonacci word.

Our last example (Figure 3) illustrates the behavior of the number of closed factors of the Sturmian word $x_r \in \{0, 1\}^{\mathbb{N}}$ whose directive sequence begins with $00100000111110111110101101110011000011 \dots$.

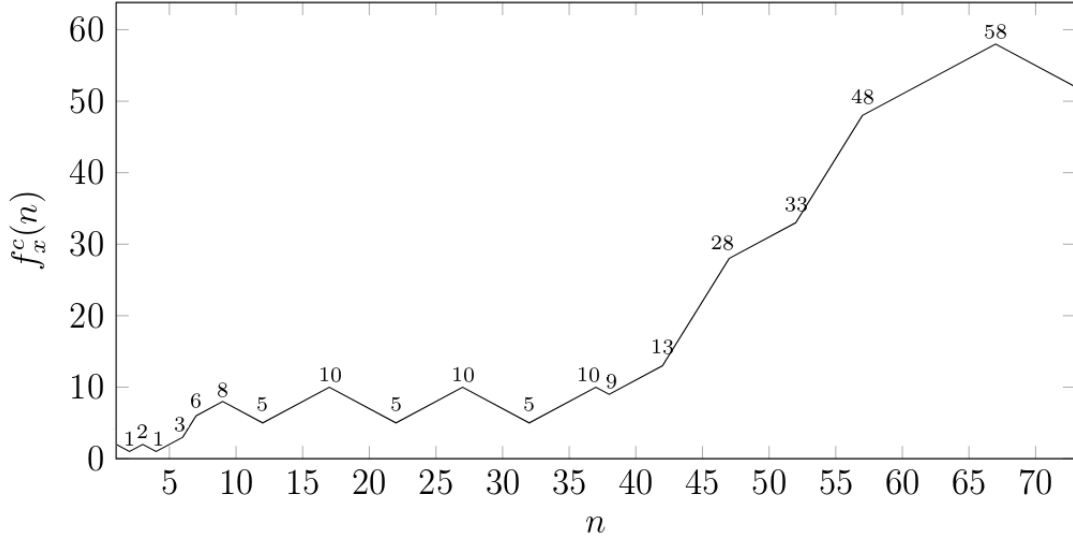


Figure 3: The number of closed factors in the word x_r .

The above examples suggest that the function $f_x^c(n)$ tends to infinity, although it need not be monotone and may contain plateaus and inflection points. Our next result establishes this fact:

Corollary 2.8. *If $x \in \mathbb{A}^{\mathbb{N}}$ is an Arnoux-Rauzy word, then*

$$\liminf_{n \rightarrow \infty} f_x^c(n) = +\infty.$$

Proof. For each $k \in \omega$ set $I_k = [b_k - p_k + 2, b_k + p_k]$, i.e., $I_0 = [1, 1]$ and $I_k = I_{a_k, k}$ for $k \geq 1$. Since $p_k \geq 1$ for each $k \in \mathbb{N}$ it follows that $\mathbb{N} = \bigcup_{k \in \omega} I_k$. Given $m \in \mathbb{N}$, pick j such that $p_j - 1 > 2m$, and put $N = b_j + 2$. We will show that $f_x^c(n) \geq m$ for every $n \geq N$. Notice that since $b_k = b_{k-1} + p_k$, the left hand endpoint of the interval I_{j+1} is $b_j + 2$. Thus for each $n \geq N$ there exists a positive integer $k \geq j$, such that $n \in I_{k+1}$. We have $|I_{k+1}| = 2p_{k+1} - 2 \geq 2p_j - 2 > 4m$. If $d(n, I_{k+1}) \geq m$, then it follows from Theorem 1 that $f_w^c(n) \geq m$. Otherwise we must have either i) $b_{k+1} + p_{k+1} - n < m$ or ii) $n - (b_k + 2) < m$. In case i) we have

$$n - b_{k+1} > p_{k+1} - m \geq p_j - m > m + 1. \quad (5)$$

Since $m + 1 \geq 2$, we have that n also belongs to I_{k+2} . We will show that $d(n, I_{k+2}) \geq m$. By (5) we have $n - (b_{k+1} + 2) \geq m$. Also

$$b_{k+2} + p_{k+2} - n \geq b_{k+2} + p_{k+2} - (b_{k+1} + p_{k+1}) = 2p_{k+2} - p_{k+1} \geq p_{k+2} \geq p_j > 2m + 1.$$

Thus $d(n, I_{k+2}) \geq m$ and hence by Theorem 1 $f_x^c(n) \geq m$.

In case ii) $n < b_k + 2 + m \leq b_k + 2m + 1 < b_k + p_k$, and hence $n \in I_k$. We will show that $d(n, I_k) \geq m$. In fact, $b_k + p_k - n > p_k - 2 - m \geq p_j - 2 - m \geq m$. Moreover, since $n \in I_{k+1}$, we have that $n - (b_{k-1} + 2) \geq b_k + 2 - (b_{k-1} + 2) = p_k \geq p_j > 2m + 1$, and thus $d(n, I_k) \geq m$. \square

While the previous result applies to Arnoux-Rauzy words, for a general aperiodic word x the limit inferior of the function $f_x^c(n)$ need not be infinite. For example, in the case of the regular paperfolding word one has that $\liminf_{n \rightarrow \infty} f_x^c(n) = 0$. In fact, in [23] the authors exhibit an 11-state automaton which accepts the base 2 representation of those n for which there is a closed factor of the regular paperfolding word of length n (see Figure 1 in [23]). As another perhaps simpler example, let x be the fixed point beginning in a of the 2-uniform morphism φ on the alphabet $\{a, b, c, d\}$ given by $\varphi : a \mapsto ac, b \mapsto ad, c \mapsto bc, d \mapsto bd$. Then it is easily shown that all factors of x of length 2^n ($n \in \mathbb{N}$) are open. We remark that this last example is closely related to the regular paperfolding word. In fact, the regular paperfolding word is the image of the fixed point of φ under the mapping which sends a, c to 0 and b, d to 1.

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