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## Open and closed factors in Arnoux-Rauzy words \*

Olga Parshina<sup>1,2</sup> and Luca Zamboni<sup>1</sup>

 <sup>1</sup>Université de Lyon, Université Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F69622 Villeurbanne Cedex, France
 <sup>2</sup>Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences, 4 Acad. Koptyug avenue, 630090 Novosibirsk, Russia {parshina,zamboni}@math.univ-lyon1.fr

In Memory of the Late Professor Aldo de Luca

**Abstract:** Given a finite non-empty set  $\mathbb{A}$ , let  $\mathbb{A}^+$  denote the free semigroup generated by  $\mathbb{A}$  consisting of all finite words  $u_1u_2\cdots u_n$  with  $u_i \in \mathbb{A}$ . A word  $u \in \mathbb{A}^+$ is said to be closed if either  $u \in \mathbb{A}$  or if u is a complete first return to some factor  $v \in \mathbb{A}^+$ , meaning u contains precisely two occurrences of v, one as a prefix and one as a suffix. We study the function  $f_x^c : \mathbb{N} \to \mathbb{N}$  which counts the number of closed factors of each length in an infinite word x. We derive an explicit formula for  $f_x^c$  in case x is an Arnoux-Rauzy word. As a consequence we prove that  $\liminf_{n\to\infty} f_x^c(n) = +\infty$ . **Keywords:** Arnoux-Rauzy word, first return, complexity, return word, Sturmian word, closed word

### 1 Introduction

Throughout this paper we let  $\mathbb{N} = \{1, 2, 3, \dots\}$  and  $\omega = \{0, 1, 2, 3, \dots\}$  be the smallest transfinite ordinal. Given a finite non-empty set  $\mathbb{A}$ , we let  $\mathbb{A}^+$  denote the

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free semigroup generated by A consisting of all words  $u_1u_2 \cdots u_n$  with  $u_i \in A$ , and  $\mathbb{A}^{\mathbb{N}}$  denote the set of (right) infinite words  $x = x_1x_2x_3\cdots$  with  $x_i \in A$ . For each infinite word  $x = x_1x_2x_3 \cdots \in \mathbb{A}^{\mathbb{N}}$ , the factor complexity  $p_x(n)$  counts the number of distinct blocks (or factors)  $x_ix_{i+1}\cdots x_{i+n-1}$  of length n occurring in x. First introduced by Hedlund and Morse in their seminal 1938 paper [19] under the name of block growth, the factor complexity provides a useful measure of the extent of randomness of x. Periodic words have bounded factor complexity while digit expansions of normal numbers have maximal complexity. A celebrated theorem of Morse and Hedlund in [19] states that every aperiodic (meaning not ultimately periodic) word contains at least n + 1 distinct factors of each length n. Sturmian words are those aperiodic words of minimal factor complexity:  $p_x(n) = n + 1$  for each  $n \ge 1$ .

Several notions of complexity have been successfully used in the study of infinite words and their combinatorial properties. They include Abelian complexity [1, 7, 8, 9, 20, 21, palindrome complexity [1], cyclic complexity [7], privileged complexity [20], group complexity [8] and maximal pattern complexity [17] to name just a few. In this paper we introduce and study two new complexity functions based on the notions of open and closed words [13]. A word  $u \in \mathbb{A}^+$  is said to be *closed* if either  $u \in \mathbb{A}$  or if u is a complete first return to some proper factor  $v \in \mathbb{A}^+$ , meaning u has precisely two occurrences of v, one as a prefix and one as a suffix. If u is not closed then u is said to be open. Thus a word  $u \in \mathbb{A}^+ \setminus \mathbb{A}$  is closed if and only if it is bordered and its longest border only occurs in u as a prefix and as a suffix. The longest border of a closed word is called *frontier*. For example, *aabaaabaa* is closed and its frontier is equal to *aabaa*. In contrast, *ab* is open as it is unbordered while abaabbaabbaabaa is open since its frontier aba occurs internally in u. It is easily seen that all privileged words [20] are closed and hence so are all palindromic factors of rich words [14]. The terminology open and closed was first introduced by the authors in [5] although the notion of a closed word had already been introduced earlier by A. Carpi and A. de Luca in [6]. For a nice overview of open and closed words we refer the reader to the recent survey article by G. Fici [13].

