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THE ONE DIMENSIONAL SEMI-CLASSICAL BOGOLIUBOV-DE GENNES HAMILTONIAN

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Abstract

We present a method for computing first order asymptotics of semiclassical spectra for 1-D Bogoliubov-de Gennes (BdG) Hamiltonian in the Theory of Supraconductivity. A more rigorous approach taking also into account tunneling corrections, will be discussed elsewhere. Our results have been announced in [IfaRou].

1 Introduction and statement of the result

In many situations, finding the spectrum for a semi-classical Hamiltonian reduces locally to a 1-D problem. This follows for instance from adiabatic approximation, separating the transverse modes from the longitudinal ones. Thus its spectrum is given in a good approximation by Bohr-Sommerfeld quantization rule. This approach is relevant for graph-like systems, as arises in the modelisation of semi-conductors or metallic nanomaterials. In this paper we are interested in a model of supraconductivity, accounting for Andreev reflection between SNS junctions (supraconducting contacts). A similar model when replacing the Normal Metal by a Ferromagnetic material, and the spin of the quasi-particles is taken into account, is called SFS junction [CaMo]. [1]

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1.1 Bogoliubov-de Gennes Hamiltonian

BdG Hamiltonians describe the dynamics of a quasi-particle (pair electron-hole) within Supraconductivity. Our framework, that we briefly recall below, is the dynamics of a quasi-particle without spin (SNS junction). It is described in [ChtLesBla], [DuGy] and based on earlier work by [An] and [deJoBe]. For a detailed insight into the theoretical and experimental setting in supraconductivity, see [BCS], [deGe], [KeSo], [Lé].

Consider a narrow metallic lead, with few transverse channels, connecting two superconducting contacts. For simplicity, the lead is identified with a 1-D structure, the interval $x \in [-L, L]$. The reference energy in the lead is taken as the Fermi level E_F, and the longitudinal problem reduces to describing the dynamics of a quasi-particle (hole/electron) in the effective chemical potential $\mu(x) = E_F - E^{\perp}(x)$, where $E^{\perp}(x)$ denotes the transverse energy of the channel, obtained from adiabatic approximation. We shall ignore channel mixing between different transverse modes, and consider only one transverse mode.

Interaction with the supraconductor bulk is modeled through the complex order parameter, or superconducting gap, $\Delta_0 e^{\pm i\phi/2}$, which extends smoothly in the metal to a function $\Delta(x)e^{i\phi(x)/2}$, $0 \leq \Delta(x) \leq \Delta_0$, accounting for the "dirty junction". Here $\phi(x) = \mathrm{sgn}(x)\phi$, $\Delta(x) \equiv 0$ in $|x| \leq L - \ell$, and we assume $\ell \ll L$. Note also that the phases $\phi(x)$ and the gap function $\Delta(x)$ should satisfy some consistency relations due to coupling between supraconductors and the lead. This difficult problem related to the so-called BCS gap equation, will not be discussed here.

As is shown in [ChtLesBla], in case of a "clean junction", namely when $\Delta(x)$ is sharp (discontinuous at $x = \pm L$), the dynamics (in particular, scattering properties) of the quasi-particle is fairly well described by the semi-classical BdG Hamiltonian

$$\mathcal{P}(x,\xi) = \begin{pmatrix} \xi^2 - \mu(x) & \Delta(x)e^{i\phi(x)/2} \\ \Delta(x)e^{-i\phi(x)/2} & -\xi^2 + \mu(x) \end{pmatrix} \tag{1}$$

This Hamiltonian was extended in [DuGy] in higher dimensions for a smooth $\Delta(x)$, allowing also for a magnetic field. The chemical potential $\mu(x)$ and order parameter $\Delta(x)$ are assumed to be constant outside the lead, namely $\mu(x) = \mu_0$, and $\Delta(x) = \Delta_0$ for $|x| \geq L$, and ℓ should be sufficiently large with respect to the typical wave-length h, the "renormalized Planck constant" h in (1), such that $h^2 = \hbar^2/2m$.

We use throughout Weyl h-quantization $\mathcal{P}(x, hD_x)$ on $L^2(\mathbf{R}) \otimes \mathbf{C}^2$. We assume that $\mathcal{P}(x, hD_x)$ enjoys time-reversal and PT symmetries : If $\mathcal{I}$ denotes complex conjugation $\mathcal{I}u(x) = \overline{u(x)}$, i.e. $\mathcal{I}$ quantizes the reflection on the ξ axis, and ${}^{\vee}$ the reflection ${}^{\vee}u(x) = u(-x)$, we have

$$\mathcal{I}\mathcal{P}(x, hD_x) = \mathcal{P}(x, hD_x)\mathcal{I}, \qquad {}^{\vee}\mathcal{I}\mathcal{P}(x, hD_x) = \mathcal{P}(x, hD_x)\mathcal{I}^{\vee} \tag{2}$$

Thus, $\mathcal{P}(x, hD_x)$ shares (formally) some features with Dirac operators, such as PT symmetry, and negative energies.

Electrons (e^-) and holes (e^+) with energy $E < \inf_{\mathbf{R}} \mu(x)$, $E < \Delta_0$ form so-called Andreev states sensitive to the variation of phase parameter ϕ between the superconducting banks.

The energy surface $\Sigma_E = \{\det(\mathcal{P} - E) = 0\} = \{(\xi^2 - \mu(x))^2 + \Delta(x)^2 = E^2\}$ is foliated by two periodic curves $\Lambda_E^> \subset \{\xi > 0\}$ and $\Lambda_E^< \subset \{\xi < 0\}$ symmetric with respect to $\xi = 0$. Because of the smoothness of Δ, the reflections occur in $]-L, L[$, we denote by $(\pm x_E, \xi_E) \in \Lambda_E^>$, the one-parameter family of focal points, where Σ_E turns vertical, i.e. $\partial_\xi \det(\mathcal{P} - E) = 0$, or $\xi^2 = \mu(x)$, which implies $\Delta(x) = E$. Due to PT symmetry (Δ is an even function), this gives $\Delta(\pm x_E) = E$ with $x_E \in]x_0 - \varepsilon_1, x_0 + \varepsilon_1[$ say. We do not consider the problem of "clustering" of eigenvalues as $E \to 0$ (Fermi level).

In the "step potential" limit $\alpha \to \infty$, for x near x_0, $\Delta(x)$ can be safely approximated by a linear function such that $\Delta(x_0) = E_0$, and $\mu(x)$ by a constant μ, for simplicity we assume $\mu = \mu_0$. So near x_0 we have

$$\phi(x) = \phi, \quad \mu(x) = \mu > E, \quad \Delta(x) = E + \alpha(x - x_E) \tag{3}$$

The condition $(x_E, \xi_E) \in \Sigma_E$ gives $\xi_E^2 = \mu > E$, $\Delta(x_E) = E$. Since the electron/hole states become semi-classically undistinguishable at $a_E = (x_E, \xi_E)$, we call it sometimes a "branching point". Contrary to the standard (scalar) potential well problem, with a turning point at $\xi_E = 0$, here the kinetic energy, common at a_E to the hole and the electron, is non zero, which truly accounts for a "current" between the superconducting banks.

This is what we call the *Normal-Supraconductor (NS) junction model.* For the dynamics of the quasi-particle associated with (1), the mechanism goes roughly as follows.

An electron e^- moving in the metallic lead, say, to the right, with energy $0 < E < \Delta_0$ below the gap and kinetic energy $K_+(x) = \mu(x) + \sqrt{E^2 - \Delta(x)^2}$ is reflected back as a hole e^+ from the right bank of the supraconductor, injecting a Cooper pair into the superconducting contact. The hole has kinetic energy $K_-(x) = \mu(x) - \sqrt{E^2 - \Delta(x)^2}$, and a momentum of the same sign as this of the electron.

When $\inf_{[-L,L]} K_-(x) > 0$ it bounces along the lead to the left hand side and picks up a Cooper pair in left bank of the supraconductor, transforming again to the original electron state, a process known as Andreev reflection. Since $\mathcal{P}(x, hD_x)$ is (formally) self-adjoint, there is of course also an electron moving to the left, and a hole moving to the right, for no net transfer of charge can occur through the lead in absence of thermalisation.

Nevertheless, when $\phi \neq 0$, this process yields so called phase-sensitive Andreev states, carrying supercurrents proportional to the ϕ-derivative of the "eigen-energies" of $\mathcal{P}(x, hD_x)$.

Actually, the hole can propagate throughout the lead only if $\inf_{[-L,L]} \mu(x) \geq E$. Otherwise, it is reflected from the potential $\mu(x)$ in the junction, and Andreev levels are quenched at higher energies, i.e. transform into localized electronic states. In this work, we shall focus on the case of the supercurrent (Andreev reflection) i.e. $\mu(x) > E$ for all $x \in [-L, L]$.

This very simplified model doesn't take into account some basic parameters in the Theory of Supraconductivity, such as the *coherence length* $\overline{\xi} = \hbar v_F / \pi \Delta_0$ (v_F being Fermi velocity),

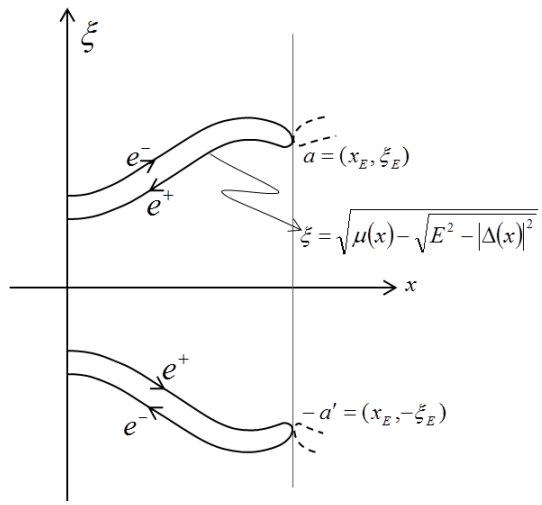

Figure 1: Phase-space picture: The Lagrangian manifolds $\Lambda_E^>$ and $\Lambda_E^<$ when $\inf_{[-L,L]} \mu(x) > E$

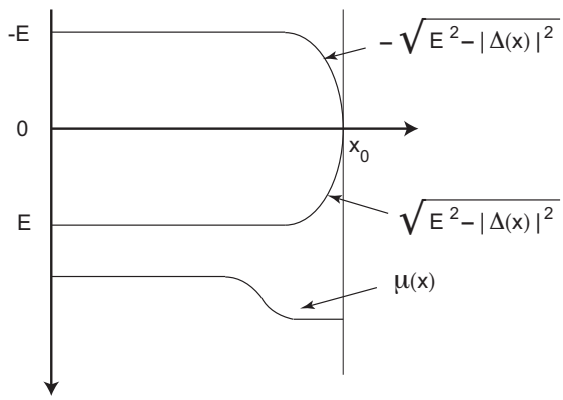

Figure 2: Energy picture: The potential curves when $\inf_{[-L,L]} \mu(x) > E$

or the *penetration depth*. However when the junction is long enough, i.e. when L is large compared to $\overline{\xi}$, it is shown is [ChtLesBla] there are relatively many quantum levels in the system, which is precisely the situation we are interested in. This holds if we think of $\overline{\xi}$ to be comparable with (or smaller than) ℓ, and assume both are much smaller than L, and h is much smaller than $\overline{\xi}$ and ℓ. See [DuGy], Eq.(5.94), and [ChtLesBla] for other conditions in case Δ is sharp but $\mu(x)$ is smooth. At least from a mathematical point of vue, (1) gives a fairly good insight into the scattering process, as we shall see below.

We denote by $\Lambda_\rho^>(E) = \Lambda_\pm^>(E)$ the branches of $\Lambda_E^>$ defined by $\xi = \sqrt{\mu(x) \pm \sqrt{E^2 - \Delta(x)^2}}$ (plus sign for e^-, minus sign for e^+), and similarly for $\Lambda_E^<$. Actually it is misleading to consider $\Lambda_E^>$ as a whole, since it belongs to different objects: no Maslov index will occur in the quantization rules.

We shall consider $[-x_E, x_E]$ essentially as the "classically allowed region" at energy E, so that classical action integrals, computed over this interval, give Bohr-Sommerfeld quantization rules at leading order, for the semi-classical spectrum. Nevertheless, we keep track of the

complex germs of the microlocal solutions at $\pm x_E$. These microlocal solutions have a complex phase outside $[-x_E, x_E]$, contrary to the scalar case (Schrödinger operator) where the phase becomes purely imaginary.

Since the pseudo-particle is no longer governed by BdG equations inside the supraconducting bulk, letting $|x| \to \infty$ makes sense only when the typical wave-length h is much smaller than ℓ.

1.2 Qualitative aspects of the spectrum

We do not attempt here at deriving rigorous spectral properties for BdG Hamiltonian. Observe that $\mathcal{P}(x, hD_x)$ is a symmetric operator with PT symmetry, but maybe not essentially self-adjoint, the difficulty being similar in Dirac equation, see e.g. [Wo]. Since we are mostly concerned with the construction of quasi-modes, we shall therefore refer to the *pseudo-spectrum* of $\mathcal{P}(x, hD_x)$ rather than to its spectrum. The (semi-classical) pseudo-spectrum of $\mathcal{P} = \mathcal{P}(\phi)$ is *a priori* not invariant under $E \mapsto -E$, but $\mathcal{P}(\phi)$ is mapped onto

$$\mathcal{I}\sigma_y \mathcal{P}(\phi)\sigma_y \mathcal{I} = -\mathcal{P}(\phi), \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The energy surface $\Sigma_E = \{\det(\mathcal{P}(x,\xi) - E) = 0\}$ depends only on E^2, which we explain physically by the existence of "negative energies". For $0 < E < \Delta_0$, Σ_E is compact, so we expect the real part of the pseudo-spectrum to be discrete in this interval. On the other hand we know [ChtLesBla] that the spectral dynamics is also conveniently described within the scattering matrix formalism in the "clean junction limit" $\ell = 0$. This is reminiscent of the fact that $0 < E < \mu$ is a scattering level for $\xi^2 - \mu(x)$: because of the relation $-E < E < \mu(x) = -V(x)$, we have $\xi^2 + V(x) = E$ and $V(x) < E$. So the pseudo-spectrum near $]0, \Delta_0[$ consists actually in "pseudo-resonances", but we will not intend to make this more precise. Instead we content ourselves to compute by Bohr-Sommerfeld quantization rules their leading order asymptotics, which actually turns out to be real. Due to the symmetry $\mathcal{I}\Lambda_E^> = \Lambda_E^<$ inducing tunneling properties, they should actually come up in pairs, with small (real or complex) splitting.

1.3 An outlook at monodromy operator and scattering matrix

To understand the monodromy properties of solutions for BdG equation, it is useful to make the analogy with the scattering process in the scalar case, see [Ar,Sect.5]. We shall however follow another route, and mention this approach only as a guideline.

1.3.1 Schrödinger operator on the real line

Consider the operator $P = -h^2\Delta + V$ together with the eigenvalue equation

$$-h^2 u''(x) + V(x)u(x) = Eu(x) \tag{4}$$

with a compactly supported potential $V(x)$, and assume the energy $E = k^2$ of the particle is strictly positive.

To the left of the support of V, (4) coincides with the equation

$$-h^2 u''(x) = k^2 u(x) \tag{5}$$

for the *free particle* whose solution span a 2-D complex vector space $\mathcal{Z} \approx \mathbf{C}^2$, called the *state space* of the free particle.

Hence Schrödinger equation has 2 solutions which coincide with $f_1 = e^{ikx}$ and $f_2 = e^{-ikx}$ to the left of the support, called *incoming to the right* and *outgoing to the left*. In the same way, there exist 2 solutions which coincide with e^{ikx} and e^{-ikx} to the right of the support, called *outgoing to the right* and *incoming to the left*, respectively. It is easy to see that the particle cannot be totally reflected to the left, but can depart totally to the right (this is obviously the case if $V \equiv 0$).

Since (5) has real coefficients, its solutions also span a 2-D real vector space $\mathcal{Z}_R \approx \mathbf{R}^2$. Real solutions $e_1 = \cos kx$, $e_2 = \sin kx$ are connected with complex solutions f_1, f_2 by $f_1 = e_1 + ie_2$ and $f_2 = e_1 - ie_2$.

The *monodromy operator* $M(k)$ of (4) with a potential of compact support is a linear operator mapping the state space of a free particle with energy $E = k^2$ into itself. It is defined in the following way. To a solution u_- of (5) in $\mathcal{Z}$ we assign a solution u of (4) coinciding with u_- to the left of the support; in turn we assign to u its value $u_+ \in \mathcal{Z}$ to the right of the support, and set $u_+ = M(k)u_-$. In other terms, the monodromy operator acts according to the formula $f_1 + Bf_2 \mapsto Af_1$, when (4) has a solution equal to $f_1 + Bf_2$ to the left of the support and to Bf_1 to the right of the support. We call $|A|^2$ the *transmission coefficient* and $|B|^2$ the *reflection coefficient*.

Considering the real solutions of (4), (5) we can show that the phase flow of (4) preserves area. It follows that $\mathcal{M}(k) \in \mathrm{SU}(1,1)$, the group of complex 2×2 matrices with determinant 1 preserving the Lorenz form $|z_1|^2 - |z_2|^2$.

Since (4) defines a self-adjoint operator with real coefficients, the monodromy operator takes the form

$$M(k) = \begin{pmatrix} 1/\overline{A} & -\overline{B}/\overline{A} \\ -B/A & 1/A \end{pmatrix} \in SU(1,1) \tag{6}$$

In particular, $|A|^2 + |B|^2 = 1$.

Along with the passage from the left to the right of the support of V, we can consider the passage from the right to the left. The corresponding solution v of (4) is $e^{-ikx/h} + B_2 e^{ikx/h}$ to the right of $\operatorname{supp} V$, and $A_2 e^{-ikx/h}$ to the left. The *scattering matrix* is defined as

$$S(k) = \begin{pmatrix} A & B \\ -\overline{B}A/\overline{A} & A \end{pmatrix} \in U(2)$$

Resonances of (4) are then defined as $E = k^2 \in \mathbf{C}$, where k is a pole of S, and *physical*

resonances those with $\operatorname{Im} k > 0$. Thus E is a resonance iff the solution of (4) is purely outgoing as $x \to +\infty$ and $x \to -\infty$.

The poles coincide with the poles of meromorphic extension of the resolvent $(P - k^2)^{-1}$ from the physical half-plane $\operatorname{Im} E < 0$ to the second sheet $\operatorname{Im} E > 0$.

1.3.2 BdG equation

Consider now BdG equation $(\mathcal{P}(x, hD_x) - E)U = 0$ for large $|x|$, i.e. when $\Delta(x) = \Delta_0$, $\mu(x) = \mu_0 > E$, and look for solutions of the form

$$U(x;h) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{ikx/h} \\ e^{i\ell x/h} \end{pmatrix}$$

The secular equation decouples into 2 identical systems for (a, c) and (b, d), which have non trivial solutions iff $\ell = \overline{k}$ with $k = \sqrt{\mu_0 + i\sqrt{\Delta_0^2 - E^2}}$. So eigenfrequencies come in opposite and complex conjugate pairs $(\pm k, \pm\overline{k})$, and the corresponding solutions are determined as follows:

Let $\mathcal{Z}$ be the 2-D complex line bundle spanned by $F_1^{\pm}(x) = \binom{e^{i\phi(x)/2}}{-i} e^{\pm ikx/h}$, $\phi(x) = \operatorname{sgn}(x)\phi$ (associated with the scattering process $e^+ \to e^-$), and $\overline{\mathcal{Z}}$ be the 2-D complex line bundle spanned by $F_2^{\pm}(x) = \binom{e^{i\phi(x)/2}}{i} e^{\pm i\overline{k}x/h}$ (associated with the scattering process $e^- \to e^+$). Then the space of solutions of exponential type to $(\mathcal{P}(x, hD_x) - E)U = 0$ for large $|x|$, consists in the 4-D complex vector space $\mathcal{Z} \oplus \overline{\mathcal{Z}}$, and $\mathcal{Z}, \overline{\mathcal{Z}}$ are orthogonal for the usual pointwise Hermitian product in $\mathbf{C}^2$.

According to the general scheme (see [ReSi]), we declare that $E \in \mathbf{C}$ is a *$\mathcal{Z}$-resonance* iff the $\mathcal{Z}$-component of the wave function solving BdG equation is outgoing at infinity, i.e.

$$U(x,h) = A\begin{pmatrix} e^{i\phi/2} \\ -i \end{pmatrix} e^{ikx/h}, x \to +\infty, \quad U(x,h) = B\begin{pmatrix} e^{-i\phi/2} \\ -i \end{pmatrix} e^{-ikx/h}, x \to -\infty$$

Similarly we say that E is a *$\overline{\mathcal{Z}}$-resonance* iff the $\overline{\mathcal{Z}}$-component of the wave function is outgoing at infinity, i.e.

$$U(x,h) = A\begin{pmatrix} e^{i\phi/2} \\ i \end{pmatrix} e^{-i\overline{k}x/h}, x \to +\infty, \quad U(x,h) = B\begin{pmatrix} e^{-i\phi/2} \\ i \end{pmatrix} e^{i\overline{k}x/h}, x \to -\infty$$

So for both sets of resonances, the corresponding solution is simultaneously decaying, and outgoing at $\pm\infty$. These sets of resonances need not coincide (although they come up in pairs), but their real parts are actually given by Bohr-Sommerfeld quantization rules.

We define next the monodromy operator $M^{\mathcal{Z}}(k)$ acting on $\mathcal{Z}$ according to the formula

$$\begin{pmatrix} e^{-i\phi/2} \\ -i \end{pmatrix} e^{ikx/h} + B\begin{pmatrix} e^{-i\phi/2} \\ -i \end{pmatrix} e^{-ikx/h} \mapsto A\begin{pmatrix} e^{i\phi/2} \\ -i \end{pmatrix} e^{ikx/h}$$

and similarly for $M^{\overline{\mathcal{Z}}}(k)$.

It is no longer true however that $M^{\mathcal{Z}}(k)$, $M^{\overline{\mathcal{Z}}}(k)$ are in SU(1,1), but we can expect them instead to be in U(1,1). Scattering matrices $S^{\mathcal{Z}}(k)$, $S^{\overline{\mathcal{Z}}}(k)$ can be defined as in the scalar case, and expected also to have a meromorphic extension to the complex plane, their poles defining the resonances $E^{\mathcal{Z}}$ and $E^{\overline{\mathcal{Z}}}$.

Similar ideas should apply for the SFS junction, see Sect.8.

In fact we shall proceed differently: assuming already existence of (pseudo-)resonances, we construct "relative monodromy operators" $\mathcal{M}_\rho^{a,a'}(E)$, $\rho = \pm$ labelling the quasi-particle, in the interval $[-x_E, x_E]$ (the "classically allowed region") which belong to U(1,1) for the "flux norm", the Lorenzian metric defined by $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

As in Fig.1, let $\gamma^{>,\rho}(E) \subset \Sigma_E$ ($\rho = 1$ for e^-, $\rho = -1$ for e^+) be the branch of $\Lambda_E^> = \Sigma_E \cap \{\xi > 0\}$ starting at $(-x_E, \xi_E)$ and ending at (x_E, ξ_E). Consider similarly $\gamma^{>,\rho}(E)$.

Bohr-Sommerfeld (BS) quantization rules will be derived from $\mathcal{M}_\rho^{a,a'}(E)$, through the associated Gram matrices $\mathcal{G}_\rho^{(a,a')}(E)$ (see Definition 1.1) and their determinants (Jost function). Thus the real part of $E^{\mathcal{Z}}$ and $E^{\overline{\mathcal{Z}}}$, which we expect to be the resonances of the problem determined by the procedure above, will be determined at leading order in h, as the zeroes of Jost functions.

Using complex WKB solutions, as we sketch in Sect. 5.2.1, we could also construct relative monodromy operators $\mathcal{M}_\rho^{+\infty,a}(E)$ and $\mathcal{M}_\rho^{a',-\infty}(E)$ in the intervals $[x_E, +\infty[$ and $]-\infty, -x_E]$. We expect that $\mathcal{M}_\rho(E) = \mathcal{M}_\rho^{+\infty,a}(E)\mathcal{M}_\rho^{a,a'}(E)\mathcal{M}_\rho^{a',-\infty}(E)$ stands for the "global monodromy operator", as (6) in the case of Schrödinger operator. Matrix $\mathcal{M}_\rho(E)$ would also provide a Jost function, that should vanish at the resonances, but since it can be computed only mod $\mathcal{O}(h)$, this is not sufficient to account for exponentially small effects. Thus we would need yet another argument, based on Grushin problem, see [Sj], [HeSj], [Ro], [IfaLouRo].

Thus matrices $\mathcal{M}_\rho(E)$ appear as "branches" of a "global" monodromy operator $\mathcal{M}(E)$ we know very little about, but since $\mathcal{M}_\rho(E)$ account for the interaction at the branching points, we can consider they carry the main relevant information.

There is actually a conserved quantity for BdG equation, defined as follows. Let $\mathcal{U}(x) = (u_1(x), u_2(x), u_3(x), u_4(x))$ with

$$u_2(x) = hD_x u_1(x),\ hD_x u_2(x) = (\mu(x) + E)u_1(x) - \Delta(x)e^{i\phi(x)/2}u_3(x)$$
$$u_4(x) = hD_x u_3(x),\ hD_x u_4(x) = (\mu(x) - E)u_3(x) + \Delta(x)e^{-i\phi(x)/2}u_1(x)$$

By the discussion above, there exists a system of fundamental solutions associated with BdG equation $(\mathcal{P}(x, hD_x) - E)U(x) = 0$, namely $\big(\mathcal{U}^1(x), \mathcal{U}^2(x), \mathcal{U}^3(x), \mathcal{U}^4(x)\big)$. So we can form

$$W(x) = \det\big(\mathcal{U}^1(x), \mathcal{U}^2(x), \mathcal{U}^3(x), \mathcal{U}^4(x)\big)$$

that is simply the Wronskian of the system. It is easy to check that $W(x) = \text{Const.}$, but we will not further elaborate on these general facts, and focus on computing Bohr-Sommerfeld quantization conditions instead.

1.4 Main result

Extending the method of positive commutators elaborated in [Sj], [HeSj], [Ro] and [IfaLouRo], we obtain Bohr-Sommerfeld quantization rules for the quasi-particle, accounting for Andreev currents.

The full justification of BS, that is beyond the scope of this paper, would consist in solving a Grushin problem, which leads to construct a matrix $E_{-+}(E)$, singular precisely at the spectral values: (1) in the case of 1-D Schrödinger operator with periodic orbits [IfaLouRo], $E_{-+}(E)$ is just a scalar, the determinant of Gram matrix ; (2) in the case of 1-D Schrödinger operator with homoclinic orbit [Sj], $E_{-+}(E)$ is a 2×2 matrix ; (3) in the case of 1-D Born-Oppenheimer 2×2 Hamiltonian with crossing of modes [Ro], $E_{-+}(E)$ is a 4×4 matrix.

Let us now make our statements more precise. Consider Weyl h-quantization $\mathcal{P}(x, hD_x)$ on $L^2(\mathbf{R})\otimes \mathbf{C}^2$ of BdG Hamiltonian (1), where we recall that $\mu(x) > 0$ and $\Delta(x) \geq 0$ are smooth, even functions on the real line, verifying $\mu(x) = \mu_0$, $\Delta(x) = \Delta_0$ when $|x| > L$, and $\Delta(x) \equiv 0$ in $|x| \leq L - \ell$, with L/ℓ large enough. Recall also $\phi(x) = \mathrm{sgn}(x)\phi$. Let $E_0 > 0$ be a "scattering energy" namely $E_0 < \inf_{\mathbf{R}} \mu(x)$, $E_0 < \Delta_0$, and $x_0 \in]L - c_1\ell, L - c_2\ell[$ for some $0 < c_2 < c_1 < 1$ such that $\Delta(x_0) = E_0$. In $]L - c_1\ell, L - c_2\ell[$ we assume $\mu(x) = \mu$ is a constant, and $\Delta(x)$ to vary linearly as in (3), where the slope α of the same order of magnitude as L/ℓ: this hypothesis will be crucial to reduce our analysis to Weber equation near the focal points. For E near E_0, let $(x_E, \xi_E) \in \Sigma_E$ be the focal points in $x > 0, \xi > 0$ with $x_E \in]L-c_1\ell, L-c_2\ell[$, $\xi_E^2 = \mu > E$, and $\Delta(x_E) = E$.

Recall $\gamma^{>,\rho}(E), \gamma^{<,\rho}(E) \subset \Sigma_E$ from Sect. 1.3.2. Ignoring tunneling effects in momentum, we only consider $\gamma^{>,\rho}(E)$ and denote it for short γ_E^ρ or simply by ρ.

So far we have only discussed "pseudo-resonances" at a heuristic level, but it is more convenient to define them somewhat abstractly.

Definition 1.1. *Consider the 2×2 Gram matrix (see (175) below)*

$$\mathcal{G}_\rho^{(a,a')}(E) = \begin{pmatrix} \big(U_1|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) & \big(U_2|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) \\ \big(U_1|F_{\varepsilon,\omega}^{\nu,\rho,a'}\big) & \big(U_2|F_{\varepsilon,\omega}^{\nu,\rho,a'}\big) \end{pmatrix} \tag{7}$$

Here $F_{\varepsilon,\omega}^{j,a,\rho}$ (resp. $F_{\varepsilon,\omega}^{j,a',\rho}$) denote some basis of the co-kernel of $\mathcal{P}(x, hD_x)$ microlocalized on γ_E^ρ near a (resp. a'), and U_1, U_2 some vectors (WKB solutions) in the kernel of $\mathcal{P}(x, hD_x)$ we shall compute in Sect. 5 and 6. Indices $\nu, -\nu-1, \varepsilon, \omega$ parametrize solutions of Weber Eq. as in Sect.2.1 and 2.2, and $\mathcal{G}_\rho^{(a,a')}(E)$ doesn't depend on the choice of ε, ω when $\varepsilon\,\omega = 1$. We call Jost function the determinant $\det \mathcal{G}_\rho^{(a,a')}(E)$, and define, mod $\mathcal{O}(h)$, the pseudo-resonances E of $\mathcal{P}(x, hD_x)$ belonging to ρ, as the zeroes of Jost function.

Here $\mathcal{G}_\rho^{(a,a')}(E)$ is computed from the "relative monodromy matrix" $\mathcal{M}_\rho^{a,a'}(E) \in U(1,1)$ (see (144) below), that relates the expressions of U_1, U_2 near a and near a', when moving along ρ.

This definition is extrapolated from the case of Schrödinger operator, where Gram matrix is associated with the entire loop γ_E over $x \in [-x_E, x_E]$, see [IfaLouRo]. Vanishing of Jost function in that case means that the bundle $K_h(E)$ of asymptotic solutions has trivial holonomy over γ_E. In the present case, however, since $\rho = +1$ and $\rho = -1$ belong to different symmetries, considering $\Lambda_E^> = \gamma^{>,+}(E) \cup \gamma^{>,-}(E)$ as periodic orbits of $\det(\mathcal{P}(x,\xi) - E)$ is semi-classically meaningless.

So the condition $\det \mathcal{G}_\rho^{(a,a')}(E) = 0$ only means that the bundles $K_h^\rho(E)$ of microlocal solutions over $x \in [-x_E, x_E]$ that we shall construct below, are one-dimensional.

The relative monodromy matrix contains some action integrals. Along ρ we define the *semi-classical momentum* $\xi^\rho(x;h) = \xi^\rho(x) + \mathcal{O}(h)$ in $\{\xi > 0\}$, as a "deformation" of the classical momentum $\xi^\rho(x) = \sqrt{\mu(x) + \rho\sqrt{E^2 - \Delta(x)^2}}$ solution of $\det(\mathcal{P}(x,\xi) - E) = 0$. It can be obtained in two different ways, which agree on $]-x_E, x_E[$: (1) by diagonalizing $\mathcal{P}^w(x, hD_x)$ outside the focal points. This adds a sub-principal term $h\lambda_\rho^{(1)}(x,\xi)$ to the principal symbol $\lambda_\rho(x,\xi)$ and yields a family of deformations of the lagrangian manifold $\Lambda_\rho^>(E)$ defined as the level sets $\xi = \xi^\rho(x;h)$ of $\lambda_\rho(x,\xi) + h\lambda_\rho^{(1)}(x,\xi)$ (see (140) below) ; (2) by computing the microlocal kernel near the focal point (x_E, ξ_E), represented by oscillating integrals defined by h-dependent phase and amplitude in Fourier representation (see Proposition 3.1). We retrieve $\xi^\rho(x;h) = \partial_x S_\rho(x;h)$, in the spatial representation, see (171).

