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Singular elastostatic fields near the notch vertex of a Mooney-Rivlin hyperelastic body

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Abstract

Within the framework of fully nonlinear fracture mechanics under plane deformation conditions, boundary value problem equations for the V-notch vertex problem in an incompressible Mooney-Rivlin material are deduced. Using an asymptotic procedure, the deformation, the Lagrange multiplier and stress fields near the notch vertex are computed and their principal properties are illustrated. First, it is shown that the singularity order depends on the V-notch angle. Second, a change of the singular term for a critical angle was revealed and a logarithmic singular term appears. Based on the results obtained, the deformed V-notch vertex lips near the tip was drawn. The V-notch vertex is shown to open, but not necessarily in a symmetric manner. Analysis of the Cauchy stress tensor shows that the component σ_{22} dominates the stress field. Some discrepancy with the linear theory have emerged.

Keywords: Asymptotic, Singular, Notch, Wedge, Vertex, Hyperelasticity, Incompressible.

Introduction

Nowadays, rubber and rubber-like elastomeric materials are in increasing use in different industrial products and engineering applications. In order to improve the design of such materials, it is of paramount importance to examine the deformation and stress fields around cracks, corners, voids of inclusions, and other material or geometrical imperfections and their effects in the fracture process. Nevertheless, in contrast to metallic materials, the deformation and stress fields with the fracture properties of rubber-like materials are more complicated due to their geometrical and material nonlinearity properties, which make investigating fracture behaviours relatively difficult and multiply the analysis methods : experimental (Breidenbach and Lake, 1981), (Quigley, 1990), (Thomas, 1994), (Balankin,

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1997), (Borret, 1998) and (Hamdi et al, 2007), theoretical (Knowles and Sternberg, 1973) (Podio-Guidolgie and Caffarelli, 1991) and (Bourdin et al, 2008) and numerical (Lund and Westmann, 1990), (Loppin, 2001) and (Legrain et al, 2005) among others.

Following these analyses, physical experiments as well as numerical analysis by the finite elements method (FEM) computations show that very high stress concentrations can occur in the vicinity of cracks, edges, corners and near interfaces where the material parameters are discontinuous (Borret, 1998). In fracture mechanics, the different fracture criteria are based on stress field in the body and therefore a good knowledge of its analytical form is important. Moreover, it is well known that the convergence rates of standard FEM decrease at the presence of stress singularities (Strang and Fix, 1973). Here, the a-priori knowledge of the stress field can be used to develop improved algorithms where e.g. special singular functions are included in the FE-spaces (Strang and Fix, 1973), (Destuynder et al, 1982) and (Moës et al, 1999).

Thus, the objective of this paper is to investigate and analyse the elastostatic fields at the notch vertex for plane transformation of an incompressible Mooney-Rivlin hyperelastic material. The choice of the notch geometrical configuration, with a notch angle ω , enable us to reproduce: a wedge for $0 < \omega < \frac{\pi}{2}$, a notch for $\frac{\pi}{2} < \omega < \pi$ and a crack for $\omega = \pi$.

A literature overview provides several methods for the evaluation and the prediction of the elastostatic fields: deformation and stress fields. Namely, three methods shall be drawn: asymptotic development, complex variables and transform methods. Particular attention will be given to the asymptotic development method which is used in this work.

In the linear elastic (elliptic) problems, the elastostatic fields occurring in the vicinity of the vertex of a two-dimensional notch was analysed by these three methods. The asymptotic development method is used to study the local homogeneous and composite notch problem under different boundary conditions. It was shown that the solutions can be made by a separation of variables (also called power type singularities) for a class of geometrical configurations, one or multi-materials configurations and/or homogeneous or inhomogeneous boundary conditions configurations (Wieghardt, 1907), (Westergaard, 1939), (Williams, 1959, 1957, 1952,). This only power type singularities method is not valid for some configurations and the power logarithm type singularities can occur (Bogy, 1972), (Dempsey, 1995). It was shown by Grisvard (1992) that the general solution of the linear boundary value problems is an asymptotic development composed by a linear combination of a power and logarithm types singularities and this solution generalises and unifies the proposed solutions of the mechanical community. The complex potential approach was proposed by (Williams, 1956) to give a complete solution.

A third technique, proposed by (Bogy, 1968), is devoted to the application of the Mellin transform. The analysis of a homogeneous linear elastic notch problem, done with traction-free surfaces, pointed out that as long as the notch is convex (namely, in the case of a wedge) there is no stress singularity, and that when the notch is concave stress singularities usually exist except for the case in which the original problem is antisymmetric about the symmetric axis of the notch (mode II problem) and the notch angle ω is less than 0.72π (Xiaolin, 1986) and (Sweryn and Molski, 1996).

For nonlinear power-type constitutive laws behaviour with small deformation, the pioneer works of (Hutchinson, 1968) and (Rice and Roengren, 1968) for a crack with traction-free surfaces showed that the asymptotic development is made by a power-type singularities. The same conclusion was deduced for a notch problem (Xia and Wang, 1993), (Yang et al, 1992) and (Yuan and Lin, 1994). Nevertheless, to our knowledge a mathematical proof of this conjecture is unknown until nowadays. The Linear Elastic Fracture Mechanics (LEFM) and the Elasto-Plastic Fracture Mechanics (EPFM) approaches described below played a prominent role in the investigation and comprehension of the crack, defect and singular problems. However, these approaches are based on the kinematic assumption of small deformations which is in contradiction with the unbounded strain field deduced.

Within the framework of finite deformation (Ogden, 1997), in the paste five decades, only few work have been focused on the analysis of the strain and stress fields around a crack, notch, defect, This is due to the formidable complexity of the mathematical problem (Ogden, 1997) which makes the boundary-value problem equations highly nonlinear and very difficult to solve analytically or even numerically. We note that (Wong and Shield, 1996) carried the first analysis of an infinite Neo-Hookean sheet containing a finite crack.

In the early **1970s**, (Knowles and Sternberg , 1974, 1973) analysed the asymptotic deformation field near the tip of a Mode-I plane strain crack for generalized Blatz-Ko compressible hyperelastic solids. Their analysis of the crack problem within the framework of nonlinear elasticity is considered as a fundamental work. Among others researchers, (Stephenson, 1982) gave the most clarified presentation of the local structure of the elastostatic fields near the crack tip of a generalised Mooney-Rivlin solid under plane deformation condition and mixed boundary conditions at infinity (Mode I and II). He showed that the crack-faces tip will open symmetrically, under Mode II conditions, contrary to the predictions of linear theory. In other words, the nonlinear global crack problem cannot admit an antisymmetric solution. Using a similar approach, (Le, 1992), (Le and Stumpf, 1993), (Geubelle and Knauss, 1994) , (Geubelle, 1995) and (Tarantino, 1996) performed a similar study as Knowles and Sternberg (1973,1974) with others incompressible or compressible hyperelastic potentials and plane

deformation or stress conditions. They arrived to the same conclusion as (Stephenson, 1982) that the crack faces tips are expected to open symmetrically both in Mode I and II. The condition of the crack-faces tip opening or penetrating was analyzed by (Knowles, 1981), (Chow et al, 1986) and (Ru, 2002) and they showed its dependence on the material behaviour. To our knowledge, the first finite-strain analysis of the singularities near the vertex of an arbitrary wedge are those of (Ru, 1997) in plane deformation for compressible harmonic material combining the asymptotic development with the complex variables methods and Tarantino (1997, 1998) in plane stress with the asymptotic development method. At least, we note also the work of Gao and Gao (1996) who used an alternative approach by dividing the singular field into shrinking and expanding sectors for which the asymptotic equations are derived separately

The analysis proposed in the present work is closely related to the work developed by Knowles and Sternberg (1974, 1973) and Stephenson (1982) for crack problem. The V-notch problem is formulated and solved for an incompressible hyperelastic material under plane deformation condition in a fully nonlinear context. In order to calculate the deformation and stress fields near the notch vertex an asymptotic analysis is carried out. Finally, the structure of the singular deformation field is examined in detail. Emphasis is placed on describing the notch-profile after deformation, proving Stephenson's (1982) conjecture in our context and evaluating the asymptotic order of elastostatic fields. The most important differences with respect to the predictions of the linear theory are evidenced and discussed.

1. Formulation of the global notch problem

Consider an isotropic homogenous incompressible hyperelastic body \mathcal{B} which, in its undeformed configuration, occupies an infinite region \mathcal{R}_0

$$\mathcal{R}_0 = \{\underline{x} \mid (x_1, x_2) \in \Omega_0, -\infty < x_3 < +\infty\}, \quad (1)$$

where \underline{x} is the position vector of the particle in the undeformed configuration and Ω_0 denotes a cross-section of \mathcal{R}_0 . Then, the plane domain Ω_0 can be described by a polar coordinates system

$$\Omega_0 = \{(r, \theta) \mid r \in [0, +\infty[, \theta \in [-\omega, \omega], \omega \in]0, \pi]\}. \quad (2)$$

Here (r, θ) are the material polar coordinates of the particle and ω is the notch angle, figure 1. Three geometrical configurations may be reproduced

- A wedge for $\omega \in]0, \frac{\pi}{2}[$.
- A notch for $\omega \in]\frac{\pi}{2}, \pi[$.
- A crack for $\omega = \pi$.

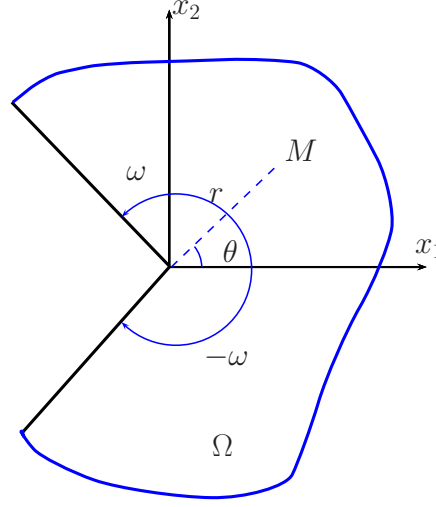


Figure 1: The notch problem

We assume that the three-dimensional body \mathcal{B} is subjected to an invertible plane transformation so that the position of material point \underline{x} (x_1, x_2, x_3) after transformation is mapped to \underline{y} (y_1, y_2, y_3) such as

$$y_\alpha = y_\alpha(x_1, x_2) \quad \forall (x_1, x_2) \in \Omega_0, \quad x_\alpha = x_\alpha(r, \theta) \quad \alpha \in \{1, 2\} \text{ and } y_3 = x_3. \quad (3)$$