For each infinite word  $x \in \mathbb{A}^{\mathbb{N}}$  we consider the functions  $f_x^c, f_x^o : \mathbb{N} \to \mathbb{N}$  which count the number of closed and open factors of x of each length  $n \in \mathbb{N}$ . In this note we investigate the function  $f_x^c$  where x is an Arnoux-Rauzy word. Arnoux-Rauzy words were first introduced in [2] in the special case of a 3-letter alphabet. They are a natural generalization of Sturmian words to alphabets of cardinality greater that two. If  $x \in \mathbb{A}^{\mathbb{N}}$  is an Arnoux-Rauzy word, then  $p_x(n) = (|\mathbb{A}| - 1)n + 1$  for each  $n \in \mathbb{N}$ . Moreover each factor u of x has precisely  $|\mathbb{A}|$  distinct complete first returns in x.

Our main result in Theorem 1 below provides an explicit formula for the closed complexity function  $f_x^c(n)$  for an Arnoux-Rauzy word x on a *t*-letter alphabet  $\mathbb{A}$ . Since for any word  $x \in \mathbb{A}^{\mathbb{N}}$  we have that  $f_x^c(n) + f_x^o(n) = p_x(n)$ , a formula for  $f_x^c(n)$ also yields a formula for  $f_x^o(n)$ . Our formula is expressed in terms of two related sequences associated to x. The first is the sequence  $(b_k)_{k\geq 0}$  of the lengths of the bispecial factors  $\varepsilon = B_0, B_1, B_2, \ldots$  of x, ordered according to increasing length. The second is the sequence  $(p_a^{(k)})_{a\in\mathbb{A}}^{k\in\omega}$  where for each  $k \in \omega$ , the t coordinates of  $(p_a^{(k)})_{a\in\mathbb{A}}$  are the lengths of the t first returns to  $B_k$  in x. Both sequences have been extensively studied in the literature. For each  $k \in \omega$ , the coordinates of  $(p_a^{(k)})_{a\in\mathbb{A}}$  are coprime and each is a period of the word  $B_k$ . Moreover, each  $B_k$  is an extremal Fine and Wilf word, i.e., any word u having periods  $(p_a^{(k)})_{a\in\mathbb{A}}$  and of length greater than  $b_k$  is a constant word, i.e.,  $u = a^n$  for some n (see [24]).

**Theorem 1.** Let  $x \in \mathbb{A}^{\mathbb{N}}$  be an Arnoux-Rauzy word. For each  $k \in \omega$  and  $a \in \mathbb{A}$  set  $I_{k,a} = [b_k - 2p_k + p_a^{(k)} + 2, b_k + p_a^{(k)}]$  where  $p_k = \min_{b \in \mathbb{A}} \{p_b^{(k)}\}$ . Let

$$F(a,n) = \sum_{\substack{k \in \omega \\ n \in I_{k,a}}} (d(n, I_{k,a}) + 1),$$
(1)

where for  $n \in I_{k,a}$ , the quantity  $d(n, I_{k,a})$  denotes the minimal distance from n to the endpoints of the interval  $I_{k,a}$ . Then the number of closed factors of x for each length n is  $f_x^c(n) = \sum_{a \in \mathbb{A}} F(a, n)$ .

For each fixed  $n \in \mathbb{N}$  and  $a \in \mathbb{A}$ , the sum in (1) is finite since it only involves those k for which  $n \in I_{k,a}$ .