The off-diagonal elements of the "relative monodromy matrices" (see Sect.7.1) have a phase of the form $\int_{-x_E}^{x_E} \xi^\rho(y;h)\,dy + b(E_1';h)$ where $b(E_1';h)$ is a "boundary term" (involving Andreev reflection) which is expressed in term of the h-dependent phase function parametrizing the microlocal kernel, evaluated at the focal points $\pm x_E$ (see (164). Here E_1' is the rescaled energy parameter as in (60)-(65).

Theorem 1.1. *Under hypotheses above, there is $\varepsilon_0 > 0$ such that the real zeroes of Jost function in $]E_0 - \varepsilon_0, E_0 + \varepsilon_0[$ verify the following "Bohr-Sommerfeld quantization rule"*

$$\int_{-x_E}^{x_E} \xi^\rho(y;h)\,dy + h\,\frac{\phi}{2} + h\frac{\pi}{4} + b(E_1';h) \equiv 0, \quad \mathrm{mod}\,\pi h\mathbf{Z} \tag{8}$$

Note that the integral $\int_{-x_E}^{x_E} \xi^\rho(x;h)\,dx$ represents half of the semi-classical action of the quasi-particle in Andreev reflection. By time reversal symmetry, the other half is obtained by changing ξ to $-\xi$. Thus the two connecteds components $\Lambda_E^{>,\rho}$ and $\Lambda_E^{<,\rho}$ merge together to a "close orbit", carrying (classically) Andreev current. Taking into account the tunneling properties between $\Lambda_E^>$ and $\Lambda_E^<$ would allow to relate the branches of microlocal solutions over $[-x_E, x_E]$ by complex monodromy, leading to resonances for the scattering process discussed in Sect.1. We stress that (8) is not the standard BS rule: it contains the additional phase $h\phi$, and the "boundary term" $b(E_1';h)$ computed at the junction, and is computed mod $\pi h\mathbf{Z}$ instead of $2\pi h\mathbf{Z}$, so that no Maslov index occurs. Note also that $b(E_1';h)$ is again an odd function of E, see (165) below.

1.5 Outline of the paper

In Sect.2 we investigate integral representations of the parabolic cylinder functions D_ν and $D_{-\nu-1}$ for real positive ν, in the form given in [WhWa] (see also [Ol]). These functions, conveniently normalized, provide the basic ingredient for microlocal solutions of $(\mathcal{P}-E)U=0$ near the branching points. Their complex branches in the "shadow zone", with different growth properties, will play a crucial role in computing the monodromy matrices.

In Sect.3 we describe the set of h-Fourier transforms $\widehat{U}$ near the branching points. They take the form of semi-classical *spinors*, and are obtained from the solutions of Weber Eq. The detailed computation of $\widehat{U}$ is postponed to Appendix B.

In Sect.4 we normalize $\widehat{U}$ by means of *microlocal Wronskians*, or positive commutators, elaborating concepts introduced in [Sj], [HeSj], [Ro] for homoclinic Lagrangian manifolds, and extended later to periodic orbits [SjZw], [IfaLouRo]. Though BdG does not really enter any of these frameworks, our approach still allows to endow the vector bundle of microlocal solutions with a Lorenzian structure, from which will merge the U(1,1) symmetry group.

In Sect.5 we convert these normalized microlocal solutions to the spatial representation, and analyse their growth in the "shadow zone".

In Sect.6 we construct WKB solutions in $]-x_E, x_E[$.

In Sect.7 we write connexion formulas relating the microlocal solutions at a_E with those at a'_E through the intermediate WKB solutions in $]-x'_E, x_E[$. This give the relative monodromy operators on each branch $\rho=\pm1$ corresponding to the electron and the hole respectively. Following the method elaborated in [IfaLouRo], we built up Gram matrices $\mathcal{G}_\rho^{a,a'}(E)$ of solutions microlocalized on each branch $\rho=\pm1$. Their determinant vanishes precisely at Andreev levels $E_n(h)$.

In Sect.8 we sketch an approach to the SFS junction as in [CaMo], carrying the discussion on the classical level.

In Appendix A, we make more precise Helffer-Sjöstrand normal form for a 1-D h-PDO near a non degenerate potential well.

In Appendix B, we construct from the normal form the microlocal solutions used in the main text.

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2 Parabolic cylinder functions

Since the semi-classical harmonic oscillator $P_0(\eta, hD_\eta)=\frac{1}{2}\big((hD_\eta)^2+\eta^2-h\big)$ plays a crucial rôle in BdG Hamiltonian at the branching points, we proceed first to discuss Weber equation.

For later purposes, let us recall from [Hö,Thm 7.7.5] the Theorem of asymptotic stationary phase, that we shall essentially use at leading order.

Theorem 2.1. *If $f : \mathbf{R}^d \to \mathbf{C}$, with $\operatorname{Im} f \geq 0$ has a non-degenerate critical point at x_0, then*

$$\int_{\mathbf{R}^d} e^{if(x)/h}\, u(x)\, dx \sim e^{if(x_0)/h} \left(\det(\frac{f''(x_0)}{2i\pi h})\right)^{-1/2} \sum_j h^j\, L_j u(x_0) \tag{9}$$

where L_j are differential operators, $L_0 u = u$, and

$$L_1 u(x_0) = \sum_{n=0}^{2} \frac{2^{-(n+1)}}{in!(n+1)!} \langle (f''(x_0))^{-1} D_x, D_x\rangle^{n+1} \big((\Phi_{x_0})^n u\big)(x_0)$$

where $\Phi_{x_0}(x) = f(x) - f(x_0) - \frac{1}{2}\langle f''(x_0)(x-x_0), x-x_0\rangle$ vanishes of order 3 at x_0 (here $D_x = \frac{1}{i}\partial_x$ denotes the derivation operator, and n the algebraic power).

For any real ν, Weber equation $P_0 v = \nu h v$, through the change of variables $\eta = (h/2)^{1/2}\zeta$, $\widetilde{v}(\zeta) = v(\eta)$ can be written in the form

$$-\widetilde{v}'' + \frac{1}{4}\zeta^2 \widetilde{v} = \big(\nu + \frac{1}{2}\big)\widetilde{v} \tag{10}$$

Fundamental solutions of (10) are expressed in term of *parabolic cylinder* functions D_ν, see [WhWat pp.347-349&p.245], [Ol]. We review below their asymptotics for large ν, giving also asymptotics for solutions of $P_0(\eta, hD_\eta)u = \nu h u$.

For any complex number ν, $D_\nu(\zeta)$ is an entire function in the complex plane, normalized by specifying its asymptotic expansion for large ζ (Whitakker normalization) :

$$D_\nu(\zeta) = e^{-\zeta^2/4}\zeta^\nu\big(1 - \frac{\nu(\nu-1)}{2\zeta^2} + \cdots\big), \quad |\arg\zeta| < \frac{3\pi}{4} \tag{11}$$

This normalization, however, will be modified in Sect.4. When ν is a positive integer, $D_\nu(\zeta) = e^{-\zeta^2/4} H_\nu(\zeta/\sqrt{2})$ with H_ν an Hermite function. For $\nu \neq -2, -3, \cdots$, we have the following integral representations :

$$D_\nu(\zeta) = \frac{\Gamma(\nu+1)}{2i\pi} e^{-\zeta^2/4} \int_\infty^{(0^+)} e^{-\zeta s - s^2/2}(-s)^{-\nu}\, \frac{ds}{s} \tag{12}$$

where the integration contour encircles the positive real axis in the direct sense. Note that the integral in (12) stands for the inverse Laplace transform of $e^{-s^2/2}(-s)^{-\nu-1}$, a multivalued function of s when ν is not an integer; see [DePh] for a discussion of such transforms. We restrict to $\nu + 1 \geq 0$, but allow for integer values of ν, which give poles to $\Gamma(-\nu)$.

Equation (10) being invariant under the change $\zeta \mapsto -\zeta$, and also changing simultaneously ζ to $\pm i\zeta$ and ν to $-\nu - 1$, when $\nu \neq 1, 2, \cdots$, it follows that $D_\nu(-\zeta)$ and $D_{-\nu-1}(\pm i\zeta)$ are still solutions, with

$$D_{-\nu-1}(i\zeta) = \frac{\Gamma(-\nu)}{2i\pi} e^{\zeta^2/4} \int_\infty^{(0^+)} e^{-i\zeta s - s^2/2}(-s)^{\nu+1}\, \frac{ds}{s} \tag{13}$$

The systems $\big(D_\nu(\pm\zeta), D_{-\nu-1}(\pm i\zeta)\big)$ are fundamental solutions of (10) for any choice of $\pm$. When ν is not an integer, both systems

$$\big(D_\nu(\zeta), D_\nu(-\zeta)\big), \quad \big(D_{-\nu-1}(i\zeta), D_{-\nu-1}(-i\zeta)\big)$$

are fundamental solutions of (10). Conversely, ν is an eigenvalue of $\widetilde{P}_0 = (D_\zeta)^2 + \frac{1}{4}\zeta^2 - \frac{1}{2}$ iff $\nu \in \mathbf{N}$. Here we construct asymptotic solutions of $(\widetilde{P}_0 - \nu h)u = 0$ by evaluating (12) or (13) as semi-classical distributions ($h \to 0$) by stationary phase formula (9).

2.1 The semi-classical distribution D_ν

For $\varepsilon = \pm 1$ and $\nu \neq -2, -3, \cdots$, (12) writes

$$D_\nu\big(\varepsilon(h/2)^{-1/2}\eta\big) = \frac{\Gamma(\nu+1)}{-2i\pi\sqrt{h}} h^{E^2/4h} \int_\infty^{(0^+)} \exp\big[i\Phi_\varepsilon^\nu(s;\eta)/h\big]\, ds \tag{14}$$

where we have set

$$\Phi_\varepsilon^\nu(s;\eta) = i\big(\frac{\eta^2}{2} + \sqrt{2}\,\varepsilon\,\eta s + \frac{s^2}{2} + \frac{E^2}{2}\log(-s)\big), \quad E = \sqrt{2(\nu+1)h} \tag{15}$$

2.1.1 Asymptotics for $|\eta| < E$

To begin with, we consider the classically allowed region. We evaluate (14) by stationary phase (9). The critical points of $s \mapsto \Phi_\varepsilon^\nu(s;\eta)$ are the roots $s_{\varepsilon,\omega}^\nu$ of the quadratic equation $s^2 + \sqrt{2}\,\varepsilon\,\eta s + \frac{E^2}{2} = 0$, namely

$$-\sqrt{2}\,\varepsilon\, s_{\varepsilon,\omega}^\nu = \eta + i\omega\sqrt{E^2 - \eta^2}, \quad \text{any } \omega = \pm 1 \tag{16}$$

Together with ε, this introduces a new index ω ; they will be eventually related by $\varepsilon\,\omega = 1$. An elementary computation shows that the corresponding critical values are

$$\frac{E^2}{4i}\big(1 - \log\frac{E^2}{2}\big) + \frac{1}{2}\big[\omega\eta\sqrt{E^2-\eta^2} - E^2\check{\Theta}_{\varepsilon,\omega}(\eta)\big]$$

where

$$\check{\Theta}_{\varepsilon,\omega}(\eta) = \arg\big(\frac{\varepsilon}{\sqrt{2}}(\eta + i\omega\sqrt{E^2-\eta^2})\big) \in]-\pi, \pi[\tag{17}$$

To simplify notations, we remove the constant term from the phase, which gives the additional factor $\big(\frac{2e}{E^2}\big)^{E^2/4h}$ in (14), and denote by

$$\Phi_{\varepsilon,\omega}^\nu(\eta) = \frac{1}{2}\big(\omega\eta\sqrt{E^2-\eta^2} - E^2\check{\Theta}_{\varepsilon,\omega}(\eta)\big) \tag{18}$$

the critical value. We restrict mainly to the classically allowed region $|\eta| \leq E$, but need also know the germ of the analytic continuations of $\Phi_\varepsilon^\nu(s;\eta)$ at $\eta = \pm E$.

The Hessian of $\Phi_\varepsilon^\nu(s;\eta)$ at the critical points $s_{\varepsilon,\omega}^\nu$ is

$$\big(\frac{\partial^2\Phi_\varepsilon^\nu}{\partial s^2}\big)\big(s_{\varepsilon,\omega}^\nu(\eta);\eta\big) = \frac{\omega\sqrt{2}\sqrt{E^2-\eta^2}}{\varepsilon\, s_{\varepsilon,\omega}^\nu(\eta)} = \frac{-2\,\omega\,\eta\sqrt{E^2-\eta^2}}{E^2} + i\frac{2\big(E^2-\eta^2\big)}{E^2}$$

hence $s^\nu_{\varepsilon,\omega}$ is non degenerate when η is not a turning point $\pm E$, and defines the Jacobian (independent of ε)

$$J^\nu_\omega(\eta) = \frac{1}{i}\frac{\partial^2 \Phi^\nu_\varepsilon}{\partial s^2}(s^\nu_{\varepsilon,\omega};\eta) \tag{19}$$

Furthermore, $\mathrm{Im}\,\frac{\partial^2\Phi^\nu_\varepsilon}{\partial s^2}(s^\nu_{\varepsilon,\omega};\eta) = 2E^{-2}(E^2-\eta^2) > 0$. We choose a "good contour" of integration in the s-plane encircling the positive real axis and intersecting the imaginary axis at the conjugate points $s^\nu_{\varepsilon,\omega}(\eta)$, $\omega = \pm 1$.

Applying (9) to the contributions of $s^\nu_{\varepsilon,\omega}$, we find :

$$D_\nu\big(\varepsilon(h/2)^{-1/2}\eta\big) = \widetilde{C}^\nu_h (E^2-\eta^2)^{-1/4} \sum_{\omega=\pm 1} \big(i\,\varepsilon\,\omega s^\nu_{\varepsilon,\omega}(\eta)\big)^{1/2} \exp\big[i\Phi^\nu_{\varepsilon,\omega}(\eta)/h\big] + \mathcal{O}(h) \tag{20}$$

with

$$\widetilde{C}^\nu_h = \frac{\Gamma(\nu+1)}{-i\sqrt{2\pi}2^{1/4}}\big(\frac{2eh}{E^2}\big)^{E^2/4h}$$

Making use of the relation $\arg(z) - \arg(-z) = \pi\,\mathrm{sgn}(\mathrm{Im}\,z)$, we find

$$\check{\Theta}_{-,+}(\eta) - \check{\Theta}_{+,+}(\eta) = -\pi, \quad \check{\Theta}_{-,-}(\eta) - \check{\Theta}_{+,-}(\eta) = \pi \tag{21}$$

Remark: From this we recover easily the quantization condition for the harmonic oscillator. Namely, comparing the values of (20) for $\varepsilon = \pm$, we observe that the only dependence on ε consists in phase factors $e^{\pm i\pi E^2/2h}$. Thus, the functions $D_\nu\big(\varepsilon(h/2)^{-1/2}\eta\big)$ are (semi-classically) linearly dependent for $\varepsilon = \pm 1$ only if $e^{-i\pi E^2/2h} = e^{i\pi E^2/2h}$ or $e^{i\pi E^2/h} = 1$, i.e. $\nu \in \mathbf{N}$ (we exclude $\nu = -1$ because $\sqrt{E^2-\eta^2}$ needs to be defined for small η, $\eta \neq 0$). According to the parity of ν, $D_\nu\big(\pm(h/2)^{-1/2}\eta\big)$ are equal or opposite. The value $\nu = 0$ gives the ground state. Let $\Theta(\eta) = \check{\Theta}_{+,+}(\eta)$. So far, taking (21) in account we have shown that for any $\nu \in \mathbf{N}$

$$D_\nu\big((h/2)^{-1/2}\eta\big) = \widetilde{C}^\nu_h (E^2-\eta^2)^{-1/4} \cos\big(\frac{\pi}{4} - \frac{\Theta(\eta)}{2} + \frac{E^2\Theta(\eta) - \eta\sqrt{E^2-\eta^2}}{2h}\big) + \mathcal{O}(h)$$

uniformly on compact sets in $|\eta| < E = \sqrt{2(\nu+1)h}$.

2.1.2 Asymptotics for $|\eta| > E$

Consider now the classically forbidden region $|\eta| > E$. Recall that the "complex geometry" of the problem is given by Stokes lines. We adopt the convention of [DeDiPh] for Stokes lines (the fastest way $\exp i\Phi/h$ has to decrease towards the turning point in the complex domain), and take advantage of the existence of explicit solutions for Weber equation. There are 3 Stokes lines, tied to each of the turning points $\eta = \pm E$ and bordering *Stokes regions*.

The (real) critical points of $s \mapsto \Phi^\nu_\varepsilon(s;\eta)$ are given by

$$-\sqrt{2}\,\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta) = \varepsilon\,\eta + \omega'\sqrt{\eta^2 - E^2}\,; \quad \text{any } \omega' = \pm 1 \tag{22}$$

The condition $-\sqrt{2}\,\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta) > 0$ (the contour encircles the positive half-line) requires

$$\varepsilon\,\mathrm{sgn}(\eta) = 1 \tag{23}$$

A straightforward computation shows that the second derivative at the critical points is given by

$$\big(\frac{\partial^2 \Phi^\nu_\varepsilon}{\partial s^2}\big)\big(\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta);\eta\big) = \frac{2\,i\,\sqrt{\eta^2-E^2}}{\varepsilon\;\omega'\,\eta+\sqrt{\eta^2-E^2}} \tag{24}$$

Since $\widetilde{s}^\nu_{\varepsilon,+}(\eta)-\widetilde{s}^\nu_{\varepsilon,-}(\eta)=2\sqrt{\eta^2-E^2}$ is a small real number, there is no single contour containing both $\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta)$ that would contribute, by stationary phase, to $D_\nu\big(\varepsilon(h/2)^{-1/2}\eta\big)$, but instead two contours giving the exponentially decaying/ growing branch.

Such a contour $\gamma^\nu_{\varepsilon,\omega'}(\eta)$ can be parametrized near the critical point $\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta)$ by

$$s=\widetilde{s}^\nu_{\varepsilon,\omega'}+t\exp[i(\varepsilon\,\mathrm{sgn}(\eta)\frac{\pi}{4}+\delta)],\quad t\in\mathbf{R}$$

For s near $\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta)$, Taylor expansion of second order gives

$$\Phi^\nu_\varepsilon(s;\eta)=\Phi^\nu_\varepsilon\big(s^\nu_{\varepsilon,\omega'}(\eta);\eta\big)+\frac{t^2}{2}\exp[i(\varepsilon\,\mathrm{sgn}(\eta)\frac{\pi}{2}+2\delta)]\big(\frac{\partial^2\Phi^\nu_\varepsilon}{\partial s^2}\big)\big(\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta);\eta\big)+\mathcal{O}(t^3)$$

Let

$$\Psi^\nu_{\varepsilon,\omega'}(\eta)=\Phi^\nu_\varepsilon(s;\eta)-\Phi^\nu_\varepsilon\big(s^\nu_{\varepsilon,\omega'}(\eta);\eta\big)$$

We have

$$\mathrm{Im}\,\big(\Psi^\nu_{\varepsilon,\omega'}(s;\eta)\big)=-\frac{\epsilon\,\mathrm{sgn}(\eta)\,\sqrt{\eta^2-E^2}}{\epsilon\,\omega'\,\eta+\sqrt{\eta^2-E^2}}\,\sin(2\delta)\,t^2+\mathcal{O}(t^3)$$

which is positive for small enough t if $-\omega'\sin(2\delta)>0$. It is also well known by the method of steepest descent that we can find (globally) $\gamma^\nu_{\varepsilon,\omega'}(\eta)$ such that $\mathrm{Im}\,\big(\Psi^\nu_{\varepsilon,\omega'}(s;\eta)\big)\geq 0$ everywhere on $\gamma^\nu_{\varepsilon,\omega'}(\eta)$, and that the integral (14) defining $D_\nu\big(i\,\varepsilon(h/2)^{-1/2}\eta\big)$ depends only on the critical point $s^\nu_{\varepsilon,\omega'}(\eta)$, modulo exponentially smaller terms (uniform in $|\eta|$).

Now we examine the behavior of $\Phi^\varepsilon_\nu\big(\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta);\eta\big)$ to see whether $D_\nu\big(i\,\varepsilon(h/2)^{-1/2}\eta\big)$ decays or grows exponentially when leaving the classically allowed region. The critical value is

$$\Phi^\nu_\varepsilon\big(\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta);\eta\big)=\frac{i}{2}\,\big[-\varepsilon\;\omega'\,\eta\,\sqrt{\eta^2-E^2}-\frac{E^2}{2}+E^2\,\log\big(\frac{1}{\sqrt{2}}\,(\varepsilon\;\eta+\omega'\,\sqrt{\eta^2-E^2})\big)\big]$$

When $\eta>E>0$, we set $\eta=E+\xi$, and get by Taylor expansion at $\xi=0_+$

$$\Phi^\nu_\varepsilon\big(\widetilde{s}^{(\nu)}_{\varepsilon,\omega'}(\eta);\eta\big)=-i\;\varepsilon\;\omega'\,\frac{E^2}{2}\,(\frac{2\,\xi}{E})^{3/2}-i\,\frac{\xi^2}{2}+i\,\frac{E^2}{2}\,\log(\frac{\varepsilon\;E}{\sqrt{2\,e}})+\cdots \tag{25}$$

When $\eta<-E<0$, we set $\eta=-E-\xi$, and get by Taylor expansion at $\xi=0_+$

$$\Phi^\nu_\varepsilon\big(\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta);\eta\big)=i\;\varepsilon\;\omega'\,\frac{E^2}{2}\,(\frac{2\,\xi}{E})^{3/2}-i\,\frac{\xi^2}{2}+i\,\frac{E^2}{2}\,\log(-\frac{\varepsilon\;E}{\sqrt{2\,e}})+\cdots \tag{26}$$

So $\mathrm{Im}\,\big(\Phi^\nu_\varepsilon\big(\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta);\eta\big)\big)$ decays/grows as $(\frac{2\,\xi}{E})^{3/2}$, depending on $\varepsilon,\omega',\mathrm{sgn}(\eta)$.

The next step is to choose ω' consistently with the former choice of ω in the classically allowed region to define $D_\nu\big(\varepsilon(h/2)^{-1/2}(\eta)\big)$. Recall from (16) the expression of the critical point when $|\eta|<E$.

We know that $D_\nu(z)$ is a one-valued function of z throughout the z-plane, but semi-classically the situation is different because of so-called Stokes phenomena (see [DePh]).

The function $\sqrt{E^2-\eta^2}$ has an analytic continuation on the complex plane where the half lines $\eta > E$ and $\eta < -E$ have been removed, which we denote by $f(\eta) = (\sqrt{E^2-\eta^2})_{\text{ext}}$. Thus, for $\eta > E$ or $\eta < -E$ we have

$$f(\eta \pm i0) = \mp i\,\mathrm{sgn}(\eta)\sqrt{\eta^2 - E^2} \tag{27}$$

with the positive square root, and (16) gives

$$-\sqrt{2}s^\nu_{\varepsilon,\omega}(\eta \pm i0) = \eta \pm \varepsilon\,\omega\,\mathrm{sgn}(\eta)\sqrt{\eta^2-E^2}$$

Comparing with (22) ($s^\nu_{\varepsilon,\omega} = \widetilde{s}^\nu_{\varepsilon,\omega'}$) we get $\omega' = \pm\,\varepsilon\,\omega\,\mathrm{sgn}(\eta)$, or $\omega' = \pm\omega$ for the boundary value $\eta \pm i0$ by (23). Let

$$\Phi^\nu_{\varepsilon,\omega}(\eta) = \big(\Phi^\nu_\varepsilon\big(\widetilde{s}^\nu_{\varepsilon,\omega'}(\eta);\eta\big)\big)$$

and $\Phi^\nu_{\varepsilon,\omega}(\eta \pm i0)$ its extensions in the upper/lower half-plane. We proved:

Lemma 2.1. *For $\eta > E$, let $\eta = E+\xi$, and for $\eta < -E$, let $\eta = -E-\xi$. Assume $\varepsilon\,\mathrm{sgn}(\eta) = 1$, then*

$$\Phi^\nu_{\varepsilon,\omega}(\eta \pm i0) = \mp i\omega\frac{E^2}{2}\big(\frac{2\xi}{E}\big)^{3/2} + \mathcal{O}(\xi^2) + \text{Const.}, \quad \xi > 0 \tag{28}$$

so there is always an analytic branch of D_ν decaying exponentially (evanescent mode), the other one being exponentially increasing.

2.2 The semi-classical distribution $D_{-\nu-1}$

We shall compute similarly the semi-classical distributions $D_{-\nu-1}$. For $\varepsilon = \pm 1$ and $\nu \neq 1, 2, \cdots$, (13) gives

$$D_{-\nu-1}\big(i\,\varepsilon(h/2)^{-1/2}\eta\big) = \frac{\Gamma(-\nu)}{2i\pi}h^{-E^2/4h}\int_\infty^{(0^+)}\exp\big[i\Phi^{-\nu-1}_\varepsilon(s;\eta)/h\big]\,\frac{ds}{s} \tag{29}$$

where we have set

$$\Phi^{-\nu-1}_\varepsilon(s;\eta) = -i\big(\frac{\eta^2}{2} - i\sqrt{2}\,\varepsilon\,\eta s - \frac{s^2}{2} + \frac{E^2}{2}\log(-s)\big), \quad E = \sqrt{2(\nu+1)h} \tag{30}$$

2.2.1 Asymptotics for $|\eta| < E$

The critical points of $s \mapsto \Phi^{-\nu-1}_\varepsilon(s;\eta)$ are the roots $s^{-\nu-1}_{\varepsilon,\omega}$ of the quadratic equation $s^2 + i\sqrt{2}\,\varepsilon\,\eta s - \frac{E^2}{2} = 0$, namely

$$-\sqrt{2}\,\varepsilon\,s^{-\nu-1}_{\varepsilon,\omega} = i\eta + \omega\sqrt{E^2-\eta^2}, \quad \text{any } \omega = \pm 1 \tag{31}$$

so that the points $s^{-\nu-1}_{\varepsilon,\omega}$ are rotated from $\overline{s^\nu_{\varepsilon,\omega}}$ as in (16) by $\pi/2$. As above we examine the classically allowed region $|\eta| < E$; the corresponding critical values are given by

$$\Phi^{-\nu-1}_\varepsilon(s^{-\nu-1}_{\varepsilon,\omega};\eta) = i\frac{E^2}{4}\big(1 - \log\frac{E^2}{2}\big) + \frac{1}{2}\big[\omega\eta\sqrt{E^2-\eta^2} + E^2\widehat{\Theta}_{\varepsilon,\omega}(\eta)\big]$$

Here

$$\widehat{\Theta}_{\varepsilon,\omega}(\eta) = \arg\big(\frac{\varepsilon}{\sqrt{2}}(i\eta + \omega\sqrt{E^2-\eta^2})\big) \in]-\pi,\pi[\tag{32}$$

is defined similarly with $\check{\Theta}_{\varepsilon,\omega}(\eta)$. As in (21) we have

$$\widehat{\Theta}_{-,+}(\eta) - \widehat{\Theta}_{+,+}(\eta) = -\pi\,\mathrm{sgn}(\eta),\ \widehat{\Theta}_{\varepsilon,\omega}(\eta) = -\widehat{\Theta}_{-\varepsilon,-\omega}(\eta) \tag{33}$$

As in (18) we remove the constant term from $\Phi_\varepsilon^{-\nu-1}(s_{\varepsilon,\omega}^{-\nu-1};\eta)$, and denote by

$$\Phi_{\varepsilon,\omega}^{-\nu-1}(\eta) = \frac{1}{2}\big(\omega\eta\sqrt{E^2-\eta^2} + E^2\widehat{\Theta}_{\varepsilon,\omega}(\eta)\big) \tag{34}$$

the resulting phase. The Jacobian is independent of ε and given by

$$J_\omega^{-\nu-1}(\eta) = \frac{1}{i}\frac{\partial^2\Phi_\varepsilon^{-\nu-1}}{\partial s^2}(s_{\varepsilon,\omega}^{-\nu-1};\eta) = \frac{-i\omega\sqrt{2}\sqrt{E^2-\eta^2}}{\varepsilon\, s_{\varepsilon,\omega}^{-\nu-1}} = \frac{2(E^2-\eta^2)}{E^2} - 2\omega i\frac{\eta\sqrt{E^2-\eta^2}}{E^2} \tag{35}$$

which shows that $s_{\varepsilon,\omega}^{-\nu-1}$ are non degenerate when η is not a turning point. Applying again Theorem 2.1, letting

$$\widetilde{C}_h^{-\nu-1} = \frac{\Gamma(-\nu)\sqrt{h}}{i\sqrt{2\pi}2^{1/4}}\Big(\frac{2eh}{E^2}\Big)^{-E^2/4h} \tag{36}$$

we find

$$\begin{aligned} D_{-\nu-1}\big(i\,\varepsilon(h/2)^{-1/2}\eta\big) &= \widetilde{C}_h^{-\nu-1}\sum_\omega (J_\omega^{-\nu-1}(\eta))^{-1/2}\exp\big[i\Phi_{\varepsilon,\omega}^{-\nu-1}/h\big](s_{\varepsilon,\omega}^{-\nu-1})^{-1}(1+\mathcal{O}(h)) = \\ &\widetilde{C}_h^{-\nu-1}(E^2-\eta^2)^{-1/4}\sum_\omega\big(-\varepsilon\,\omega s_{\varepsilon,\omega}^{-\nu-1}\big)^{-1/2}\exp[i\big(\omega\eta\sqrt{E^2-\eta^2} + E^2\widehat{\Theta}_{\varepsilon,\omega}(\eta)/2h](1+\mathcal{O}(h)) \end{aligned} \tag{37}$$

Except for the fact that the normalization factor $\widetilde{C}_h^{-\nu-1}$ has a pole at $\nu \in \mathbf{N}^*$, the same argument as above gives that $D_{-\nu-1}\big(i\,\varepsilon(h/2)^{-1/2}\eta\big)$, $\varepsilon = \pm 1$, are colinear on $|\eta| < E$ iff $\nu \in \mathbf{N}$ (in fact equal or opposite, according to ν is even or odd). In this case, $D_{-\nu-1}\big(i(h/2)^{-1/2}\eta\big)$ is colinear to $D_\nu\big((h/2)^{-1/2}\eta\big)$, up to $\mathcal{O}(h^\infty)$, uniformly on any compact set inside $]-E,E[$.

2.2.2 Asymptotics for $|\eta| > E$

The critical points of $\Phi_\varepsilon^{-\nu-1}$ are given by

$$i\sqrt{2}\widetilde{s}_{\varepsilon,\omega'}^{-\nu-1}(\eta) = \varepsilon\,\eta + \omega'\sqrt{\eta^2-E^2}),\ \omega' = \pm 1 \tag{38}$$

So both critical points lie on the negative (resp. positive) imaginary axis, $\varepsilon\,\mathrm{sgn}(\eta) = \pm 1$, and as before each of those gives a branch of $D_{-\nu-1}\big(i\,\varepsilon(h/2)^{-1/2}\eta\big)$ with exponential growth or decay when leaving the classically allowed region. Second derivatives

$$\frac{\partial^2\Phi_\varepsilon^{-\nu-1}}{\partial s^2}(\widetilde{s}_{\varepsilon,\omega'}^{-\nu-1}) = \frac{2i\sqrt{\eta^2-E^2}}{\varepsilon\,\omega'\eta + \sqrt{\eta^2-E^2}} \tag{39}$$

have the same expression as in (24) so we choose the contours $\gamma_{\varepsilon,\omega'}^{-\nu-1}(\eta)$ near $\widetilde{s}_{\varepsilon,\omega'}^{-\nu-1}(\eta)$ like $\gamma_{\varepsilon,\omega'}^{\nu}(\eta)$ near $\widetilde{s}_{\varepsilon,\omega'}^{\nu}(\eta)$, $\omega' = \pm 1$.