Then the domain Ω_0 is transmuted to a domain Ω of the same plane. Assume that the transformation y_α is continuous and twice continuously differentiable at least on Ω_0 , ($y_\alpha \in C^2(\Omega_0)$). To describe the geometry deformation, a two-dimensional second order tensor, the transformation gradient $\underline{\underline{F}}(\underline{x})$, is introduced :

$$\underline{\underline{F}}(\underline{x}) = \underline{\underline{\nabla}}_{\underline{x}} \underline{y}(\underline{x}), \quad F_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta} \quad \{\alpha, \beta\} \in \{1, 2\} \quad \text{on } \Omega_0, \quad (4)$$

where $\underline{\underline{\nabla}}_{\underline{x}}(\bullet)$ is the gradient operator with respect to material coordinates. Then the incompressibility constraint leads to

$$J = \det(\underline{\underline{F}}) = 1 \Leftrightarrow J = \frac{1}{r} \left\{ \frac{\partial y_1}{\partial r} \frac{\partial y_2}{\partial \theta} - \frac{\partial y_2}{\partial r} \frac{\partial y_1}{\partial \theta} \right\} = 1 \quad \text{on } \Omega_0. \quad (5)$$

Remark 1

We refer to Abeyaratne (1980) and Stephenson (1982) for a concise introduction to the boundary value problems of plane transformation within the framework of incompressible hyperelasticity

For hyperelastic material, the existence of an elastic potential function W per unit undeformed volume is assumed. We introduce a particular class of polyconvex incompressible material governed by the

Mooney-Rivlin potential (Ogden, 1997). In the case of plane transformation, this potential take the following expression (Stephenson, 1982) :

$$W(I) = \frac{\mu}{2}(I - 2), \quad \begin{cases} \mu > 0 \\ I = \text{tr}(\underline{\underline{F}}^T \underline{\underline{F}}) \end{cases} \quad \text{on } \Omega_0, \quad (6)$$

where $(\bullet)^T$ and $\text{tr}(\bullet)$ denote the transpose and the trace operators, respectively, and μ is the shear modulus. Then, the first Piola-Kirchhoff two-dimensionnal stress tensor $\underline{\underline{\tau}}$ can be deduced (Ogden, 1997) :

$$\underline{\underline{\tau}} = \frac{\partial W}{\partial \underline{\underline{F}}} - q \underline{\underline{F}}^{-T} = \mu \underline{\underline{F}} - q \underline{\underline{F}}^{-T} \quad \text{on } \Omega_0. \quad (7)$$

In the deformed configuration, the Cauchy stress tensor in terms of potential energy can be inferred :

$$\underline{\underline{\sigma}} = \underline{\underline{\tau}} \underline{\underline{F}}^T = \frac{\partial W}{\partial \underline{\underline{F}}} \underline{\underline{F}}^T - q \underline{\underline{I}} = \mu \underline{\underline{F}} \underline{\underline{F}}^T - q \underline{\underline{I}} \quad \text{on } \Omega \quad (8)$$

In (7) and (8), q denotes an unknown Lagrange multiplier field resulting from the constraint of incompressibility and $\underline{\underline{I}}$ denotes the unit second order tensor. By neglecting body forces, the equilibrium problem in two-dimensional is governed by the following equation :

$$\text{Div}(\underline{\underline{\tau}}) = \underline{\underline{0}} \quad \text{on } \Omega_0 \Leftrightarrow \begin{cases} \frac{\partial q}{\partial r} = \mu \frac{\partial y_\alpha}{\partial r} \Delta y_\alpha \\ \frac{\partial q}{\partial \theta} = \mu \frac{\partial y_\alpha}{\partial \theta} \Delta y_\alpha \end{cases} \quad \text{on } \Omega_0, \text{ sum on } \alpha \quad (9)$$

where $\text{Div}(\bullet)$ and $\Delta(\bullet)$ are the divergence and the Laplace operators with respect to material coordinates, respectively, and with the convention of implicit sum on repeated indices.

The local notch problem may now be defined as follows: given an elastic potential $W(I)$ (6), we seek a plane transformation $\underline{\underline{y}}$ on Ω satisfying the incompressibility constraint (5), as well as a first Piola-Kirchhoff stress field $\underline{\underline{\tau}}$ and a Lagrange multiplier field q such that (7) holds, while $\underline{\underline{\tau}}$ satisfies the equilibrium equations (9); in addition the transformation $\underline{\underline{y}}$ should verify the boundaries conditions at infinity (far from the notch vertex) and at the notch vertex faces.

At infinity, the boundary conditions prescribed on the transformation $\underline{\underline{y}}$ should be compatible with the kinematic loading conditions (general mixed-mode loading) as follows :

$$\underline{\underline{y}}(\underline{\underline{x}}) = \underline{\underline{F}}^\infty \underline{\underline{x}} + O(1) \quad \text{as } \|\underline{\underline{x}}\| \rightarrow \infty, \quad (10)$$

where $\underline{\underline{F}}^\infty$ designates a known constant tensor which can be characterised by a combination of modes I and II loadings conditions

$$\begin{bmatrix} F_{\alpha\beta}^\infty \end{bmatrix} = \begin{bmatrix} \lambda^{\infty-1} & 0 \\ 0 & \lambda^{\infty-1} \end{bmatrix} \quad \text{on mode I}, \quad (11)$$

$$\begin{bmatrix} F_{\alpha\beta}^{\infty} \end{bmatrix} = \begin{bmatrix} 1 & k^{\infty} \\ 0 & 1 \end{bmatrix} \quad \text{on mode II} \quad (12)$$

and $\lambda^{\infty} > 0$ and $k^{\infty} > 0$ are respectively the principal stretches and the amount of shear prescribed at infinity.

To satisfy traction-free boundaries conditions at the notch faces, the following conditions on stress tensors is assumed

$$\underline{\tau} \underline{n}_0 = \underline{0} \quad \Leftrightarrow \quad \underline{\sigma} \underline{n} = \underline{0} \quad (13)$$

Where \underline{n} and \underline{n}_0 denote respectively the unit normal vector to notch vertex faces in the deformed and the undeformed configurations. The traction free boundaries conditions (13) lead

$$\text{For } \theta = \pm\omega \quad \begin{cases} \tau_{r\theta} = 0 \\ \tau_{\theta\theta} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial y_{\alpha}}{\partial r} \frac{\partial y_{\alpha}}{\partial \theta} = 0 \\ \frac{\mu}{r^2} \frac{\partial y_{\alpha}}{\partial \theta} \frac{\partial y_{\alpha}}{\partial \theta} = q \end{cases}, \text{sum on } \alpha \quad (14)$$

where $\tau_{r\theta}$ and $\tau_{\theta\theta}$ are the first Piola-Kirchoff stress components in the cylindrical basis. Coupling the two equilibrium equations of (9) and using the incompressibility equations (5), the local structure of the Lagrange multiplier field can be determined by

$$\mu \nabla^2 y_1 = \frac{1}{r} \left(\frac{\partial q}{\partial r} \frac{\partial y_2}{\partial \theta} - \frac{\partial q}{\partial \theta} \frac{\partial y_2}{\partial r} \right). \quad (15)$$

The equilibrium equation (9), Lagrange multiplier (15) equation and the boundary condition (14) form a boundary value problem which is very hard to solve (Ogden, 1997). In this case, the elastostatic fields near the notch vertex are sought. It is commonly known in the mechanical community that such points may cause stress singularities, which means that the stresses can be unbounded there.

Remark 2

The exact asymptotic solution of the nonlinear boundary value problem established below is not well known. On the contrary, many results was shown and established for linear elliptic boundary value problem for nonsmooth domains (Grisvard, 1992). Unfortunately, similar theorems and results of nonlinear elastic materials on nonsmooth domains are much less known (Borsuk and Konratiev, 2006). It is also an open problem whether the elastostatic fields can be completely expressed by an asymptotic expansion like in the linear case or not.

In this case, the elastostatic fields near the notch vertex are sought as

$$y_{\alpha}(r, \theta) = \sum_{k=1}^K \left(r^{m_k} U_{\alpha}^{(k)}(\theta) + r^{m_k} \ln(r) V_{\alpha}^{(k)}(\theta) \right) + o(r^{m_K}), \quad (16a)$$

$$q(r, \theta) = \sum_{k=1}^K \left(r^{l_k} P_k(\theta) + r^{l_k} \ln(r) Q_k(\theta) \right) + o(r^{l_K}), \quad (16b)$$

where the transformation and the Lagrange multiplier fields do not necessary have the same asymptotic expansion order. In other hand, the unknown angular functions appearing in (16a) and (16b) are at least twice and once continuously differentiable on $[-\omega, \omega]$ respectively.

Remark 3

The asymptotic expressions of the elastostatic fields proposed by (16a) and (16b) are slightly different from the exact analytical form for the linear elliptic equation (Grisvard, 1992). In fact, inspired by the work of Knowles and Sternberg (1974, 1973) and Stephenson (1982) where it was shown that the transformation fields have a linear logarithm term dependence for a critical hardening parameter, the asymptotic forms (16a) and (16b) are assumed and the advantages of this form will be illustrated in the sequel. One notes that a non-integer power dependence of the transformation fields on logarithm term was shown in Long et al (2011).

The local notch problem can now be announced as follows: the transformation \underline{y} and the Lagrange multiplier q must satisfy the field equations and the notch vertex boundary conditions that the surfaces of the notch are free.

In this work, the boundary conditions at infinity are not taken into account in the formulation of the local notch problem which make the problem not completely defined. However, the physical sense of the elastostatic fields deduced from the local formulation is related to the singular elastostatic field behaviour in the vertex region, namely, as $r \rightarrow 0$.

Finally, let \mathfrak{S} be the class of all $\{\underline{y}, \underline{\sigma}, q\}$ that satisfy the boundary value problems. Then it is easy to prove that :

$$\{\underline{y}, \underline{\sigma}, q\} \subset \mathfrak{S} \Leftrightarrow \{\underline{Q} \underline{y}, \underline{Q} \underline{\sigma} \underline{Q}^T, q\} \subset \mathfrak{S}, \quad (17)$$

where \underline{Q} is an orthogonal second order tensor. This is assured by the objectivity of the constitutive equation (7) and (13) and by the form of the boundary conditions. This property will be used to better understand the nature of the local transformation field that would be deduced later by Stephenson(1982).

2. First order asymptotic analysis of the elastostatic field near the notch vertex

First, we assume that the global notch problem admits a nontrivial solution. In order to characterize the singularity induced by the vertex presence, we assume that the first order development of the transformation admit the following form :

$$y_\alpha(r, \theta) = r^{m_1} (U_\alpha(\theta) + Ln(r) V_\alpha(\theta)) + o(r^{m_1}) \quad U_\alpha, V_\alpha \in C^2([-\omega, \omega]). \quad (18)$$

Here $U_\alpha(\theta)$ and $V_\alpha(\theta)$ are a real-valued functions on $[-\omega, \omega]$ and do not vanish identically for boundary conditions at infinity prescribed by (10). The exponent m_1 must be a real constant to avoid the appearance of oscillations arising in the linearized local solution and have to satisfy the inequality

$$0 \leq m_1 < 1, \quad (19)$$

in order to ensure bounded displacement at the notch vertex, only unbounded gradients are allowed. As suggested and shown by Stephenson (1982) and assumed by Knowles and Sternberg (1973), the exponent m_1 in (18) replaces, without loss of generality, an eventual pair of exponents $m_1^{(\alpha)}$. This is justified by the objectivity of the constitutive equations Stephenson (1982).