As a corollary of Theorem 1, we show that if x is an Arnoux-Rauzy word, then lim inf  $f_x^c(n) = +\infty$ . In contrast, it follows from [23] that if x is the regular paperfolding word, then lim inf  $f_x^c(n) = 0$ , in other words, for infinitely many n, all factors of x of length n are open.

We end this section by recalling a few basic notions in combinatorics on words relevant to the paper. Throughout this text A will denote a finite non-empty set (the alphabet). For  $n \in \mathbb{N}$ , let  $\mathbb{A}^n$  denote the set of all words  $a_1a_2\cdots a_n$  with  $a_i \in \mathbb{A}$ . For  $u = a_1a_2\cdots a_n \in \mathbb{A}^n$ , we let  $\overline{u} \in \mathbb{A}^n$  denote the reversal of u, i.e.,  $\overline{u} = a_na_{n-1}\cdots a_1$ . Let  $\mathbb{A}^+ = \bigcup_{n\in\mathbb{N}} \mathbb{A}^n$  denote the free semigroup generated by A. For  $u = a_1a_2\cdots a_n \in \mathbb{A}^+$  the quantity n is called the length of u and denoted |u|. We set  $\mathbb{A}^* = \mathbb{A}^+ \cup \{\varepsilon\}$  where  $\varepsilon$  is the empty word (of length equal to 0). We let  $\mathbb{A}^\mathbb{N}$  denote the set of all infinite words  $a_1a_2a_3\cdots$  with  $a_i \in \mathbb{A}$ . For  $x \in \mathbb{A}^+ \cup \mathbb{A}^\mathbb{N}$  and  $v \in \mathbb{A}^*$  we say that v is a factor of x if x = uvy for some  $u \in \mathbb{A}^*$  and  $y \in \mathbb{A}^* \cup \mathbb{A}^\mathbb{N}$ . We let  $\operatorname{Fac}(x)$  denote the set of all factors of x. A factor v of x is called right (resp. left) special if va and vb (resp. av and bv) are each factors of x for some choice of distinct  $a, b \in \mathbb{A}$ . A factor which is both right and left special is said to be bispecial. Given factors u and v of x, we say that u is a first return to v in x if uv is a factor of x having precisely two first occurrences of v, one as a prefix and one as a suffix. In which case the word uv is called a complete first return to v.

### 2 Counting closed factors in Arnoux-Rauzy words

Throughout this section we let  $\mathbb{A}$  denote a finite set of cardinality  $t \geq 2$ . A recurrent word  $x \in \mathbb{A}^{\mathbb{N}}$  is called an Arnoux-Rauzy word if x contains, for each  $n \geq 0$ , precisely one right special factor  $R_n$  of length n and one left special factor  $L_n$  of length n. Furthermore,  $R_n$  is a prefix of t-many distinct factors of x of length n + 1 while  $L_n$  is a suffix of t-many distinct factors of x of length n + 1. In particular one has  $p_x(n) = (t-1)n + 1$  and each factor u of x has precisely t distinct complete first returns. In the special case of a binary alphabet, we see that x is a Sturmian word. Arnoux-Rauzy words constitute a special class of episturmian words (see [3, 12, 16]) and hence each factor u of an Arnoux-Rauzy word is (palindromically) rich, i.e., ucontains exactly |u| + 1 many distinct palindromic factors (including the empty word  $\varepsilon$ ). We will make use of the following alternative characterisation of rich words given in [12].

**Lemma 2.1.** [Proposition 3 in [12]] A word  $u \in \mathbb{A}^+$  is rich if and only if for every prefix v of u, the longest palindromic suffix of v is uni-occurrent in v.