The expression for $\Phi_\varepsilon^{-\nu-1}(\widetilde{s}_{\varepsilon,\omega'}^{-\nu-1}(\eta))$ simplifies to

$$\Phi_\varepsilon^{-\nu-1}(\widetilde{s}_{\varepsilon,\omega'}^{-\nu-1}(\eta)) = \frac{i}{2}\big[\varepsilon\,\omega'\eta\sqrt{\eta^2-E^2} - E^2\log(\varepsilon\,\omega'\eta+\sqrt{\eta^2-E^2}) + \frac{E^2}{2} - E^2\log\big(\frac{i\omega'}{\sqrt{2}}\big)\big] \quad (40)$$

so it is odd (modulo Const.) in $\varepsilon\,\omega'\operatorname{sgn}\eta$. Assume $\eta = E+\xi > E$, we have

$$\Phi_\varepsilon^{-\nu-1}(\widetilde{s}_{\varepsilon,\omega'}^{-\nu-1}(\eta)) = \frac{i}{2}\varepsilon\,\omega' E^2\big(\frac{2\xi}{E}\big)^{3/2} + \mathcal{O}(\xi^2) + \text{Const.}, \quad \eta > E \quad (41)$$

Similarly, (40) shows that for these ε, ω', and $\eta = -E-\xi < E$

$$\Phi_\varepsilon^{-\nu-1}(\widetilde{s}_{\varepsilon,\omega'}^{-\nu-1}(\eta)) = -\frac{i}{2}\varepsilon\,\omega' E^2\big(\frac{2\xi}{E}\big)^{3/2} + \mathcal{O}(\xi^2) + \text{Const.}, \quad \eta < -E \quad (42)$$

so $\operatorname{Im}\Phi_\varepsilon^{-\nu-1}(\widetilde{s}_{\varepsilon,\omega'}^{-\nu-1}(\eta))$ decays or grows as $\big(\frac{2\xi}{E}\big)^{3/2}$, depending on $\varepsilon, \omega', \operatorname{sgn}(\eta)$ (this time we do not impose any condition on $\varepsilon\operatorname{sgn}\eta$).

The next step is to choose ω' consistently with the choices of ω in the classically allowed region as above, so to define $D_{-\nu-1}\big(i\,\varepsilon(h/2)^{-1/2}(\eta)\big)$.

First we relate (38) with the analytic continuation of the critical points given by

$$-\sqrt{2}\,\varepsilon\, s_{\varepsilon,\omega}^{-\nu-1} = i\eta + \omega\sqrt{E^2-\eta^2} \quad (43)$$

By (27) and (43)

$$i\sqrt{2}s_{\varepsilon,\omega}^{-\nu-1}(\eta \pm i0) = \varepsilon\,\eta \mp \varepsilon\,\omega\operatorname{sgn}(\eta)\sqrt{\eta^2-E^2} \quad (44)$$

Comparing with (38) ($\widetilde{s}_{\varepsilon,\omega'}^{-\nu-1} = s_{\varepsilon,\omega}^{-\nu-1}$), we get $\omega' = \mp\varepsilon\,\omega\operatorname{sgn}(\eta)$, so that (41) and (42) give:

Lemma 2.2. *Define $\Phi_{\varepsilon,\omega}^{-\nu-1}(\eta \pm i0)$ similarly with $\Phi_{\varepsilon,\omega}^{\nu}(\eta \pm i0)$ as in Lemma 2.1. Then (even without the condition $\varepsilon\operatorname{sgn}(\eta) = 1$), we have, as in (28):*

$$\Phi_{\varepsilon,\omega}^{-\nu-1}(\eta \pm i0) = \mp i\,\omega\,\frac{E^2}{2}\,(\frac{2\,\xi}{E})^{3/2} + \mathcal{O}(\xi^2) + \text{Const.},\ \xi > 0 \quad (45)$$

2.3 Relating D_ν and $D_{-\nu-1}$

We present some relations between the critical points and critical values of the phase functions defining D_ν and $D_{-\nu-1}$ in the classically allowed region, which will be useful in the sequel. First we have

$$\begin{aligned} s_{\varepsilon,\omega}^{-\nu-1}(\eta) &= i\,\overline{s_{\varepsilon,\omega}^{\nu}}(\eta) \\ J_\omega^{-\nu-1}(\eta) &= \overline{J_\omega^{\nu}}(\eta) \end{aligned} \quad (46)$$

The difference between critical values (18) and (34) is given by

$$\widehat{\Theta}_{\varepsilon,\omega}(\eta) + \check{\Theta}_{\varepsilon,\omega}(\eta) = \pi\zeta_{\varepsilon,\omega}(\eta) \quad (47)$$

where index $\zeta_{\varepsilon,\omega}(\eta)$ is a half-integer defined by

$$\zeta_{\varepsilon,\omega}(\eta) = \frac{1}{2} \text{ when } \varepsilon\,\omega = 1, \quad \zeta_{\varepsilon,\omega}(\eta) = -(\omega\operatorname{sgn}(\eta) + \frac{1}{2}) \text{ otherwise} \quad (48)$$

In $|\eta| < E$, the phases are related (at the critical point) by

$$\Phi_{\varepsilon,\omega}^{-\nu-1} = \frac{\pi E^2}{4} + \Phi_{\varepsilon,\omega}^{\nu} \tag{49}$$

The normalization constants are related by (when $\nu \notin \mathbf{Z}$)

$$\widetilde{C}_h^{\nu}\widetilde{C}_h^{-\nu-1} = -\frac{\sqrt{2h}}{4\sin\pi\nu} \tag{50}$$

3 Microlocal solutions in Fourier representation

3.1 The normal form of BdG near the branching points

Here we recall some notations from Sect.1 and collect the relevant information from Appendix. Eigenvalues of classical BdG Hamiltonian $\mathcal{P}(x,\xi)$ are of the from

$$\lambda_\rho(x,\xi) = \rho\sqrt{\Delta(x)^2 + (\xi^2 - \mu(x))^2} \tag{51}$$

The energy surface $\Sigma_E = \{\det(\mathcal{P} - E) = -(\xi^2 - \mu(x))^2 - \Delta(x)^2 + E^2 = 0\}$, foliated by two smooth Lagrangian connected manifolds $\Lambda_E^> \subset \{\xi > 0\}$ and $\Lambda_E^< \subset \{\xi < 0\}$, is invariant under the PT symmetries $\mathcal{I} : (x,\xi) \mapsto (x,-\xi)$ and ${}^\vee : (x,\xi) \mapsto (-x,\xi)$.

Denote by $a_E = (x_E, \xi_E)$ and $a'_E = (-x_E, \xi_E)$ the focal (or branching) points in $\Lambda_E^>$, defined by $\Delta(x_E) = E$, which fixes x_E close to $x_0 > 0$, and $\xi_E^2 = \mu(x_E)$. Since we assumed $x \mapsto \mu(x)$ to be a constant near x_0, we set $\mu(x_E) = \mu$.

To start with, we consider the family of quasi-modes supported on $\Lambda_E^>$. Those supported on $\Lambda_E^<$ are implied by $\mathcal{I}$. Thus we denote $\Lambda_E^{>,\rho}$ by Λ_E^ρ, or also simply by ρ, when no confusion could occur.

We shall work (locally) in h-Fourier representation and introduce an "effective Hamiltonian" (scalar differential operator), whose normal form is given in Appendix A. Recall the h-Fourier transform

$$\mathcal{F}_h u(\xi) = (2\pi h)^{-1/2} \int e^{-ix\xi/h} u(x)\, dx$$

Near $a = a_E$, the local Hamiltonian $\mathcal{P}^a = \mathcal{F}_h \mathcal{P} \mathcal{F}_h^{-1}$ takes the form :

$$\mathcal{P}^a(-hD_\xi, \xi) = \begin{pmatrix} \xi^2 - \mu & e^{i\phi/2}(E - \alpha h D_\xi - \alpha x_E) \\ e^{-i\phi/2}(E - \alpha h D_\xi - \alpha x_E) & -\xi^2 + \mu \end{pmatrix} \tag{52}$$

and by PT symmetry (2), the corresponding local Hamiltonian near a' reads :

$$\mathcal{P}^{a'} = \mathcal{I}\mathcal{P}^a\mathcal{I} = \begin{pmatrix} \xi^2 - \mu & e^{-i\phi/2}(E + \alpha h D_\xi - \alpha x_E) \\ e^{i\phi/2}(E + \alpha h D_\xi - \alpha x_E) & -\xi^2 + \mu \end{pmatrix} \tag{53}$$

so that we only have to consider $\mathcal{P}^a(-hD_\xi,\xi)$. By definition of $a=(x_E,\xi_E)$, we have

$$\det(\mathcal{P}^a(x_E,\xi_E)-E)=0 \tag{54}$$

Consider the system $\big(\mathcal{P}^a(-hD_\xi,\xi)-E\big)\widehat{U}^a=0$, $\widehat{U}^a=\binom{\widehat{\varphi}_1}{\widehat{\varphi}_2}$ will be refered henceforth as the *microlocal solution* near a.

By the first equation we can express $\widehat{\varphi}_1$ as

$$\widehat{\varphi}_1(\xi)=-e^{i\phi/2}(\xi^2-\mu-E)^{-1}(E-\alpha x_E-\alpha hD_\xi)\widehat{\varphi}_2(\xi) \tag{55}$$

then take the hD_ξ derivatives of $\widehat{\varphi}_1$, and replace into the second equation, we find :

$$\begin{aligned}&(hD_\xi)^2\widehat{\varphi}_2-2\big[\alpha^{-1}(E-\alpha x_E)-i(\xi^2-\mu-E)^{-1}h\xi\big]hD_\xi\widehat{\varphi}_2\\&+\alpha^{-2}\big[(E-\alpha x_E)^2+(\xi^2-\mu)^2-E^2-2ih\alpha(E-\alpha x_E)(\xi^2-\mu-E)^{-1}\xi\big]\widehat{\varphi}_2=0\end{aligned} \tag{56}$$

We make the substitution $\widehat{\varphi}_2(\xi)=\exp[i\int^\xi g(s)ds/h]u(\xi)$, where we choose $g(\xi)$ so that the hD_ξ term drops out, i.e. $g(\xi)=\alpha^{-1}(E-\alpha x_E)-ih(\xi^2-\mu-E)^{-1}\xi$. This gives the integrating factor $\exp[i\int^\xi g(s)ds/h]=\text{Const.}(\xi^2-\mu-E)^{1/2}e^{i(E-\alpha x_E)\xi/\alpha h}$, and a little computation shows that u verifies

$$P^a(-hD_\xi,\xi,h)u(\xi)=\frac{E^2}{\alpha^2}u(\xi) \tag{57}$$

where

$$P^a(-hD_\xi,\xi,h)=(hD_\xi)^2+\alpha^{-2}(\xi^2-\mu)^2+h^2(\xi^2-\mu-E)^{-2}(2\xi^2+\mu+E)$$

We then recover the second component of the system as

$$\widehat{\varphi}_2(\xi)=(\xi^2-\mu-E)^{1/2}e^{i(E-\alpha x_E)\xi/\alpha h}u(\xi) \tag{58}$$

and the first one using (55). We make a number of E-dependent scalings. Let $\omega\in\mathbf{S}^1$ ("moduli space") and parametrize

$$\xi=2\xi_E\omega\beta\xi'+\xi_E,\quad \beta=\sqrt{\alpha}(2\xi_E)^{-3/2} \tag{59}$$

defining a "local (complex) momentum" ξ' and a corresponding "local (complex) time" variable. We define also scaled "Planck constant" h' and energy parameter

$$h'=\beta^2h,\quad E_1=(2\xi_E)^{-2}E \tag{60}$$

and restrict to $E_1<\frac{1}{4}$ to allow the harmonic approximation as is explained in Appendix A. This takes (57) to

$$P^a_\omega(-hD_{\xi'},\xi',h)u_\omega(\xi')=\big(\frac{E_1\omega}{\beta}\big)^2u_\omega(\xi') \tag{61}$$

where $P^a_\omega(-hD_{\xi'},\xi';h)=(-hD_{\xi'})^2+\omega^4(\xi'+\omega\beta\xi'^2)^2+h^2(\omega\beta)^2f(\omega\beta\xi')$ is the double well Schrödinger operator (with lower order term $\mathcal{O}(h^2)$) of the form (183) with

$$f(z)=(2z^2+2z+\frac{3}{4}+E_1)(z^2+z-E_1)^{-2} \tag{62}$$

Note that f has a pole in $\Lambda_E^>$ at $\xi' = \xi'_f = \frac{-1+\sqrt{1+4E_1}}{2\omega\beta}$, $(0, \xi'_f)$ being one of the turning points at energy $\left(\frac{E_1\omega}{\beta}\right)^2$ of the classical Hamiltonian $p^a_\omega(x', \xi') = (-x')^2 + \omega^4(\xi' + \omega\beta\xi'^2)^2$. In the spatial representation, this pole corresponds to the point x_f on the characteristic variety of $\mathcal{P}^a - E$ such that $\Delta_0(x_f) = 0$, where the linear approximation of the gap function breaks down. There we need to use standard WKB solutions for original $\mathcal{P}(x, hD_x)$. So we restrict ξ' to a neighborhood of 0 not containing ξ'_f. We rescale the phase-space variables as

$$\xi_1 = \beta\xi', \quad x_1 = (2\xi_E)^{-2}\alpha(x - x_E) \tag{63}$$

and set

$$x^0_E = (2\xi_E)^{-3}(E - \alpha x_E) = (2\xi_E)^{-1}(E_1 - 2x_E\xi_E\beta^2) \tag{64}$$

Passage from $P^a_\omega(-hD_{\xi'}, \xi', h)$ to its harmonic approximation changes as in (205) parameter $\frac{\omega E_1}{\beta}$ to E'_1, a non linear function of E_1 (since the period of oscillations depends on E), which are related by (see App.A)

$$E_1'^{\,2} = E_1^{\,2} + \frac{3}{2}E_1^{\,4} + \frac{35}{4}E_1^{\,6} + \cdots \Longleftrightarrow E_1 = E'_1 - \frac{3}{4}E_1'^{\,3} - \frac{77}{32}E_1'^{\,5} - \cdots \tag{65}$$

This defines the frequency ν as in (15) by

$$E'_1 = \beta\sqrt{2(\nu+1)h} \tag{66}$$

Parameter ω "explores" the domain of complex momenta as in the "radar method" [DePh]. Complex values of ω are quite artificial when ignoring tunneling effects, so we shall assume henceforth $\omega = \pm 1$, which plays the rôle of parameter ω in Sect.2. Note that operators $P^a_{\pm 1}$ are unitarily equivalent. Due to (55) and (58), there are natural isomorphisms

$$\iota^a_\omega : \mathrm{Ker}_h(P^a_\omega - (\frac{E_1\omega}{\beta})^2) \to \mathrm{Ker}_h(\mathcal{P}^a - E) \tag{67}$$

where Ker_h denotes the microlocal kernel. The same holds near a', and actually $P^a_\omega = P^{a'}_\omega$ are denoted simply by P_ω. We shall endow the RHS of (67) with a Lorenzian structure, and "diagonalize" ι^a_ω in some orthogonal subspaces.

As long as we focus on a single branching point, we drop the superscript a.

3.2 The microlocal kernel near a branching point

Following [DuGy], [Ro,Sect.4] we first introduce a class of semi-classical spinors:

Definition 3.1. *We call* spinor *an oscillatory integral (Lagrangian distribution)*

$$I(\vec{a}, \varphi)(x, h) = (2\pi h)^{-d/2} \int_{\mathbf{R}^d} e^{i\varphi(x,\theta,h)/h} \vec{a}(x, \theta; h)\, d\theta$$

with the following properties (all functions being defined locally) :

(1) $\varphi(x,\theta,h)$ denotes a non degenerate phase-function and

$$\vec{a}(x,\theta;h)=\vec{a}_0(x,\theta;h)+h\vec{a}_1(x,\theta;h)+\cdots$$

a $\mathbf{C}^2$-valued amplitude (i.e. a classical symbol in h), $\vec{a}_k=\binom{e^{i\phi(x)/2}X_k}{Y_k}$ possibly depending on h (with the property that $\phi(x)=\mathrm{sgn}(x)\phi$).

(2) For $k=0$, $\binom{X_0}{Y_0}=\lambda(x,\theta;h)\binom{X_0'}{Y_0'}$, $\lambda\in\mathbf{C}$, is proportional to a real vector $\binom{X_0'}{Y_0'}$, depending also on $(x,\theta;h)$.

Actually we allow $I(\vec{a},\varphi)(x,h)$ to be a 2-microlocal object, in the sense that $\varphi(x,\theta,h)=\varphi_0(x,\theta,h)+\beta^{-2}\varphi_1(x,\theta,h)$ where β as in (60) is a large parameter, but this point is not crucial and will be omitted.

If $u(x,h)=I(\vec{a},\varphi)(x,h)$ is a spinor, so is its h-Fourier transform.

It turns out that the microlocal solutions of $\big(\mathcal{P}^a(-hD_\xi,\xi)-E\big)\widehat{U}^a=0$ are spinors in the sense of Definition 3.1. They are constructed from the parabolic cylinder function of Sect.2 via the normal form of P^a_ω given in [HeSj] (the same holds of course with a'.)

We collect here a number of notations and results from Appendix, in a form that will be directly used in the sequel.

A basis of solutions $\{u^\nu_{\varepsilon,\omega},u^{-\nu-1}_{\varepsilon,\omega}\}$ of (61) is constructed in Appendix B from the solutions of Weber equation (10) in the following way: (1) apply to D_ν, and $D_{\nu-1}$ a h-FIO A_ω of the form (190), microlocally unitary near a, with leading amplitude $c_{0\omega}(\xi',\eta,\theta)$; (2) compute $A_\omega D_\nu, A_\omega D_{-\nu-1}$ using a contour integral parametrized by variables (θ,η,s); (3) specify (correction) the extension of $c_{0\omega}(\xi',\eta,\theta)$ from Γ'_κ (valid for both $D_\nu,D_{-\nu-1}$), in such a way that $\{u^\nu_{\varepsilon,\omega},u^{-\nu-1}_{\varepsilon,\omega}\}$ solves (61).

Index $\varepsilon=\pm1$ is the same as in $D_\nu(\varepsilon\,\zeta)$ or $D_{-\nu-1}(i\,\varepsilon\,\zeta)$, see Sect.2, and $(\theta^j_\omega,\eta^j_\omega,s^j_{\varepsilon,\omega})$ denote the critical points, with

$$\begin{aligned}&-\sqrt{2}\,\varepsilon\,s^\nu_{\varepsilon,\omega}(\theta_\omega)=i\theta_\omega+\omega({E'}_1^2-\theta_\omega^2)^{1/2}\\&-\sqrt{2}\,\varepsilon\,s^{-\nu-1}_{\varepsilon,\omega}(\theta_\omega)=\theta_\omega+i\omega({E'}_1^2-\theta_\omega^2)^{1/2}=-i\sqrt{2}\,\varepsilon\,\overline{s^\nu_{\varepsilon,\omega}(\theta_\omega)}\end{aligned}\tag{68}$$

Here $\varepsilon\,s^j_{\varepsilon,\omega}(\theta_\omega)$ depends on ω,j, but not on ε, $\theta_\omega=\pm\widehat{\theta}_\omega(\xi_1)$, $\widehat{\theta}_\omega(\xi_1)>0$ depends analytically on ξ_1 near 0 and on ${E'}_1^2$, but not on ε,j. We have the relation

$$\theta_\omega(\xi_1)=\theta_{-\omega}(-\xi_1)\tag{69}$$

Recall from (17) and (32) the angles $\check{\Theta}_{\varepsilon,\omega}(\theta)$ and $\widehat{\Theta}_{\varepsilon,\omega}(\theta)$, from (219) and (231) the phases

$$\begin{aligned}&T^\nu_{\varepsilon,\omega}\big(\xi_1,\theta_\omega(\xi_1)\big)=\big[\xi_1\theta_1+h_\omega(\xi_1,\theta_1)-\frac{1}{2}\omega\theta_1({E'}_1^2-\theta_1^2)^{1/2}-\frac{1}{2}{E'}_1^2\widehat{\Theta}_{\varepsilon,\omega}(\theta_1)\big]|_{\theta_1=\theta_\omega(\xi_1)}\\&T^{-\nu-1}_{\varepsilon,\omega}\big(\xi_1,\theta_\omega(\xi_1)\big)=\big[\xi_1\theta_1+h_\omega(\xi_1,\theta_1)-\frac{1}{2}\omega\theta_1({E'}_1^2-\theta_1^2)^{1/2}+\frac{1}{2}{E'}_1^2\check{\Theta}_{\varepsilon,\omega}(\theta_1)\big]|_{\theta_1=\theta_\omega(\xi_1)}\end{aligned}\tag{70}$$

with $h_\omega(\xi_1,\theta_1)$ an analytic function.

Note the analogy of (68) with (16) and (38), as well as (70) with (18) and (34), but with the rôles of D_ν and $D_{-\nu-1}$ interchanged.

Eventually ω and ε will be related by $\varepsilon\,\omega = 1$. Recall from (238) that $T^{\nu}_{\varepsilon,\omega}\big(\xi_1, \theta_\omega(\xi_1)\big)$ and $T^{-\nu-1}_{\varepsilon,\omega}\big(\xi_1, \theta_\omega(\xi_1)\big)$ differ only by a piecewise constant term, so the ξ_1-derivative of $T^{j}_{\varepsilon,\omega}$ doesn't depend on ε and j. The phase functions associated with our spinors are

$$\Phi^{j}_{\varepsilon,\omega}\big(\xi_1, \theta_\omega(\xi_1)\big) = x^0_E\xi_E + 2\omega x^0_E\xi_E\xi_1 + T^{j}_{\varepsilon,\omega}\big(\xi_1, \theta_\omega(\xi_1)\big) = x^0_E\xi + T^{j}_{\varepsilon,\omega}\big(\xi_1, \theta_\omega(\xi_1)\big) \tag{71}$$

and $\Phi^{\nu}_{\varepsilon,\omega}$ and $\Phi^{-\nu-1}_{\varepsilon,\omega}$ differ only by a constant. Next we examine the leading part of the amplitudes.

From (223) and (234) we recall the amplitudes $a^{j}_{\varepsilon,\omega}$, with principal part (independent of ε)

$$a^{\nu}_{\omega}\big(\xi_1, \theta_\omega(\xi_1)\big) = a^{-\nu-1}_{\omega}\big(\xi_1, \theta_\omega(\xi_1)\big) \tag{72}$$

and the phases R^{j}_{ω} with

$$R^{\nu}_{\omega}\big(\theta_\omega(\xi_1)\big) = R^{-\nu-1}_{\omega}\big(\theta_\omega(\xi_1)\big) = -\frac{1}{2}\check{\Theta}_{\operatorname{sgn}\theta_\omega,\omega}(\theta_\omega) \tag{73}$$

It is convenient to rewrite the amplitudes in the polar representation

$$a^{\nu}_{\omega}\big(\xi_1, \theta_\omega(\xi_1), h'\big) = |a^{\nu}_{\omega}|\exp\big[iR^{\nu}_{\omega}\big(\theta_\omega(\xi_1)\big)\big] \tag{74}$$

We apply Theorem 2.1 to the Lagrangian distributions defining $u^{j}_{\varepsilon,\omega}$ as in App.A, extending computations in Sect.2 with the previous normalization of D_ν and $D_{-\nu-1}$. This gives the second component $\widehat{\varphi}_2(\xi)$ of the spinor $\widehat{U}^{j}_{\varepsilon,\omega}$. Next we use (55) to compute the first component $\widehat{\varphi}_1(\xi)$. Let

$$X^{j}_{\varepsilon,\omega}\big(\xi_1, \theta_\omega(\xi_1); h'\big) = (2\xi_E)^3\big[\frac{1}{2\omega\xi_E}\partial_{\xi_1}T^{j}_{\varepsilon,\omega} + \frac{h'}{2i\omega\xi_E}\frac{\partial_{\xi_1}a^{j}_{\varepsilon,\omega}}{a^{j}_{\varepsilon,\omega}} + \frac{h'}{i}\xi(\xi^2-\mu-E)^{-1}\big]_{\theta_1=\theta_\omega(\xi_1)} \tag{75}$$

with 0-th order term $X^{j}_{\omega}\big(\xi_1, \theta_\omega(\xi_1)\big)$ independent of ε. Using (67) we can state the main result of this Section:

Proposition 3.1. *For any $\varepsilon, \omega = \pm 1$, $K^{a}_{h}(E) = \operatorname{Ker}_h(\mathcal{P}^{a}(-hD_\xi, \xi) - E)$ (in Fourier representation) is spanned by the spinors $\widehat{U}^{j,a}_{\varepsilon,\omega} = \widehat{U}^{j}_{\varepsilon,\omega} = \binom{\widehat{\varphi}_1}{\widehat{\varphi}_2}^{j}_{\varepsilon,\omega}$, $j \in \{\nu, -\nu-1\}$, of the form:*

$$\begin{aligned}\widehat{U}^{\nu}_{\varepsilon,\omega} = C^{\nu}_{h'}\sum_{\theta_\omega=\pm\widehat{\theta}_\omega(\xi_1)}&\begin{pmatrix} e^{i\phi/2}(\xi^2-\mu-E)^{-1/2}X^{\nu}_{\varepsilon,\omega}(\xi_1,\theta_\omega(\xi_1)) \\ (\xi^2-\mu-E)^{1/2}\end{pmatrix}\times \\ &|a^{\nu}_{\omega}(\xi_1,\theta_\omega(\xi_1);h')|\exp[i\big(\Phi^{\nu}_{\varepsilon,\omega}(\xi_1,\theta_\omega(\xi_1)) + h'R^{\nu}_{\omega})(\theta_\omega(\xi_1))\big)/h']\end{aligned} \tag{76}$$

$$\begin{aligned}\widehat{U}^{-\nu-1}_{\varepsilon,\omega} = \frac{\sqrt{2}}{E'_1}C^{-\nu-1}_{h'}\sum_{\theta_\omega=\pm\widehat{\theta}_\omega(\xi_1)}&\varepsilon\operatorname{sgn}(\theta_\omega)\begin{pmatrix} e^{i\phi/2}(\xi^2-\mu-E)^{-1/2}X^{-\nu-1}_{\varepsilon,\omega}(\xi_1,\theta_\omega(\xi_1)) \\ (\xi^2-\mu-E)^{1/2}\end{pmatrix}\times \\ &|a^{-\nu-1}_{\omega}(\xi_1,\theta_\omega(\xi_1);h')|\exp[i\big(\Phi^{-\nu-1}_{\varepsilon,\omega}(\xi_1,\theta_\omega(\xi_1)) + h'R^{-\nu-1}_{\omega}(\theta_\omega(\xi_1))\big)/h']\end{aligned} \tag{77}$$

where we recall (72)-(73), and the constants $C^{j}_{h'}$ (computed from Whittaker normalization in Sect.2) are related by

$$C^{\nu}_{h'}C^{-\nu-1}_{h'} = \big((2\sqrt{h'})^3\pi^2\sin\pi\nu\big)^{-1} \tag{78}$$

provided $\nu \notin \mathbf{Z}$.

Remarks:

1) Writing

$$\frac{1}{h'}\Phi^j_{\varepsilon,\omega}\big(\xi_1,\theta_\omega(\xi_1)\big) = -\frac{x_E\xi}{h} + \frac{E\xi}{\alpha h} + \frac{1}{h'}T^j_{\varepsilon,\omega}\big(\xi_1,\theta_\omega(\xi_1)\big)$$

we recognize that $\widehat{U}^j_{\varepsilon,\omega}$ are indeed 2-microlocal spinors.

2) Since $D_\nu(\zeta)$ and $D_{-\nu-1}(i\zeta)$ are linearly independent, the vectors $\widehat{U}^\nu_{\varepsilon,\omega}$, $\varepsilon = \pm 1$ are linearly independent from the vectors $\widehat{U}^{-\nu-1}_{\varepsilon,\omega}$, $\varepsilon = \pm 1$.

3) In the next Section, we will remove the spurious factors $C^\nu_{h'}$ and $C^{-\nu-1}_{h'}$ by changing the normalization of the parabolic cylinder functions (11). The new spinors then make sense for integer ν by a continuity argument.

3.3 Some symmetries

Using symmetries in App.B, Sect.1.3 we find that

$$\begin{aligned}
&X^{-\nu-1}_\omega\big(\xi_1,\theta_\omega(\xi_1);h'\big) = X^\nu_\omega\big(\xi_1,\theta_\omega(\xi_1);h'\big)\\
&X^j_\omega\big(\xi_1,-\theta_\omega(\xi_1);h'\big) = -\overline{X^j_\omega\big(\xi_1,\theta_\omega(\xi_1);h'\big)}\\
&X^j_{-\omega}\big(-\xi_1,\theta_{-\omega}(-\xi_1);h'\big) = -\overline{X^j_\omega\big(\xi_1,\theta_\omega(\xi_1);h'\big)} \bmod \mathcal{O}(h')
\end{aligned} \tag{79}$$

The following Proposition will also be crucial to select the indices ε,ω.

Proposition 3.2. *Spinors $U^j_{\varepsilon,\omega}$ verify (at least at leading order) the symmetry*

$$^\dagger\widehat{U}^j_{-\varepsilon,-\omega} = \widehat{U}^j_{\varepsilon,\omega} \tag{80}$$

for the "local time" reversal operator $^\dagger u(\xi_1) = u(-\xi_1)$.

Proof. We use symmetry $h_\omega(\xi_1,\theta_1) = h_{-\omega}(-\xi_1,-\theta_1)$ proved in Proposition A.1, relations (17),(32), and symmetry $\theta_\omega(\xi_1) = \theta_{-\omega}(-\xi_1)$ to show that

$$T^j_{-\varepsilon,-\omega}\big(-\xi_1,-\theta_{-\omega}(-\xi_1)\big) = T^j_{\varepsilon,\omega}\big(\xi_1,\theta_\omega(\xi_1)\big)$$

and thus

$$\Phi^j_{-\varepsilon,-\omega}\big(-\xi_1,-\theta_{-\omega}(-\xi_1)\big) = \Phi^j_{\varepsilon,\omega}\big(\xi_1,\theta_\omega(\xi_1)\big)$$

which gives (80) at the level of phase function. We have

$$s^{-\nu-1}_{-\varepsilon,-\omega}\big(-\theta_{-\omega}(-\xi_1)\big) = s^{-\nu-1}_{\varepsilon,\omega}\big(\theta_\omega(\xi_1)\big)$$

and since the amplitudes $c^j_{\varepsilon,\omega}(\xi_1;h)$ constructed in Theorem A.2 are again invariant changing ξ_1 to $-\xi_1$ and simultaneously θ_ω to $-\theta_\omega$, it follows that $a^\nu_{-\omega}\big(-\xi_1,-\theta_{-\omega}(-\xi_1)\big) = a^\nu_\omega\big(\xi_1,\theta_\omega(\xi_1)\big)$, and so $X^j_{-\omega}\big(-\xi_1,-\theta_{-\omega}(-\xi_1)\big) = -\overline{X^j_\omega\big(\xi_1,\theta_\omega(\xi_1)\big)}$. Then changing ξ_1 to $-\xi_1$ together with (ε,ω) to $(-\varepsilon,-\omega)$ simply permutes the terms of the sum in (76)-(77); thus (80) holds. □

4 Normalization

4.1 The microlocal Wronskian

We extend here to BdG Hamiltonian the algebraic and microlocal framework for computing 1-D quantization rules: It is based on the classical "positive commutator method" using normalization of the microlocal solutions and conservation of some quantity called a "quantum flux". This normalization will replace Whittaker normalization for parabolic-cylinder functions, that diverges when ν takes integer values.

We have seen in (76)-(77) that $\gamma^{>,\rho}(E)$ is parametrized near $a = a_E$ by the phase functions $\Phi^j_{\varepsilon,\omega}$, that differ asymptotically in $x < x_E$ from each other when $j = \nu, -\nu - 1$ only by a constant.

We choose the orientation on $\gamma^{>,\rho}(E)$ according to this of Hamilton vector field. Let $\chi^a \in C_0^\infty(\mathbf{R}^2)$ be a smooth cut-off equal to 1 near a, and ω^a_ρ be a small neighborhood of $\mathrm{supp}\{\mathcal{P}, \chi^a\} \cap \Lambda_E$ near ρ; we shall write $\mathcal{P}(x, hD_x)$ (spatial representation) as well as $\mathcal{P}(-hD_\xi, \xi)$ (Fourier representation).