Remark 4

The requirement (19), imposed in Knowles and Sternberg (1973) for nonlinear problem, originated by Williams (1959) for linear (Hooke constitutive law) problems. For nonlinear constitutive laws there is not any special requirement on m_1 . As an example, the requirement (19) is relaxed by Tarantino (1997) in the case of constrained hyperelastic potentials to achieve a non singular solution, namely, the transformation gradient is no longer unbounded.

Finally, we suppose that the Lagrange multiplier field associated with the elastostatic field solution satisfies

$$\begin{cases} q(r, \theta) = r^{l_1} [P_1(\theta) + Ln(r)Q_1(\theta)] + o(r^{l_1}), \\ P_1(\theta), Q_1(\theta) \in C^1([-\omega, \omega]), \end{cases} \quad (20)$$

where l_1 being another real constant exponent.

So the problem here is to determine the smallest exponent $m_1 \in [0, 1[$ and the functions U_α and V_α appearing in (18) being consistent with the incompressibility constraint (5), the governing field equations (9) and boundary conditions (13).

The expression of the invariant I (6), together with the asymptotic development transformation (18), gives

$$\begin{aligned} I(r, \theta) = & r^{2(m_1-1)} [\dot{U}_1^2 + \dot{U}_2^2 + m_1^2 (U_1^2 + U_2^2) + V_1^2 + V_2^2 \\ & + (\dot{V}_1^2 + \dot{V}_2^2 + m_1^2 (V_1^2 + V_2^2)) Ln^2(r)] + o(r^{2(m_1-1)}), \end{aligned} \quad (21)$$

where the dot is the differentiation operation with respect to θ . Since the invariant I , defined by the relation (6), is strictly positive, we shall henceforth take for granted that U_1 , U_2 , V_1 and V_2 do not have a common multiple zero on $[-\omega, \omega]$ so that the coefficient of $r^{2(m_1-1)}$ in (21) does not vanish on $[-\omega, \omega]$.

The incompressibility constraint (5), together with the asymptotic development transformation (18), gives

$$\begin{aligned} J(r, \theta) = & r^{2m_1-2} [(\dot{U}_2(m_1 U_1 + V_1) - \dot{U}_1(m_1 U_2 + V_2)) \\ & + (m_1 V_1 \dot{U}_2 + m_1 U_1 \dot{V}_2 + V_1 \dot{V}_2 \\ & - m_1 V_2 \dot{U}_1 - m_1 U_2 \dot{V}_1 - V_2 \dot{V}_1) Ln(r) \\ & + m_1 (V_1 \dot{V}_2 - V_2 \dot{V}_1) Ln^2(r)] + o(r^{2m_1-2}) = 1. \end{aligned} \quad (22)$$

For $m_1 < 1$, when dividing the previous identity by $r^{2(m_1-1)}$ and proceeding to the limit as $r \rightarrow 0$ we obtain the following differential equation system :

$$m_1 U_1 \dot{U}_2 + V_1 \dot{U}_2 - m_1 U_2 \dot{U}_1 - V_2 \dot{U}_1 = 0, \quad (23a)$$

$$m_1 V_1 \dot{U}_2 + m_1 U_1 \dot{V}_2 + V_1 \dot{V}_2 - m_1 V_2 \dot{U}_1 - m_1 U_2 \dot{V}_1 - V_2 \dot{V}_1 = 0, \quad (23b)$$

$$V_1 \dot{V}_2 - V_2 \dot{V}_1 = 0. \quad (23c)$$

Equation (23c) leads to

$$V_1(\theta) = b_1 V(\theta) \quad \text{and} \quad V_2(\theta) = b_2 V(\theta), \quad (24)$$

where b_1 and b_2 are real constants. Equations (23a) and (23b) give

$$U_1(\theta) = \frac{b_1}{m_1} \left(V(\theta) \operatorname{Ln} |V(\theta)| + d_1 V(\theta) \right), \quad (25)$$

$$U_2(\theta) = \frac{b_2}{m_1} \left(V(\theta) \operatorname{Ln} |V(\theta)| + d_1 V(\theta) \right) + \frac{c}{b_1} V(\theta). \quad (26)$$

The constraints $U_\alpha(\theta) \in C^2([-\omega, \omega])$ and $V_\alpha(\theta) \in C^2([-\omega, \omega])$ impose that $b_1 = 0$ and $b_2 = 0$. So in this case, the logarithms terms must not appear in the expressions of y_1 and y_2 . Thus

$$U_\alpha = a_\alpha U \quad (\alpha = 1, 2) \quad \text{on} \quad [-\omega, \omega] \quad \text{and} \quad a^2 = a_1^2 + a_2^2 \neq 0 \quad (27)$$

in which a_α are real constants and U is an unknown real function.

Remark 5

It can be verified that a more general transformation development than (18) with nonlinear dependence on logarithm terms, like the linear elliptic boundary value problem proposed in (Grisvard, 1992), will give the same results, i.e. the absence of the logarithmic terms.

Combining the equilibrium equation (9) with the asymptotic development transformation (18) leads to

$$\begin{cases} \frac{\partial q}{\partial r} &= \mu m_1 r^{2m_1-3} a^2 U \left(m_1^2 U + \ddot{U} \right) + o(r^{2m_1-3}), \\ \frac{\partial q}{\partial \theta} &= \mu r^{2m_1-2} a^2 \dot{U} \left(m_1^2 U + \ddot{U} \right) + o(r^{2m_1-2}). \end{cases} \quad (28)$$

In (28) only dominating terms have been taken into account. On the other hand, the traction free local boundary conditions (13) with the asymptotic development transformation (18) yield

$$U \dot{U} = 0, \quad q = \mu a^2 r^{2(m_1-1)} \dot{U}^2 + o(r^{2(m_1-1)}) \quad \text{at} \quad \theta = \pm\omega. \quad (29)$$

Since U does not have a common multiple zero on $[-\omega, \omega]$ to ensure the strict positivity of the invariant I given by equation (21), we obtain from equations (28) and (29) that U satisfies

$$m_1^2 U + \ddot{U} = 0 \quad \text{on } [-\omega, \omega] \quad \text{and} \quad \dot{U}(\theta) = 0 \quad \text{at } \pm \omega. \quad (30)$$

The solution of the eigenvalue problem (30) is given by

$$U(\theta) = \sin(m_1 \theta) \quad \text{with} \quad m_1 = \frac{\pi}{2\omega}. \quad (31)$$

In the other part, the inequality $m_1 < 1$ implies

$$\omega > \frac{\pi}{2}. \quad (32)$$

Namely, the result (31) is singular for concave notch problems only. Consequently, the problem in the case of re-entrant notches (wedge, for $\omega \leq \frac{\pi}{2}$) does not admit a singular solutions. Therefore, the first order asymptotic solution for transformation is

$$y_\alpha(r, \theta) = a_\alpha r^{\frac{\pi}{2\omega}} \sin\left(\frac{\pi}{2\omega} \theta\right) + o\left(r^{\frac{\pi}{2\omega}}\right) \quad \text{as } r \rightarrow 0 \quad \text{and} \quad \omega > \frac{\pi}{2} \quad (33)$$

Nevertheless, such a solution provides the following weak estimate for the Jacobian of the transformation and the Lagrange multiplier

$$J(r, \theta) \sim o\left(r^{\left(\frac{\pi}{\omega}-2\right)}\right) \quad q(r, \theta) \sim o\left(r^{\left(\frac{\pi}{\omega}-2\right)}\right). \quad (34)$$

therefore inadequate. In fact, the Jacobian (34) is unbounded which reflects the degenerate character. The solution does not preserve the volume of the asymptotic approximation of the transformation (33) locally as $(r \rightarrow 0)$. The Lagrange multiplier in (34) has also a degenerate character. One concludes that the first asymptotic approximation (18) leads to a solution that presents a number of mathematical and physical inconsistencies and is therefore inadequate. Then a developement to a higher order should be achieved.

3. Second order asymptotic analysis of the elastostatic field near the notch vertex

The first order approximation of the local transformation at the vicinity of the notch vertex does not constitute an invertible mapping. Consequently, we should refine (18) by developing the following two terms approximation

$$y_\alpha(r, \theta) = a_\alpha r^{m_1} U(\theta) + r^{m_2} (V_\alpha(\theta) + L n(r) W_\alpha(\theta)) + o(r^{m_2}) \quad V_\alpha, W_\alpha \in C^2, \quad (35)$$

where $m_2 > m_1$, $V_\alpha(\theta)$ and $W_\alpha(\theta)$ are as yet undetermined, whereas m_1 and U are now given by (31). From the incompressibility constraint (5) and the asymptotic development transformation (35), one

can write

$$J = r^{m_1+m_2-2} \left\{ \left[m_1 U \dot{\Psi}_2 - m_2 \dot{U} \Psi_2 - \dot{U} \Phi_2 \right] + \left[m_1 U \dot{\Phi}_2 - m_2 \dot{U} \Phi_2 \right] L n(r) \right\} + o(r^{m_1+m_2-2}), \quad (36)$$

where

$$\Psi_2 = a_1 V_2 - a_2 V_1 \quad \text{and} \quad \Phi = a_1 W_2 - a_2 W_1. \quad (37)$$

Consequently

$$m_1 + m_2 - 2 \leq 0. \quad (38)$$

Analysis of the equation (36) leads to

$$m_1 U \dot{\Phi}_2 - m_2 \dot{U} \Phi_2 = 0 \quad \text{on} \quad [-\omega, \omega] \quad \text{if} \quad m_1 < m_2 \leq 2 - m_1, \quad (39a)$$

$$m_1 U \dot{\Psi}_2 - m_2 \dot{U} \Psi_2 - \dot{U} \Phi_2 = 0 \quad \text{on} \quad [-\omega, \omega] \quad \text{if} \quad m_1 < m_2 < 2 - m_1, \quad (39b)$$

$$m_1 U \dot{\Psi}_2 - m_2 \dot{U} \Psi_2 - \dot{U} \Phi_2 = 1 \quad \text{on} \quad [-\omega, \omega] \quad \text{if} \quad m_1 < m_2 = 2 - m_1. \quad (39c)$$

The boundary conditions can be obtained from (39b) and (39c)

$$\dot{\Phi}_2(\pm\omega) = 0 \quad \text{if} \quad m_1 < m_2 \leq 2 - m_1, \quad (40a)$$

$$\dot{\Psi}_2(\pm\omega) = 0 \quad \text{if} \quad m_1 < m_2 < 2 - m_1, \quad (40b)$$

$$\dot{\Psi}_2(\pm\omega) = \frac{1}{m_1 U(\pm\omega)} \quad \text{if} \quad m_2 = 2 - m_1. \quad (40c)$$