Let us now fix an Arnoux-Rauzy word  $x \in \mathbb{A}^{\mathbb{N}}$ . Recall that for each length  $n \in \omega$ an Arnoux-Rauzy word contains either zero or one bispecial factor of length n. Let  $\varepsilon = B_0, B_1, B_2, \ldots$  be the sequence of bispecial factors of x ordered according to increasing length. Put  $b_k = |B_k|$  so that  $0 = b_0 < b_1 < b_2 < \cdots$ . We recall the following characterization of the bispecial factors  $B_k$  of x in terms of palindromic closures (see [10, 12]). For each  $k \in \mathbb{N}$  there exists a unique  $a_k \in \mathbb{A}$  such that  $B_{k-1}a_k$ is a left special factor of x. The sequence  $(a_k)_{k\in\mathbb{N}}$  is called the directive sequence of x. It follows that  $B_{k-1}a_k$  is a prefix of  $B_k$  but in fact  $B_k$  is the palindromic closure of  $B_{k-1}a_k$ , i.e., the shortest palindrome beginning in  $B_{k-1}a_k$ . More precisely, if we let  $S_k$  denote the longest palindromic suffix of  $B_{k-1}a_k$  and write  $B_{k-1}a_k = x_kS_k$  with  $x_k \in \mathbb{A}^*$ , then  $B_k = x_kS_k\overline{x_k}$  (see for instance Lemma 5 in [10] in case the alphabet  $\mathbb{A}$  is binary).

**Lemma 2.2.** For each  $k \in \mathbb{N}$  we have that  $S_k$  is a uni-occurrent factor of  $B_k$ . In particular  $S_k$  is not a factor of  $B_{k-1}$ .

*Proof.* Clearly  $S_k$  is a factor of  $B_k$ . To see that it is uni-occurrent, suppose that  $S_k$  occurs more than once in  $B_k$ . Since  $S_k$  and  $B_k$  are each palindromes and  $B_k = x_k S_k \overline{x_k}$ , it follows that  $S_k$  occurs at least twice in  $x_k S_k = B_{k-1}a_k$ . But this contradicts Lemma 2.1 since  $S_k$  was defined as the longest palindromic suffix of  $B_{k-1}a_k$ .  $\Box$ 

Define  $\varphi$ : Fac $(x) \to \omega$  by  $\varphi(v)$  is the least  $k \in \omega$  such that v is a factor of  $B_k$ . In particular  $\varphi(v) = 0 \Leftrightarrow v = \varepsilon$ .

**Lemma 2.3.** Let  $k \in \mathbb{N}$  and  $v \in \mathbb{A}^+$ . Then  $v \in \varphi^{-1}(k)$  if and only if v is a factor of  $B_k$  containing  $S_k$  as a factor. In particular each  $v \in \varphi^{-1}(k)$  is uni-occurrent in  $B_k$ .

*Proof.* Suppose v is a factor of  $B_k$  containing  $S_k$  as a factor. Then by Lemma 2.2, v is not a factor of  $B_{k-1}$  and hence not a factor of any  $B_j$  with j < k. Hence  $\varphi(v) = k$ . Conversely suppose that  $\varphi(v) = k$ . Then v is a factor of  $B_k$  but not of  $B_{k-1}$ . Since

$$B_k = x_k S_k \overline{x_k} = B_{k-1} a_k \overline{x_k} = x_k a_k B_{k-1},$$

it follows that v must contain  $S_k$  as a factor. Having established that each  $v \in \varphi^{-1}(k)$  contains  $S_k$ , it follows by Lemma 2.2 that v is uni-occurrent in  $B_k$ .