Definition 4.1. *Let $\mathcal{P}$ be (formally) self-adjoint, and $U^a, V^a \in K_h(E)$ be supported on $\Lambda^>_E$. We call the sesquilinear form*

$$\mathcal{W}^a_\rho(U^a, V^a) = \big(\frac{i}{h}[\mathcal{P}, \chi^a]_\rho U^a | V^a\big) = \big(\frac{i}{h}[\mathcal{P}, \chi^a]_\rho \widehat{U}^a | \widehat{V}^a\big) \tag{81}$$

the microlocal Wronskian of $(U^a, \overline{V^a})$ in ω^a_ρ. Here $\frac{i}{h}[\mathcal{P}, \chi^a]_\rho$ denotes the part of the commutator supported microlocally on ω^a_ρ.

To understand this terminology, let us consider instead the scalar Schrödinger operator $P = -h^2\Delta + V$, $x_E = 0$ and change χ to Heaviside unit step-function $\chi(x)$, depending on x alone. Then in distributional sense, we have $\frac{i}{h}[P, \chi] = -ih\delta' + 2\delta h D_x$, where δ denotes the Dirac measure at 0, and δ' its derivative, so that $\big(\frac{i}{h}[P, \chi]u|u\big) = -ih\big(u'(0)\overline{u(0)} - u(0)\overline{u'(0)}\big)$ is the usual Wronskian of $(u, \overline{u})$.

For regular BS quantization rules in the scalar case, the key formula

$$\big(\frac{i}{h}[P, \chi^a]u^a | v^a\big) = \big(\frac{i}{h}[P, \chi^a]\widehat{u}^a | \widehat{v}^a\big) = 0 \tag{82}$$

and its corollary, namely that $\rho = \pm$ give opposite contributions to the scalar products, result easily from the fact that $\widehat{u}^a, \widehat{u}^a$ are of WKB type near the focal point a. Because of Proposition 3.1, the latter property fails near the branching point a of our system, so we will have to compute separately $\big(\frac{i}{h}[\mathcal{P}, \chi^a]_\rho \widehat{U}^a | \widehat{V}^a\big)$ for $\rho = \pm$. Nevertheless (82) turns out to be true, see Lemma 4.1 below.

We note that $\mathcal{W}^a_\rho(U^a, V^a)$ still doesn't depend, at least modulo $\mathcal{O}(h)$, of the choice of χ^a. Namely, Proposition 3.1 shows that $\widetilde{U}^a, \widetilde{V}^a$ are smooth away from a, so if $\chi^a, \widetilde{\chi}^a \in C_0^\infty$ equals 1 near a, we can expand the commutator $\frac{i}{h}[\mathcal{P}, \chi^a - \widetilde{\chi}^a]_\rho$ as above and find that the two quantities $\mathcal{W}^a_\rho$, defined by any of these cut-off, are equal mod $\mathcal{O}(h')$ (at least, since the microlocal solutions in Proposition 3.1 have been computed only up to this accuracy).

4.2 Normalization of spinors in Fourier representation

We compute, at leading order, the matrix elements $\mathcal{W}^a_\rho(U^j_{\varepsilon,\omega}, U^k_{\varepsilon,\omega})$, $j,k \in \{\nu, -\nu-1\}$. Since they are independent of the choices of χ^a as above, we are free to choose $\chi^a(x,\xi) = \chi_1(x)\chi_2(\xi)$, with supp (χ_2) so small that $\chi_1(x) \equiv 1$ on ω^a_ρ. In Sect.7 however, without changing the normalization, we shall deform χ^a to $\widetilde{\chi}^a$ so that $\frac{i}{h}[\mathcal{P}^a, \widetilde{\chi}^a)$ is supported very close to a'.

By the functional calculus of h-PDO's we have microlocally in ω^a_ρ

$$\frac{i}{h}[\mathcal{P}^a, \chi^a](-hD_\xi, \xi) = \begin{pmatrix} 0 & -\alpha e^{i\phi/2}\chi_2'(\xi) \\ -\alpha e^{-i\phi/2}\chi_2'(\xi) & 0 \end{pmatrix} \tag{83}$$

To simplify the notations, we omit superscript a, as well as ξ_1 from the argument $\big(\xi_1, \theta_\omega(\xi_1)\big)$. Let χ_{ω_ρ} be equal to 1 near ω_ρ. We have

$$\begin{aligned} \mathcal{W}_\rho(U^\nu_{\varepsilon,\omega}, U^\nu_{\varepsilon,\omega}) &= -\alpha e^{i\phi/2}\big(\chi_{\omega_\rho}\chi_2'(\xi)(\widehat{\varphi}_2)^\nu_{\varepsilon,\omega}|(\widehat{\varphi}_1)^\nu_{\varepsilon,\omega}\big) - \alpha e^{-i\phi/2}\big(\chi_{\omega_\rho}\chi_2'(\xi)(\widehat{\varphi}_1)^\nu_{\varepsilon,\omega}|(\widehat{\varphi}_2)^\nu_{\varepsilon,\omega}\big) \\ &= -2\alpha\,\mathrm{Re}\big[e^{i\phi/2}\big(\chi_{\omega_\rho}\chi_2'(\xi)(\widehat{\varphi}_2)^\nu_{\varepsilon,\omega}|(\widehat{\varphi}_1)^\nu_{\varepsilon,\omega}\big)\big] \end{aligned} \tag{84}$$

Recall we have denoted by $\widehat{\theta}_\omega$ the critical point with $\theta_\omega > 0$. By (242) we have

$$\Phi^\nu_{\varepsilon,\omega}\big(\xi_1, \widehat{\theta}_\omega(\xi_1)\big) - \Phi^\nu_{\varepsilon,\omega}\big(\xi_1, -\widehat{\theta}_\omega(\xi_1)\big) = 2T^\nu_{\varepsilon,\omega}\big(\xi_1, \theta_\omega(\xi_1)\big) \tag{85}$$

and using also (79)

$$\begin{aligned} &-\frac{1}{\alpha}|C^\nu_{\beta,h'}|^{-2}e^{i\phi/2}\big(\chi_{\omega_\rho}\chi_2'(\xi)(\widehat{\varphi}_2)^\nu_{\varepsilon,\omega}|(\widehat{\varphi}_1)^\nu_{\varepsilon,\omega}\big) = \\ &\big(\chi_{\omega_\rho}\chi_2'(\xi)a^\nu_\omega(\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(\widehat{\theta}_\omega)/h'}|X^\nu_\omega(\widehat{\theta}_\omega)a^\nu_\omega(\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(\widehat{\theta}_\omega)/h'}\big)+ \\ &\big(\chi_{\omega_\rho}\chi_2'(\xi)a^\nu_\omega(-\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(-\widehat{\theta}_\omega)/h'}|X^\nu_\omega(-\widehat{\theta}_\omega)a^\nu_\omega(-\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(-\widehat{\theta}_\omega)/h'}\big)+ \\ &\big(\chi_{\omega_\rho}\chi_2'(\xi)a^\nu_\omega(\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(\widehat{\theta}_\omega)/h'}|X^\nu_\omega(-\widehat{\theta}_\omega)a^\nu_\omega(-\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(-\widehat{\theta}_\omega)/h'}\big)+ \\ &\big(\chi_{\omega_\rho}\chi_2'(\xi)a^\nu_\omega(-\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(-\widehat{\theta}_\omega)/h'}|X^\nu_\omega(\widehat{\theta}_\omega)a^\nu_\omega(\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(\widehat{\theta}_\omega)/h'}\big) \end{aligned} \tag{86}$$

we rewrite the sum of the first 2 terms of the RHS of (86) as :

$$-2i\int_{\omega_\rho}\chi_2'(\xi)|a^\nu_\omega(\widehat{\theta}_\omega)|^2\,\mathrm{Im}\big(X^\nu_\omega(\widehat{\theta}_\omega)\big)\,d\xi \tag{87}$$

which is purely imaginary (still mod $\mathcal{O}(h')$). The other terms are oscillating integrals with phases

$$\Phi^\nu_{\varepsilon,\omega}(\theta_\omega) - \Phi^\nu_{\varepsilon,\omega}(-\theta_\omega) = 2T^\nu_{\varepsilon,\omega}(\xi_1, \theta_\omega(\xi_1))$$

due to (85), whose derivative are given by

$$\partial_{\xi_1}T^\nu_{\varepsilon,\omega}(\xi_1, \theta_\omega(\xi_1)) = \theta_\omega + \partial_{\xi_1}h_\omega(\xi_1, \theta_\omega(\xi_1)) \tag{88}$$

In Appendix B, we show that for $\omega = \pm 1$

$$x^\nu_\omega(\xi_1) = -\frac{(2\xi_E)^2}{\alpha}\partial_{\xi_1}T^\nu_{\varepsilon,\omega}(\xi_1, \theta_\omega(\xi_1)) = -\frac{(2\xi_E)^2}{\alpha}(E_1' - (E_1')^2 + \cdots)$$

So $\partial_{\xi_1} T_{\varepsilon,\omega}(\xi_1, \theta_\omega(\xi_1)) \neq 0$ and the last 2 terms of (86) are $\mathcal{O}(h')$ because the phase is non-stationary. So by (84)

$$\mathcal{W}_\rho(U^\nu_{\varepsilon,\omega}, U^\nu_{\varepsilon,\omega}) = |C^\nu_{h'}|^2 \mathcal{O}(h') \tag{89}$$

at least. We proceed to compute

$$\begin{aligned}\mathcal{W}_\rho(U^{-\nu-1}_{\varepsilon,\omega}, U^{-\nu-1}_{\varepsilon,\omega}) &= -\alpha e^{i\phi/2}\big(\chi_{\omega_\rho}\chi_2'(\xi)(\widehat{\varphi}_2)^{-\nu-1}_{\varepsilon,\omega}|(\widehat{\varphi}_1)^{-\nu-1}_{\varepsilon,\omega}\big)\\ &- \alpha e^{-i\phi/2}\big(\chi_{\omega_\rho}\chi_2'(\xi)(\widehat{\varphi}_1)^{-\nu-1}_{\varepsilon,\omega}|(\widehat{\varphi}_2)^{-\nu-1}_{\varepsilon,\omega}\big)\end{aligned}$$

The computation carried over in (86) repeats identically, and yields

$$\mathcal{W}_\rho(U^{-\nu-1}_{\varepsilon,\omega}, U^{-\nu-1}_{\varepsilon,\omega}) = |C^{-\nu-1}_{h'}|^2 \mathcal{O}(h') \tag{90}$$

We are left to the mixed terms

$$\mathcal{W}_\rho(U^{\nu}_{\varepsilon,\omega}, U^{-\nu-1}_{\varepsilon,\omega}) = -\alpha e^{i\phi/2}\big(\chi_{\omega_\rho}\chi_2'(\xi)(\widehat{\varphi}_2)^{\nu}_{\varepsilon,\omega}|(\widehat{\varphi}_1)^{-\nu-1}_{\varepsilon,\omega}\big) - \alpha e^{-i\phi/2}\big(\chi_{\omega_\rho}\chi_2'(\xi)(\widehat{\varphi}_1)^{\nu}_{\varepsilon,\omega}|(\widehat{\varphi}_2)^{-\nu-1}_{\varepsilon,\omega}\big) \tag{91}$$

The first term on the RHS of (91) times $-\frac{1}{\alpha}\big(C^\nu_{h'}\overline{C^{-\nu-1}_{h'}}\big)^{-1}$ equals

$$\begin{aligned}&\big(\chi_{\omega_\rho}\chi_2'(\xi)a^\nu_\omega(\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(\widehat{\theta}_\omega)/h'}|X^{-\nu-1}_\omega(\widehat{\theta}_\omega)a^{-\nu-1}_\omega(\widehat{\theta}_\omega)e^{i\Phi^{-\nu-1}_{\varepsilon,\omega}(\widehat{\theta}_\omega)/h'}\big)+\\ &\big(\chi_{\omega_\rho}\chi_2'(\xi)a^\nu_\omega(-\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(-\widehat{\theta}_\omega)/h'}|X^{-\nu-1}_\omega(-\widehat{\theta}_\omega)a^{-\nu-1}_\omega(-\widehat{\theta}_\omega)e^{i\Phi^{-\nu-1}_{\varepsilon,\omega}(-\widehat{\theta}_\omega)/h'}\big)+\\ &\big(\chi_{\omega_\rho}\chi_2'(\xi)a^\nu_\omega(-\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(-\widehat{\theta}_\omega)/h'}|X^{-\nu-1}_\omega(\widehat{\theta}_\omega)a^{-\nu-1}_\omega(\widehat{\theta}_\omega)e^{i\Phi^{-\nu-1}_{\varepsilon,\omega}(\widehat{\theta}_\omega)/h'}\big)+\\ &\big(\chi_{\omega_\rho}\chi_2'(\xi)a^\nu_\omega(\widehat{\theta}_\omega)e^{i\Phi^\nu_{\varepsilon,\omega}(\widehat{\theta}_\omega)/h'}|X^{-\nu-1}_\omega(-\widehat{\theta}_\omega)a^{-\nu-1}_\omega(-\widehat{\theta}_\omega)e^{i\Phi^{-\nu-1}_{\varepsilon,\omega}(-\widehat{\theta}_\omega)/h'}\big)\end{aligned} \tag{92}$$

Using again symmetries (79), (255) from Appendix B, together with relation (47) we rewrite the sum of the first 2 terms of (92) as :

$$\begin{aligned}&\exp[-i\pi {E_1'}^2\zeta_{\varepsilon,\omega}(\widehat{\theta}_\omega)/2h']\int_{\omega_\rho}\chi_2'(\xi)a^\nu_{\varepsilon,\beta}(\widehat{\theta}_\omega)\overline{a^{-\nu-1}_{\omega\beta}(\widehat{\theta}_\omega)X^{-\nu-1}_\omega(\widehat{\theta}_\omega)}\,d\xi+\\ &\exp[-i\pi {E_1'}^2\zeta_{\varepsilon,\omega}(-\widehat{\theta}_\omega)/2h']\int_{\omega_\rho}\chi_2'(\xi)\overline{a^\nu_{\varepsilon,\beta}(\widehat{\theta}_\omega)}a^{-\nu-1}_{\omega\beta}(\widehat{\theta}_\omega)X^{-\nu-1}_\omega(\widehat{\theta}_\omega)\,d\xi\end{aligned} \tag{93}$$

Recall E_1' from (64) and $\zeta_{\varepsilon,\omega}(\widehat{\theta}_\omega)$ from (48). We observe that $\exp[-i\pi {E_1'}^2\zeta(\pm\widehat{\theta}_\omega,\varepsilon,\omega)/2h'] = \exp[-i\pi {E_1'}^2/4h']$ when $\varepsilon\,\omega = 1$; otherwise, $\exp[-i\pi {E_1'}^2\zeta(\pm\widehat{\theta}_\omega,\varepsilon,\omega)/2h'] = \exp[i\pi {E_1'}^2(\pm\omega + \frac{1}{2})/2h']$ takes the values $\exp[-i\pi {E_1'}^2/4h']$ or $\exp[3i\pi {E_1'}^2/4h']$. In the latter case (93), as a function of ${E_1'}^2/h'$, will have many cancellations due to the oscillating phases.

Thus we restrict henceforth to the choice $\varepsilon\,\omega = 1$. The term $\exp[-i\pi {E_1'}^2/4h']$ factors out in (93), which takes now the form

$$2\exp[-i\pi {E_1'}^2/4h']\,\mathrm{Re}\int_{\omega_\rho}\chi_2'(\xi)a^\nu_{\varepsilon,\beta}(\widehat{\theta}_\omega)\overline{a^{-\nu-1}_{\omega\beta}(\widehat{\theta}_\omega)X^{-\nu-1}_\omega(\widehat{\theta}_\omega)}\,d\xi \tag{94}$$

The last 2 terms in (92) are $\mathcal{O}(h')$ because they are given by oscillatory integrals with non-stationary phase as before. For the second term on the RHS of (91) times $-\frac{1}{\alpha}\big(C^\nu_{h'}\overline{C}^{-\nu-1}_{h'}\big)^{-1}$, we find similarly, under the same condition $\varepsilon\,\omega = 1$

$$2\exp[-i\pi {E_1'}^2/4h']\,\mathrm{Re}\int_{\omega_\rho}\chi_2'(\xi)\overline{a^\nu_{\varepsilon,\omega}(\widehat{\theta}_\omega)}a^{-\nu-1}_{\varepsilon,\omega}(\widehat{\theta}_\omega)\overline{X^\nu_\omega(\widehat{\theta}_\omega)}\,d\xi$$

which adds to (94), giving

$$\mathcal{W}_\rho(U^\nu_{\varepsilon,\omega}, U^{-\nu-1}_{\varepsilon,\omega}) = -\alpha \exp[-i\pi {E'_1}^2/4h'] C^\nu_{h'} \overline{C^{-\nu-1}_{h'}}$$
$$\times\, 2\,\mathrm{Re} \int_{\omega_\rho} \chi'_2(\xi) a^\nu_{\varepsilon,\omega}(\widehat{\theta}_\omega) \overline{a^{-\nu-1}_{\varepsilon,\omega}(\widehat{\theta}_\omega)} \big[X^\nu_\omega(\widehat{\theta}_\omega) + \overline{X^{-\nu-1}_\omega(\widehat{\theta}_\omega)}\big]\, d\xi$$

Since $\varepsilon\,\omega = 1$, this expression simplifies to

$$\mathcal{W}_\rho(\widehat{U}^\nu_{\varepsilon,\omega}, \widehat{U}^{-\nu-1}_{\varepsilon,\omega}) = -2\sqrt{2}\alpha(2\xi_E)^2 C^\nu_{h'} \overline{C^{-\nu-1}_{h'}} \exp[-i\pi {E'_1}^2/4h'] \int_{\omega_\rho} \chi'_2(\xi) \mathcal{F}_\omega(\xi_1, \widehat{\theta}_\omega(\xi_1))\, d\xi \quad (95)$$

where

$$\mathcal{F}_\omega(\xi_1, \widehat{\theta}_\omega(\xi_1)) = \left[\frac{c^2_{0,\omega}(\xi_1, \eta_1, \theta_1)\big(\theta_1 + \partial_{\xi_1} h_\omega(\xi_1, \theta_1)\big)}{\theta_1 + \omega({E'_1}^2 - \theta_1^2)^{1/2} \partial^2_{\theta_1} h_\omega(\xi_1, \theta_1)} \right]_{\eta_1 = \eta_\omega(\widehat{\theta}_1), \theta_1 = \widehat{\theta}_\omega(\xi_1),} \quad (96)$$

and $c_{0,\omega}(\xi_1, \eta_1, \theta_1)$ is the amplitude associated with FIO A_ω defined in App.A. The next step is to check that $\mathcal{F}_\omega(\xi_1, \widehat{\theta}_\omega(\xi_1))$ is a constant, which would be automatically satisfied in the scalar case, see the discussion after (82). A careful inspection of the normal form of [HeSj] for P_ω carried in Appendix A allows to compute precisely $c^2_{0,\omega}(\xi_1, \eta_1, \theta_1)$, from which we obtain the following Lemma (at least at the level of Taylor expansions).

Lemma 4.1. *With the notations above, and the proper choice of the extension of $c_{0\omega}(\xi', \eta, \theta)$ from Γ'_κ giving the microlocal solutions, we have :*

$$\mathcal{F}_\omega(\xi_1, \widehat{\theta}_\omega(\xi_1)) = \frac{\theta_\omega(\xi_1) + 4\,\omega\,\xi_1\,\theta_\omega(\xi_1) + \cdots}{\theta_\omega(\xi_1) + 4\,\omega\,\xi_1\,\theta_\omega(\xi_1) + \cdots} = 1$$

Remark: Without the correction on the amplitude $c_{0,\omega}$ carried in Theorem A.2, i.e. replacing $c_{0,\omega}(\xi_1, \eta_1, \theta_1)$ by $c_{0,\omega}|_{\Gamma'_\kappa}$ (evaluated on the critical set), the RHS of (96) takes the form

$$\frac{\theta_\omega(\xi_1) + 4\,\omega\,\xi_1\,\theta_\omega(\xi_1) + \frac{13}{2}\xi_1^2\,\theta_\omega(\xi_1) + \cdots}{\theta_\omega(\xi_1) + 4\,\omega\,\xi_1\,\theta_\omega(\xi_1) + \frac{23}{2}\xi_1^2\,\theta_\omega(\xi_1) \cdots} = 1 + \mathcal{O}(\xi_1^2)$$

and we cannot perform the integration in (96). The same subtelty occurs in Proposition 7.2 below.

Now because $\int_{\omega_\rho} \chi'_2(\xi) = -\rho$, we get from (95)

$$\mathcal{W}_\rho(\widehat{U}^\nu_{\varepsilon,\omega}, \widehat{U}^{-\nu-1}_{\varepsilon,\omega}) = 2\sqrt{2}\rho\alpha(2\xi_E)^2 C^\nu_{h'} \overline{C^{-\nu-1}_{h'}} \exp[-i\pi {E'_1}^2/4h'] \quad (97)$$

which is independent of ω (still with $\varepsilon\,\omega = 1$). Taking complex conjugate, this gives also $\mathcal{W}_\rho(\widehat{U}^{-\nu-1}_{\varepsilon,\omega}, \widehat{U}^\nu_{\varepsilon,\omega})$.

Collecting (89), (90) and (97), and identifying the microlocal Wronskian in Fourier representation with its matrix $\mathcal{W}_\rho$ in the basis $\widehat{U}^j_{\varepsilon,\omega}, j = \nu, -\nu-1$, we have shown that, for $\varepsilon\,\omega = 1$

$$\rho\mathcal{W}_\rho = 2\sqrt{2}\alpha(2\xi_E)^2 \times$$
$$\begin{pmatrix} |C^\nu_{h'}|^2 \mathcal{O}(h') & C^\nu_{h'} \overline{C^{-\nu-1}_{h'}} \exp[-i\pi {E'_1}^2/4h'] \big(1 + \mathcal{O}(h')\big) \\ \overline{C^\nu_{h'}} C^{-\nu-1}_{h'} \exp[i\pi {E'_1}^2/4h'] \big(1 + \mathcal{O}(h')\big) & |C^{-\nu-1}_{h'}|^2 \mathcal{O}(h') \end{pmatrix} \quad (98)$$

does not depend on ρ, and makes of $\mathcal{W}_\rho$ a non degenerate (1,1)-Hermitian (or Lorenzian) form on $K_h^a(E) = \mathrm{Ker}_h(\mathcal{P}^a - E)$.

We want to find a new basis of microlocal solution so that (98) has a canonical form, at least with the current accuracy $\mathcal{O}(h')$. The prefactor in (98) can be safely ignored. So the first step in normalizing, modulo $\mathcal{O}(h')$ terms, amounts in dividing $\widehat{U}^j_{\varepsilon,\omega}$ by $C^j_{h'}$, $j \in \{\nu, -\nu-1\}$. Because of (50), this is harmless so long as $\nu \notin \mathbf{Z}$, in which case (at least) one of the "new" $\widehat{U}^\nu_{\varepsilon,\omega}, \widehat{U}^{-\nu-1}_{\varepsilon,\omega}$ would vanish identically, and the pair fail to be linearly independent. Dividing also $\widehat{U}^\nu_{\varepsilon,\omega}$ (resp. $\widehat{U}^{-\nu-1}_{\varepsilon,\omega}$) by the phase factor $\exp[i\pi {E'_1}^2/4h']$, (resp. $\exp[i\pi {E'_1}^2/2h']$) we can remove $\exp[-i\pi {E'_1}^2/4h']$ from (98). Let $N^j_{\varepsilon,\omega}\widehat{U}^j_{\varepsilon,\omega}$ denote the spinors normalized this way. In this new basis $\rho\mathcal{W}_\rho$ eventually takes the canonical form

$$\rho\mathcal{W}_\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mathcal{O}(h') \tag{99}$$

By Plancherel theorem (81) $\rho\mathcal{W}_\rho$ assumes the same value in the spatial representation,

We keep in mind that we have been working near $a = a_E$ by writing $N^{a,j}_{\varepsilon,\omega}U^{a,j}_{\varepsilon,\omega}$ instead of $N^j_{\varepsilon,\omega}U^j_{\varepsilon,\omega}$ and $\rho\mathcal{W}^a_\rho$ instead of $\rho\mathcal{W}^a_\rho$. We proved:

Proposition 4.1. *The microlocal Wronskians $\rho\mathcal{W}^a_\rho$ supplies $K^a_h(E)$ with a (1,1)-Hermitean form. All vectors $N^{a,j}_{\varepsilon,\omega}U^{a,j}_{\varepsilon,\omega} \in K^a_h(E)$, $(j,\varepsilon,\omega) \in \{\nu, -\nu-1\} \times \{-1,1\}^2$ are isotropic (modulo $\mathcal{O}(h')$). Those subject to the condition $\varepsilon\,\omega = 1$ form a normalized basis (modulo $\mathcal{O}(h')$) in which $\rho\mathcal{W}^a_\rho$ takes the form (99).*

Since $K^a_h(E)$ is 2-D, there follows simple colinearity relations (modulo $\mathcal{O}(h')$) between the $U^j_{\varepsilon,\omega}$'s and their complex conjugates. For simplicity we keep denoting the normalized spinors $N^{a,j}_{\varepsilon,\omega}\widehat{U}^{a,j}_{\varepsilon,\omega}$ by $\widehat{U}^{a,j}_{\varepsilon,\omega}$.

4.3 Lorenzian structure on the microlocal kernel

We want to extend $\rho\mathcal{W}^a_\rho$ to a globally defined Hermitean form on $K_h(E)$, and start to study some symmetries. Let $\widehat{U}^{a',j}_{\varepsilon,\omega} = \mathcal{I}\widehat{U}^{a,j}_{\varepsilon,\omega}$, $j \in \{\nu, -\nu-1\}$ solve

$$\big(\mathcal{P}^{a'} - E\big)\widehat{U}^{a',j}_{\varepsilon,\omega} = 0 \tag{100}$$

with $\mathcal{P}^{a'}$ as in (53). The normalized functions $N^{a',j}_{\varepsilon,\omega}\widehat{U}^{a',j}_{\varepsilon,\omega}$ are defined in the same way. Let also $\chi^{a'} = \mathcal{I}\chi^a\mathcal{I}$, and define the spinors

$$\widehat{F}^{a,j}_{\varepsilon,\omega} = \frac{i}{h}[\mathcal{P}^a, \chi^a]\widehat{U}^{a,j}_{\varepsilon,\omega} \tag{101}$$

Then

$$\widehat{F}^{a',j}_{\varepsilon,\omega} = \frac{i}{h}[\mathcal{P}^{a'}, \chi^{a'}]\widehat{U}^{a',j}_{\varepsilon,\omega} = -\mathcal{I}\widehat{F}^{a,j}_{\varepsilon,\omega} \tag{102}$$

Equations (101) and (102) hold also in spatial representation, removing the "hats". Considering the microlocal Wronskians, we have

$$\mathcal{W}^a_\rho(U^{a,j}_{\varepsilon,\omega}, U^{a,k}_{\varepsilon,\omega}) = \big(\widehat{F}^{a,j}_{\varepsilon,\omega}|\widehat{U}^{a,k}_{\varepsilon,\omega}\big), \quad j,k \in \{\nu, -\nu-1\} \tag{103}$$

and similarly with a' instead of a. By (102):

$$\mathcal{W}_\rho^{a'}\big(U_{\varepsilon,\omega}^{a',j}, U_{\varepsilon,\omega}^{a',k}\big) = -\overline{\mathcal{W}_\rho^{a}\big(U_{-\varepsilon,-\omega}^{a,j}, U_{-\varepsilon,-\omega}^{a,k}\big)}, \quad j,k \in \{\nu, -\nu-1\} \tag{104}$$

Let $K_h(E)$ be the microlocal kernel $\mathrm{Ker}_h(\mathcal{P}(x,hD_x)-E)$, which splits into $K_h^{a'}(E)$ and $K_h^a(E)$.

The forms $\mathcal{W}_\rho^a, \mathcal{W}_\rho^{a'}$ provide a natural duality between $K_h(E)$ and the span of $\{F_{\varepsilon,\omega}^{a,j}, F_{\varepsilon,\omega}^{a',j}\}$ which identifies with the microlocal co-kernel $K_h^*(E) = K_h^{a*}(E) + K_h^{a'*}(E)$ of $\mathcal{P}-E$ on $]-x_E, x_E[\times\mathbf{R}_+$. Actually, varying $j \in \{\nu, -\nu-1\}$ (depending on a or a'), *a priori* $\dim K_h^a(E) = \dim K_h^{a*}(E) = 2$ for general E, so $\dim K_h(E) = \dim K_h^*(E) = 4$, but we shall see how to break $K_h(E)$ into two orthogonal subspaces (see Sect.1) so that this dimension drops to 2, and eventually to 1 when E is an eigenvalue of either sub-system. Let $\mathcal{W}^a = \frac{1}{2}(\mathcal{W}_+^a - \mathcal{W}_-^a)$, and $\mathcal{W}^{a'} = \frac{1}{2}(-\mathcal{W}_+^{a'} + \mathcal{W}_-^{a'})$. These indices which account for the jump of Maslov index at a or a', are both equal to 1. Thus we have proved :

Proposition 4.2. *Under PT symmetry ${}^\vee\mathcal{I}\mathcal{P}(x,hD_x) = \mathcal{P}(x,hD_x)\mathcal{I}^\vee$ which we recall from (2), the microlocal Wronskians $\mathcal{W}_\rho^a$ provide $K_h^a(E)$ with an (1,1)-Hermitian form $\mathcal{W}_\rho^a = -\mathcal{W}_{-\rho}^a$. The same holds true at a', and by (104) the corresponding structures on $K_h^a \times K_h^{a*}$ and $K_h^{a'} \times K_h^{a'*}$ are anti-isomorphic.*

The group of automorphisms preserving $\mathcal{W}_\rho^a$ and $\mathcal{W}_\rho^{a'}$ (still modulo $\mathcal{O}(h')$) is therefore U(1,1). In Sect.5 & 6, we shall extend the 2 families of spinors $U_{\varepsilon,\omega}^{a,j}$ and $U_{\varepsilon,\omega}^{a',j}$ along $\gamma^{>,\rho}(E)$ and show in Sect.7 how they are related by the relative monodromy operator.

5 Spinors in the spatial representation near a branching point

We apply to our microlocal solutions in Fourier representation, the inverse h-Fourier transform $\mathcal{F}_h^* u(\xi,h) = (2\pi h)^{-1/2}\int e^{ix\xi/h}u(x)\,dx$. Like Airy functions, they have complex branches at a, a', so it is necessary to distinguish carefully between the germs of complex phase functions in $|x| > x_E$, even if one contents with the region $|x| < x_E$.