These boundary conditions are not physical and come from the first order differential equations (39). In order to obtain others conditions for the functions $\Psi_2(\theta)$ and $\Phi_2(\theta)$, we recall that the function $U(\theta)$ is $C^\infty([-\omega, \omega])$ and then $\Psi_2(\theta)$ and $\Phi_2(\theta)$ are $C^\infty([-\omega, \omega])$, du to the ellipticity of the boundary value problem (Knowles and Sternberg, 1974, 1973; Stephenson, 1982; Le, 1992 and Le and Stumpf, 1993). By inserting (35) into the equilibrium field equations (9), and recalling that U satisfies relations in (30), one confirms that

$$\frac{\partial q}{\partial r} = \mu m_1 U r^{m_1+m_2-3} \left[(m_2^2 \chi_2 + \ddot{\chi}_2) + 2m_2 \gamma_2 + L n(r) (m_2^2 \gamma_2 + \ddot{\gamma}_2) \right], \quad (41)$$

$$\frac{\partial q}{\partial \theta} = \mu \dot{U} r^{m_1+m_2-2} \left[(m_2^2 \chi_2 + \ddot{\chi}_2) + 2m_2 \gamma_2 + L n(r) (m_2^2 \gamma_2 + \ddot{\gamma}_2) \right], \quad (42)$$

in which

$$\chi_2 = a_1 V_1 + a_2 V_2, \quad \gamma_2 = a_1 W_1 + a_2 W_2 \quad \text{on} \quad [-\omega, \omega] \quad (43)$$

In the other hand, the traction free boundary conditions (13) leads to :

$$\dot{\chi}_2(\pm\omega) = 0 \quad \text{and} \quad \dot{\gamma}_2(\pm\omega) = 0 \quad (44)$$

As a consequence, we can prove from equilibrium equations (41) and (42) that γ_2 and χ_2 must be solution to the following differential equations

$$\begin{cases} \ddot{\gamma}_2 + m_2^2 \gamma_2 = 0 \\ \ddot{\chi}_2 + m_2^2 \chi_2 + 2m_2 \gamma_2 = 0 \end{cases} \quad \text{on } [-\omega, \omega] \quad \text{if } m_1 < m_2 \leq 2 - m_1. \quad (45)$$

Equation (45) together with (44) constitute an eigenvalue problems for χ_2 and γ_2 with m_2 as eigenvalue parameter. The determination of the local structure of the Lagrange multiplier field, which has been assumed to admit the representation (20), is based on equations (15), (20) and (35) :

$$\begin{aligned} & -\mu \frac{r^{m_2-2}}{a} \left[m_2^2 \Psi_2 + \ddot{\Psi}_2 + 2m_2 \Phi_2 + Ln(r) \left(m_2^2 \Phi_2 + \ddot{\Phi}_2 \right) \right] + o(r^{m_2-2}) \\ & = \\ & ar^{l_1+m_1-2} \left[l_1 P_1 \dot{U} - m_1 \dot{P}_1 U + Q_1 \dot{U} + Ln(r) \left(l_1 Q_1 \dot{U} - m_1 \dot{Q}_1 U \right) \right] + o(r^{l_1+m_1-2}) \end{aligned} \quad (46)$$

By virtue of the boundary conditions (13), along with (35) :

$$\begin{aligned} & r^{l_1} P_1(\pm\omega) + r^{l_1} Ln(r) Q_1(\pm\omega) + o(r^{l_1}) \\ & = \\ & \frac{\mu r^{2(m_2-1)}}{a^2} \left[\dot{\Psi}_2^2(\pm\omega) - 2Ln(r) \dot{\Psi}_2(\pm\omega) \dot{\Phi}_2(\pm\omega) \right. \\ & \quad \left. + Ln^2(r) \dot{\Phi}_2^2(\pm\omega) \right] + o(r^{2(m_2-1)}). \end{aligned} \quad (47)$$

Remark 6

It can be checked that the boundary value problem for the second order asymptotic development has no solution in the absence of the logarithmic terms included in (35) for particular notch angle. In the same way, the **remark 5** is still available for this second order asymptotic development.

3.1. Results in the case $m_1 < m_2 < 2 - m_1$

Now U is a known function (33). The resolution of (39a) and (39b) with the condition that the functions Ψ_2 and Φ_2 are $C^\infty([-\omega, \omega])$ yields to

$$\Phi_2 = 0 \quad \text{and} \quad \Psi_2 = b_2 (|U|)^{\frac{m_2}{m_1}} \quad \text{on } [-\omega, \omega], \quad (48)$$

where b_2 is a real constant. The assumption that Ψ_2 possesses continuous derivatives of all orders implies that $\frac{m_2}{m_1}$ must be a positive integer. Since $m_2 > m_1$ the first positive integer that satisfies this assumption is 2. Then one can draw

$$m_2 = 2m_1 = \frac{\pi}{\omega}. \quad (49)$$

Finally

$$\Phi_2 = 0 \quad \text{and} \quad \Psi_2 = b_2 \left(\sin \left(\frac{\pi}{2\omega} \theta \right) \right)^2 \quad \text{on } [-\omega, \omega]. \quad (50)$$

The eigenvalue problems for χ_2 and γ_2 defined by the equations (45) and (44) with m_2 as parameter done by relation (49) furnishes:

$$\gamma_2(\theta) = 0, \quad \chi_2(\theta) = b_1 \cos\left(\frac{\pi}{\omega}\theta\right) \quad \text{on} \quad [-\omega, \omega]. \quad (51)$$

The equations determining the Lagrange multiplier q can be deduced from (46) and (47) with the use of the expressions of Ψ_2 and Φ_2 (50)

$$\begin{cases} l_1 Q_1 \dot{U} - m_1 \dot{Q}_1 U & = 0 \\ l_1 P_1 \dot{U} - m_1 \dot{P}_1 U + Q_1 \dot{U} & = -\frac{\mu}{a^2} [m_2^2 \Psi_2 + \ddot{\Psi}_2] \end{cases} \quad \text{on} \quad [-\omega, \omega], \quad (52)$$

$$P_1(\pm\omega) = 0 \quad \text{and} \quad Q_1(\pm\omega) = 0. \quad (53)$$

Hence, the Lagrange multiplier expression take the form,

$$q(r, \theta) = r^{l_1} P_1(\theta) + o(r^{l_1}), \quad \begin{cases} P_1(\theta) & = -\frac{2\mu b_2}{a^2} \cos(l_1 \theta) \\ Q_1(\theta) & = 0 \end{cases} \quad l_1 = m_1 = \frac{\pi}{2\omega}. \quad (54)$$

We note that the asymptotic development of the transformation and the Lagrange multiplier fields q obtained in this section are valuable in the case of $m_1 < m_2 < 2 - m_1$ corresponding to $\omega \in]\frac{3\pi}{4}, \pi]$ and $m_1 \in [\frac{1}{2}, \frac{2}{3}]$. Nevertheless, such a solution provides the following Jacobian weak estimate :

$$J \sim o\left(r^{\frac{3\pi}{2\omega}-2}\right). \quad (55)$$

Approximation (55) presents a number of mathematical and physical inconsistencies and is therefore inadequate. In fact, the Jacobian is unbounded which reflects the degenerate character of the asymptotic approximation of the transformation given by the equations (35), (50) and (51), which is not locally volume-preserving.

3.2. Case $m_2 = 2 - m_1 > m_1$ and $\frac{\pi}{2} < \omega \leq \frac{3\pi}{4}$

In the case of notch angle $\omega \in]\frac{\pi}{2}, \frac{3\pi}{4}]$ and $m_2 = 2 - m_1$. The differential equations (39a) and (39c), with the condition that the functions Ψ_2 and Φ_2 are $C^\infty([-\omega, \omega])$, yields to

$$\begin{cases} \Psi_2(\theta) & = U^{\frac{m_2}{m_1}} \left\{ A - \frac{1}{m_1} \int_{|\theta|}^{\omega} \frac{d\varphi}{U(\varphi)^{\frac{m_2}{m_1}+1}} \right\} \\ \Phi_2(\theta) & = 0 \end{cases} \quad \text{on} \quad [-\omega, \omega]. \quad (56)$$

The integral in (56), with the condition that the functions Ψ_2 is $C^\infty([-\omega, \omega])$, is evaluated in terms of hypergeometric functions in the same way as done by (Stephenson, 1982), taking account that $\frac{\pi}{2} < \omega < \frac{3\pi}{4}$ ($m_1 \neq \frac{2}{3}$). In the case of $\omega = \frac{3\pi}{4}$, equation (39c) necessitates a special consideration which require separate attention until next section. Then Φ_2 and Ψ_2 take the following form :

$$\begin{aligned}\Psi_2(\theta) &= -\frac{1}{m_1 m_2} F\left(\frac{1}{2} - \frac{1}{m_1}, \frac{1}{2}; \frac{3}{2} - \frac{1}{m_1}; \sin^2(m_1 \theta)\right) \\ &= -\frac{1}{m_1 m_2} + \frac{1}{m_1^2} \sum_{n=1}^{\infty} \frac{\prod_{k=1}^n (2k-1)}{\prod_{k=1}^n 2k} \frac{\sin^{2n}(m_1 \theta)}{2n+1 - \frac{2}{m_1}}\end{aligned}\quad (57)$$

$$\Phi_2(\theta) = 0 \quad \text{on} \quad [-\omega, \omega],$$

in which F is a hypergeometric function (Seaborn, 1991). One deduces from (57) that

$$\Psi_2(0) = -\frac{1}{m_1 m_2}, \quad (58)$$

$$\Psi_2(\pm\omega) = -\frac{1}{m_1 m_2} F\left(\frac{1}{2} - \frac{1}{m_1}, \frac{1}{2}; \frac{3}{2} - \frac{1}{m_1}; 1\right). \quad (59)$$

We note that the hypergeometric function defined by (57) is not defined for $m_1 = \frac{2}{3}$ (For this value $\omega = \frac{3\pi}{4}$) which confirms that equation (39c) requires a special treatment for this case.

In the case of $m_2 = 2 - m_1$, the eigenvalue problems for χ_2 and γ_2 defined by the equations (45) and (44) with m_2 as a known parameter fail to admit a nonzero solution :

$$\chi_2(\theta) = 0 \quad \text{and} \quad \gamma_2(\theta) = 0 \quad \text{on} \quad [-\omega, \omega]. \quad (60)$$

The equations determining the Lagrange multiplier can be deduced from (46) and (47) with the use of the expressions of Ψ_2 and Φ_2 (57) :

$$\begin{cases} l_1 Q_1 \dot{U} - m_1 \dot{Q}_1 U &= 0 \\ l_1 P_1 \dot{U} - m_1 \dot{P}_1 U + Q_1 \dot{U} &= -\frac{\mu}{a^2} [m_2^2 \Psi_2 + \ddot{\Psi}_2] \end{cases} \quad \text{on} \quad [-\omega, \omega], \quad (61)$$

$$Q_1(\pm\omega) = 0 \quad \text{and} \quad P_1(\pm\omega) = \frac{\mu}{a^2} [\dot{\Psi}_2^2(\pm\omega)], \quad (62)$$

The boundary-value problem consisting of (61) and (62) fails to admit a solution that verify the boundary conditions (62). A non-degenerate Lagrange multiplier field would presumably necessitate a higher-order asymptotic analysis of the deformation field. Fortunately, there is no need of the Lagrange multiplier field to investigate the singular stress components as would be shown in section 4.