For each  $k \in \omega$  and  $a \in \mathbb{A}$ , let  $R_a^{(k)}$  denote the complete first return to  $B_k$  in x beginning in  $B_k a$  and put  $p_a^{(k)} = |R_a^{(k)}| - b_k$ . In other words  $p_a^{(k)}$  is the length of the first return to  $B_k$  determined by  $R_a^{(k)}$ . We note that  $R_a^{(0)} = a$  for each  $a \in \mathbb{A}$ . The sequence  $(p_a^{(k)})_{a\in\mathbb{A}}^{k\in\omega}$  is computed recursively as follows :  $p_a^{(0)} = 1$  for each  $a \in \mathbb{A}$ . For  $k \ge 1$ , we have  $p_{a_k}^{(k)} = p_{a_k}^{(k-1)}$ , and  $p_b^{(k)} = p_b^{(k-1)} + p_{a_k}^{(k-1)}$  for  $b \in \mathbb{A} \setminus \{a_k\}$ . It is easily verified by induction that

$$b_k = \frac{\sum_{a \in \mathbb{A}} p_a^{(k)} - t}{t - 1}$$

For each  $k \in \mathbb{N}$ , we set  $p_k = p_{a_k}^{(k)}$ . Since  $B_k$  is a complete first return to  $B_{k-1}$  beginning in  $B_{k-1}a_k$  i.e.,  $R_{a_k}^{(k-1)} = B_k$ , it follows that

$$p_k = p_{a_k}^{(k)} = p_{a_k}^{(k-1)} = |R_{a_k}^{(k-1)}| - b_{k-1} = b_k - b_{k-1}.$$
 (2)

It follows immediately from our recursive definition of the  $p_a^{(k)}$  that

$$p_k = \min\{p_a^{(k)} | a \in \mathbb{A}\}$$

**Lemma 2.4.** Let  $k \in \mathbb{N}$  and let  $J_k$  denote the interval  $[b_k - 2p_k + 2, b_k]$ . If  $v \in \varphi^{-1}(k)$  then  $|v| \in J_k$  and, for each  $m \in J_k$ , the set  $\varphi^{-1}(k)$  contains precisely  $d(m, J_k) + 1$  distinct words of length m, where  $d(m, J_k)$  is the minimal distance between m and the two boundary points of the interval  $J_k$ . In particular  $|\varphi^{-1}(k)| = p_k^2$ .

Proof. In view of Lemma 2.3 we have that  $v \in \varphi^{-1}(k)$  if and only if v is a factor of  $B_k$  which contains  $S_k$  as a subfactor It follows that  $|S_k| \leq |v| \leq |B_k|$ . Also, since  $B_k = B_{k-1}a_k\overline{x_k}$ , by (2) we deduce that  $p_k = b_k - b_{k-1} = |x_k| + 1$ . Furthermore, as  $B_k = x_kS_k\overline{x_k}$  we have  $|S_k| = |B_k| - 2|x_k| = b_k - 2(p_k - 1) = b_k - 2p_k + 2$ . Hence  $b_k - 2p_k + 2 \leq |v| \leq b_k$ . Now suppose  $m \in J_k$ . To see that  $\varphi^{-1}(k)$  contains  $d(m, J_k) + 1$ distinct words of length m we simply use the fact that each  $v \in \varphi^{-1}(k)$  contains  $S_k$ and is uni-occurrent in  $B_k$  (see Lemma 2.3). Finally,

$$|\varphi^{-1}(k)| = 1 + 2 + \dots + (p_k - 1) + p_k + (p_k - 1) + \dots + 2 + 1 = 2\left(\frac{p_k(p_k - 1)}{2}\right) + p_k = p_k^2.$$

Let  $k \in \mathbb{N}$  and  $v \in \varphi^{-1}(k)$ . As a consequence of Lemma 2.3, there exists a unique decomposition  $B_k = u_1 v u_2$  with  $u_1, u_2 \in \mathbb{A}^*$ . In particular,  $v u_2$  is right special in x and  $u_1 v$  is left special in x. Now suppose u is a closed factor of x with frontier v. In particular u begins and ends in v. Since x is recurrent and aperiodic, it follows that  $v u_2$  is a prefix of u and  $u_1 v$  is a suffix of u, whence  $u_1 u u_2$  is a complete first return to  $B_k$ . In fact,  $u_1 u u_2$  begins and ends in  $B_k$  and does not admit other occurrences of  $B_k$  for otherwise v would occur in u internally (meaning not as a prefix or as a suffix). Thus  $u_1 u u_2 = R_a^{(k)}$  for some  $a \in \mathbb{A}$ .