5.1 Formulas in the spatial representation

For the moment, we work near x_E, and omit henceforth the label a. By inverse Fourier transform $\mathcal{F}_h^*$ to (76)-(77) we get (before normalization of Sect.4), at leading order in h

$$U_{\varepsilon,\omega}^\nu(x) = 2\omega\xi_E C_{h'}^\nu e^{ix\xi_E/h} \sum_{\theta_\omega(\xi_1)=\pm\widehat{\theta}_\omega(\xi_1)} (2\pi h)^{-1/2}\int \begin{pmatrix} e^{i\phi/2}(\xi^2-\mu-E)^{-1/2}X_\omega^\nu(\xi_1,\theta_\omega(\xi_1)) \\ (\xi^2-\mu-E)^{1/2}\end{pmatrix}$$
$$\times\, |a_\omega^\nu\big(\xi_1,\theta_\omega(\xi_1)\big)|\exp[iR_\omega^\nu(\theta_\omega(\xi_1))]\exp[i\widetilde{\Psi}_{\varepsilon,\omega}^\nu(x_1,\xi_1)/h']\,d\xi_1 \tag{105}$$

and

$$U_{\varepsilon,\omega}^{-\nu-1}(x) = \frac{2\sqrt{2}}{E_1'}\omega\xi_E C_{h'}^{-\nu-1} e^{ix\xi_E/h}$$
$$\sum_{\theta_\omega(\xi_1)=\pm\widehat{\theta}_\omega(\xi_1)} \varepsilon \operatorname{sgn}\theta_\omega (2\pi h)^{-1/2} \int \begin{pmatrix} e^{i\phi/2}(\xi^2-\mu-E)^{-1/2} X_\omega^{-\nu-1}(\xi_1,\theta_\omega(\xi_1)) \\ (\xi^2-\mu-E)^{1/2} \end{pmatrix} \tag{106}$$
$$\times |a_\omega^{-\nu-1}\big((\xi_1,\theta_\omega(\xi_1)\big)| \exp[iR_\omega^{-\nu-1}(\theta_\omega(\xi_1))] \exp[i\widetilde{\Psi}_{\varepsilon,\omega}^{-\nu-1}(x_1,\xi_1)/h']\, d\xi_1$$

with

$$\widetilde{\Psi}_{\varepsilon,\omega}^{j}(x_1,\xi_1) = x_E^0\xi_E + \omega(E_1+x_1)\xi_1 + T_{\varepsilon,\omega}^{j}(\xi_1,\theta_\omega(\xi_1)) \tag{107}$$

Because θ_ω is the common θ-projection of the critical point of the phase defining $A_\omega D_\nu$ (resp. $A_\omega D_{-\nu-1}$) as in (206) (resp. (226)), the critical points of $\xi_1 \mapsto \widetilde{\Psi}_{\varepsilon,\omega}^{j}(x_1,\xi_1)$ satisfy $G_\omega(x_1,\xi_1;E_1')=0$, with

$$G_\omega(x_1,\xi_1;E_1') = \omega(E_1+x_1) + \theta_\omega(\xi_1) + \frac{\partial h_\omega}{\partial \xi_1}(\xi_1,\theta_\omega(\xi_1)) \tag{108}$$

In other words, the critical points of $\xi_1 \mapsto \widetilde{\Psi}_{\varepsilon,\omega}^{j}(x_1,\xi_1)$ belong to the ξ_1-component of the set of critical points of $\widetilde{H}_\omega : (\xi_1,\theta_1) \mapsto \omega(E_1+x_1)\xi_1 + H_\omega(\xi_1,\theta_1;E_1')$, with

$$H_\omega(\xi_1,\theta_1;E_1') = \xi_1\theta_1 + h_\omega(\xi_1,\theta_1) - \omega\int ({E_1'}^2-\theta_1^2)^{1/2}\, d\theta_1 \tag{109}$$

as in (214). We notice that $G_\omega(\xi_1,\theta_1;E_1')$ and $H_\omega(\xi_1,\theta_1;E_1')$ are independent of j and ε. Let us look at the critical points $G_\omega(x_1,\xi_1;E_1')=0$, and the critical values $\widetilde{\Psi}_{\varepsilon,\omega}^{j}(x_1,\xi_1)$, as functions of x_1.

The critical point of $\widetilde{H}_\omega$ is given by

$$\begin{aligned} &\omega(x_1+E_1) + \theta_1 + \partial_{\xi_1} h_\omega(\xi_1,\theta_1) = 0 \\ &\xi_1 + \partial_{\theta_1} h_\omega(\xi_1,\theta_1) - \omega({E_1'}^2-\theta_1^2)^{1/2} = 0 \end{aligned} \tag{110}$$

Second equation (110) gives the critical point $\theta_1 = \theta_\omega(\xi_1)$. Recall from Lemma A.3 that this is actually an equation for θ_1^2 and the positive root is denoted by $\widehat{\theta}_\omega(\xi_1)$. To show that $(x_1,\xi_1)=0$ is a critical point for $\widetilde{H}_\omega$, and hence $G_\omega(0,0)=0$, we must prove that the first Eq.(110) holds for $(x_1,\xi_1)=0$. The following Lemma will be checked by formal calculus in Sect.5.1.

Lemma 5.1. *: For $\omega=\pm1$ we have $\omega E_1 + \theta_\omega(0) + \partial_{\xi_1} h_\omega(0,\theta_\omega(0)) = 0$ provided $\theta_\omega(\xi_1) = -\omega\widehat{\theta}_\omega(\xi_1)$.*

Now we have

$$\partial_{\xi_1} G_\omega(x_1,\xi_1;E_1') = (\partial_{\xi_1}\theta_\omega(\xi_1)\left[1 + (\frac{\partial^2 h_\omega}{\partial\theta_1\partial\xi_1})\big(\xi_1,\theta_\omega(\xi_1)\big)\right] + (\frac{\partial^2 h_\omega}{\partial {\xi_1}^2})\big(\xi_1,\theta_\omega(\xi_1)\big) \tag{111}$$

In Sect. 5.1 we prove by formal calculus

Lemma 5.2. *: We have*

$$\begin{aligned}\partial_{\xi_1} G_\omega(0,0;E_1') &= 0\\ \partial_{\xi_1}^2 G_\omega(0,0;E_1') &= \omega\big(\frac{1}{E'_1}+\frac{3}{4}\,E'_1+\cdots\big)\neq 0\end{aligned}\tag{112}$$

There follows the following factorization near $(x_1,\xi_1)=0$

$$\begin{aligned}G_\omega(x_1,\xi_1,E_1') &= A_\omega(x_1,\xi_1,E_1')(\xi_1^2+a_\omega(x_1,E_1')\xi_1+b_\omega(x_1,E_1'))\\ &= A_\omega(x_1,\xi_1,E_1')(\xi_1-\xi_\omega^+(x_1,E_1'))(\xi_1-\xi_\omega^-(x_1,E_1'))\end{aligned}\tag{113}$$

where $A_\omega(x_1,\xi_1,E_1'), a_\omega(x_1,E_1'), b_\omega(x_1,E_1')$ are analytic near (0,0), A_ω is elliptic, and $a_\omega(0,E_1')=b_\omega(0,E_1')=0$. Thus $G_\omega(x_1,\xi_1;E_1')=0$ has locally 2 solutions $\xi_1=\xi_\omega^\rho(x)$ (the critical points of $\xi_1\mapsto\widetilde{\Psi}^j_{\varepsilon,\omega}(x_1,\xi_1)$) normalized by $\rho\xi_\omega^\rho(x)\geq 0$ for small $x_1<0$.

Having computed the critical points of $\widetilde{\Psi}^j_{\varepsilon,\omega}(x_1,\xi_1)$, we perform the integration with respect to ξ_1 in (105), and apply Theorem 2.1. Denote by

$$\Psi^{j,\rho}_{\varepsilon,\omega}(x)=\widetilde{\Psi}^j_{\varepsilon,\omega}(x_1,\xi_\omega^\rho)\tag{114}$$

the critical values. Since

$$\Psi^{-\nu-1,\rho}_{\varepsilon,\omega}(x)=\Psi^{\nu,\rho}_{\varepsilon,\omega}(x)+\frac{\pi E_1'^{\,2}}{2}\,\zeta_{\varepsilon,\omega}\big(\theta_\omega\big(\xi_\omega^\rho(x)\big)\big)\tag{115}$$

(where index $\zeta_{\varepsilon,\omega}$ defined in (48) is 1/2 when $\varepsilon\,\omega=1$) they differ only by a constant, and the densities $K^{j,\rho}_\omega(x)$ (Hessian at the critical point) take the same value

$$K^\rho_\omega(x)=\frac{1}{i}\frac{\partial^2\Psi^\nu_{\varepsilon,\omega}}{\partial\xi_1^2}(\xi_1,\theta_1)|_{\xi_1=\xi_\omega^\rho(x)}=\frac{1}{i}\partial_{\xi_1}G_\omega(x_1,\xi_1;E_1')|_{\xi_1=\xi_\omega^\rho(x)}\tag{116}$$

where we recall $\partial_{\xi_1}G_\omega$ from (111). By (112) we have $K^\rho_\omega(x_E)=0$.

The $\omega\widehat{\theta}_\omega(\xi_1)$ term in (105) has non stationary phase, and contributes to an error term $\mathcal{O}(h')$, while the $\theta_\omega(\xi_1)=-\omega\widehat{\theta}_\omega(\xi_1)$ term contributes by two critical points, labelled by ρ, and located on the corresponding segments $\rho=\Lambda^>_\rho$. Applying the asymptotic stationary phase we find (before normalization of Sect.4), and up to the phase factor $e^{ix_E^0\xi_E/h'}$ which doesn't change normalization:

$$\begin{aligned}U^\nu_{\varepsilon,\omega}(x,h) &= 2\omega\beta\xi_E C^\nu_{h'}e^{ix\xi_E/h}\sum_\rho\big(K^\rho_\omega(x)\big)^{-1/2}\\ &\begin{pmatrix}e^{i\phi/2}(\xi^2-\mu-E)^{-1/2}X^\nu_\omega(\xi_1,\theta_\omega(\xi_1))\\ (\xi^2-\mu-E)^{1/2}\end{pmatrix}|_{\xi_1=\xi_\omega^\rho(x)}|a^\nu_\omega(\xi_1,\theta_\omega(\xi_1))|\exp[iR^\nu_\omega(\theta_\omega(\xi_1))]|_{\xi_1=\xi_\omega^\rho(x)}\\ &\exp[i\Psi^{\nu,\rho}_{\varepsilon,\omega}(x)/h'](1+\mathcal{O}(h'))\end{aligned}\tag{117}$$

and

$$\begin{aligned}U^{-\nu-1}_{\varepsilon,\omega}(x,h) &= 2\omega\beta\xi_E C^{-\nu-1}_{h'}e^{ix\xi_E/h}\sum_\rho\varepsilon\,\mathrm{sgn}(\theta_\omega(\xi_\omega^\rho(x))\big(K^\rho_\omega(x)\big)^{-1/2}\\ &\begin{pmatrix}e^{i\phi/2}(\xi^2-\mu-E)^{-1/2}X^{-\nu-1}_\omega(\xi_1,\theta_\omega(\xi_1))\\ (\xi^2-\mu-E)^{1/2}\end{pmatrix}|_{\xi_1=\xi_\omega^\rho(x)}\\ &|a^{-\nu-1}_\omega(\xi_1,\theta_\omega(\xi_1))|\exp[iR^{-\nu-1}_\omega(\theta_\omega(\xi_1))]|_{\xi_1=\xi_\omega^\rho(x)}\exp[i\Psi^{-\nu-1,\rho}_{\varepsilon,\omega}(x)/h'](1+\mathcal{O}(h'))\end{aligned}\tag{118}$$

We summarize (117) and (118) as

$$U^j_{\varepsilon,\omega}(x,h) = \sum_\rho U^{j,\rho}_{\varepsilon,\omega}(x,h) = C^j_{\omega,h'} \sum_\rho \widetilde{U}^{j,\rho}_{\varepsilon,\omega}(x,h) \exp[i\big(x\xi_E + \frac{(2\xi_E)^3}{\alpha}\Psi^{j,\rho}_{\varepsilon,\omega}(x)/h] \tag{119}$$

where $U^{\nu,\rho}_{\varepsilon,\omega}(x,h)$ and $U^{-\nu-1,\rho}_{\varepsilon,\omega}(x,h)$ are colinear (outside the branching points the space of microlocal solutions is 1-D). In particular the common phase factor for $U^{j,\rho}_{\varepsilon,\omega}(x,h)$, up to a multiplicative constant, and including the lower order term in h' (sub-principal 1-form) is given by $\exp[ix\xi_E/h]\exp\big[i(\Psi^{\nu,\rho}_{\varepsilon,\omega}(x) + h'R^{\nu,\rho}_{\varepsilon,\omega}(\theta_\omega)/h'\big]$. To stress that Jacobians $K^{\nu,\rho}_\omega(x)$ are purely imaginary we set

$$K^{\nu,\rho}_\omega(x) = K^{-\nu-1,\rho}_\omega(x) = \frac{1}{i}L^\rho_\omega(x) \tag{120}$$

with $L^\rho_\omega(x)$ real and vanishing at x_E. Notice that $\varepsilon \operatorname{sgn}\theta_\omega = -\varepsilon\,\omega = -1$ since $\theta_\omega(\xi_1)$ has sign opposite to ω.

5.2 Asymptotics of the phase functions near the branching point

Here we study the microlocal solutions near a_E, in particular in the region $x > x_E$ where the phase (107) is complex ; our aim is to sort out among $j, \varepsilon, \omega, \rho$'s, the exponentially decreasing (recessive) and exponentially increasing (dominant) branches of solutions. We chosed already $\varepsilon\,\omega = 1$. To this end, we shall relate the picture of Stokes lines and Stokes regions of phase (107) to the corresponding one for the harmonic oscillator $P_0(\theta, hD_\theta) = \frac{1}{2}\big((hD_\theta)^2 + \theta^2 - h\big)$, close to the turning points, as explained in Sect.2.

Note that "local time" reversal invariance property (80) shows that $U^j_{\varepsilon,\omega}$ continued across Stokes line $x > x_E$ in the forbidden region has the same asymptotics as $U^j_{-\varepsilon,-\omega}$; this still would allow for exponentially small corrections, called Stokes phenomenon (see [DeDiPh], [Sil] for the Schrödinger case).

5.2.1 Extension of $U^j_{\varepsilon,\omega}$ across Stokes lines

In this Sect. we sketch the construction of the complex germs of microlocal solutions outside $[-x_E, x_E]$. Since we do not really consider the problem of resonances, we only use these construction to sort out between exponentially growing and exponentially decaying branches. This will determine the monodromy matrices $\mathcal{M}^{a,a'}_\rho(E)$ in Sect.7. By (114) for x_1 in a neighborhood of 0 we have

$$\Psi^{\nu,\rho}_{\varepsilon,\omega}(x_1) = x^0_E\xi_E + \omega(E_1 + x_1)\xi_1 + T^\nu_{\varepsilon,\omega}(\xi_1, \theta_\omega(\xi_1))|_{\xi_1 = \omega\xi^\rho_1(x_1)} =$$
$$T^\nu_{\varepsilon,\omega}(\xi_1,\theta_1) = \xi_1\theta_1 + h_\omega(\xi_1,\theta_1) - \frac{1}{2}\omega\theta_\omega(\xi_1)({E'_1}^2 - \theta_1)^{1/2} - \frac{{E'_1}^2}{2}\widehat{\Theta}_{\varepsilon,\omega}(\theta_1)$$

with $h_\omega(\xi_1,\theta_1)$ holomorphic in a neighborhood of (0,0), and where we recall from Lemma A.3

$$\theta^2_\omega(\xi_1) = ({E'_1}^2 - 4{E'_1}^4 + \cdots) - \xi_1(4{E'_1}^2 + \cdots) + \cdots$$

is very close to ${E'_1}^2$. Define ν_1 by $E'_1 = \sqrt{2(\nu_1+1)h}$. We can rewrite this phase as

$$\Psi^{\nu,\rho}_{\varepsilon,\omega}(x_1) = \big[\mathrm{Hol}(\xi_1) + \Phi^{-\nu_1-1}_{\varepsilon,\omega}(-\theta_\omega(\xi_1))\big]|_{\xi_1=\omega\xi^\rho_1(x_1)} \tag{121}$$

where $\Phi^{-\nu_1-1}_{\varepsilon,\omega}$ is as (34), and $\xi_1 \mapsto \mathrm{Hol}(\xi_1)$ is holomorphic near 0. Let

$$\eta^\rho_\omega(x_1) = -\theta_\omega(\xi_1)|_{\xi_1=\omega\xi^\rho_1(x_1)}$$

this is an analytic function of $\sqrt{E_1 x_1}$ near 0, and $\eta^\rho_+(x_1) > 0$, and for $x_1 = 0$ lies on the real axis and is very close, to the left, of the turning point $E = E'_1$ of the harmonic oscillator of Sect.2. We have the expansion

$$\eta^\rho_\omega(x_1) = {E'}_1(1 - 2{E'}_1^2 - \cdots) - 2i{E'}_1\rho\sqrt{2E_1x_1} + \cdots \tag{122}$$

so that $\eta^\rho_\omega(x_1)$ moves to the axis $[{E'}_1, +\infty[$.

When $x_1 > 0$, and $\rho = +1$, $\eta^\rho_+(x_1)$ moves upwards towards the Stokes line making an angle $2\pi/3$ with the positive real axis at E.

In computing the integral (105) for positive x_1 we take a good contour $\gamma_+ \subset \{\mathrm{Im}\,\xi_1 > 0\}$ passing through $\xi^\rho_\omega(x_1)$, $\rho = +1$, and parametrized locally by $\xi_1 = \xi^\rho_+(x_1) + t$, $t \in \mathbf{R}$. It clearly suffices to show that $\mathrm{Im}\,\Psi^\nu_{\varepsilon,\omega}(x,\xi_1) \geq 0$ along γ_+; since the "regular part" $\mathrm{Hol}(\xi_1)$ varies slowly it suffices in turn to check that $t \mapsto \Phi^{-\nu-1}_{\varepsilon,\omega}(\eta^\rho_+(x_1) + t)$ has positive imaginary part. Similar considerations hold for $U^{-\nu-1}_{\varepsilon,\omega}$.

5.2.2 Taylor expansion of the phase functions

We write (114) as

$$\begin{aligned}\Psi^{j,\rho}_{\varepsilon,\omega}(x) &= \Psi^{j,\rho}_{\varepsilon,\omega}(x_E) + \int_{x_E}^{x} \frac{\partial}{\partial y}(\Psi^{j,\rho}_{\varepsilon,\omega}(y))\,dy \\ &= \Psi^{j,\rho}_{\varepsilon,\omega}(x_E) + \omega(2\xi_E)^{-2}\alpha\int_0^{x_1} \xi^\rho_\omega(y_1)\,dy_1\end{aligned} \tag{123}$$

where we recall from Lemma 5.1 that $\xi^\rho_\omega(0) = 0$. The expansion of $\xi^\rho_\omega(x_1)$ near 0 is given in the following Lemma, that we prove by formal calculus in Sect.5.3.

Lemma 5.3. *The critical points $\xi^\rho_\omega(x_1)$, $\rho = \pm$, are given by the branches of an analytic function*

$$\omega\xi^\rho_\omega(x_1) = f_1(x_1, E_1) + \rho\sqrt{2E_1}\sqrt{-x_1}f_2(x_1, E_1) \tag{124}$$

(independent of ω) with

$$\begin{aligned}f_1(x_1, E_1) &= 2E_1\,x_1 + x_1^2 + \cdots \\ f_2(x_1, E_1) &= 1 - x_1\big(4E_1 - \frac{1}{4E_1} + \cdots\big) + \cdots\end{aligned} \tag{125}$$

where we recall E_1 and E'_1 are related by (65). In particular we have the following symmetry between critical points

$$\xi^\rho_{-\omega}(x_1; E_1) = -\xi^\rho_\omega(x_1; E_1) \tag{126}$$

Returning to the original variables (x,ξ) by $x_1=(2\xi_E)^{-2}\,\alpha\,(x-x_E)$, $\omega\xi_1=(2\xi_E)^{-1}(\xi-\xi_E)$, for these values of $\xi_1=\xi^\rho_\omega(x_1)$, we have of course $\det(\mathcal{P}(x,\xi)-E)=0$.

Now we compute the asymptotics of $\Psi^{\nu,\rho}_{\varepsilon,\omega}(x_E)$. Recall from (71) and (70) that

$$\begin{aligned}\Psi^{\nu,\rho}_{\varepsilon,\omega}(x_E)&=x^0_E\xi_E+T^\nu_{\varepsilon,\omega}\big(0,\theta_\omega(0)\big)\\ &x^0_E\xi_E+h_\omega(0,\theta_\omega(0))-\frac{1}{2}\omega\theta_\omega(0)({E'}_1^2-\theta_\omega(0)^2)^{1/2}-\frac{1}{2}{E'}_1^2\widehat{\Theta}_{\varepsilon,\omega}(\theta_\omega(0))\end{aligned}\tag{127}$$

and from (110)

$$\partial_{\theta_1}h_\omega(\xi_1,\theta_\omega(0))=\omega({E'_1}^2-\theta_1^2)^{1/2}$$

We also have

$$\widehat{\Theta}_{\epsilon,\omega}(\theta_1)=\arg\big(\frac{\varepsilon}{\sqrt{2}}\big(i\theta_1+\omega({E'_1}^2-\theta_1^2)^{1/2}\big)\big)$$

We discuss according to the cases $\varepsilon=\omega=1$ and $\varepsilon=\omega=-1$. In the first case,

$$\widehat{\Theta}_{+,+}(\theta_+(0))=-\frac{\pi}{2}+\arcsin\frac{({E'_1}^2-\theta_+(0)^2)^{1/2}}{E'_1}$$

and by a Taylor expansion using Appendix A

$$\big({E'_1}^2-\theta_+^2(0)\big)^{1/2}=(\partial_{\theta_1}h_+)(0,\theta_+(0))=2\theta_+^2(0)+\frac{25}{2}\theta_+^4(0)+\cdots\sim 2{E'_1}^2+\frac{9}{2}{E'_1}^4+\cdots$$

we get

$$\widehat{\Theta}_{+,+}(\theta_+(0))\sim-\frac{\pi}{2}+\arcsin\big(2\,E'_1+\frac{9}{2}\,{E'_1}^3\big)=-\frac{\pi}{2}+2E'_1+\frac{35}{6}{E'_1}^3+\cdots$$

and using also (65)

$$\Psi^{\nu,\rho}_{+,+}(x_E)\sim\frac{\pi}{4}\,{E'_1}^2-\frac{2}{3}\,{E'_1}^3-\cdots=\frac{\pi}{4}(E_1^2+\frac{3}{2}E_1^4+\cdots)-\frac{2}{3}\,{E_1}^3-\cdots$$

The same expression holds when $\varepsilon=\omega=-1$. In fact, for all ε,ω

$$\Psi^{\nu,\rho}_{\varepsilon,\omega}(x_E)\sim\frac{\pi\,\varepsilon\,\omega}{4}\,{E'_1}^2-\frac{2}{3}\,{E'_1}^3-\cdots=\frac{\pi\,\varepsilon\,\omega}{4}(E_1^2+\frac{3}{2}E_1^4+\cdots)-\frac{2}{3}\,{E_1}^3-\cdots\tag{128}$$

To compute $\Psi^{-\nu-1,\rho}_{\varepsilon,\omega}(x_E)$, we use (115), which gives, when $\varepsilon\,\omega=1$

$$\Psi^{-\nu-1,\rho}_{\varepsilon,\omega}(x_E)\sim\frac{\pi}{2}\,{E'_1}^2-\frac{2}{3}\,{E'_1}^3-\cdots=\frac{\pi}{2}(E_1^2+\frac{3}{2}E_1^4+\cdots)-\frac{2}{3}\,{E_1}^3-\cdots\tag{129}$$

since $\zeta_{\varepsilon,\omega}(\theta_\omega(0))=\frac{1}{2}$. Note that the leading term $\frac{\pi}{4}\,{E'_1}^2$ (resp. $\frac{\pi}{2}\,{E'_1}^2$) in $\Psi^{\nu,\rho}_{\varepsilon,\omega}(x_E)$ (resp. $\Psi^{-\nu-1,\rho}_{\varepsilon,\omega}(x_E)$) drops out after normalization in Sect.4. Integrating ξ^ρ_ω given by Lemma 5.3, we substitute in (123) and get :

Lemma 5.4. *We have*

$$\Psi^{\nu,\rho}_{\varepsilon,\omega}(x)=\Psi^{\nu,\rho}_{\varepsilon,\omega}(x_E)+E_1x_1^2g_1(x_1,E_1)-\frac{2}{3}\rho\sqrt{2E_1}(-x_1)^{-3/2}g_2(x_1,E_1)\tag{130}$$

(independent of ω) with

$$\begin{aligned}g_1(x_1,E_1)&=1+\frac{x_1}{3E_1}+\cdots\\ g_2(x_1,E_1)&=1+\frac{3}{5}x_1\big(\frac{1}{E_1}-4E_1+\cdots\big)+\cdots\end{aligned}\tag{131}$$

and similarly fo $\Psi^{-\nu-1,\rho}_{\varepsilon,\omega}(x)$.

Of course, $\Psi^{j,\rho}_{\varepsilon,\omega}(x)$ looks much the same as the phase of an Airy function.

5.3 Appendix: Formal proof of Lemmas 5.1-5.3

We content ourselves with proving the Lemma at the level of Taylor expansions.

5.3.1 Proof of Lemma 5.1

Recall (65) and the expansion of $\theta_\omega(\xi_1)$. In particular $-\omega\theta_\omega(0) = E_1' - 2{E_1'}^3 - 11{E_1'}^5 - \cdots$, and

$$(\partial_{\xi_1} h_\omega)\big(0, \theta_\omega(0)\big) = \frac{5}{4}\big(\theta_\omega(0)\big)^3 + \frac{515}{32}\big(\theta_\omega(0)\big)^5 \sim -\frac{5}{4}{E_1'}^3 - \frac{275}{32}{E_1'}^5$$

There follows, with $\theta_\omega(0) = -\omega\widehat{\theta}_\omega(0)$, $E_1 + \theta_+(0) + \partial_{\xi_1} h_+)\big(0, \theta_+(0)\big) \sim 0$ and $-E_1 + \theta_-(0) + \partial_{\xi_1} h_-)\big(0, \theta_-(0)\big) \sim 0$ up to this accuracy. This gives Lemma 5.1 ♣

5.3.2 Proof of Lemma 5.2

We prove the Lemma analogously: check first 112) using (111). Again we work at the level of Taylor expansions in E_1', specifically up to order 5. We have

$$(\frac{\partial^2 h_\omega}{\partial\theta_1\partial\xi_1})\big(0, \theta_\omega(0)\big) = \frac{15}{4}\,\big(\theta_\omega(0)\big)^2 + \frac{2575}{32}\,\big(\theta_\omega(0)\big)^4 + \cdots$$

and

$$(\frac{\partial^2 h_\omega}{\partial\xi_1^2})\big(0, \theta_\omega(0)\big) = \omega\,\big(2\,\theta_\omega(0) + \frac{23}{2}\,\big(\theta_\omega(0)\big)^3 + \frac{3439}{16}\,\big(\theta_\omega(0)\big)^5 + \cdots\big)$$

For $\omega = 1$, choosing the negative root $\theta_+(0)$ we get

$$\begin{aligned}
&(\theta_+(0))^3 = -{E_1'}^3 + 6\,{E'}_1^{\,5} + \cdots\\
&(\frac{\partial\theta_+}{\partial\xi_1})(0) = 2\,E_1' - 7\,{E_1'}^5 - \cdots\\
&(\frac{\partial^2 h_+}{\partial\theta_+\partial\xi_1})\big(0, \theta_1(0)\big) = \frac{15}{4}\,{E'}_1^2 + \frac{2095}{32}\,{E'}_1^4 + \cdots\\
&(\frac{\partial^2 h_+}{\partial\xi_1^2})\big(0, \theta_+(0)\big) = -2\,E_1' - \frac{15}{2}\,{E'}_1^3 - \frac{1983}{16}\,{E'}_1^5 - \cdots
\end{aligned}$$

so

$$(\frac{\partial G_+}{\partial\xi_1})(0, 0; E_1') = 0$$

Another computation shows that

$$(\frac{\partial^2 G_+}{\partial\xi_1^2})(0, 0; E_1') = \frac{1}{{E'}_1} + \frac{3}{4}\,{E'}_1 + \cdots \neq 0$$

For $\omega = -1$, we choose instead the positive root $\theta_-(0) > 0$, and show

$$(\frac{\partial G_-}{\partial\xi_1})(0, 0; E_1') = 0, \quad (\frac{\partial^2 G_-}{\partial\xi_1^2})(0, 0; E_1') \neq 0$$

which takes the proof of Lemma 5.2 to an end. ♣

5.3.3 Proof of Lemma 5.3

Here we compute the critical point $\xi_\omega^\rho(x_1, E_1')$ for $x_1 < 0$. We have

$$\begin{aligned} G_\omega(x_1,\xi_1;E_1') &= \omega\,(E_1+x_1) + \theta_\omega(\xi_1) + 2\,\omega\,\xi_1\,\theta_\omega(\xi_1) + \frac{9}{4}\,\xi_1^2\,\theta_\omega(\xi_1) + \frac{5}{4}\,\big(\theta_\omega(\xi_1)\big)^3 \\ &+ \omega\left(9\,\xi_1^3\,\theta_\omega(\xi_1) + \frac{23}{2}\,\xi_1\big(\theta_\omega(\xi_1)\big)^3\right) + \frac{515}{32}\,\big(\theta_\omega(\xi_1)\big)^5 + \frac{1073}{16}\,\xi_1^2\,\big(\theta_\omega(\xi_1)\big)^3 \\ &+ \frac{1015}{32}\,\xi_1^4\,\theta_\omega(\xi_1) + \omega\left(\frac{3439}{16}\,\xi_1\,\big(\theta_\omega(\xi_1)\big)^5 + \frac{765}{2}\,\xi_1^3\,\big(\theta_\omega(\xi_\omega)\big)^3 + \frac{2037}{16}\,\xi_1^5\,\theta_\omega(\xi_1)\right) + \cdots \end{aligned}$$

For $\omega = 1$, we choose the branch $\theta_+(0) < 0$, and we get

$$G_+(x_1,\xi_1;E_1') = x_1 + \big(\frac{1}{2\,E'_1} + \frac{3}{8}\,E'_1 + \frac{95}{64}\,{E'_1}^3 + \cdots\big)\,\xi_1^2 + \big(\frac{1}{E_1'} + \frac{3}{4}\,E_1' + \frac{95}{32}\,{E'_1}^3 + \cdots\big)\,\xi_1^3 + \cdots \tag{132}$$

Using Lemma 5.1, in particular $a_\omega(0;E_1') = 0$, we expand

$$\begin{aligned} a_+(x_1;E_1') &= a_+(0;E_1') + (\frac{\partial a_+}{\partial x_1})(0;E_1')\,x_1 + \cdots = (\frac{\partial a_+}{\partial x_1})(0;E_1')x_1 + \cdots \\ b_+(x_1;E_1') &= b_+(0;E_1') + (\frac{\partial b_+}{\partial x_1})(0;E_1')\,x_1 + \cdots = (\frac{\partial b_+}{\partial x_1})(0;E_1')\,x_1 + \cdots \\ A_+(x_1,\xi_1;E_1') &= A_+(0,0;E_1') + (\frac{\partial A_+}{\partial x_1})(0,0;E_1')\,x_1 + (\frac{\partial A_+}{\partial \xi_1})(0,0;E_1')\,\xi_1 + \cdots \end{aligned}$$

By identification

$$\begin{aligned} A_+(0,0;E_1') &= \frac{1}{2E_1'} + \frac{3}{8}E_1' + \frac{95}{64}\,{E'_1}^3 + \cdots \\ (\frac{\partial A_+}{\partial \xi_1})(0,0;E_1') &= \frac{1}{E_1'} + \frac{3}{4}\,E'_1 + \frac{95}{32}\,{E'_1}^3 + \cdots \\ (\frac{\partial A_+}{\partial\, x_1})(0,0;E_1') &= 0 \\ (\frac{\partial a_+}{\partial x_1})(0;E_1') &= -4\,E'_1 + 3\,{E'_1}^3 + \frac{77}{8}\,{E'_1}^5 + \cdots \\ (\frac{\partial b_+}{\partial x_1})(0;E_1') &= 2\,E'_1 - \frac{3}{2}\,{E'_1}^3 - \frac{77}{16}\,{E'_1}^5 - \cdots \end{aligned}$$

It follows

$$\begin{aligned} G_+(x_1,\xi_1;E_1') &= A_+(x_1,\xi_1;E_1')\big(\xi_1^2 + \big(-4\,E_1' + 3\,{E'_1}^3 + \frac{77}{8}\,{E'_1}^5 + \cdots\big)\,x_1\,\xi_1 + \\ &\big(2\,E_1' - \frac{3}{2}\,{E'_1}^3 - \frac{77}{16}\,{E'_1}^5 - \cdots\big)\,x_1\big) \end{aligned} \tag{133}$$

and the expression for $\xi_+^{\rho,\omega}$ follows easily, by solving the quadratic equation.