3.3. Results in the case $m_2 = 2 - m_1 > m_1$ and $\omega = \frac{3\pi}{4}$

This case corresponds to $m_1 = \frac{2}{3}$ and $\omega = \frac{3\pi}{4}$. The differential equations (39a) and (39c) with the condition that the functions Ψ_2 and Φ_2 are $C^\infty[-\omega, \omega]$ yields to

$$\Phi_2(\theta) = -\frac{1}{2m_1} U^2(\theta) \quad \text{on} \quad [-\omega, \omega], \quad (63)$$

and

$$\begin{aligned} \Psi_2(\theta) &= b_2 \sin^2(m_1 \theta) - \frac{1}{m_1 m_2} \\ &+ \frac{1}{m_1^2} \sum_{n=2}^{\infty} \frac{\prod_{k=1}^n (2k-1)}{\prod_{k=1}^n 2k} \frac{\sin^{2n}(m_1 \theta)}{2n+1-\frac{2}{m_1}} \quad \text{on } [-\omega, \omega]. \end{aligned} \quad (64)$$

The equilibrium equations (45) and the boundary conditions (44) furnish

$$m_2^2 \gamma_2 + \ddot{\gamma}_2 = 0 \quad \text{and} \quad m_2^2 \chi_2 + \ddot{\chi}_2 + 2m_2 \gamma_2 = 0, \quad (65)$$

$$\dot{\gamma}_2(\pm\omega) = 0 \quad \text{and} \quad \dot{\chi}_2(\pm\omega) = 0. \quad (66)$$

In case $m_2 = 2 - m_1 = \frac{4}{3}$ and $m_1 = \frac{2}{3}$ equations (65) and (66) give

$$\gamma_2(\theta) = 0 \quad \text{and} \quad \chi_2(\theta) = b_1 \cos(m_2 \theta) \quad \text{on } [-\omega, \omega]. \quad (67)$$

From the Lagrange multiplier field equation (46)

$$l_1 = m_2 - m_1 = m_1 = \frac{2}{3}, \quad (68)$$

$$\begin{aligned} l_1 Q_1 \dot{U} - m_1 \dot{Q}_1 U &= -\frac{\mu}{a^2} (m_2^2 \Phi_2 + \ddot{\Phi}_2) \\ l_1 P_1 \dot{U} - m_1 \dot{P}_1 U + Q_1 \dot{U} &= -\frac{\mu}{a^2} (m_2^2 \Psi_2 + \ddot{\Psi}_2 + 2m_2 \Phi_2), \end{aligned} \quad (69)$$

$$Q_1(\pm\omega) = 0, \quad P_1(\pm\omega) = \frac{\mu \dot{\Psi}_2^2(\pm\omega)}{a^2} = \frac{\mu}{(am_1)^2}. \quad (70)$$

The problem (69) and (70) does not admit a solution that verify the boundary condition (70). In section 4 it will be shown that there is no need for the Lagrange multiplier field to investigate the singular stress components. A non-degenerate Lagrange multiplier field would presumably necessitate a higher-order asymptotic analysis of the deformation field.

4. Third order asymptotic analysis of the elastostatic field near the notch vertex for $\omega \in]\frac{3\pi}{4}, \pi[$

Recalling that the second order asymptotic analysis of this case leads to a weak estimate of the Jacobian (34). With a view to refining these estimates, when $m_2 < 2 - m_1$, we first replace (35) by

$$\begin{aligned} y_\alpha(r, \theta) &= a_\alpha r^{m_1} U(\theta) + r^{m_2} V_\alpha(\theta) + r^{m_3} R_\alpha(\theta) \\ &+ r^{m_3} L_n(r) T_\alpha(\theta) + o(r^{m_3}), \quad R_\alpha, T_\alpha \in C^\infty[-\omega, \omega], \end{aligned} \quad (71)$$

where $m_3 > m_2 > m_1$, $R_\alpha(\theta)$ and $T_\alpha(\theta)$ denote undetermined functions, where $U(\theta)$ and $V_\alpha(\theta)$ are already known functions. Combining (71) with (9) and invoking the boundary conditions (13) one finds the solution for $\omega \in]\frac{3\pi}{4}, \pi[$:

$$m_3 = 2 - m_1 = 2 - \frac{\pi}{2\omega}, \quad (72)$$

$$R_1 = \frac{1}{a^2} [a_1 \chi_3 - a_2 \Psi_3] \quad \text{and} \quad R_2 = \frac{1}{a^2} [a_2 \chi_3 + a_1 \Psi_3] \quad \text{on} \quad [-\omega, \omega], \quad (73)$$

$$T_1 = \frac{1}{a^2} [a_1 \gamma_3 - a_2 \Phi_3] \quad \text{and} \quad T_2 = \frac{1}{a^2} [a_2 \gamma_3 + a_1 \Phi_3] \quad \text{on} \quad [-\omega, \omega], \quad (74)$$

with Ψ_3 and Φ_3 defined

$$\Psi_3(\theta) = -\frac{1}{m_1 m_3} F\left(\frac{1}{2} - \frac{1}{m_1}, \frac{1}{2}; \frac{3}{2} - \frac{1}{m_1}; \sin^2(m_1 \theta)\right), \quad (75)$$

$$T_1 = 0 \quad T_2 = 0 \quad \text{on} \quad [-\omega, \omega] \quad (76)$$

and

$$\chi_3 = 0 \quad \text{on} \quad [-\omega, \omega]. \quad (77)$$

We now suppose that the Lagrange multiplier field conforms to

$$q(r, \theta) = r^{l_1} P_1(\theta) + r^{l_2} P_2(\theta) + r^{l_2} L n(r) Q_2(\theta) + o(r^{l_2}), \quad l_2 > l_1, \quad (78)$$

where l_2 , P_2 and Q_2 are unknown. Their determination needs a higher order development in transformation field.

The solution for $\omega = \pi$ was given by (Stephenson, 1982) and will be analyzed later.

5. Discussion of the deformation and stresses near the notch vertex

This section contains first, a discussion of the asymptotic transformation field described by (17), then a study of the structure of the associated notch vertex stress fields. The structure of the two-term or three-term asymptotic transformation approximation deduced in sections (2), (3) and (4) is too much complex to be analyzed. For this purpose, we use the propriety of invariance of the local field under rigid-body rotation (17) to render the intrinsic structure of the two-term or three-term asymptotic transformation approximation deduced in sections (2), (3) and (4) more transparent. To this end, we apply the objectivity principle (17) by using a particular form of $\underline{\underline{Q}}$ (Stephenson, 1982)

$$[Q_{\alpha\beta}] = \begin{bmatrix} \frac{a_2}{a} & \frac{-a_1}{a} \\ \frac{a_1}{a} & \frac{a_2}{a} \end{bmatrix} \quad (79)$$

to (71). One arrives at

$$\begin{cases} y_1(r, \theta) = -\frac{r^{m_2}}{a} \Psi_2(\theta) - \frac{r^{m_2} L n(r)}{a} \Phi_2(\theta) - \frac{r^{m_3}}{a} \Psi_3(\theta) \\ \quad - \frac{r^{m_3}}{a} L n(r) \Phi_3(\theta) + o(r^{m_3}), \\ y_2(r, \theta) = a r^{m_1} U(\theta) + \frac{r^{m_2}}{a} \chi_2(\theta) + \frac{r^{m_2} L n(r)}{a} \gamma_2(\theta) + \frac{r^{m_3}}{a} \chi_3(\theta) \\ \quad + \frac{r^{m_3}}{a} L n(r) \gamma_3(\theta) + o(r^{m_3}), \end{cases} \quad (80)$$

where functions $U(\theta)$, $\Psi_k(\theta)$, $\Phi_k(\theta)$, $\gamma_k(\theta)$ and $\chi_k(\theta)$ ($k = 2, 3$) are deduced in section (2), (3) and (4). Our initial objective here is to analyse the physical sense of the asymptotic results for the transformation field established in earlier sections. To this end, we summarize the different expressions of transformations fields. Thus for $\omega = \pi$ (solution given by Stephenson (1982))

$$\begin{aligned} y_1(r, \theta) = & -\frac{b_2 r}{a} \sin^2\left(\frac{\theta}{2}\right) - \frac{r^{\frac{3}{2}}}{a} \left\{ c_2 \sin^3\left(\frac{\theta}{2}\right) + \frac{2b_1 b_2}{a} \sin\left(\frac{\theta}{2}\right) \right. \\ & \left. - 4 \sin^2\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) - \frac{4}{3} \cos^3\left(\frac{\theta}{2}\right) \right\} + o\left(r^{\frac{3}{2}}\right), \\ y_2(r, \theta) = & ar^{\frac{1}{2}} \sin\left(\frac{\theta}{2}\right) + \frac{b_1 r}{a} \cos(\theta) + \frac{r^{\frac{3}{2}}}{a} \left\{ c_1 \sin\left(\frac{3\theta}{2}\right) \right. \\ & \left. - \frac{b_2^2}{2a} \sin\left(\frac{\theta}{2}\right) \right\} + o\left(r^{\frac{3}{2}}\right), \end{aligned} \quad (81)$$

while for $\frac{3\pi}{4} < \omega < \pi$

$$\begin{aligned} y_1(r, \theta) = & -\frac{b_2 r^{\frac{\pi}{\omega}}}{a} \sin^2\left(\frac{\pi}{2\omega} \theta\right) \\ & + \frac{r^{2-\frac{\pi}{2\omega}}}{a^{\frac{\pi}{2\omega}} (2-\frac{\pi}{2\omega})} F\left(\frac{1}{2} - \frac{2\omega}{\pi}, \frac{1}{2}; \frac{3}{2} - \frac{2\omega}{\pi}; \sin^2\left(\frac{\pi}{2\omega} \theta\right)\right) \\ & + o\left(r^{(2-\frac{\pi}{2\omega})}\right), \\ y_2(r, \theta) = & ar^{\frac{\pi}{2\omega}} \sin\left(\frac{\pi}{2\omega} \theta\right) + \frac{b_1}{a} r^{\frac{\pi}{\omega}} \cos\left(\frac{\pi}{\omega} \theta\right) + o\left(r^{\frac{\pi}{\omega}}\right). \end{aligned} \quad (82)$$

In the case of $\frac{\pi}{2} < \omega < \frac{3\pi}{4}$

$$\begin{aligned} y_1(r, \theta) = & \frac{r^{(2-\frac{\pi}{2\omega})}}{a^{\frac{\pi}{2\omega}} (2-\frac{\pi}{2\omega})} F\left(\frac{1}{2} - \frac{2\omega}{\pi}, \frac{1}{2}; \frac{3}{2} - \frac{2\omega}{\pi}; \sin^2\left(\frac{\pi}{2\omega} \theta\right)\right) + o\left(r^{(2-\frac{\pi}{2\omega})}\right), \\ y_2(r, \theta) = & ar^{\frac{\pi}{2\omega}} \sin\left(\frac{\pi}{2\omega} \theta\right) + o\left(r^{\frac{\pi}{2\omega}}\right). \end{aligned} \quad (83)$$

while for $\omega = \frac{3\pi}{4}$

$$\begin{aligned} y_1(r, \theta) = & \frac{r^{\frac{4}{3}}}{a} \left\{ \frac{9}{8} - b_2 \sin^2\left(\frac{2}{3} \theta\right) - \frac{9}{4} \sum_{n=2}^{\infty} \frac{\prod_{k=1}^n (2k-1)}{\prod_{k=1}^n 2k} \frac{\sin^{2n}\left(\frac{2}{3} \theta\right)}{2n-2} \right\} \\ & + \frac{3r^{\frac{4}{3}} \text{Ln}(r)}{4a} \sin^2\left(\frac{2}{3} \theta\right) + o\left(r^{\frac{4}{3}}\right), \\ y_2(r, \theta) = & ar^{\frac{2}{3}} \sin\left(\frac{2}{3} \theta\right) + o\left(r^{\frac{2}{3}}\right). \end{aligned} \quad (84)$$

The asymptotic transformation approximation given by equations (81)-(84), involves different real constants, which determination eludes the local analysis carried out here, and depends on the boundary conditions at infinity, the geometry and the material properties. We note that the transformation field y_α involves a logarithm term in the case of the transition notch angle $\omega = \frac{3\pi}{4}$.