**Definition 2.5.** Let u be a closed factor of x and  $a \in \mathbb{A}$ . We say u is of type a if and only if either u = a or, if u is closed with frontier  $v \in \mathbb{A}^+$ , then  $u_1 u u_2 = R_a^{(k)}$  where  $k = \varphi(v)$  and  $B_k = u_1 v u_2$ .

If u is a closed factor of x of type  $a \in \mathbb{A}$  and frontier  $v \in \mathbb{A}^+$ , then

$$|u| - |v| = |R_a^{(k)}| - (|u_1| + |u_2| + |v|) = |R_a^{(k)}| - |B_k| = p_a^{(k)},$$
(3)

where  $k = \varphi(v)$ . We observe that the equality  $|u| - |v| = p_a^{(k)}$  in (3) also holds in case  $u \in \mathbb{A}$  taking  $v = \varepsilon$  and k = 0.

Let C(x) denote the set of all closed factors of x and for each  $u \in C(x)$  let  $\operatorname{fr}(u) \in \mathbb{A}^*$  denote its frontier. By convention we define  $\operatorname{fr}(a) = \varepsilon$  for each  $a \in \mathbb{A}$ . For each  $k \in \omega$  and  $a \in \mathbb{A}$  we let  $C_{k,a}(x)$  denote the set of all closed factors u of x of type a whose frontier  $\operatorname{fr}(u)$  belongs to  $\varphi^{-1}(k)$ .

**Lemma 2.6.** The sets  $\{C_{k,a}(x) : k \in \omega, a \in \mathbb{A}\}$  define a partition of C(x) and  $fr: C_{k,a}(x) \to \varphi^{-1}(k)$  is a bijection.

Proof. Each closed factor  $u \in C(x)$  has a unique type and its frontier fr(u) belongs to  $\varphi^{-1}(k)$  for a unique value of  $k \in \omega$ . Whence each closed factor u of x belongs to a unique  $C_{k,a}(x)$ . By definition, if  $u \in C_{k,a}(x)$  then  $fr(u) \in \varphi^{-1}(k)$ . Moreover u is uniquely determined by its frontier fr(u) and its type. In fact, if  $u \in C_{k,a}(x)$ then  $u_1uu_2 = R_a^{(k)}$  where  $u_1, u_2$  are determined by the (unique) factorization  $B_k =$  $u_1fr(u)u_2$ . This proves fr is injective. To see that fr is also surjective, let  $v \in \varphi^{-1}(k)$ . Then we can write  $B_k = u_1vu_2$  for some  $u_1, u_2 \in \mathbb{A}^*$ . Hence  $R_a^{(k)}$  begins in  $u_1$  and ends in  $u_2$ . It follows that  $u = u_1^{-1}R_a^{(k)}u_2^{-1}$  is a closed factor of x of type a and fr(u) = v.

*Proof of Theorem 1.* Fix  $n \in \mathbb{N}$ . By Lemma 2.6 we have

$$f_x^c(n) = |C(x) \cap \mathbb{A}^n| = \sum_{\substack{k \in \omega \\ a \in \mathbb{A}}} |C_{k,a}(x) \cap \mathbb{A}^n|.$$

Now assume  $u \in C_{k,a} \cap \mathbb{A}^n$  and put  $v = \operatorname{fr}(u) \in \varphi^{-1}(k)$ . Then by (3) we have that  $n = |u| = |v| + p_a^{(k)}$ . By Lemma 2.4,  $|v| = n - p_a^{(k)} \in J_k = [b_k - 2p_k + 2, b_k]$ . By Lemma 2.6 the number of words  $u \in C_{k,a}(x) \cap \mathbb{A}^n$  is equal to the number of words  $v \in \varphi^{-1}(k)$  of length  $n - p_a^{(k)}$  which by Lemma 2.4 is equal to  $d(n - p_a^{(k)}, J_k) + 1 = d(n, I_{k,a}) + 1$  where  $I_{k,a} = [b_k - 2p_k + p_a^{(k)} + 2, b_k + p_a^{(k)}]$ . This completes the proof of Theorem 1.