Now for $\omega = -1$, we choose $\theta_-(\xi_1) > 0$, and get

$$G_-(x_1,\xi_1;E_1') = -x_1 - \big(\frac{1}{2E'_1} + \frac{3}{8}\,E'_1 + \frac{95}{64}\,{E'_1}^3 + \cdots\big)\,\xi_1^2 + \big(\frac{1}{E_1'} + \frac{3}{4}\,E_1' + \frac{95}{32}\,{E'_1}^3 + \cdots\big)\,\xi_1^3 + \cdots \tag{134}$$

We notice that

$$G_+(x_1,-\xi_1;E_1') = -G_-(x_1,\xi_1;E_1') \tag{135}$$

and get similarly the expression for ξ_-^ρ. This proves Lemma 5.3. ♣

6 Spinors in the spatial representation outside the branching points

We diagonalize $\mathcal{P} - E$ outside the branching points and construct WKB solutions extending microlocal solutions of Sect.4 as WKB solutions along simple bicharacteristics. The situation is close to the "pseudo-scalar" case set up in [Ro], because there is no longer "strong mixing" between electronic and hole wave-functions. See also [Iv] for general systems. Recall the Lagrangian manifold $\Lambda_E^>$ consists in 2 branches $\Lambda_E^\rho = \Lambda_E^{>,\rho}$, $\rho = \pm 1$ so that $\rho = +1$ belongs to the electronic state ($\xi_1 > 0$ in the local coordinates near a above), resp. $\rho = -1$ to the hole state ($\xi_1 < 0$). These states mix up when $\Delta(x) \neq 0$, but we can still sort them out semiclassically, outside a, a'. Call the vector space of $\mathbf{C}^2$ generated by $\binom{1}{0}$ the space of (pure) *electronic states*, or *electronic spinors*, and this by $\binom{0}{1}$ the space of (pure) *hole states*, or *hole spinors*.

Recall from (51) that the positive eigenvalue of $\mathcal{P}(x,\xi)$ is $\lambda(x,\xi) = \sqrt{\Delta(x)^2 + (\xi^2 - \mu(x))^2}$, and the corresponding normalized eigenvector (for the pointwise hermitian norm in $\mathbf{C}^2$)

$$Y(x,\xi) = \big(\Delta^2 + (-\xi^2 + \mu + \lambda)^2\big)^{-1/2} \begin{pmatrix} \Delta e^{i\phi/2} \\ -\xi^2 + \mu + \lambda \end{pmatrix} \tag{136}$$

Let $\lambda_\rho(x,\xi) = \rho\lambda(x,\xi)$. On Λ_E^ρ we consider the line bundle with fiber

$$Y_\rho(x,\xi) = \big(\Delta^2 + (-\xi^2 + \mu + \lambda_\rho)^2\big)^{-1/2} \begin{pmatrix} \Delta e^{i\phi/2} \\ -\xi^2 + \mu + \lambda_\rho \end{pmatrix} \tag{137}$$

The vectors $Y_\rho(x,\xi)$ are orthogonal, and $Y_+(x,\xi) = Y(x,\xi)$. When $\Delta(x) = 0$, $\Lambda_E^+ = \{\xi^2 = \mu(x) + E\}$ carries a pure electronic state, spanned by $Y_+(x,\xi)$ while $\Lambda_E^- = \{\xi^2 = \mu(x) - E\}$ carries a pure hole state, spanned by $Y_-(x,\xi)$. Because of the normalization factor in (137), Y_ρ has a discontinuity on Λ_E^ρ when Δ vanishes, while $Y_{-\rho}$ is continuous. To analyse the discontinuity, we use the identity

$$-\xi^2 + \mu + \rho\sqrt{\Delta^2 + (\xi^2 - \mu)^2} = -\Delta^2\big(-\xi^2 + \mu - \rho\sqrt{\Delta^2 + (\xi^2 - \mu)^2}\big)^{-1}$$

to show

$$Y_\rho(x,\xi) = \Big(1 + \frac{\Delta^2}{\big(-\xi^2 + \mu - \rho\sqrt{\Delta^2 + (\xi^2 - \mu)^2}\big)^2}\Big)^{-1/2} \begin{pmatrix} e^{i\phi(x)/2} \\ \Delta\big(\xi^2 - \mu + \rho\sqrt{\Delta^2 + (\xi^2 - \mu)^2}\big)^{-1} \end{pmatrix} \tag{138}$$

Because of the jump of $x \mapsto \phi(x)$ at $x = 0$, $Y_\rho(x,\xi)$ is only piecewise smooth on Λ_E^ρ, and $Y_\rho(x,\xi) = \binom{e^{i\,\mathrm{sgn}(x)\phi/2}}{0}$ where $\Delta = 0$, while $Y_{-\rho}(x,\xi)$ is everywhere smooth on Λ_E^ρ, and $Y_{-\rho}(x,\xi) = \binom{0}{1}$ where $\Delta = 0$.

Thus it can be convenient to view Y_ρ as a smooth section over Λ_E (up to the branching points a, a') valued in the projective space $\mathrm{P}^1(\mathbf{C})$, where $\binom{e^{i\,\mathrm{sgn}(x)\phi/2}}{0}$ is identified with $\binom{1}{0}$. At a they consist of the most "hybrid" superposition of electronic and hole states $\frac{1}{\sqrt{2}}\binom{e^{i\phi/2}}{1}$, and similarly at a'. Moreover it satisfies the symmetry $\mathcal{I}^\vee Y_\rho = Y_\rho$.

Let $\xi^\rho_\omega(x,h')$ be the critical point computed in Lemma 5.3 (which depends on ω only by its sign), we identify the section $Y_\rho(x,\xi)$ over Λ_E with a function of x, as

$$Y_\rho(x,h') = Y_\rho(x,\xi)|_{\xi_1=\xi^\rho_\omega(x,h')}$$

Choose $(x_\rho,\xi_\rho)\in\Lambda^\rho_E\setminus\{a,a'\}$, and let us construct WKB solutions starting from (x_ρ,ξ_ρ). The vectors Y_ρ make a unitary matrix with (piecewise) smooth coefficients, $\mathcal{A}_\rho(x,\xi)$, defined along the piece ρ of Λ_E containing (x_ρ,ξ_ρ), and up to the branching points, such that

$$\mathcal{A}^*_\rho(\mathcal{P}-E)\mathcal{A}_\rho = \begin{pmatrix}\lambda_\rho - E & 0\\ 0 & \lambda_{-\rho}-E\end{pmatrix} \tag{139}$$

When looking at an electronic state, we choose $\rho=+1$ so that $\lambda_\rho(x_\rho,\xi_\rho)-E=0$, while $\lambda_{-\rho}(x_\rho,\xi_\rho)-E$ is elliptic. When looking at a hole state instead, we choose $\rho=-1$ so that $\lambda_{-\rho}(x_\rho,\xi_\rho)-E=0$, while $\lambda_\rho(x_\rho,\xi_\rho)-E$ is elliptic.

It is standard (see e.g. [HeSj], [Iv], [Ro,Sect.4]) that we can implement this diagonalization at the level of operators, in particular the first 2 terms of the Weyl symbol of $\lambda_\rho(x,hD_x,h)-E$ are computed in [HeSj2,Sect 6]. Of course, due to the discontinuity of $x\mapsto\phi(x)$ at $x=0$, the corresponding operators should be considered separately for $x>0$ and $x<0$, so we change the phase factor $e^{i\phi/2}$ on $x>0$ to $e^{-i\phi/2}$ on $x<0$, so that the spinors verify Definition 3.1. It is convenient to keep the (real valued) subprincipal symbol $\lambda^{(1)}_\rho(x,\xi)$ together with the principal symbol $\lambda_\rho(x,\xi)$, so let

$$\widehat{\lambda}_\rho(x,\xi,h) = \lambda_\rho(x,\xi) + h\lambda^{(1)}_\rho(x,\xi) \tag{140}$$

In the "quasi-particle local frame", we look for a WKB solution $Z^\rho(x,h)$ of

$$\mathcal{A}^*_\rho(\mathcal{P}-E)\mathcal{A}_\rho Z^\rho(x,h)=0$$

of the form

$$Z^+(x,h) = e^{iS_+(x,h)/h}\begin{pmatrix}w^+(x,h)\\0\end{pmatrix},\quad Z^-(x,h)=e^{iS_-(x,h)/h}\begin{pmatrix}0\\w^-(x,h)\end{pmatrix} \tag{141}$$

where S_ρ solves the modified eikonal equation $\widehat{\lambda}_\rho(x,\partial_xS_\rho,h)=E$, and the symbol $w^\rho = w^\rho_0 + hw^\rho_1+\cdots$ is uniquely determined up to a constant factor by solving transport equations along ρ. In particular $w^\rho_0(x,h)$ satisfies :

$$\partial_\xi\widehat{\lambda}_\rho(x,\partial_xS_\rho(x,h),h)\partial_xw^\rho_0 + \frac{1}{2}\frac{\partial^2}{\partial x\partial\xi}\widehat{\lambda}_\rho(x,\partial_xS_\rho,h)w^\rho_0=0$$

so w^ρ_0 can be chosen to be real. More invariantly,

$$\mathcal{L}_{H_{\lambda_\rho}|\Lambda_\rho(E)}(w^\rho_0(x,h)|dx|^{1/2})=0 \tag{142}$$

where $\mathcal{L}_{H_{\lambda_\rho}}$ denotes the Lie derivative acting on half-densities. Applying $\mathcal{A}_\rho$ we get

$$W^\rho(x,h)=\mathcal{A}_\rho Z^\rho(x,h) = e^{iS_\rho(x,h)/h}\big(w^\rho_0(x,h)Y_\rho(x,\partial_xS_\rho)+\mathcal{O}(h)\big) = e^{iS_\rho(x,h)/h}\widetilde{W}^\rho(x,h) \tag{143}$$

We summarize our constructions in the:

Proposition 6.1. *Let $K_h^\rho(E) = \mathrm{Ker}_h^\rho(\mathcal{P} - E)$ be the microlocal kernel of $\mathcal{P}$ along Λ_E^ρ, with $\mathrm{Ker}_h(\mathcal{P} - E) = K_h^+(E) \cup K_h^-(E)$. Then $K_h^\rho(E)$ is one-dimensional and consists of spinors $W^\rho(x,h)$ defined in (143).*

7 Connexion formulas and BS rules

First we look for connexion formulas between microlocal solutions near the branching points, and WKB solutions along the pieces of bicharacteristics.

The set $K_h(E)$ of asymptotic solutions to $(\mathcal{P} - E)U = 0$ as a vector bundle over $\mathbf{R}$ with compact base $[-x_E, x_E]$, projects microlocally on 2 sub-bundles over $\gamma^{>,\rho}(E)$, $\rho = \pm$ (joining smoothly at a_E, a'_E, where the electron and the hole state are indistinguishable), or equivalently of the 2 sub-bundles over $\gamma^{<,\rho}(E)$ obtained from $\gamma^{>,\rho}(E)$ by the complex involution $\mathcal{I}$. For short we identity $\gamma^{>,\rho}(E)$ with ρ.

We shall determine 2 sequences of eigenvalues $E = E_n^\rho(h)$, $\rho = \pm$, in such a way that $K_h(E)$ consists of 2 orthogonal line bundles (modulo small corrections), carrying the electron state for $\rho = 1$, resp. the hole state for $\rho = -1$.

7.1 Relative monodromy matrices

The key point in our argument is that along simple characteristics, $K_h^\rho(E)$ is one dimensional; in particular the phase function $S_\rho(x;h)$ (including the h-correction) and the leading order, real valued amplitude (as $|a_\omega^\nu|$ in Proposition 3.1) are uniquely defined (see [HeSj,App.a.3] for a general discussion).

Recall from (53) and (54) BdG Hamiltonians near a, a' respectively. Because of Proposition 4.2 for each $\omega, \varepsilon, \rho = \pm 1$, the normalized microlocal solutions $U_{\varepsilon,\omega}^{j,\rho,a'}$ are related to the extension $U_{\varepsilon,\omega,\mathrm{ext}}^{k,\rho,a'}$ of the normalized microlocal solutions $U_{\varepsilon,\omega}^{k,\rho,a}$ along ρ by the relative monodromy matrix over $[-x_E, x_E]$

$$\mathcal{M}_\rho^{a,a'} = \begin{pmatrix} d_{11}^\rho & d_{12}^\rho \\ d_{21}^\rho & d_{22}^\rho \end{pmatrix} \in U(1,1) \tag{144}$$

where we claim that $d_{ij}^\rho \in \mathbf{C}$ does not depend on (ε, ω). We also claim (see Proposition 4.1) that $\mathcal{M}_\rho^{a',a}(E) \in U(1,1)$ obtained by extending from the left to the right, and applying PT symmetry, takes the form

$$\mathcal{M}_\rho^{a',a}(E) = \mathcal{I}(\mathcal{M}_\rho^{a,a'}(E))^{-1}\mathcal{I} = \overline{\left(\mathcal{M}_\rho^{a,a'}(E)\right)^{-1}} \tag{145}$$

We are to compute the coefficients d_{ij}^ρ at first order in h by connecting the microlocal solutions from a to a'. Considerations made in Sect.5.2.1 concerning the behavior of the complex germs of microlocal solutions $U_{\varepsilon,\omega}^j$ beyond the branching points, fix the choice of the d_{ij}^ρ as follows.

Extend a microlocal solution near a of the form $\alpha_1 U^{\nu,a}_{\varepsilon,\omega} + \alpha_2 U^{-\nu-1,a}_{\varepsilon,\omega}$ towards a'. We have

$$\big(\alpha_1 U^{\nu,a}_{\varepsilon,\omega} + \alpha_2 U^{-\nu-1,a}_{\varepsilon,\omega}\big)_{\text{ext}} = \beta_1 U^{\nu,a'}_{\varepsilon,\omega} + \beta_2 U^{-\nu-1,a'}_{\varepsilon,\omega} \tag{146}$$

By definition of $\mathcal{M}^{a,a'}(E)$

$$\begin{aligned} \big(U^{\nu,a}_{\varepsilon,\omega}\big)_{\text{ext}} &= d_{11} U^{\nu,a'}_{\varepsilon,\omega} + d_{21} U^{-\nu-1,a'}_{\varepsilon,\omega} \\ \big(U^{-\nu-1,a}_{\varepsilon,\omega}\big)_{\text{ext}} &= d_{12} U^{\nu,a'}_{\varepsilon,\omega} + d_{22} U^{-\nu-1,a'}_{\varepsilon,\omega} \end{aligned} \tag{147}$$

Projecting microlocally onto ρ, (147) takes the form

$$\begin{aligned} \big(U^{\nu,a}_{\varepsilon,\omega}\big)^{\rho}_{\text{ext}} &= d^{\rho}_{11} U^{\nu,a'}_{\varepsilon,\omega} + d^{\rho}_{21} U^{-\nu-1,a'}_{\varepsilon,\omega} \\ \big(U^{-\nu-1,a}_{\varepsilon,\omega}\big)^{\rho}_{\text{ext}} &= d^{\rho}_{12} U^{\nu,a'}_{\varepsilon,\omega} + d^{\rho}_{22} U^{-\nu-1,a'}_{\varepsilon,\omega} \end{aligned} \tag{148}$$

Similarly

$$\begin{aligned} \big(U^{\nu,a'}_{\varepsilon,\omega}\big)^{\rho}_{\text{ext}} &= e^{\rho}_{11} U^{\nu,a}_{\varepsilon,\omega} + e^{\rho}_{21} U^{-\nu-1,a}_{\varepsilon,\omega} \\ \big(U^{-\nu-1,a'}_{\varepsilon,\omega}\big)^{\rho}_{\text{ext}} &= e^{\rho}_{12} U^{\nu,a'}_{\varepsilon,\omega} + e^{\rho}_{22} U^{-\nu-1,a'}_{\varepsilon,\omega} \end{aligned} \tag{149}$$

For each scattering process, the dominant branch (i.e. exponentially growing outside $[-x_E, x_E]$) of the microlocal solution at either branching point a, a' becomes recessive (i.e. exponentially decaying outside $[-x_E, x_E]$) at the other point. There are actually 2 possible cases. First:

$$d^{\rho}_{11} = d^{\rho}_{22} = 0 \tag{150}$$

The relation $\mathcal{M}^{a,a'}_{\rho}(E) \in U(1,1)$ expresses as $d^{\rho}_{12}\overline{d^{\rho}_{21}} = 1$ and by (145), we have $\mathcal{M}^{a',a}_{\rho}(E) = \mathcal{M}^{a,a'}_{\rho}(E)$. Second:

$$d^{\rho}_{12} = d^{\rho}_{21} = 0 \tag{151}$$

and $d^{\rho}_{11}\overline{d_{22}}^{\rho} = 1$, so $\mathcal{M}^{a',a}_{\rho}(E) = \sigma_x \mathcal{M}^{a,a'}_{\rho} \sigma_x$, with $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

By (149) we have $\big(U^{\nu,a,\rho}_{\varepsilon,\omega}\big)_{\text{ext}} = d^{\rho}_{21} U^{-\nu-1,a',\rho}_{\varepsilon,\omega}$ and $\big(U^{-\nu-1,a,\rho}_{\varepsilon,\omega}\big)_{\text{ext}} = d^{\rho}_{12} U^{\nu,a',\rho}_{\varepsilon,\omega}$ in case (150), resp. $\big(U^{\nu,a,\rho}_{\varepsilon,\omega}\big)_{\text{ext}} = d^{\rho}_{11} U^{-\nu-1,a',\rho}_{\varepsilon,\omega}$ and $\big(U^{-\nu-1,a,\rho}_{\varepsilon,\omega}\big)_{\text{ext}} = d^{\rho}_{22} U^{-\nu-1,a',\rho}_{\varepsilon,\omega}$ in case (151).

We show in Remark 7.1 below that Gram matrix is meaningless in case (151), so shall only consider $\mathcal{M}^{a,a'}_{\rho}(E)$ in case (150).

The extension determines the normalization in the sense of Sect.4 along the bicharacteristics $\rho = \pm$. Namely, choose a pair of normalized solutions (U_1, U_3) with $U_1 = U^{\nu,a}_{\varepsilon,\omega}$ and $U_3 = U^{-\nu-1,a}_{\varepsilon,\omega}$ near a, so that $\big(\frac{i}{h}[\mathcal{P}, \chi^a]_{\rho} U_1 | U_3\big) = 1$ (still at leading order in h) and deform continuously χ^a into $\chi \in C^{\infty}_0(\mathbf{R}^2)$ so that χ equals 1 near $\Lambda^{>}_E \setminus \{a'_E\}$. By the discussion after Definition 4.1, the normalization remain unchanged along ρ, namely

$$\rho\big(\frac{i}{h}[\mathcal{P}, \chi^a]_{\rho} U_1 | U_3\big) = \rho\big(\frac{i}{h}[\mathcal{P}, \chi]_{\rho} d^{\rho}_{21} U^{-\nu-1,a'}_{\varepsilon,\omega} | d^{\rho}_{12} U^{\nu,a'}_{\varepsilon,\omega}\big) = 1 \tag{152}$$

since $d^{\rho}_{21}\overline{d^{\rho}_{12}} = 1$ and $\rho\big(\frac{i}{h}[\mathcal{P}, \chi^{a'}]_{\rho} U^{-\nu-1,a'}_{\varepsilon,\omega} | U^{\nu,a'}_{\varepsilon,\omega}\big) = 1$. So by (145) we have

$$\mathcal{M}^{a,a'}_{\rho} = \begin{pmatrix} 0 & d^{\rho}_{12} \\ d^{\rho}_{21} & 0 \end{pmatrix}, \quad \mathcal{M}^{a',a}_{\rho} = \begin{pmatrix} 0 & (\overline{d^{\rho}_{21}})^{-1} \\ d^{\rho}_{21} & 0 \end{pmatrix} = \begin{pmatrix} e^{\rho}_{11} & e^{\rho}_{12} \\ e^{\rho}_{21} & e^{\rho}_{22} \end{pmatrix} \tag{153}$$

Let also U_2 be a normalized solution, such that $U_2^{a'} = U_{\varepsilon,\omega}^{-\nu-1,a'}$ near a', and

$$U_2^a = \big(U_{\varepsilon,\omega}^{-\nu-1,a'}\big)_{\text{ext}} = e_{12}^\rho U_{\varepsilon,\omega}^{\nu,a} = d_{12}^\rho U_{\varepsilon,\omega}^{\nu,a} \tag{154}$$

near a. We have

$$\rho\big(\frac{i}{h}[\mathcal{P},\chi^a]_\rho U_1|U_2\big) = \rho\big(\frac{i}{h}[\mathcal{P},\chi]_\rho d_{21}^\rho U_{\varepsilon,\omega}^{-\nu-1,a'}|U_{\varepsilon,\omega}^{-\nu-1,a'}\big) = 0 \tag{155}$$

Similarly, let U_4 be a normalized solution, such that $U_4^{a'} = U_{\varepsilon,\omega}^{\nu,a'}$ near a', and

$$U_4^a = \big(U_{\varepsilon,\omega}^{\nu,a'}\big)_{\text{ext}} = e_{21}^\rho U_{\varepsilon,\omega}^{-\nu-1,a} = e_{21}^\rho U_{\varepsilon,\omega}^{-\nu-1,a} \tag{156}$$

near a. We have

$$\rho\big(\frac{i}{h}[\mathcal{P},\chi^a]_\rho U_3|U_4\big) = \rho\big(\frac{i}{h}[\mathcal{P},\chi]_\rho U_{\varepsilon,\omega}^{-\nu-1,a}|e_{21}^\rho U_{\varepsilon,\omega}^{-\nu-1,a}\big) = 0 \tag{157}$$

Moreover, formula (152) with U_1, U_3 carries to U_2, U_4.

So we only have to determine d_{21}^ρ. Note that $\rho\big(\frac{i}{h}[\mathcal{P},\chi^a]_\rho U_1|U_3\big) = 0 \bmod \mathcal{O}(h)$.

7.2 Computing d_{21}^ρ

Arguing essentially as in [IfaLouRo], we identify the WKB solutions of Sect.6 with the microlocal solutions along ρ. Recall from Proposition 6.1 that the WKB solutions are of the form

$$W^\rho(x,h) = \widetilde{C}_\rho e^{i\phi/2} e^{iS_\rho(x,h)/h}\big(w_0^\rho(x,h)Y_\rho(x,\partial_x S_\rho)+\mathcal{O}(h)\big) = \widetilde{C}_\rho e^{i\phi/2} e^{iS_\rho(x,h)/h}\widetilde{W}^\rho(x,h) \tag{158}$$

where the factor $e^{i\phi/2}$ arises as an initial data for the transport equation in the region where $\Delta(x) = 0$, when extending U_1 from a_E to a'_E. By (151) the normalization of U_1 near a determines uniquely $\widetilde{C}_\rho > 0$. Near a', we have

$$U_1 = W^\rho(x,h) = d_{21}^\rho U_{\varepsilon,\omega}^{-\nu-1,a',\rho}(x,h) \tag{159}$$

Computing in two different ways $\big(\frac{i}{h}[\mathcal{P}^{a'},\chi^{a'}]_\rho U_1|U_{\varepsilon,\omega}^{\nu,a'}\big)$ gives d_{21}^ρ. Namely on the one hand

$$\begin{aligned}
&\big(\frac{i}{h}[\mathcal{P}^{a'},\chi^{a'}]_\rho U_1|U_{\varepsilon,\omega}^{\nu,a'}\big) = d_{21}^\rho\big(\frac{i}{h}[\mathcal{P}^{a'},\chi^{a'}]_\rho U_{\varepsilon,\omega}^{-\nu-1,a'}|U_{\varepsilon,\omega}^{\nu,a'}\big) = d_{21}^\rho \mathcal{W}_\rho^{a'}\big(U_{\varepsilon,\omega}^{-\nu-1},U_{\varepsilon,\omega}^{\nu}\big) = \\
&= d_{21}^\rho \mathcal{W}_\rho^{a'}\big(\widehat{U}_{\varepsilon,\omega}^{-\nu-1},\widehat{U}_{\varepsilon,\omega}^{\nu}\big) \quad \text{(Plancherel)} \\
&= -d_{21}^\rho \overline{\mathcal{W}_\rho^{a}\big(\widehat{U}_{-\varepsilon,-\omega}^{-\nu-1},\widehat{U}_{-\varepsilon,-\omega}^{\nu}\big)} = -d_{21}^\rho \mathcal{W}_\rho^{a}\big(\widehat{U}_{-\varepsilon,-\omega}^{\nu},\widehat{U}_{-\varepsilon,-\omega}^{-\nu-1}\big) \quad \text{(by (104))} \\
&= -\rho d_{21}^\rho \quad \text{(by (99))}
\end{aligned} \tag{160}$$

Here we used microlocal solutions normalized as in Sect.4. On the other hand, by (159)

$$\big(\frac{i}{h}[\mathcal{P}^{a'},\chi^{a'}]_\rho U_1|U_{\varepsilon,\omega}^{\nu,a'}\big) = \big(\frac{i}{h}[\mathcal{P}^{a'},\chi^{a'}]_\rho W^\rho(x,h)|U_{\varepsilon,\omega}^{\nu,a'}\big)$$

Identifying with (160) gives

$$-\rho d_{21}^\rho = \big(\frac{i}{h}[\mathcal{P}^{a'},\chi^{a'}]_\rho W^\rho(x,h)|U_{\varepsilon,\omega}^{\nu,a'}\big) \tag{161}$$

Proposition 7.1. *We have*

$$\big(\frac{i}{h}[\mathcal{P}^{a'},\chi^{a'}]_\rho W^\rho|U^{\nu,a',\rho}_{\varepsilon,\omega}\big) = 2\,\widetilde{C}_\rho\, e^{i\pi/4}e^{i\phi/2}\int \exp\big[i\widetilde{S}_\rho(x;h)/h\big]\beta^\nu_\omega(x,h)\,(\chi_1^{a'})'(x)\,dx + \mathcal{O}(h) \tag{162}$$

the phase $\widetilde{S}_\rho(x;h)$ and amplitude $\beta^\nu_\omega(x,h)$ being defined in (169) below. Actually $\widetilde{S}_\rho(x;h)$ is independent of x, and takes the form

$$\widetilde{S}_\rho(x,h) = \tau_\rho(h) = -\int_{-x_E}^{x_E}\xi^\rho(y;h)\,dy + b(E_1';h) \tag{163}$$

The boundary term

$$b(E_1';h) = \frac{(2\,\xi_E)^3}{\alpha}\,\Psi^{\nu,a,\rho}_{-\varepsilon,-\omega}(x_E) + h\,R^\nu_{-\omega}\big(\theta_{-\omega}(0)\big) \tag{164}$$

has the following expansion (independent of ρ and with only odd powers of E_1')

$$b(E_1';h) = -\frac{(2\,\xi_E)^3}{\alpha}\,\big(\frac{2}{3}\,{E_1'}^3 + \frac{7}{6}\,{E_1'}^5 + \cdots\big) + h\,\big(E_1' + \frac{35}{12}\,{E_1'}^3 + \cdots\big) \tag{165}$$

Moreover the amplitude $\beta^\nu_\omega(x,h)$, real mod $\mathcal{O}(h)$, is again independent of x.

Proof. Recall Y_ρ from (138), of the form

$$Y_\rho(x,\xi) = A_\rho(x,\xi)\begin{pmatrix} e^{i\phi/2} \\ B_\rho(x,\xi)\end{pmatrix}$$

In the spatial representation (recall $\mu > E$ is a constant near a')

$$\mathcal{P}^{a'}(x,hD_x) = \begin{pmatrix} (hD_x)^2-\mu & \Delta(-x)\,e^{-i\phi/2} \\ \Delta(-x)\,e^{i\phi/2} & -(hD_x)^2+\mu\end{pmatrix}$$

We choose $\chi^{a'}(x,\xi) = \chi_1(x)\chi_2(\xi)$ where, contrary to the choice made in Fourier representation, $\mathrm{supp}(\chi_1)$ is so small that $\chi_2(\xi)\equiv 1$ on $\omega^{a'}_\rho$ (recall that normalization does not depend on $\chi^{a'}$). Computing the commutator, (158) shows

$$\frac{i}{h}\,[\mathcal{P}^{a'},\chi^{a'}]_\rho W^\rho = 2\widetilde{C}_\rho\, e^{i\phi/2}e^{iS_\rho/h}\,\partial_x S_\rho(x;h)\begin{pmatrix} e^{i\phi/2}z_\rho(x;h) \\ -t_\rho(x;h)\end{pmatrix}\chi_1'(x) + \mathcal{O}(h')$$

with

$$\begin{aligned} z_\rho(x;h) &= w_0^\rho(x;h)\,A_\rho\big(x,\partial_x S_\rho(x;h)\big) \\ t_\rho(x;h) &= w_0^\rho(x;h)\,A_\rho\big(x,\partial_x S_\rho(x;h)\big)\,B_\rho\big(x,\partial_x S_\rho(x;h)\big)\end{aligned}$$

With the notation of (119) we recall

$$\begin{aligned} &U^{\nu,a',\rho}_{\varepsilon,\omega}(x;h) = \widetilde{U}^{\nu,a',\rho}_{\varepsilon,\omega}(x)\,\exp\big[\frac{i}{h}\,\big(x\,\xi_E + \frac{(2\,\xi_E)^3}{\alpha}\Psi^{\nu,a',\rho}_{\varepsilon,\omega}(x)\big)\big] = \\ &\overline{U^{\nu,a,\rho}_{-\varepsilon,-\omega}(-x;h)} = \overline{\widetilde{U}^{\nu,a,\rho}_{-\varepsilon,-\omega}(-x)}\,\exp\big[\frac{i}{h}\,\big(x\,\xi_E - \frac{(2\,\xi_E)^3}{\alpha}\overline{\Psi^{\nu,a,\rho}_{-\varepsilon,-\omega}(-x)}\big)\big]\end{aligned}$$

Thus the critical values of the phase in the spatial representation near a' are given by

$$\Psi^{\nu,a',\rho}_{\varepsilon,\omega}(x) = -\overline{\Psi^{\nu,a,\rho}_{-\varepsilon,-\omega}(-x)} = -\Psi^{\nu,a,\rho}_{-\varepsilon,-\omega}(-x) \tag{166}$$

and

$$\widetilde{U}^{\nu,\rho,a'}_{\varepsilon,\omega}(x) = \overline{\widetilde{U}^{\nu,\rho,a}_{-\varepsilon,-\omega}(-x)}$$

where, with $\xi = \xi^{\rho,a}(x)$ as in (59) , $\xi_1 = \xi^{\rho,a}_{\omega}(x)$

$$\widetilde{U}^{\nu,a,\rho}_{\varepsilon,\omega}(x) = a^{\nu}_{\omega}\big(\xi_1,\theta_\omega(\xi_1)\big)\,\big(\frac{L^{\rho}_{\omega}(x)}{i}\big)^{-1/2}\begin{pmatrix} e^{i\phi/2}\,\big(\xi^2-\mu-E\big)^{-1/2}\,X^{\nu}_{\omega}\big(\xi_1,\theta_\omega(\xi_1)\big) \\ \big(\xi^2-\mu-E\big)^{1/2}\end{pmatrix} \tag{167}$$

Let $\widetilde{\Psi}^{\nu,\rho,a'}_{\varepsilon,\omega}(x) = x\,\xi_E + \frac{(2\,\xi_E)^3}{\alpha}\,\Psi^{\nu,a',\rho}_{\varepsilon,\omega}(x)$. Recall a^{ν}_{ω} from (74), with $\xi_1 = \xi^{\rho}_{\omega}(x)$. Modulo $\mathcal{O}(h)$, we have

$$\begin{aligned}\big(\frac{i}{h}[\mathcal{P}^{a'},\chi^{a'}]_\rho W^\rho|U^{\nu,\rho,a'}_{\varepsilon,\omega}\big) &= 2\widetilde{C}_\rho\,e^{i\pi/4}\,e^{i\phi/2}\int e^{i\big(S_\rho(x;h)-\widetilde{\Psi}^{\nu,a',\rho}_{\varepsilon,\omega}(x)\big)/h}\\ &e^{i\phi/2}\big[z_\rho(x;h)(\xi^2-\mu-E)^{-1/2}X^{\nu}_{-\omega}-t_\rho(x;h)(\xi^2-\mu-E)^{1/2}\big]\\ &\partial_x S_\rho(x;h)\,a^{\nu}_{-\omega}\,\big(L^{(\rho)}_{-\omega}(-x)\big)^{-1/2}\,(\chi^{a'}_1)'(x)\,dx\end{aligned} \tag{168}$$

the factor $e^{i\pi/4}$ arising from (167). Let

$$\begin{aligned}\widetilde{S}^{\nu,\rho,a'}_{\varepsilon,\omega}(x;h) &= S_\rho(x;h) - \Big(\widetilde{\Psi}^{\nu,\rho,a'}_{\varepsilon,\omega}(x) - h\,R^{\nu}_{-\omega}\big(\theta_{-\omega}(\xi^{\rho}_{-\omega}(-x))\big)\Big)\\ \beta^{\nu}_{\omega}(x;h) &= \big(z_\rho(x;h)(\xi^2-\mu-E)^{-1/2}\,X^{\nu}_{-\omega}-t_\rho(x;h)\,(\xi^2-\mu-E)^{1/2}\big)\partial_x S_\rho\,|a^{\nu}_{-\omega}|\big(L^{(\rho)}_{-\omega}(-x)\big)^{-1/2}\end{aligned} \tag{169}$$

Then (168) takes the form

$$\big(\frac{i}{h}[\mathcal{P}^{a'},\chi^{a'}]_\rho W^\rho|U^{\nu,\rho,a'}_{\varepsilon,\omega}\big) = 2\widetilde{C}_\rho e^{i\pi/4}\,e^{i\phi/2}\int e^{i\widetilde{S}^{\nu,\rho,a'}_{\varepsilon,\omega}(x;h)/h}\,\beta^{\nu}_{\omega}(x;h)\,(\chi^{a'}_1)'(x)\,dx \tag{170}$$

We are left to show that $\widetilde{S}^{\nu,\rho,a'}_{\varepsilon,\omega}(x;h)$ and $\beta_\omega(x;h)$ are constant. Recall the h-dependent action $\xi^\rho(x;h) = \partial_x S_\rho(x;h)$ is well-defined all along ρ, in particular

$$\xi^\rho(x;h) = \xi_\rho(x) - h\frac{\partial}{\partial x}\Big(R^{\nu}_{-\omega}\big(\theta_{-\omega}(\xi^{\rho}_{-\omega}(-x))\big)\Big) \tag{171}$$

We recall from (59), (63) and (113), that $\xi^\rho_0(x) = \xi_E + 2\omega\xi_E\xi^\rho_\omega(x)$ satisfies $(x,\xi^\rho_0(x)) \in \gamma^\rho(E)$. Integrating (171) along $[-x_E, x_E]$, we find that

$$\tau_\rho(h) = \widetilde{S}^{\nu,\rho,a'}_{\omega}(x;h) = -\int_{-x_E}^{x_E}\xi^\rho(y;h)\,dy + \frac{(2\,\xi_E)^3}{\alpha}\,\Psi^{\nu,a,\rho}_{-\varepsilon,-\omega}(x_E) + h\,R^{\nu}_{-\omega}\big(\theta_{-\omega}(0)\big) \tag{172}$$

is a constant. The boundary term $b(E'_1;h) = \frac{(2\,\xi_E)^3}{\alpha}\,\Psi^{\nu,a,\rho}_{-\varepsilon,-\omega}(x_E) + h\,R^{\nu}_{-\omega}\big(\theta_{-\omega}(0)\big)$ can be computed with asymptotics (165). On the other hand, by (152) and Lemma 4.1, it turns out that $\beta^{\nu}_{\omega}(x;h)$ is indeed a real constant that we denote by $\widetilde{D}_\rho$. This takes the proof to an end. □

Thus Proposition 7.1 shows that

$$\big(\frac{i}{h}[\mathcal{P}^{a'},\chi^{a'}]_\rho W^\rho|U^{\nu,a',\rho}_{\varepsilon,\omega}\big)=2\widetilde{C}_\rho\widetilde{D}_\rho e^{i\tau_\rho(h)/h}e^{i\pi/4}\,e^{i\phi/2}\int(\chi_1^{a'})'(x)\,dx=-2\widetilde{C}_\rho\widetilde{D}_\rho e^{i\tau_\rho(h)/h}e^{i\pi/4}\,e^{i\phi/2} \tag{173}$$

Comparison with (162) eventually gives, still at leading order in h

$$-\rho d^\rho_{21}=(2\widetilde{C}_\rho\widetilde{D}_\rho)^{-1}e^{i\tau_\rho(h)/h}e^{i\phi/2}e^{i\pi/4} \tag{174}$$

7.3 Gram matrix and Bohr-Sommerfeld quantization rules

Recall from (101) for $j=\nu,-\nu-1$

$$F^{j,a}_{\varepsilon,\omega}=\frac{i}{h}[\mathcal{P}^a,\chi^a]\widehat{U}^{j,a}_{\varepsilon,\omega}$$

in spatial representation, and similarly with a' instead of a. It is convenient to view $F^{-\nu-1,\rho,a}_{\varepsilon,\omega}$ and $F^{\nu,\rho,a'}_{\varepsilon,\omega}$ as belonging to the microlocal co-kernel of $\mathcal{P}-E$ in the sense that they are not annihilated by $\mathcal{P}-E$. Everything will be computed mod $\mathcal{O}(h)$.