Remark 7

Following the linear elastic (Xiaolin, 1986) and (Seweryn and Molski, 1993) and elastoplastic (Kuang and Xu, 1987; Chao and Yang, 1992; Yang and Yuan, 1992; Xia and Wang, 1993 ; Yuan and Lin, 1994) theories in small transformation, the order of the singularity depend on the boundary conditions at infinity, i.e. symmetric or antisymmetric loading. This is in contrast with the results obtained here where the singularity is the same in both cases. In other hand, the asymptotic development obtained with these theories exhibit only a power singularity and there is no logarithm singularity with Neumann boundary conditions.

The asymptotic approximation of the displacement field deduced by orthogonal transformation, given by equations (81)-(84), can satisfy the symmetry property by an appropriate choice of the different real constants. Nevertheless this approximation fails to have an antisymmetric solution about the plane of the notch under the nonlinear theory of elasticity of finite plane strain with Mooney-Rivlin hyperelastic potential. To deeply investigate this property; let us find the image of the notch faces $\theta = \pm\omega$ in the vicinity of the undeformed configuration (material) end of the notch vertex faces.

Thus for $\omega = \pi$

$$\begin{aligned} y_1(r, \pm\pi) &= -\frac{b_2 r}{a} \mp \frac{r^{\frac{3}{2}}}{a} \left(c_2 + \frac{2b_1 b_2}{a} \right) + o\left(r^{\frac{3}{2}}\right), \\ y_2(r, \pm\pi) &= \pm ar^{\frac{1}{2}} - \frac{b_1 r}{a} + \frac{r^{\frac{3}{2}}}{a} \left(\mp c_1 \mp \frac{b_2^2}{2a} \right) + o\left(r^{\frac{3}{2}}\right), \end{aligned} \quad (85)$$

while $\frac{3\pi}{4} < \omega < \pi$

$$\begin{aligned} y_1(r, \pm\omega) &= -\frac{b_2 r^{\frac{\pi}{\omega}}}{a} + \frac{r^{2-\frac{\pi}{\omega}}}{a^{\frac{\pi}{2\omega}} (2 - \frac{\pi}{2\omega})} F\left(\frac{1}{2} - \frac{2\omega}{\pi}, \frac{1}{2}; \frac{3}{2} - \frac{2\omega}{\pi}; 1\right) + o\left(r^{2-\frac{\pi}{2\omega}}\right), \\ y_2(r, \pm\omega) &= \pm ar^{\frac{\pi}{2\omega}} - \frac{b_1 r^{\frac{\pi}{\omega}}}{a} + o\left(r^{\frac{\pi}{\omega}}\right). \end{aligned} \quad (86)$$

In the case of $\frac{\pi}{2} < \omega < \frac{3\pi}{4}$

$$\begin{aligned} y_1(r, \pm\omega) &= \frac{r^{2-\frac{\pi}{2\omega}}}{a^{\frac{\pi}{2\omega}} (2 - \frac{\pi}{2\omega})} F\left(\frac{1}{2} - \frac{2\omega}{\pi}, \frac{1}{2}; \frac{3}{2} - \frac{2\omega}{\pi}; 1\right) + o\left(r^{2-\frac{\pi}{2\omega}}\right), \\ y_2(r, \pm\omega) &= \pm ar^{\frac{\pi}{2\omega}} + o\left(r^{\frac{\pi}{2\omega}}\right). \end{aligned} \quad (87)$$

while for $\omega = \frac{3\pi}{4}$

$$\begin{aligned} y_1\left(r, \pm \frac{3\pi}{4}\right) &= -\frac{r^{\frac{4}{3}}}{a} \left\{ b_2 - \frac{9}{8} + \frac{9}{4} \sum_{n=2}^{\infty} \frac{\prod_{k=1}^n (2k-1)}{\prod_{k=1}^n 2k} \frac{1}{2n-2} \right\} \\ &\quad + \frac{3r^{\frac{4}{3}} \ln(r)}{a} + o\left(r^{\frac{4}{3}}\right), \\ y_2\left(r, \pm \frac{3\pi}{4}\right) &= \pm ar^{\frac{2}{3}} + o\left(r^{\frac{2}{3}}\right). \end{aligned} \quad (88)$$

Elimination of undeformed coordinate r between $y_1(r, \pm\omega)$ and $y_2(r, \pm\omega)$ in the image of the notch faces equations (85)-(88) leads to approximate description of the curves into which the notch-faces at $\theta = \pm\omega$ are deformed.

Crack case

For crack case ($\omega = \pi$) and in the non-degenerate case $b_2 \neq 0$, one deduces from (85)

$$y_1(r, \pm\pi) = -\frac{b_2}{a^3} y_2^2(r, \pm\pi) \quad \text{with} \quad \begin{cases} y_2 \geq 0 & \text{for } \theta = \pi, \\ y_2 \leq 0 & \text{for } \theta = -\pi. \end{cases} \quad (89)$$

This description shows that each crack face is locally transformed by a first order approximation into two arcs of the same parabola whose tangent at the crack is perpendicular to the crack-axis. The concavity or convexity of the parabola (89) is governed by the undetermined constant b_2 sign : concave for $b_2 > 0$ and convex for $b_2 < 0$ as shown in table 1. In the degenerate case $b_2 = 0$, one deduce from (85)

$$y_1(r, \pm\pi) = \mp \frac{c_2}{a^4} (\pm y_2(r, \pm\pi))^3 \quad \text{with} \quad \begin{cases} y_2 \geq 0 & \text{for } \theta = \pi \\ y_2 \leq 0 & \text{for } \theta = -\pi \end{cases} \quad (90)$$

The second order approximation of the crack faces transformation shows that they have a common tangent at the tip of the crack. Figure 1 illustrates the shape of the deformed crack-faces based in formula (90) and shown the crack-faces deformed into **S-shaped curve**. Accordingly, the special case $b_2 = 0$ marks the transition from conditions in which the deformed crack-faces is concave near the crack tip corresponding to $b_2 > 0$ to conditions in which the deformed crack faces is convex $b_2 < 0$. A literature survey shows that it is not clear, whether or not there is a global loading and a special

crack geometries that induce these different cases $b_2 > 0$, $b_2 = 0$ or $b_2 < 0$. The three scenarios of the crack faces deformation was spotted analytically by (Knowles and Sternberg, 1983) for interface crack in bimaterial case. Borret(1998) showed that the **S-shaped curve** for crack face deformation is plausible.

Vertex case $\frac{3\pi}{4} < \omega < \pi$

While for $\frac{3\pi}{4} < \omega < \pi$, one deduces from (86)

$$y_1(r, \pm\omega) = -\frac{b_2}{a^3} y_2^2(r, \pm\omega) \quad \text{with} \quad \begin{cases} y_2 \geq 0 & \text{for } \theta = \omega \\ y_2 \leq 0 & \text{for } \theta = -\omega \end{cases} \quad (91)$$

In the degenerate case $b_2 = 0$, one deduce from (86) :

$$y_1(r, \pm\omega) = \frac{F\left(\frac{1}{2} - \frac{2\omega}{\pi}, \frac{1}{2}; \frac{3}{2} - \frac{2\omega}{\pi}; 1\right)}{a^{\frac{4\omega}{\pi}} \frac{\pi}{2\omega} \left(2 - \frac{\pi}{2\omega}\right)} (\pm y_2(r, \pm\omega))^{\frac{4\omega}{\pi} - 1} \quad (92)$$

with $\begin{cases} y_2 \geq 0 & \text{for } \theta = \omega \\ y_2 \leq 0 & \text{for } \theta = -\omega \end{cases}$ and $F\left(\frac{1}{2} - \frac{2\omega}{\pi}, \frac{1}{2}; \frac{3}{2} - \frac{2\omega}{\pi}; 1\right) > 0$.

For notch vertex same scenarios of deformation as crack case are identified, except that the degenerate case $b_2 = 0$ presents the same behaviour as $b_2 < 0$ table 2 and marks a brutal transition from conditions in which the deformed notch vertex-faces is concave near the notch vertex corresponding to $b_2 > 0$ to conditions in which the deformed notch vertex-faces is convex. Note that from a behaviorist viewpoint this confirm the singular character of $\omega = \frac{3\pi}{4}$ geometry. This description shows that each notch vertex faces is locally transformed, by a first order approximation, into two arcs of the same concave parabola whose tangent at the notch vertex is perpendicular to the notch-axis (Table 3)

Vertex case $\frac{\pi}{2} < \omega < \frac{3\pi}{4}$

In the case of $\frac{\pi}{2} < \omega < \frac{3\pi}{4}$, one deduces from (87) :

$$y_1(r, \pm\omega) = \frac{F\left(\frac{1}{2} - \frac{2\omega}{\pi}, \frac{1}{2}; \frac{3}{2} - \frac{2\omega}{\pi}; 1\right)}{a^{\frac{4\omega}{\pi}} \frac{\pi}{2\omega} \left(2 - \frac{\pi}{2\omega}\right)} (\pm y_2(r, \pm\omega))^{\frac{4\omega}{\pi} - 1}$$

$$\text{with } \begin{cases} y_2 \geq 0 & \text{for } \theta = \omega \\ y_2 \leq 0 & \text{for } \theta = -\omega \end{cases} \quad \text{and} \quad F\left(\frac{1}{2} - \frac{2\omega}{\pi}, \frac{1}{2}; \frac{3}{2} - \frac{2\omega}{\pi}; 1\right) < 0. \quad (93)$$

Vertex case $\omega = \frac{3\pi}{4}$

Whereas for $\omega = \frac{3\pi}{4}$, one deduces from (88)

$$y_1\left(r, \pm\frac{3\pi}{4}\right) = \frac{9}{8a^3} \left(y_2\left(r, \pm\frac{3\pi}{4}\right)\right)^2 \text{Ln}\left(\pm \frac{y_2\left(r, \pm\frac{3\pi}{4}\right)}{a}\right)$$

$$\text{with } \begin{cases} y_2 \geq 0 & \text{for } \theta = \frac{3\pi}{4} \\ y_2 \leq 0 & \text{for } \theta = -\frac{3\pi}{4} \end{cases} \quad (94)$$

This description shows that each notch vertex face is locally transformed, by a first order approximation into two arcs of the same concave parabola whose tangent at the notch vertex is perpendicular to the notch-axis (see Table 4)

From (81), (82), (83) and (84), it follows that particles near the notch vertex faces in the undeformed body lie to the right of this curve after transformation. We conclude on the basis of (89)-(94) that the notch vertex is bound to open but not in a symmetrically manner, independently of the magnitude and the mode of the boundary conditions at infinity. This conclusion is in contradiction with the predictions of linear elastic (Xiaolin, 1986), (Seweryn and Zwoliński, 1996) and elastoplastic (Kuang and Xu, 1987), (Chao and Yang, 1992), (Yang and Chao, 1992), (Xia and Wang, 1993), (Yuan and Lin, 1994), (Yang and Yuan, 1996) constitutive theory with small deformation hypothesis for a Mode II loading.