In case  $|\mathbb{A}| = 2$ , i.e., x is Sturmian, each bispecial factor  $B_k$  has precisely two first returns, the shortest one is of length  $p_k$ , and we let  $q_k$  denote the length of the other first return. So for fixed  $a \in \mathbb{A}$  and  $k \in \mathbb{N}$  we have

$$p_a^{(k)} = \begin{cases} p_k, & \text{if } a = a_k; \\ q_k, & \text{otherwise.} \end{cases}$$

If  $a = a_k$  then  $I_{k,a} = [q_k, q_k + 2p_k - 2]$  and if  $a \neq a_k$  then  $I_{k,a} = [2q_k - p_k, 2q_k + p_k - 2]$ . Putting  $P_k = [q_k, q_k + 2p_k - 2]$  and  $Q_k = [2q_k - p_k, 2q_k + p_k - 2]$ , we obtain that for a Sturmian word x the number of closed factors of x of each length n is given by

$$f_x^c(n) = \sum_{\substack{k \in \omega \\ n \in P_k}} (d(n, P_k) + 1) + \sum_{\substack{k \in \omega \\ n \in Q_k}} (d(n, Q_k) + 1).$$
(4)

**Example 2.7.** Consider the Fibonacci word

fixed by the morphism  $a \mapsto ab$ ,  $b \mapsto a$ . Then  $p_k = F_k$  and  $q_k = F_{k+1}$  where the sequence  $(F_k)_{k \in \omega}$  is the Fibonacci sequence given by  $F_0 = F_1 = 1$  and  $F_{k+1} = F_k + F_{k-1}$  for  $k \ge 1$ .

Table 1 shows the number of closed factors of length  $n \leq 15$  in the Fibonacci word computed using (4).

Table 1: The number of closed factors in the Fibonacci word.

n	1	<b>2</b>	3	4	5	6	7	8	9	10	11	12	13	14	15
$\int f^{c}_{x_{fib}}(n)$	2	1	2	3	4	3	4	5	6	5	6	7	8	9	10

For example, for n = 11 we must determine those k for which either  $11 \in P_k$  or  $11 \in Q_k$ . It is easily checked that 11 only belongs to  $P_4 = [8, 16], Q_3 = [7, 11]$  and  $Q_4 = [11, 19]$ . So

$$f_{x_{fib}}^{c}(11) = d(11, P_4) + 1 + d(11, Q_3) + 1 + d(11, Q_4) + 1 = 4 + 1 + 1 = 6.$$

The graph of the function  $f_{x_{fib}}^c$  is shown in Figure 1. The function is clearly not monotone.



Figure 1: The number of closed factors in the Fibonacci word.

Figure 2 illustrates the behavior of the number of closed factors of the Tribonacci word  $x_{trib} \in \{a, b, c\}^{\mathbb{N}}$  defined as the fixed point of the morphism  $a \mapsto ab, b \mapsto ac, c \mapsto a$ .



Figure 2: The number of closed factors in the Tribonacci word.

Our last example (Figure 3) illustrates the behavior of the number of closed factors of the Sturmian word  $x_r \in \{0, 1\}^{\mathbb{N}}$  whose directive sequence begins with 00100000111101111010110011000011....



Figure 3: The number of closed factors in the word  $x_r$ .