With U_1,U_2 we form Gram matrix

$$\mathcal{G}^{(a,a')}_\rho(E)=\begin{pmatrix}\big(U_1|F^{-\nu-1,\rho,a}_{\varepsilon,\omega}\big) & \big(U_2|F^{-\nu-1,\rho,a}_{\varepsilon,\omega}\big)\\ \big(U_1|F^{\nu,\rho,a'}_{\varepsilon,\omega}\big) & \big(U_2|F^{\nu,\rho,a'}_{\varepsilon,\omega}\big)\end{pmatrix} \tag{175}$$

We have by (99), (103) and Plancherel formula

$$\big(U_1|F^{-\nu-1,\rho,a}\big)=\big(U^{\nu,a}_{\varepsilon,\omega}|F^{-\nu-1,\rho,a}_{\varepsilon,\omega}\big)=\mathcal{W}^a_\rho\big(U^{\nu,a}_{\varepsilon,\omega},U^{-\nu-1,a}_{\varepsilon,\omega}\big)=\mathcal{W}^a_\rho\big(\widehat{U}^{\nu,a}_{\varepsilon,\omega},\widehat{U}^{-\nu-1,a}_{\varepsilon,\omega}\big)=1$$

and using also (154) defining e^ρ_{12}

$$\big(U_2|F^{-\nu-1,\rho,a}_{\varepsilon,\omega}\big)=e_{12}\,\big(U^{\nu,a}_{\varepsilon,\omega}|F^{-\nu-1,\rho,a}_{\varepsilon,\omega}\big)=e_{12}\,\mathcal{W}^a_\rho\big(U^{\nu,a}_{\varepsilon,\omega},U^{-\nu-1,a}_{\varepsilon,\omega}\big)=e_{12}\,\mathcal{W}^a_\rho\big(\widehat{U}^{\nu,a}_{\varepsilon,\omega},\widehat{U}^{-\nu-1,a}_{\varepsilon,\omega}\big)=e^\rho_{12}$$

Recall the symmetry $\widehat{U}^{j,\rho,a'}_{\varepsilon,\omega}=\mathcal{I}\widehat{U}^{j,\rho,a}_{-\varepsilon,-\omega}$, where $\mathcal{I}$ denotes complex conjugation. By (104) and the definition of d^ρ_{21}

$$\frac{1}{d^\rho_{21}}\big(U_1|F^{\nu,\rho,a'}_{\varepsilon,\omega}\big)=\big(U^{-\nu-1,\rho,a'}_{\varepsilon,\omega}|F^{\nu,\rho,a'}_{\varepsilon,\omega}\big)=\mathcal{W}^{a'}_\rho\big(\widehat{U}^{-\nu-1,a'}_{\varepsilon,\omega},\widehat{U}^{\nu,a'}_{\varepsilon,\omega}\big)=-\mathcal{W}^a_\rho\big(\widehat{U}^{\nu,a}_{-\varepsilon,-\omega},\widehat{U}^{-\nu-1,a}_{-\varepsilon,-\omega}\big)=-1$$

and similarly

$$\big(U_2|F^{\nu,\rho,a'}_{\varepsilon,\omega}\big)=\big(U^{-\nu-1,\rho,a'}_{\varepsilon,\omega}|F^{\nu,\rho,a'}_{\varepsilon,\omega}\big)=-1$$

Note that we could use the basis (U_3,U_4) instead, and define Gram matrix as

$$\mathcal{G}^{(a,a')}_\rho(E)=\begin{pmatrix}\big(U_3|F^{\nu,\rho,a}_{\varepsilon,\omega}\big) & \big(U_4|F^{\nu,\rho,a}_{\varepsilon,\omega}\big)\\ \big(U_3|F^{-\nu-1,\rho,a'}_{\varepsilon,\omega}\big) & \big(U_4|F^{-\nu-1,\rho,a'}_{\varepsilon,\omega}\big)\end{pmatrix} \tag{176}$$

So in any case, Gram matrix takes the form

$$\mathcal{G}_\rho^{a,a'}(E) = \begin{pmatrix} 1 & e_{12}^\rho \\ -d_{21}^\rho & -1 \end{pmatrix} \tag{177}$$

By our Definition 1.1, E verifies Bohr-Sommerfeld quantization rule precisely when $\mathcal{G}_\rho^{a,a'}(E)$ is singular. But $\det \mathcal{G}_\rho^{a,a'}(E) = 0$ iff $e_{12}^\rho\, d_{21}^\rho = 1$, or by (174)

$$\frac{d_{21}^\rho}{\overline{d_{21}^\rho}} = 1 \iff \arg d_{21}^\rho = 0 \bmod \pi \tag{178}$$

Then (174) gives again $\sin\left(\frac{1}{h}\tau_\rho + \frac{\phi}{2} + \frac{\pi}{4}\right) = 0$, and by (172)

$$\int_{-x_E}^{x_E} \xi^\rho(y;h)\, dy + h\,\frac{\phi}{2} + h\frac{\pi}{4} + b(E_1';h) \equiv 0, \quad \bmod \pi h\mathbf{Z} \tag{179}$$

which eventually proves Theorem 1.1.

To conclude this Section, we show that case (151) is inconsistent. There are 2 cases.

(1) Let as before $U_1 = U_{\varepsilon,\omega}^{\nu,a}$ near a, and $U_2^{a'} = U_{\varepsilon,\omega}^{-\nu-1,a'}$ near a'. By (148) and (149) we have $U_1 = d_{11}^\rho U_{\varepsilon,\omega}^{-\nu-1,a',\rho}$ near a' and $U_2 = e_{22}^\rho U_{\varepsilon,\omega}^{-\nu-1,a,\rho}$ near a. Then Gram matrix becomes

$$\mathcal{G}_\rho^{(a,a')}(E) = \begin{pmatrix} \big(U_1|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) & \big(U_2|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) \\ \big(U_1|F_{\varepsilon,\omega}^{\nu,\rho,a'}\big) & \big(U_2|F_{\varepsilon,\omega}^{\nu,\rho,a'}\big) \end{pmatrix} =$$
$$\begin{pmatrix} \big(U_{\varepsilon,\omega}^{\nu,a}|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) & e_{22}\big(U_{\varepsilon,\omega}^{-\nu-1,a}|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) \\ d_{11}\big(U_{\varepsilon,\omega}^{\nu,a'}|F_{\varepsilon,\omega}^{\nu,\rho,a'}\big) & \big(U_{\varepsilon,\omega}^{-\nu-1,a}|F_{\varepsilon,\omega}^{\nu,\rho,a'}\big) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is inconsistent for it is never singular.

(2) Let as before $U_3 = U_{\varepsilon,\omega}^{-\nu-1,a}$ near a, and $U_4 = U_{\varepsilon,\omega}^{\nu,a'}$ near a'. By By (148) and (149) we have $U_3 = d_{22}^\rho U_{\varepsilon,\omega}^{-\nu-1,a'}$ near a', and $U_4 = e_{11} U_{\varepsilon,\omega}^{\nu,a}$ near a. Then Gram matrix writes

$$\mathcal{G}_\rho^{(a,a')}(E) = \begin{pmatrix} \big(U_3|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) & \big(U_4|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) \\ \big(U_3|F_{\varepsilon,\omega}^{\nu,\rho,a'}\big) & \big(U_4|F_{\varepsilon,\omega}^{\nu,\rho,a'}\big) \end{pmatrix} =$$
$$\begin{pmatrix} \big(U_{\varepsilon,\omega}^{-\nu-1,a}|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) & e_{11}\big(U_{\varepsilon,\omega}^{\nu,a}|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) \\ d_{22}\big(U_{\varepsilon,\omega}^{-\nu-1,a'}|F_{\varepsilon,\omega}^{-\nu-1,\rho,a}\big) & \big(U_{\varepsilon,\omega}^{\nu,a'}|F_{\varepsilon,\omega}^{\nu,\rho,a'}\big) \end{pmatrix} = \begin{pmatrix} 0 & e_{11} \\ d_{22} & 0 \end{pmatrix}$$

which is again inconsistent.

We can also try to modify $\mathcal{G}_\rho^{(a,a')}(E)$, according to (176), either with the pair (U_1, U_2) or (U_3, U_4). Both cases are inconsistent. So there is only one possible choice for defining $\mathcal{M}_\rho^{a,a'}(E)$.

8 An extension to the SFS junction

Following mainly [CaMo] we try to extend our constructions to the case of a SFS junction, taking into account the spins $(\uparrow, \downarrow)$ of a pair of quasi-particles in the state $(u_\uparrow, v_\downarrow)$ and the other

pair in the state $(u_\downarrow, v_\uparrow)$. Let $J(x)$ be the *spin interaction*, vanishing in the supra-conducting bulk, such that only quasi-particles of the same species (i.e electron/electron, or hole/hole) do interact through their spin. We take $J(x) \leq 0$ in the ferromagnetic case, $J(x) \geq 0$ in the antiferromagnetic case. Let also $F(x) \geq 0$ be the *exchange energy*, vanishing in the supra-conducting bulk, and $A(x)$ a smooth magnetic potential. We consider the Hamiltonian (omitting the variable x from the notations)

$$\mathcal{P}(x,\xi) = \begin{pmatrix} (\xi - A)^2 - \mu - F & \Delta e^{i\phi/2} & J & 0 \\ \Delta e^{-i\phi/2} & -(\xi - A)^2 + \mu - F & 0 & J \\ J & 0 & (\xi - A)^2 - \mu + F & \Delta e^{i\phi/2} \\ 0 & J & \Delta e^{-i\phi/2} & -(\xi - A)^2 + \mu + F \end{pmatrix} \tag{180}$$

and the corresponding operator $\mathcal{P}(x, hD_x)$ acting on vectors ${}^t(u_\uparrow, v_\downarrow, u_\downarrow, v_\uparrow) \in L^2(\mathbf{R}) \otimes \mathbf{C}^4$. Such a model for the SFS junction is discussed in [CaMo]. The term $2J(x)\,\mathrm{Re}(u_\uparrow|u_\downarrow) + 2J(x)\,\mathrm{Re}(v_\uparrow|v_\downarrow)$ stands for the spin interaction, while the term $-F(x)(|u_\uparrow|^2+|v_\downarrow|^2)+F(x)(|u_\downarrow|^2+|v_\uparrow|^2)$ accounts for the itinerant (anti-)ferromagnetism. We assume F, J to be smooth over $\mathbf{R}$ and small enough. The exchange energy should be small with respect to Fermi energy E_F, so we do not assume here that Fermi energy is taken to 0, and energy equals now $E = E_F + \varepsilon$, where ε stands for the spectral parameter.

The corresponding semi-classical Hamiltonian $\mathcal{P}(x, hD_x)$ is similar to a Dirac operator and could be described within the general framework of [Wo].

Moreover, in case $A = 0$, $\mathcal{P}(x, hD_x)$ is just the product of 2 decoupled operators of type (1) for $|x| > L$, since we assumed $J(x) = F(x) = 0$ in the supraconductor. Hence the scattering problem is essentially the same as in Sect.1.3.2.

Look now at the geometry of $\mathcal{P}(x,\xi)$. Let $q(x,\xi) = (\xi - A(x))^2 - \mu(x)$. The determinant of $\mathcal{P}(x,\xi) - E$ is given by :

$$\delta = q^4 - 2q^2(E^2+F^2+J^2-\Delta^2) + (E^2-F^2)^2 - 2\Delta^2(E^2+F^2) + 2J^2(F^2-E^2) + \Delta^4 - \Delta^2 J^2 + J^4$$

The discriminant of the quadratic equation $\delta = 0$ is $4E^2(F^2 + J^2 + \Delta^2) + \Delta^2(4F^2 - J^2)$, so that the characteristic variety Σ_E is given by

$$(\xi - A(x))^2 = \mu(x) + \varepsilon_2 \sqrt{E^2 - \Delta^2 + F^2 + J^2 + \varepsilon_1 \sqrt{4E^2(F^2 + J^2 + \Delta^2) + \Delta^2(4F^2 - J^2)}} \tag{181}$$

with $\varepsilon_j = \pm 1$. The focal points in turn are determined by $\partial_\xi(\det(\mathcal{P}(x,\xi) - E)) = 0$, or

$$4E^2(F^2 + J^2 + \Delta^2) + \Delta^2(4F^2 - J^2) = 0 \tag{182}$$

If $F^2 + J^2 + \Delta^2 > 0$ and $2F > J$ there are no focal point: itinerant (anti-)ferromagnetism is dominant, and no Andreev current occurs. For small F however, (182) is analogous to the previous relation $\Delta(x) = E$, and there is generically a one-parameter family of focal points $a_E = (x_E, \xi_E)$ in $x > 0$ (and symmetrically in $x < 0$), depending smoothly on E. Then (181)

would define 2 closed orbits in Σ_E corresponding to quasi-particles with opposite spin, and meeting tangentially (vertically) at a_E. In absence of the magnetic field, and if $\Delta\sqrt{\frac{J^2-F^2/4}{F^2+J^2+\Delta^2}}$ varies linearly in x near x_E we can expect that our previous constructions apply, and give raise to 4 monodromy matrices, labelled by $\rho = \pm, \sigma = \uparrow, \downarrow$, and thus to 4 quantization rules.

9 Appendix A: The normal form for the anharmonic oscillator

We need a refinement of the theory of [HeSj] on normal forms for a h-PDO P_ω of the form (61) acting on the semi-classical distributions D_ν and $D_{-\nu-1}$ (Gaussian wave-packets). Another normal form to Weber equation is presented in [AoYo], using Sato's microdifferential Calculus, but it holds for fixed energy E, so does not preserves unitarity.

Consider

$$P_{\omega\beta}(-hD_{\xi'}, \xi'; h) = (-hD_{\xi'})^2 + (\xi' + \omega\beta\xi'^2)^2 + h^2 f(\omega\beta\xi') \tag{183}$$

where β as in (59) can be thought of a large parameter, and f a real analytic function defined near 0 and given by (62). Here we have changed the notation P_β to $P_{\omega\beta}$ to emphasize some homogeneity properties. For $\omega = \pm 1$, $P_{\omega\beta}(-hD_{\xi'}, \xi'; h)$ is self-adjoint. Modulo a lower order term in h^2, $P_{\omega\beta}$ is the familiar Schrödinger operator with potential $V(\xi') = (\xi' + \omega\beta\xi'^2)^2$, having non degenerate minima at 0 and $-1/\omega\beta$ and a non degenerate local maximum at $\xi' = -1/(2\omega\beta)$. Moreover f has a pole at one of the turning points given by $V(\xi') = \frac{E_1^2}{\beta^2}$. We will take $P_\omega(-hD_{\xi'}, \xi'; h)$ microlocally near (0,0) to $G(P_0; h)$, by conjugating with a unitary h-FIO A_ω. The role of the principal part G_0 of G is to make the period of the Hamiltonian flow independent of energy (isochore Morse Lemma).

9.1 Isochore Morse Lemma

Here we compute G_0 (and its inverse F_0), as well as the canonical relation associated with A_ω. We shall always assume $\omega = \pm 1$. For energies below the critical level $(4\beta)^{-2}$, the variable ξ' can be restricted in a neighborhood of 0. So when $0 < E_1 < 1/4$, we consider the equation, with energy parameter rescaled as in (61)

$$\big(P_{\omega\beta} - (\frac{\omega\, E_1}{\beta})^2\big) u_\omega(\xi') = 0 \tag{184}$$

Our goal is to take the principal symbol of $P_\omega(-hD_{\xi'}, \xi'; h)$ to its normal form by means of an elliptic integral. Let $p_{\omega\beta}(x', \xi') = (-x')^2 + (\xi' + \omega\beta\xi'^2)^2$, $p_0(y, \eta) = \frac{1}{2}\big((-y)^2 + \eta^2\big)$.

Proposition A.1. *There exists an analytic diffeomorphism $t \mapsto F_0(t, \omega\beta)$ defined in a neighborhood of 0, $F_0(0) = 0$, with inverse G_0, and a real analytic phase function $\phi_{\omega\beta}(\xi', \theta)$, defined in a neighborhood of (0,0), of the form $\phi_{\omega\beta}(\xi', \theta) = \xi'\theta + g_{\omega\beta}(\xi', \theta)$, $g_{\omega\beta}(\xi', \theta) =$*

$\mathcal{O}(|\xi',\theta|^3)$, *parametrizing the canonical transformation* $\kappa_{\omega\beta} : (\partial_\theta \varphi_{\omega\beta}, \theta) \mapsto (\xi', \partial_{\xi'}\varphi_{\omega\beta})$, *such that* $F_0 \circ p_{\omega\beta} \circ \kappa_{\omega\beta} = p_0$. *At leading order we have :*

$$F_0(t) = \frac{1}{2}(t + \frac{3}{2}t^2\beta^2 + \frac{35}{4}t^3\beta^4 + \frac{1155}{16}t^4\beta^6 + \cdots) \tag{185}$$

$$G_0(s) = 2(s - 3s^2\beta^2 - 17s^3\beta^4 - \frac{375}{2}s^4\beta^6 - \frac{10689}{4}s^5\beta^8 - \cdots) \tag{186}$$

and

$$\begin{aligned} g_{\omega\beta}(\xi',\theta) = {} & \omega\beta\xi'^2\theta + \frac{2\omega\beta}{3}\theta^3 + \frac{3\beta^2}{4}\xi'^3\theta + \frac{5}{4}\beta^2\xi'\theta^3 + \frac{5}{2}\omega\beta^3\theta^5 + \frac{9}{4}\omega\beta^3\xi'^4\theta + \frac{23}{4}\omega\beta^3\xi'^2\theta^3 \\ & + \frac{515}{32}\beta^4\xi'\theta^5 + \frac{1073}{48}\beta^4\xi'^3\theta^3 + \frac{203}{32}\beta^4\xi'^5\theta + \frac{376}{16}\omega\beta^5\theta^7 + \cdots \end{aligned} \tag{187}$$

In particular, $g_{\omega\beta}$ *is invariant under the reflection* $(\xi',\theta,\omega) \mapsto (-\xi',-\theta,-\omega)$, *and odd under the reflection* $(\xi',\theta,\omega) \mapsto (-\xi',\theta,-\omega)$. *Moreover,* $(\omega\beta)^2 g_{\omega\beta}\big(\frac{\xi'}{\omega\beta}, \frac{\theta}{\omega\beta}\big)$ *is independent of* $\omega\beta$.

Proof. Consider first $F_0(t) = \oint(-x')\,d\xi'$ where the integral is computed on the energy surface $\Sigma_t = \{p_\omega = t\}$. When

$$4\beta\sqrt{t} < 1 \tag{188}$$

which is the condition for the harmonic approximation to hold near $\xi' = 0$, Σ_t has two connected components. Changing variables, the elliptic integral $F_0(t) = \frac{t}{2\pi}\int_0^{2\pi}\cos^2 u(1 + 4\beta\sqrt{t}\sin u)^{-1/2}du$ can be expanded as a (convergent) power series in $4\beta\sqrt{t}$, and equation (185) follows, which we invert according to (186), for s in a suitable neighborhood of 0.

Next, still at the level of Taylor series we determine $g_{\omega\beta}(x,\theta)$ by :

$$g_{\omega\beta}(\xi',\theta) = \sum_{n=(n_1,n_2),|n|\geq 3} a_n \xi'^{n_1}\theta^{n_2}, \quad a_n \in \mathbf{R}$$

using the relation $p_{\omega\beta} \circ \kappa_{\omega\beta}(y,\eta) = G_0(p_0)$, with

$$\kappa_{\omega\beta} : (y,\eta) = (\xi' + \partial_\theta g_{\omega\beta}(\xi',\theta), \theta) \mapsto (\xi', x') = (\xi', \theta + \partial_{\xi'} g_{\omega\beta}(\xi',\theta)) \tag{189}$$

Keeping terms up to order 3 in G_0 we obtain first, at order 5,

$$\begin{aligned} g_{\omega\beta}(\xi',\theta) = {} & \omega\beta\xi'^2\theta + \frac{2\omega\beta}{3}\theta^3 + \frac{3\beta^2}{4}\xi'^3\theta + \frac{5\beta^2}{4}\xi'\theta^3 + \frac{5}{2}\omega\beta^3\theta^5 + \frac{9}{4}\omega\beta^3\xi'^4\theta + \frac{23}{4}\omega\beta^3\xi'^2\theta^3 \\ & + a_{04}(\xi'^2 + \theta^2)^2 + 4\beta a_{04}(\xi'^5 + 2\xi'\theta^4 + 3\xi'^3\theta^2) + \cdots \end{aligned}$$

Next truncating G_0 to order 4, we expand $g_\omega(\xi',\theta)$ to order 6, this gives $a_{04} = 0$, and we find

$$\begin{aligned} g_{\omega\beta}(\xi',\theta) = {} & \beta\xi'^2\theta + \frac{2\omega\beta}{3}\theta^3 + \frac{3\beta^2}{4}\xi'^3\theta + \frac{5\beta^2}{4}\xi'\theta^3 + \frac{5}{2}\omega\beta^3\theta^5 + \frac{9}{4}\omega\beta^3\xi'^4\theta + \frac{23}{4}\omega\beta^3\xi'^2\theta^3 \\ & + \frac{515}{32}\beta^4\xi'\theta^5 + \frac{1073}{48}\beta^4\xi'^3\theta^3 + \frac{203}{32}\beta^4\xi'^5\theta + a_{06}(\xi'^2 + \theta^2)^3 + \cdots \end{aligned}$$

Finally, at order 5 for G_0, we observe that within this accuracy, the $(\xi'^2+\theta^2)^3$ term in $g_{\omega\beta}(\xi',\theta)$ vanishes, and (187) holds. □

9.2 The semi-classical normal form of Helffer-Sjöstrand

We bring next the operator $P_\omega(-hD_{\xi'}, \xi'; h)$ to its normal form $P_0(-hD_\eta, \eta) = \frac{1}{2}\big((-hD_\eta)^2 + \eta^2 - h\big)$ by conjugating with FIO's A_ω.

Let $X, Y \subset \mathbf{R}$ be open sets, $\kappa : T^*X \times Y \to T^*Y$ a smooth canonical map, parametrized by a non degenerate phase function $\varphi(x, y, \theta)$ and $\Gamma'_\kappa = \{(x, \xi; y, -\eta) : (x, \xi) = \kappa(y, \eta)\} \subset T^*(X \times Y)$ the corresponding canonical relation. There is a Lagrangian embedding $C_\varphi = \{(x, y, \theta) : \partial_\theta \varphi(x, y, \theta) = 0\} \mapsto \Gamma'_\kappa$. Let A be a h-FIO whose Schwartz kernel belongs to the class $O^0(X \times Y; \Gamma'_\kappa)$ of oscillatory integrals (or semi-classical distributions) defined via an amplitude $c(x, y, \theta; h)$. Then it is known that $c(x, y, \theta; h)$ is not uniquely defined, but only its restriction to the critical set C_φ. In the normal form of Helffer-Sjöstrand (see (191) below) the choice of $c(x, y, \theta; h)$ is constrained on another set, so that the extension is simply parametrized by a family of functions $a_\theta(u, v)$ (in local coordinates) non vanishing near $(u, v) = (0, 0)$. It is necessary to find such an extension when A acts on semi-classical distributions u of Gaussian type, such as the parabolic cylinder functions $D_\nu, D_{-\nu-1}$. Again, we content ourselves of the level of Taylor series.