In conclusion, the results of analysis done below state that the local transformation

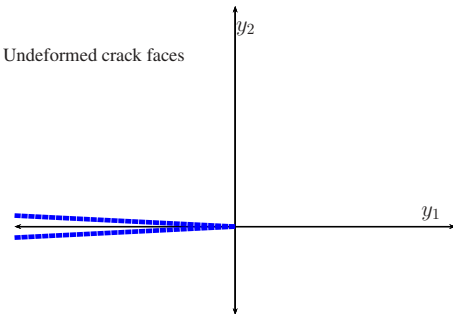
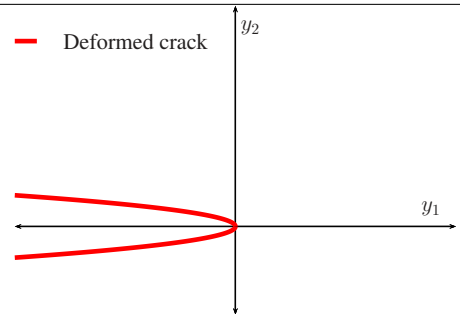
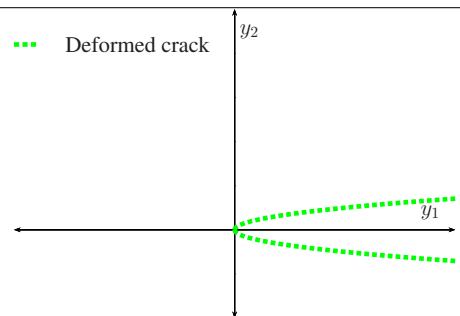
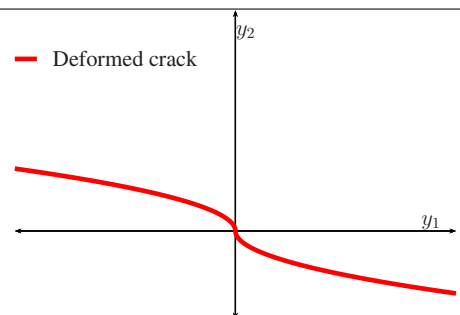
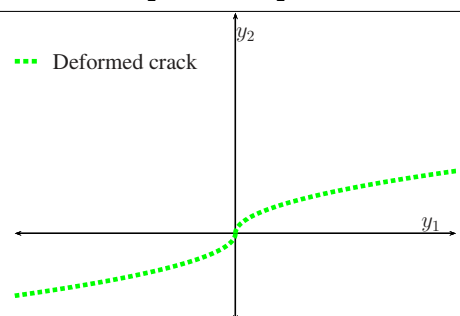
Undeformed crack	Deformed crack	
	$b_2 > 0$	
	$b_2 < 0$	
	$b_2 = 0$ and $c_2 > 0$	
	$b_2 = 0$ and $c_2 < 0$	

Table 1: Crack face image

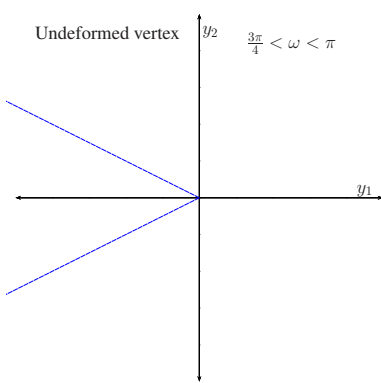
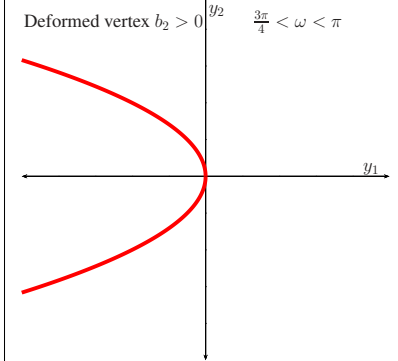
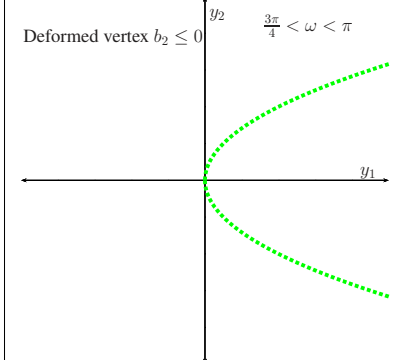
Undeformed vertex	Deformed vertex
<p>Undeformed vertex</p>  <p>$\frac{3\pi}{4} < \omega < \pi$</p>	<p>$b_2 > 0$</p> <p>Deformed vertex $b_2 > 0$</p>  <p>$\frac{3\pi}{4} < \omega < \pi$</p>
	<p>$b_2 \leq 0$</p> <p>Deformed vertex $b_2 \leq 0$</p>  <p>$\frac{3\pi}{4} < \omega < \pi$</p>

Table 2: Vertex face image $\frac{3\pi}{4} < \omega < \pi$

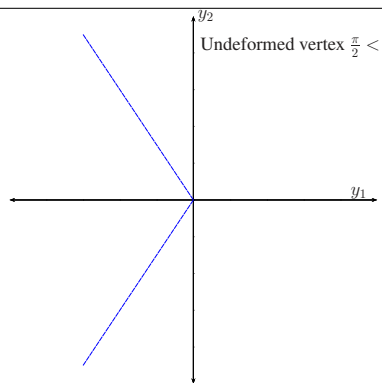
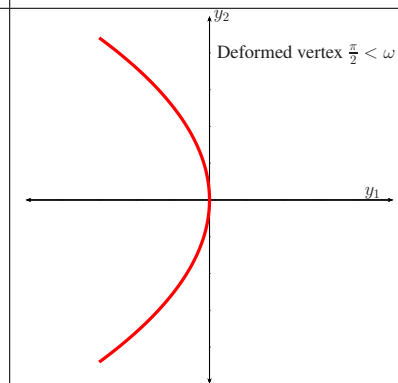
Undeformed vertex	Deformed vertex
<p>Undeformed vertex</p>  <p>$\frac{\pi}{2} < \omega < \frac{3\pi}{4}$</p>	<p>Deformed vertex</p>  <p>$\frac{\pi}{2} < \omega < \frac{3\pi}{4}$</p>

Table 3: Vertex face image $\frac{\pi}{2} < \omega < \frac{3\pi}{4}$

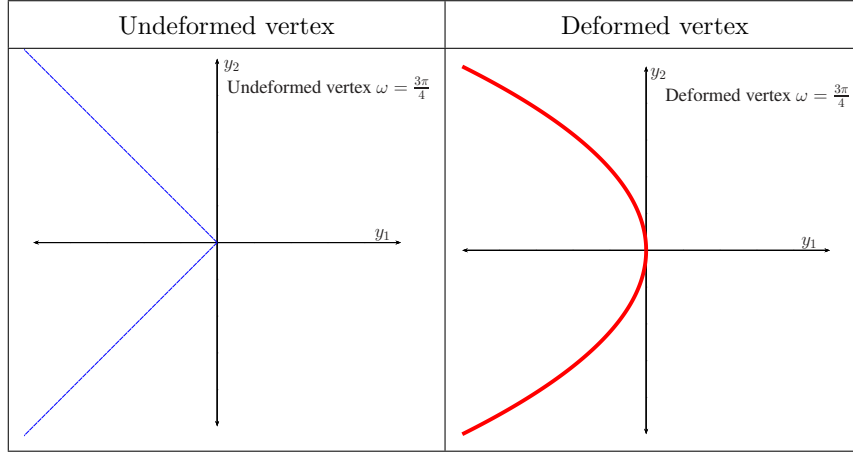


Table 4: Vertex face image $\omega = \frac{3\pi}{4}$

field (35) at the notch vertex can be viewed as a pure deformation of the body in a suitable basis, in which the notch vertex opens, followed by a rigid rotation, with $Q_{\alpha\beta}$ given by (79). The particular form of $\underline{\underline{Q}}$ (79) corresponds to a rotation angle $\arctan\left(-\frac{a_1}{a_2}\right)$ given by the mixed-mode loading of the first asymptotic term. We shall call the particular field (80) deduced from the principle of objectivity (17), with the use of the transformation third order asymptotic development (71) and the particular form of the orthogonal tensor $\underline{\underline{Q}}$ (79), a canonical field, because it is the standard representative element of the set \mathfrak{S} of local singular fields. To specify all the other elements of \mathfrak{S} , we simply apply the reverse formula of (17).

Remark 8

Following the linear (Xiaolin, 1986) and (Seweryn and Zwoliński, 1996) and nonlinear elastoplastic (Kuang and Xu, 1987), (Chao and Yang, 1992), (Yang and Chao, 1992), (Xia and Yang, 1993) and (Yuan and Lin, 1994) fracture mechanics in small transformation, the deformed notch faces fails to separate and no interaction between the upper and lower notch faces occurs under the mode II loading. This problem was analysed with different hyperelastic potentials. It was shown that the deformed crack (Stephenson, 1982), (Le, 1992), (Le and Stumpf, 1993) (Geubelle and Knauss, 1994) and (Tarantino, 1996) and notch (Tarantino, 1994), (Tarantino, 1997) and (Ru, 1997) faces open even under mode II loading for a class of hyperelastic potential. This question was more investigated by (Knowles, 1981), (Chow et al, 1986) and (Ru, 2002) and was shown that the opening or not of the crack faces under mode II loading depend essentially on the nature of the hyperelastic potential.