The above examples suggest that the function  $f_x^c(n)$  tends to infinity, although it need not be monotone and may contain plateaus and inflection points. Our next result establishes this fact:

**Corollary 2.8.** If  $x \in \mathbb{A}^{\mathbb{N}}$  is an Arnoux-Rauzy word, then

$$\liminf_{n \to \infty} f_x^c(n) = +\infty.$$

Proof. For each  $k \in \omega$  set  $I_k = [b_k - p_k + 2, b_k + p_k]$ , i.e.,  $I_0 = [1, 1]$  and  $I_k = I_{a_k,k}$  for  $k \ge 1$ . Since  $p_k \ge 1$  for each  $k \in \mathbb{N}$  it follows that  $\mathbb{N} = \bigcup_{k \in \omega} I_k$ . Given  $m \in \mathbb{N}$ , pick j such that  $p_j - 1 > 2m$ , and put  $N = b_j + 2$ . We will show that  $f_x^c(n) \ge m$  for every  $n \ge N$ . Notice that since  $b_k = b_{k-1} + p_k$ , the left hand endpoint of the interval  $I_{j+1}$  is  $b_j + 2$ . Thus for each  $n \ge N$  there exists a positive integer  $k \ge j$ , such that  $n \in I_{k+1}$ . We have  $|I_{k+1}| = 2p_{k+1} - 2 \ge 2p_j - 2 > 4m$ . If  $d(n, I_{k+1}) \ge m$ , then it follows from Theorem 1 that  $f_w^c(n) \ge m$ . Otherwise we must have either i)  $b_{k+1} + p_{k+1} - n < m$  or ii)  $n - (b_k + 2) < m$ . In case i) we have

$$n - b_{k+1} > p_{k+1} - m \ge p_j - m > m + 1.$$
(5)

Since  $m+1 \ge 2$ , we have that n also belongs to  $I_{k+2}$ . We will show that  $d(n, I_{k+2}) \ge m$ . By (5) we have  $n - (b_{k+1} + 2) \ge m$ . Also

$$b_{k+2} + p_{k+2} - n \ge b_{k+2} + p_{k+2} - (b_{k+1} + p_{k+1}) = 2p_{k+2} - p_{k+1} \ge p_{k+2} \ge p_j > 2m + 1.$$

Thus  $d(n, I_{k+2}) \ge m$  and hence by Theorem  $1f_x^c(n) \ge m$ .

In case ii)  $n < b_k + 2 + m \le b_k + 2m + 1 < b_k + p_k$ , and hence  $n \in I_k$ . We will show that  $d(n, I_k) \ge m$ . In fact,  $b_k + p_k - n > p_k - 2 - m \ge p_j - 2 - m \ge m$ . Moreover, since  $n \in I_{k+1}$ , we have that  $n - (b_{k-1} + 2) \ge b_k + 2 - (b_{k-1} + 2) = p_k \ge p_j > 2m + 1$ , and thus  $d(n, I_k) \ge m$ .

While the previous result applies to Arnoux-Rauzy words, for a general aperiodic word x the limit inferior of the function  $f_x^c(n)$  need not be infinite. For example, in the case of the regular paperfolding word one has that  $\liminf_{n\to\infty} f_x^c(n) = 0$ . In fact, in [23] the authors exhibit an 11-state automaton which accepts the base 2 representation of those n for which there is a closed factor of the regular paperfolding word of length n (see Figure 1 in [23]). As another perhaps simpler example, let xbe the fixed point beginning in a of the 2-uniform morphism  $\varphi$  on the alphabet  $\{a, b, c, d\}$  given by  $\varphi : a \mapsto ac, b \mapsto ad, c \mapsto bc, d \mapsto bd$ . Then it is easily shown that all factors of x of length  $2^n$   $(n \in \mathbb{N})$  are open. We remark that this last example is closely related to the regular paperfolding word. In fact, the regular paperfolding word is the image of the fixed point of  $\varphi$  under the mapping which sends a, c to 0 and b, d to 1.

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