In view of the principle of objectivity, we shall therefore be entitled to base the computation of the local stress distribution on (81), (82), (83), (84). At this moment, we determine the associated local true-stress field. On account of (8) one has :

$$\sigma_{\alpha\beta} = \mu F_{\alpha\gamma} F_{\beta\gamma} - q \delta_{\alpha\beta}. \quad (95)$$

We retain here only the singular terms or the dominant term. For the notch problem with $\omega \in]\frac{3\pi}{4}, \pi]$ the asymptotic Cauchy stress components are :

$$\begin{aligned} \sigma_{11} &= \mu \left(\frac{b_2 \pi}{a\omega} \right)^2 r^{\frac{2\pi}{\omega}-2} \sin^2 \left(\frac{\pi}{2\omega} \theta \right) + o \left(r^{\frac{2\pi}{\omega}-2} \right) \quad \forall \theta \in [-\omega, \omega] \setminus \{0\} \\ \sigma_{11} &= \frac{\mu b_2}{a^2} r^{\frac{\pi}{2\omega}} + o \left(r^{\frac{\pi}{2\omega}} \right) \quad \text{for } \theta = 0 \\ \sigma_{22} &= \mu \left(\frac{\pi}{2\omega} \right)^2 a^2 r^{\frac{\pi}{\omega}-2} - \mu \left(\frac{\pi}{\omega} \right)^2 b_1 r^{\frac{3\pi}{2\omega}-2} \sin \left(\frac{\pi}{2\omega} \theta \right) + o \left(r^{\frac{3\pi}{2\omega}-2} \right) \quad \forall \theta \in [-\omega, \omega] \\ \sigma_{12} &= \sigma_{21} = -2\mu \left(\frac{\pi}{2\omega} \right)^2 b_2 r^{\frac{3\pi}{2\omega}-2} \sin \left(\frac{\pi}{2\omega} \theta \right) + o \left(r^{\frac{3\pi}{2\omega}-2} \right) \quad \forall \theta \in [-\omega, \omega] \setminus \{0\} \\ \sigma_{12} &= \sigma_{21} = \frac{\mu m_2 b_1}{m_1 a^2} r^{\frac{\pi}{2\omega}} + o \left(r^{\frac{\pi}{2\omega}} \right) \quad \text{for } \theta = 0. \end{aligned} \quad (96)$$

For $\omega = \frac{3\pi}{4}$, the asymptotic Cauchy stress components are :

$$\begin{aligned}
 \sigma_{11} &= o\left(r^{\frac{2}{3}}\right) \quad \forall \theta \in [-\omega, \omega] \setminus \{0\} \\
 \sigma_{11} &= \frac{9\mu b_2}{4a} r^{\frac{2}{3}} \ln(r) + o\left(r^{\frac{2}{3}}\right) \quad \text{for } \theta = 0 \\
 \sigma_{22} &= \frac{4}{9} \mu a^2 r^{-\frac{2}{3}} - \frac{16}{9} \mu \sin\left(\frac{2}{3}\theta\right) + o(1) \quad \forall \theta \in [-\omega, \omega] \setminus \{0\} \\
 \sigma_{12} &= \sigma_{21} = \frac{2}{3} \mu \ln(r) \sin\left(\frac{2}{3}\theta\right) + o(\ln(r)) \quad \forall \theta \in [-\omega, \omega] \setminus \{0\} \\
 \sigma_{12} &= \sigma_{21} = \frac{2\mu b_1}{3a^2} r^{\frac{2}{3}} + o\left(r^{\frac{2}{3}}\right) \quad \text{for } \theta = 0.
 \end{aligned} \tag{97}$$

For $\omega \in]\frac{\pi}{2}, \frac{3\pi}{4}[$, the asymptotic Cauchy stress components are

$$\begin{aligned}
 \sigma_{11} &= o\left(r^{2-\frac{\pi}{\omega}}\right) \quad \forall \theta \in [-\omega, \omega] \\
 \sigma_{22} &= \mu \left(\frac{\pi}{2\omega}\right) a^2 r^{\frac{\pi}{\omega}-2} + o\left(r^{\frac{\pi}{\omega}-2}\right) \quad \forall \theta \in [-\omega, \omega] \\
 \sigma_{12} &= \sigma_{21} = -\mu \left\{ -\sin\left(\frac{\pi}{2\omega}\theta\right) F\left(\frac{1}{2} - \frac{2\omega}{\pi}, \frac{1}{2}; \frac{3}{2} - \frac{2\omega}{\pi}; \sin^2\left(\frac{\pi}{2\omega}\right)\right) \right. \\
 &\quad \left. + \frac{\pi}{3\pi - 4\omega} \sin\left(\frac{\pi}{2\omega}\theta\right) \cos^2\left(\frac{\pi}{2\omega}\theta\right) F\left(\frac{3}{2} - \frac{2\omega}{\pi}, \frac{3}{2}; \frac{5}{2} - \frac{2\omega}{\pi}; \sin^2\left(\frac{\pi}{2\omega}\theta\right)\right) \right\} \\
 &\quad + o(1) \quad \forall \theta \in [-\omega, \omega] \setminus \{0\}.
 \end{aligned} \tag{98}$$

The analysis of the Cauchy stress tensor components $\sigma_{\alpha\beta}$ (96)-(98) shows that, unlike the linear elastic notch field (Xiaolin, 1986), their radial and angular dependences are different for each components $\sigma_{\alpha\beta}$. The order of stress singularities of the Cauchy stress components, equations (96)-(98), depends monotonically only on the local geometry of the notch, i.e. the notch angle ω . This means that the order of stress singularities does not depend on the type of the boundary conditions at infinity. In fact, the Cauchy stress components have the same order of singularities for a mode I and mode II loading. This is still in contrast to the predictions of the linear theory (Xiaolin, 1986). The Cauchy stress component σ_{22} dominates the stress field and has the most singular term $r^{\frac{\pi}{\omega}-2}$, but it is not a function of the angular material coordinate, depending only on radial coordinate. Equations (98) show that σ_{22} is strictly positive, this make the notch vertex region $r \rightarrow 0$ under tensile loading. In the other part, σ_{22} is asymptotically dominant as one approaches the notch vertex, that is, $\frac{\sigma_{11}}{\sigma_{22}} \rightarrow 0$, $\frac{\sigma_{12}}{\sigma_{22}} \rightarrow 0$ as $r \rightarrow 0$.

In other words, every material element near the notch vertex is under uniaxial tension parallel to y_2 axis. This result is in agreement with the propriety that the notch vertex is bound to open regardless of the magnitude and nature of the particular loading at infinity. In the case of $\omega \in]\frac{3\pi}{4}, \pi]$, σ_{22} possesses two singular terms $r^{\frac{\pi}{\omega}-2}$ and $r^{\frac{3\pi}{2\omega}-2}$ coming from the exponents m_1 and m_2 contributions.

In the case of $\omega \in]\frac{\pi}{2}, \frac{3\pi}{4}[$, σ_{22} possesses one singular terms $r^{\frac{\pi}{2}-2}$ coming from the exponent m_1 contribution. The value of $\omega = \frac{3\pi}{4}$ is a particular notch vertex angle value since σ_{22} possesses one singular term $r^{\frac{\pi}{\omega}-2}$ coming from the exponent m_1 contribution and a term independent from radial material coordinates r like a "T" stress in linear elasticity. So, this particular notch angle value characterizes a transition between two regimes. The radial behaviour of the Cauchy stress component σ_{12} varies with the notch angle: radial singularity in the case of $\omega \in]\frac{3\pi}{4}, \pi[$, logarithmic singularity if $\omega = \frac{3\pi}{4}$ and no singular behaviour in the case of $\omega \in]\frac{\pi}{2}, \frac{3\pi}{4}[$. This result confirms that the particular notch angle value $\omega = \frac{3\pi}{4}$ characterizes a transition between two regimes. In contrast, σ_{11} is not singular.

Remark 9

The analyse done for the Cauchy stress tensor components provides important insights into the physics of fracture mechanics and it also gives tools for the extended finite element method (XFEM) enriched functions (Moës et al, 1999). In fact, this analysis indicates that it is necessary to add two singular functions coming from the exponents m_1 and m_2 contributions to the FEM classical polynomial basis. This is also shown by a numerical method identifying the singularity exponents (Karoui et al, 2014).

The Cauchy stress components, (96)-(98), are deduced from the canonical transformation fields (81)-(84). If $\sigma_{\alpha\beta}$ are the Cauchy stress components associated with the original field (16), one has :

$$\sigma_{\alpha\beta}^* = Q_{\gamma\alpha} Q_{\rho\beta} \sigma_{\gamma\rho} \quad (99)$$

in which $Q_{\alpha\beta}$ are the components (79) of the rotation tensor $\underline{\underline{Q}}$. In order to compare the magnitude of the order of the singularities of the cauchy stress tensor components in linear and non-linear cases, we determine the dominant order of the Cauchy stresses when the latter are function of the spatial coordinates y_α . Since such a representation of the stresses depends on the availability of an invertible estimate for the local transformation, we introduce the special coordinate $y = \sqrt{y_1^2 + y_2^2}$, evaluated along the line $\theta = 0$ and one draws from (82)-(84), (96)-(98) that :

$$\sigma_{22} \sim y^{1-\frac{2\omega}{\pi}} \quad \text{if } \omega \in \left] \frac{3\pi}{4}, \pi \right], \quad (100)$$

$$\sigma_{22} \sim y^{\frac{2\pi-4\omega}{4\omega-\pi}} \quad \text{if } \omega \in \left] \frac{\pi}{2}, \frac{3\pi}{4} \right], \quad (101)$$

$$\sigma_{22} \sim y^{-\frac{1}{2}} \quad \text{if } \omega = \frac{3\pi}{4}. \quad (102)$$

For the crack problem, $\omega = \pi$, the Cauchy stress components versus special coordinate y , given by (100), show that the most singular term has the asymptotic behaviour y^{-1} . This is a more pronounced singularity than the $y^{-\frac{1}{2}}$ singularity predicted by linear fracture mechanics (Xiaolin, 1986). For the notch problem, the order of the stress singularities, given by (100)-(102), is governed by the local geometry of the notch and increase with notch angle ω . In particular, for $\omega = \frac{3\pi}{4}$, the asymptotic behaviour of the component σ_{22} reduces to $\left(y^{-\frac{1}{2}}\right)$. On the other hand, the order of stress singularities does not depend on the type of the boundary conditions at infinity. In fact, the Cauchy stress components have the same order of singularities for a mode I and mode II loading. This is still in contrast to the predictions of the linear theory (Xiaolin, 1986).

Conclusion

Except works of (Gao, 1996) (Tarantino, 1997), (Ru, 1997) and (Tarantino, 1998), the notch vertex problem within the nonlinear elasticity framework has not received much attention. An asymptotic analysis of the notch vertex finite deformation fields in an incompressible Mooney-Rivlin material with a plane deformation context has

been presented for the first time. A deep analysis of the symmetric (mode I) and non-symmetric (mixed-mode) situations have been investigated. It was shown that the general case is asymptotically obtained by a mere rotation of the canonical fields, as it had been observed previously in the finite plane strain analysis of the crack problem (Stephenson,1982). By varying the value of the notch angle parameter appearing in the boundary value problem, the phenomenon of singularity and its effect on the stress fields are established. The nonlinear asymptotic analysis also reveals that there exists more than one singular term and that the leading singularity is stronger than that predicted by the linearized theory. Singular functions obtained by the way of this analysis can be used as enrichment function for the partition of unity finite element method (PUFEM). The obtained results involve many arbitrary constants governing the amplitude of the ensuing local elastostatic field. A precise estimate of these unknowns can be established on the basis of conservation law. The ability of this analytical solution to predict the true physical behaviour requires the determination of unknowns using a matching method (Edmunds and Willis, 1976a;1976b;1977).

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