Theorem A.2. *For real ω, there exists a (formally) unitary FIO operator $A = A_\omega$ defined microlocally near (0,0), of the form :*

$$Av(\xi', h) = (2\pi h)^{-1} \int\int e^{i\varphi_{\omega\beta}(\xi', \eta, \theta)/h} c_\omega(\xi', \eta, \theta, h) v(\eta, h)\, d\eta\, d\theta \tag{190}$$

and a real valued analytic symbol

$$F(t, \omega\beta, h) = F_0(t, \omega\beta) + hF_1(t, \omega\beta) + \cdots$$

with $F_1(t, \omega\beta) = -\frac{1}{2}$ such that

$$A^* F(P_\omega, \omega\beta, h) A = P_0 \tag{191}$$

The phase function is given by

$$\varphi_{\omega\beta}(\xi', \eta, \theta) = \phi_{\omega\beta}(\xi', \theta) - \eta\theta = (\xi' - \eta)\theta + g_{\omega\beta}(\xi', \theta) \tag{192}$$

with $g_{\omega\beta}$ parametrizing $\kappa_{\omega\beta}$, as in Proposition A.1. The analytic symbol

$$c_\omega(\xi', \eta, \theta, h) = c_0(\xi', \eta, \theta) + hc_1(\cdots) + \cdots$$

with real principal part c_0 is determined by the unitarity relations $A^*A \equiv AA^* \equiv \mathrm{Id}$, up to its extensions from Γ'_κ, which can be chosen, in suitable local coordinates (u, v), of the form ($a_\theta(u, v)$ being elliptic)

$$c(\xi', \eta, \theta) = \big(1 + \frac{\partial^2 g_{\omega\beta}}{\partial\xi'\partial\theta}(\xi', \theta)\big)^{1/2} + a_\theta(u, v)uv \tag{193}$$

Proof. F_0 has been computed in Proposition A.1; the value of F_1 is due to [HeRo]. Let us compute c_0. In order to comply with usual notations, we change for a moment the name of the variables. With $\varphi_{\omega\beta}$ as in (192) we have

$$Au(y;h) = (2\pi h)^{-1} \int\int e^{i\left((y-z)\xi + g_{\omega\beta}(y,\xi)\right)/h} c(y,z,\xi;h) u(z)\, dz\, d\xi$$

Then

$$A^* v(x;h) = (2\pi h)^{-1} \int\int e^{i\left((y-z)\xi - g_{\omega\beta}(y,\theta)\right)/h} \overline{c(y,x,\theta;h)} v(y)\, dy\, d\theta$$

and

$$A^* A u(x) = (2\pi h)^{-1} \int\int I(x,z,\xi)\, u(z)\, dz\, d\xi$$

The phase function associated with the oscillatory integral defining $I(x,z,\xi)$ is given by

$$\Phi_{\omega\beta}(y,\theta;x,z,\xi) = (x-y)\,\theta + (y-z)\,\xi + g_{\omega\beta}(y,\xi) - g_{\omega\beta}(y,\theta)$$

Writing $g_2(y,\theta,\theta) = \partial_\theta\, g_{\omega\beta}(y,\theta)$, we have

$$g_{\omega\beta}(y,\xi) - g_{\omega\beta}(y,\theta) = (\xi-\theta)\, g_2(y,\xi,\theta)$$

thus the critical points (y_c,θ_c) of $\Phi_{\omega\beta}(.,.;x,z,\xi)$ solve

$$-\theta + \xi + (\xi-\theta)\,\partial_y\, g_2(y,\xi,\theta) = 0, \quad x - y - \partial_\theta\, g_{\omega\,\beta}(y,\theta) = 0$$

or

$$\theta_c = \xi, \quad x - y_c - \partial_\xi\, g_{\omega\,\beta}(y_c,\xi) = 0$$

and the critical value is the pseudo-differential phase $\Phi_{\omega\,\beta}(y_c,\theta_c;x,z,\xi) = (x-z)\xi$. Computing Hess $\Phi_{\omega\,\beta}\big(y_c,\theta_c;x,z,\xi\big)$, and using that c_0 is obviously real, Theorem 2.1 shows that (193), microlocally near (0,0), takes the form

$$A^* A\, u(x;h) = (2\,\pi\, h)^{-1} \int\int e^{\frac{i}{h}\,(x-z)\,\xi}\, J(x,z,\xi)\, u(z)\, dz\, d\xi + \mathcal{O}(h)$$

with

$$J(x,z,\xi) = \big(1 + (\partial_y g_2)(y_c,\xi,\xi)\big)^{-1}\, c_0(y_c,x,\xi)\, c_0(y_c,z,\xi)$$

The unitarity condition gives in particular $J(x,x,\xi) = 1$, that is

$$c_0(z_c,x,\theta) = \big(1 + (\frac{\partial^2 g_{\omega\,\beta}}{\partial x\,\partial\theta})(z_c,\theta)\big)^{1/2}, \quad z_c = z_c(x,\theta) = x + \partial_\theta g_{\omega\beta}(z_c,\theta) \tag{194}$$

Computing similarly AA^* we find

$$c_0(x,y_c,\theta) = \big(1 + (\frac{\partial^2 g_{\omega\beta}}{\partial x\,\partial\theta})(x,\theta)\big)^{1/2}, \quad y_c = y_c(x,\theta) = x + \partial_\theta g_{\omega\beta}(x,\theta) \tag{195}$$

With the help of (187), we find easily

$$x_c(y,\theta) = y + 2\omega\beta\theta^2 + \omega\beta y^2 + \frac{11}{4}\beta^2 y^3 + \frac{31}{4}\beta^2 y\theta^2 + \cdots \tag{196}$$

There remains to compute $c_0(x,y,\theta)$ from (194) and (195). Changing again notations, we write (194)-(195) as

$$c(x_c(y,\theta),y,\theta) = F(y,\theta), \quad c(x,y_c(x,\theta),\theta) = G(x,\theta) \tag{197}$$

Consider the family of parametrized curves $t_\theta : x \mapsto t_\theta(x) = (x, y_c(x,\theta)$, $s_\theta : y \mapsto t_\theta(y) = (x_c(y,\theta), y)$, where $y_c(x,\theta)$ as in (195) and $x_c(y,\theta) = y + \partial_\theta g_{\omega\beta}(x_c,\theta)$ is implicitely defined as in (194). For $\theta = 0$, Proposition A.1 shows that $t_0 \cap s_0 = (0,0)$; the "velocity vectors" verify $t_0'(0) = s_0'(0) = (1,1)$, while the "acceleration vectors" verify $t_0''(0) = (0, 2\omega\beta)$ and $s_0''(0) = (2\omega\beta, 0)$. So these curves are not transverse when $\theta \neq 0$, but when $\theta \neq 0$ (but still small enough), it turns out that (t_θ, s_θ) can be straightened to new axis of coordinates. First we look for the points $M_\theta = (x_\theta, y_\theta)$ where these curves intersect. We have

$$\begin{aligned} y_\theta &= x_\theta + \frac{\partial g_{\omega\beta}}{\partial\theta}(x_\theta,\theta) \\ x_\theta &= x_c(y_\theta,\theta) = y_\theta + \frac{\partial g_{\omega\beta}}{\partial\theta}(x_c(y_\theta,\theta),\theta) \end{aligned} \tag{198}$$

Substituting for $x_\theta = x_0\theta + x_1\theta^2 + x_2\theta^3 + \cdots$, $x_\theta = y_0\theta + y_1\theta^2 + y_2\theta^3 + \cdots$, and using (196), we find $x_0 = y_0$ together with the relations

$$\begin{aligned} y_1 &= x_1 + 2\omega\beta + \omega\beta x_0^2 \\ y_2 &= x_2 + 2\omega\beta x_0 x_1 + \frac{3}{4}\beta^2 x_0^3 + \frac{15}{4}\beta^2 x_0 \\ x_1 &= y_1 + 2\omega\beta + \omega\beta y_0^2 \\ x_2 &= y_2 + 2\omega\beta y_0 y_1 + 2\beta^2 y_0^3 + \frac{15}{4}\beta^2 y_0 \end{aligned}$$

The first and third Eq. give $x_0^2 + 2 = 0$, so the intersection of t_θ and s_θ consists in 2 complex points $M_\theta, \overline{M}_\theta$; we shall only consider M_θ, with $x_\theta = i\sqrt{2}\theta + \cdots$. Substituting into the second and fourth Eq. yields $x_1 = y_1 = -\frac{7}{4}\omega\beta$. So (up to this accuracy) $M_\theta, \overline{M}_\theta$ lie on the (complex) diagonal. We check that $t_\theta'(x_\theta)$ and $s_\theta'(y_\theta)$ are transverse when $\theta \neq 0$ is small enough, since

$$\det\left(t_\theta'(x_\theta), s_\theta'(y_\theta)\right) = -\frac{35}{4}\beta^2\theta^2 + \cdots \neq 0$$

So we change to a local frame, choose origin at M_θ, and set $y = y_\theta + y'$, $x = x_\theta + x'$, $u = y - y_c(x,\theta) = y' - y_c'(x',\theta)$, $v = x - x_c(y,\theta) = x' - x_c'(y',\theta)$; here we have made use of the fact that $x_\theta = y_\theta$ (up to this accuracy). The curves t_θ and s_θ take the form $u = 0$, resp. $v = 0$. Consider the local diffeomorphism $\Phi_\theta : (x,y) \mapsto (u,v)$. Put $\widetilde{c}_\theta = c \circ \Phi_\theta^{-1}$, with c as in (197). We have

$$\begin{aligned} c(x_c(y,\theta),y,\theta) &= F(y,\theta) = \widetilde{c}_\theta(u,0) = \widetilde{F}_\theta(u) \\ c(x,y_c(x,\theta),\theta) &= G(x,\theta) = \widetilde{c}_\theta(0,v) = \widetilde{G}_\theta(v) \end{aligned} \tag{199}$$

so that by Taylor expansion at first order

$$\widetilde{c}_\theta(u,v) = \widetilde{c}(0,0) + \partial_u\widetilde{c}_\theta(0,0)u + \partial_v\widetilde{c}_\theta(0,0)v + \cdots = \partial_u\widetilde{F}_\theta(0)u + \widetilde{G}_\theta(0)v + \cdots \tag{200}$$

and we are left to compute $\partial_u \widetilde{F}_\theta(0), \partial_v \widetilde{G}_\theta(0)$. This gives only the germ of c at order 1, since we have no information on the mixed derivatives.

We may proceed a bit further by setting

$$f(x,y,\theta) = c(x,y,\theta) - \big(1 + \frac{\partial^2 g_{\omega\beta}}{\partial x \partial \theta}(x,\theta)\big)^{1/2} \tag{201}$$

We notice that f vanishes along the curves t_θ and s_θ

$$f(x, y_c(x,\theta), \theta) = 0, \quad f(x_c(y,\theta), y, \theta) = 0$$

which in the local coordinates (u,v) above centered at M_θ, takes the form

$$\widetilde{f}_\theta(u,0) = \widetilde{f}_\theta(0,v) = 0 \tag{202}$$

Generically (divisors with normal crossings) there is $a_\theta(u,v)$ elliptic at (0,0), such that

$$\widetilde{f}_\theta(u,v) = a_\theta(u,v)uv \tag{203}$$

This gives the required extension. □

The function a_θ will be chosen below by checking that $A_\omega D_\nu$ and $A_\omega D_{\nu-1}$ actually solve (184).

10 Appendix B: Constructing the microlocal solutions

10.1 Microlocal solutions with the D_ν function

Assume already we have chosen a suitable extension of the amplitude of A_ω as in (193), $\omega = \pm 1$. It follows from Theorem A.2 that equation (184) takes the form, with $v_\omega = A_\omega^* u_\omega$ and $\nu = \frac{1}{h} F(\frac{E_1^2}{\beta^2}, h)$:

$$P_0 v_\omega = F(\frac{E_1^2}{\beta^2}, h) v_\omega = h\nu v_\omega \tag{204}$$

or, after the scaling $\eta = (h/2)^{1/2}\zeta$, of Weber equation (10). In view of (185) we also set

$$\frac{{E_1'}^2}{2\beta^2} = F(\frac{(\omega E_1)^2}{\beta^2}; h) + h = (\nu+1)h \tag{205}$$

so that $0 < E_1' \sim E_1$ as $(E_1, h) \to 0$. Inserting $v_\omega(\eta) = D_\nu(\varepsilon(h/2)^{-1/2}\eta)$ (independent of β) in (190) and using the fact that A_ω is (formally) unitary, we get with $u_\omega^\nu = u_{\varepsilon,\omega}^\nu(\xi', h)$

$$\begin{aligned} u_{\varepsilon,\omega}^\nu(\xi', h) = A_\omega\big(D_\nu(\varepsilon(h/2)^{-1/2}\cdot)\big)(\xi') = \frac{\Gamma(\nu+1)}{-2i\pi}\frac{1}{2\pi h} \\ \times \int\int_\infty^{(0+)} e^{i\widetilde{\psi}_{\varepsilon,\omega}^\nu(\xi',\eta,\theta,s)/h} c(\xi',\eta,\theta,h)(-s)^{-(\nu+1)}\, ds\, d\eta\, d\theta \end{aligned} \tag{206}$$

where :

$$\widetilde{\psi}^{\nu}_{\varepsilon,\omega} = (\xi' - \eta)\theta + g_{\omega\beta}(\xi', \eta) + i\big[\frac{\eta^2}{2} + \varepsilon(h/2)^{-1/2}\eta s h + \frac{s^2}{2}h + (\nu+1)h\log(-s)\big]$$

We rescale the variables as :

$$\theta_1 = \beta\theta, \quad \eta_1 = \beta\eta, \quad s_1 = \beta\sqrt{h}s, \quad \xi_1 = \beta\xi'$$

and set $h' = \beta^2 h$ as in (60) so that

$$\begin{aligned} u^{\nu}_{\varepsilon,\omega}(\xi', h) = &-\frac{\Gamma(\nu+1)}{4i\pi^2(h')^{3/2}} \exp\big[{E'_1}^2(\log(h')/4h'\big] \\ &\int\int_{\infty}^{(0+)} \exp\big[\psi^{\nu}_{\varepsilon,\omega}(\eta_1, s_1, \theta_1, \xi_1)/h'\big]\, c_{\omega}(\xi_1, \eta_1, \theta_1; h')\, ds_1\, d\eta_1\, d\theta_1 \end{aligned} \tag{207}$$

where, in the new coordinates

$$\psi^{\nu}_{\varepsilon,\omega}(\xi_1, \eta_1, \theta_1, s_1) = (\xi_1 - \eta_1)\theta_1 + h_{\omega}(\xi_1, \theta_1) + i\big[\frac{\eta_1^2}{2} + \frac{s_1^2}{2} + \sqrt{2}\,\varepsilon\,\eta_1 s_1 + \frac{1}{2}{E'_1}^2\log(-s_1)\big] \tag{208}$$

with $h_{\omega}(\xi_1, \theta_1) = \beta^2\, g_{\omega\beta}\big(\frac{\xi_1}{\beta}, \frac{\theta_1}{\beta}\big)$ independent of β, and

$$c_{\omega}(\xi_1, \eta_1, \theta_1; h') = c\big(\frac{\xi_1}{\beta}, \frac{\eta_1}{\beta}, \frac{\theta_1}{\beta}; \frac{h'}{\beta^2}\big)$$

As in Sect.2, we apply stationary phase to (207) with small parameter h'. The critical points

$$(\eta_1, s_1, \theta_1) = (\eta_{\omega}, s_{\omega}, \theta_{\omega}) = \big((\eta^{\nu}_{\omega}(\theta_{\omega}(\xi_1)), s^{\nu}_{\varepsilon,\omega}(\theta_{\omega}(\xi_1)), \theta^{\nu}_{\omega}(\xi_1)\big) \tag{209}$$

are given by

$$\xi_1 - \eta_1 + \partial_{\theta_1} h_{\omega}(\xi_1, \theta_1) = 0,\ -\theta_1 + i\eta_1 + i\sqrt{2}\,\varepsilon\, s_1 = 0,\ \sqrt{2}\,\varepsilon\,\eta_1 + s_1 + \frac{{E'_1}^2}{2s_1} = 0 \tag{210}$$

which yields a quadratic equation for s_1 and

$$\eta_1 = \omega(E_{\beta}^2 - \theta_1^2)^{1/2} \tag{211}$$

$$-\sqrt{2}\,\varepsilon\, s_1 = i\theta_1 + ({E'_1}^2 - \theta_1^2)^{1/2} \tag{212}$$

$$\xi_1 - \omega(E_{\beta}^2 - \theta_1^2)^{1/2} + \partial_{\theta_1} h_{\omega}(\xi_1, \theta_1) = 0 \tag{213}$$

So $\theta^{\nu}_{\omega}(\xi_1)$ is a also critical point for

$$\theta_1 \mapsto H_{\beta}(\xi_1, \theta_1) = \xi_1\theta_1 + h_{\omega}(\xi_1, \theta_1) - \omega\int({E'_1}^2 - \theta_1^2)^{1/2}\, d\theta_1 \tag{214}$$

Properties of these critical points are summarized in the following :

Lemma B.1. *For $\omega = \pm 1$ and ξ_1 small enough, we have the expansion*

$$\begin{aligned} \big(\theta^{\nu}_{\omega}(\xi_1)\big)^2 = &\big({E'}_1^2 - 4{E'}_1^4 - 18{E'}_1^6 - \cdots\big) - \omega\big(4{E'}_1^2 - 8{E'}_1^4 - 58{E'}_1^6 - \cdots\big)\xi_1 + \\ &\big(-1 + \frac{25}{2}{E'}_1^2 + 19{E'}_1^4 + \cdots\big)\xi_1^2 - \omega\big(-2 + 17\,{E'}_1^2 + 58{E'}_1^4 + \cdots\big)\xi_1^3 + \big(-3 + 17{E'}_1^2 + \cdots\big)\xi_1^4 - \cdots \end{aligned} \tag{215}$$

and thus

$$\begin{aligned}
&\pm\,\theta^{\nu}_{\omega}(\xi_1) = E'_1 - 2E'^3_1 - 11E'^5_1 - \cdots - \omega\,\big(2E'_1 - 7E'^5_1 - \cdots\big)\xi_1 + \\
&\big(-\frac{1}{2E'_1} + \frac{13}{4}E'_1 + \frac{21}{2}E'^3_1 + \cdots\big)\xi_1^2 - \omega\big(2E'_1 + \frac{17}{2}E'^3_1 + \cdots\big)\xi_1^3 + \\
&\big(-\frac{1}{8E'^3_1} - \frac{1}{8E'_1} + \frac{91}{32}E'_1 + \cdots\big)\xi_1^4 - \cdots
\end{aligned} \tag{216}$$

We denote by $\widehat{\theta}^{\nu}_{\omega}(\xi_1) \in [0, E'_1]$ the positive critical point of (214). Moreover, there is the symmetry relation between the critical points $\theta^{(\nu)}_{\omega}(\xi_1) = \theta^{(\nu)}_{-\omega}(-\xi_1)$

Proof. We know that $\theta_1 = \theta^{(\nu)}_{\omega}(\xi_1)$ is a root of the equation

$$\xi_1 - \varepsilon'\,(E'^2_1 - \theta_1^2)^{1/2} + \partial_{\theta_1} h_{\omega}(\xi_1, \theta_1) = 0 \tag{217}$$

($\varepsilon' = \pm 1$). When $\xi_1 = 0$, we get $\varepsilon'(E'^2_1 - \theta_1^2)^{1/2} = \omega\big(2\theta_1^2 + \frac{25}{2}\theta_1^4 + \frac{2611}{16}\theta_1^6 + \cdots\big)$ which implies $\varepsilon' = \omega$, and

$$2\theta_1^2 + \frac{25}{2}\,\theta_1^4 + \frac{2611}{16}\theta_1^6 + \cdots = (E'^2_1 - \theta_1^2)^{1/2}$$

Squaring this Eq. we get $\theta_1^2 + 4\,\theta_1^4 + 50\theta_1^6 + \cdots = E'^2_1$, and

$$\pm\theta^{\nu}_{\omega}(0) = E'_1 - 2E'^3_1 - 11E'^5_1 - \cdots \tag{218}$$

Computing the critical point $\theta^{(\nu)}_{\omega}(\xi_1)$ from (217) with $\varepsilon' = \omega$, when $\omega = 1$, we get

$$\xi_1 + \xi_1^2 + 2\,\theta_1^2 + \frac{3}{4}\,\xi_1^3 + \frac{15}{4}\,\xi_1\,\theta_1^2 + \frac{9}{4}\,\xi_1^4 + \frac{69}{4}\,\xi_1^2\,\theta_1^2 + \frac{25}{2}\,\theta_1^4 + \frac{2575}{32}\xi_1\,\theta_1^4 + \frac{1073}{16}\xi_1^3\theta_1^2 + \frac{203}{32}\xi_1^5 + \cdots = (E'^2_1 - \theta_1^2)^{1/2}$$

and $\theta^{(\nu)}_{\omega}(\xi_1)$ by looking for a power series in ξ_1, with coefficients given in turn by power series in E'^2_1. We proceed in the same way when $\omega = -1$, so we get (215). Taking square root gives also (216). This brings the proof of Lemma 1.1 to an end. □

The Lagrangian manifold $\Lambda_E \subset \{p_{\omega\beta} = \big(\frac{E_1^2}{\beta^2}\big)\}$, containing in its interior the connected component of $(\xi_1, -x_1) = 0$ in the potential well $V(\xi_1) \le E_1$, has a simple representation in the (ξ_1, θ_1)-plane, since it becomes symmetric with respect to the ξ_1 axis. This is an ovale-shaped curve, close to the circle of radius E_1 and center at the origin, with vertical tangents at $(\xi_1^{\pm}, 0)$, $\xi_1^- = -E_1 + \cdots, \xi_1^+ = E_1 + \cdots$ (the turning points for $p_{\omega\beta}$), and an horizontal tangent at $(\xi_1^0, \pm\theta_1^0) = \big(-2E'^2_1 + \cdots, \pm E'_1(1 + 2E'^2_1 + \cdots)\big)$. But the lack of analyticity of f in (183) forces to exclude $(\xi_1^+, 0)$ from that picture.

The critical values of $\psi^{\nu}_{\varepsilon,\beta}(\xi_1, \eta_1, \theta_1, s_1)$ are of the form

$$\begin{aligned}
&\psi^{\nu}_{\varepsilon,\beta} = T^{\nu}_{\varepsilon,\beta} - i\frac{E'^2_1}{4}\big(1 - \log\frac{E'^2_1}{2}\big) \\
&T^{\nu}_{\varepsilon,\beta}(\xi_1, \theta_c(\xi_1)) = \big[\xi_1\theta_1 + h_{\omega}(\xi_1, \theta_1) - \omega\frac{1}{2}\theta_1(E'^2_1 - \theta_1^2)^{1/2} - \frac{1}{2}E'^2_1\big]|_{\theta_1 = \theta_{\omega}(\xi_1)}
\end{aligned} \tag{219}$$

where we recall from (32)

$$\widehat{\Theta}_{\varepsilon,\omega}(\theta_1) = \arg\big(\frac{\varepsilon}{\sqrt{2}}\big(i\theta_1 + \omega(E'^2_1 - \theta_1^2)^{1/2}\big)\big)$$

As in Sect.2, we expand the integral by stationary phase according to Theorem 2.1, and find, at first order in h' :

$$u^\nu_{\varepsilon,\beta}(\xi',h) = C^\nu_{h'} \sum_{\theta_\omega} \exp i\big[T^\nu_{\varepsilon,\beta}\big(\xi_1,\theta_\omega\big)/h'\big]\big(J^\nu_\omega(\xi_1,\theta_\omega)\big)^{-1/2}\big[c_{0,\omega}(\xi_1,\eta^\nu_1(\theta_\omega),\theta_\omega)+\mathcal{O}(h')\big] \quad (220)$$

where we sum over the 2 critical values $\theta_\omega = \pm\widehat{\theta}_\omega(\xi_1)$, and

$$C^\nu_{h'} = \frac{\Gamma(\nu+1)}{-i(2\pi)^{3/2}h'}\Big(\frac{2eh'}{{E'_1}^2}\Big)^{{E'_1}^2/4h'} \quad (221)$$

Now we compute the half-density $\big(J^\nu_\omega\big)^{-1/2}$ (independent of ε), where

$$J^\nu_\omega\big(\xi_1,\theta_\omega(\xi_1)\big) = \det\big(\frac{1}{i}\,\mathrm{Hess}\,\psi^\nu_{\varepsilon,\omega}(\xi_1,\theta_1,\eta_1,s_1)\big)|_{\mathrm{p.c.}}$$

and "p.c". means : "evaluated at the critical point $(\theta_\omega,\eta_\omega,s_\omega)$" given in (209). We have :

$$\frac{1}{i}\,\mathrm{Hess}\,\psi^\nu_{\varepsilon,\beta} = \frac{1}{i}\begin{pmatrix} \partial^2_{\theta_1}h_\omega & -1 & 0 \\ -1 & i & i\,\varepsilon\sqrt{2} \\ 0 & i\,\varepsilon\sqrt{2} & i\big(1-\frac{{E'_1}^2}{2s_1^2}\big)\end{pmatrix}$$

At the critical point, $1-\frac{{E'_1}^2}{2s_1^2} = \frac{2i\theta_\omega}{-\,\varepsilon\sqrt{2}s_\omega}$, so that using (211) we have :

$$\begin{aligned} \mathrm{Im}\big(J^\nu_\omega(\xi_1,\theta_\omega)\big) &= \frac{2\omega({E'_1}^2-\theta_1^2)^{1/2}}{{E'_1}^2}\big(\theta_1+\omega({E'_1}^2-\theta_1^2)^{1/2}\partial^2_{\theta_1}h_\omega(\xi_1,\theta_1)\big)|_{\theta_1=\theta_\omega} \\ \mathrm{Re}\big(J^\nu_\omega(\xi_1,\theta_\omega)\big) &= \frac{2\theta_1}{{E'_1}^2}\big(\theta_1+\omega({E'_1}^2-\theta_1^2)^{1/2}\partial^2_{\theta_1}h_\omega(\xi_1,\theta_1)\big)|_{\theta_1=\theta_\omega} \end{aligned} \quad (222)$$

In polar coordinates

$$J^\nu_\omega\big(\xi_1,\theta_\omega(\xi_1)\big) = \big|J^{(\nu)}_\omega\big(\xi_1,\theta_\omega(\xi_1)\big)\big|\,\exp\big(i\,\check{\Theta}_{\mathrm{sgn}(\theta_1),\omega}(\theta_1)\big)\big|_{\theta_1=\theta_\omega(\xi_1)}$$

where we recall $\check{\Theta}_{\mathrm{sgn}(\theta_1),\omega}$ from (17). If we set

$$a^\nu_\omega\big(\xi_1,\theta_\omega(\xi_1)\big) = \big(J^\nu_\omega(\xi_1,\theta_1)\big)\big)^{-1/2}c_{0,\omega}\big(\xi_1,\eta^\nu_\omega(\theta_1),\theta_1\big)\big|_{\theta_1=\theta_\omega(\xi_1)} = \big|a^\nu_\omega(\xi_1,\theta_1)\big|\exp\big[iR^\nu_\omega(\theta_1)\big]\big|_{\theta_1=\theta_\omega(\xi_1)} \quad (223)$$

where

$$R^\nu_\omega(\theta_1) = -\frac{1}{2}\,\check{\Theta}_{\mathrm{sgn}(\theta_1),\omega}(\theta_1) \quad (224)$$

we get eventually

$$u^\nu_{\varepsilon,\omega}(\xi';h) = C^\nu_{h'}\sum_{\theta_1=\pm\widehat{\theta}_\omega(\xi_1)}\big|a^\nu_\omega\big(\xi_1,\theta_1)\big)\big|\,\exp\big[\frac{i}{h'}\,\big(T^\nu_{\varepsilon,\omega}(\xi_1,\theta_1)+h'\,R^\nu_\omega(\theta_1)\big] \quad (225)$$

In (223), $c_{0,\omega}\big(\xi_1,\eta^\nu_\omega(\theta_1),\theta_1\big)\big|_{\theta_1=\theta_\omega(\xi_1)}$ is an extension of $c_{0,\omega}\big|_{\Gamma'_\kappa}$ as prescribed in Theorem A.2. It is obtained by writing that $u^\nu_{\varepsilon,\omega}(\xi';h)$ actually solves (184) mod $\mathcal{O}(h')$.

10.2 Microlocal solutions with the $D_{-\nu-1}$ function

We proceed now to use $D_{-\nu-1}(i\,\varepsilon\,\zeta)$ instead of $D_\nu(\varepsilon\,\zeta)$, and outline the main steps of the computation. Recall from (13) the integral representation of $D_{-\nu-1}(i\,\varepsilon\,\zeta)$, so that as in (206) $u^{-\nu-1}_{\varepsilon,\omega}(\xi',h) = A_\omega\big(D_{-\nu-1}(i\,\varepsilon(h/2)^{-1/2}\cdot)\big)(\xi')$ takes the form

$$u^{-\nu-1}_{\varepsilon,\omega}(\xi',h) = \frac{\Gamma(-\nu)}{4i\pi^2 h'}\exp-\big[{E'_1}^2(\log(h')/4h'\big]\int\int_{\infty}^{(0+)} e^{i\psi^{-\nu-1}_{\varepsilon,\omega}/h'}c_\omega(\xi',\eta,\theta,h)\,d\theta_1 d\eta_1\frac{ds_1}{s_1} \tag{226}$$

where the variables (θ_1,η_1,s_1) are chosen as before, and

$$\psi^{-\nu-1}_{\varepsilon,\beta}(\xi_1,\eta_1,\theta_1,s_1) = (\xi_1-\eta_1)\theta_1+h_\beta(\xi_1,\theta_1)+i\big[-\frac{\eta_1^2}{2}+\frac{s_1^2}{2}+\sqrt{2}i\,\varepsilon\,\eta_1 s_1-\frac{1}{2}E_\beta^2\log(-s_1)\big] \tag{227}$$

The critical points (θ_1,η_1,s_1) are given by

$$\eta_1 = \xi_1+\partial_{\theta_1}h_\omega(\xi_1,\theta_1)=0,\ \ \eta_1 = i(\theta_1+\sqrt{2}\,\varepsilon\,s_1),\ \ \sqrt{2}\,\varepsilon\,\eta_1+s_1-\frac{E_\beta^2}{2s_1}=0 \tag{228}$$

They differ from $(\theta^\nu_\omega,\eta^\nu_\omega,s^\nu_\omega)$ only by the s_1 component, and we have the relation

$$s^{-\nu-1}_\omega(\theta_\omega(\xi_1)) = i\,\overline{s^\nu_\omega(\theta_\omega(\xi_1))} \tag{229}$$

It follows in particular that the extension of $c_{0,\omega}$ carried in the case of D_ν is again valid for $D_{-\nu-1}$. The critical values of $\psi^{\nu-1}_{\varepsilon,\omega}(\xi_1,\eta_1,\theta_1,s_1)$ are of the form

$$\psi^{-\nu-1}_{\varepsilon,\omega} = T^{-\nu-1}_{\varepsilon,\omega}+i\frac{{E'_1}^2}{4}\big(1-\log\frac{{E'_1}^2}{2}\big) \tag{230}$$

where

$$T^{-\nu-1}_{\varepsilon,\omega}(\xi_1,\theta_\omega(\xi_1)) = \big[\xi_1\theta_1+h_\omega(\xi_1,\theta_1)-\frac{\omega\theta_1}{2}({E'_1}^2-\theta_1^2)^{1/2}+\frac{{E'_1}^2}{2}\check{\Theta}(\omega_1)\big]\big|_{\theta_1=\theta_\omega(\xi_1)} \tag{231}$$

We apply again asymptotic stationary phase to (226), using the density

$$J^{-\nu-1}_\omega\big(\xi_1,\theta_\omega(\xi_1)\big) = \overline{J^\nu_\omega\big(\xi_1,\theta_\omega(\xi_1)\big)} \tag{232}$$

and

$$s^{-\nu-1}_{\varepsilon,\omega}\big(\theta_\omega(\xi_1)\big) = -\frac{\epsilon\ \mathrm{sgn}\,\big(\theta_\omega(\xi_1)\big)\,E'_1}{\sqrt{2}}\ \exp\big[-2i\,R^\nu_\omega\big(\theta_\omega(\xi_1)\big)\big] \tag{233}$$

If we set

$$\begin{aligned}\widetilde{a}^{-\nu-1}_\omega(\xi_1,\theta_\omega(\xi_1)) &= \frac{\sqrt{2}}{E'_1}\big|J^{-\nu-1}_\omega\big(\xi_1,\theta_1 g\big|^{-1/2}\,c_{0,\omega}\big(\xi_1,\eta_\omega(\theta_1),\theta_1\big)\ \exp\big[i\,R^{-\nu-1}_\omega(\theta_1)\big]\big|_{\theta_1=\theta_\omega(\xi_1)}\\ &= \frac{\sqrt{2}}{E'_1}a^{-\nu-1}_\omega(\xi_1,\theta_\omega(\xi_1))\end{aligned} \tag{234}$$

and

$$R^{-\nu-1}_\omega\big(\theta_\omega(\xi_1)\big) = R^\nu_\omega\big(\theta_\omega(\xi_1)\big) \tag{235}$$

we get eventually

$$u_{\varepsilon,\omega}^{-\nu-1}(\xi';h) = C_{h'}^{-\nu-1} \sum_{\theta_\omega=\pm\widehat{\theta_\omega}} \varepsilon\,\mathrm{sgn}(\theta_\omega)\,\big|\widetilde{a}_\omega^{-\nu-1}\big(\xi_1,\theta_\omega\big)\big|\,\exp\big[\frac{i}{h'}\big(T_{\varepsilon,\omega}^{-\nu-1}\big(\xi_1,\theta_\omega\big)+h'R_\omega^{-\nu-1}(\theta_\omega)\big)\big] \tag{236}$$

where

$$C_{h'}^{-\nu-1} = -\frac{\Gamma(-\nu)}{(2\pi)^{3/2}\,i\sqrt{h'}}\,\big(\frac{2e\,h'}{E_1'^{\,2}}\big)^{-E_1'^{\,2}/4h'} \tag{237}$$

With the notation of (48), $(T^\nu, T^{-\nu-1})$, $(a^\nu, a^{-\nu-1})$ are related by

$$\begin{aligned} T_{\varepsilon,\omega}^{-\nu-1}\big(\xi_1,\theta_\omega(\xi_1)\big) &= T_{\varepsilon,\omega}^{\nu}\big(\xi_1,\theta_\omega(\xi_1)\big) + \frac{\pi\,E_1'^{\,2}}{2}\,\zeta_{\varepsilon,\omega}\big(\theta_\omega(\xi_1)\big) \\ a_\omega^\nu(\xi_1,\theta_\omega(\xi_1)) &= a_\omega^{-\nu-1}(\xi_1,\theta_\omega(\xi_1)) \end{aligned} \tag{238}$$

10.3 Some symmetries

Many symmetries occur when changing the signs of $\theta, \varepsilon, \omega$. They imply in particular relations between microlocal solutions at a_E and a'_E. First group of parities involve the reflection in θ_1. We have

$$\widehat{\Theta}_{\varepsilon,\omega}(-\theta_1) = -\widehat{\Theta}_{\varepsilon,\omega}(\theta_1) \tag{239}$$

$$\check{\Theta}_{\varepsilon,\omega}(-\theta_1) = \pi - \check{\Theta}_{\varepsilon,\omega}(\theta_1), \quad \varepsilon\,\omega = 1 \tag{240}$$

$$\zeta_{-\,\varepsilon,-\omega}(-\theta_1) = \zeta_{\varepsilon,\omega}(\theta_1) \tag{241}$$

$$T_{\varepsilon,\omega}^{\nu}(\xi_1,-\theta_1) = -T_{\varepsilon,\omega}^{\nu}(\xi_1,\theta_1) \tag{242}$$

$$T_{\varepsilon,\omega}^{-\nu-1}(\xi_1,-\theta_1) = -T_{\varepsilon,\omega}^{-\nu-1}(\xi_1,\theta_1) + \frac{\pi\,E_1'^{\,2}}{2} \tag{243}$$

$$s_{\varepsilon,\omega}^{\nu}(-\theta_1) = \overline{s_{\varepsilon,\omega}^{\nu}(\theta_1)} \tag{244}$$

$$s_\omega^{-\nu-1}(-\theta_1) = -\overline{s_{\varepsilon,\omega}^{-\nu-1}(\theta_1)} \tag{245}$$

As $h_\omega(\xi_1,\theta_1)$ is odd in θ_1, we have the following relations

$$J_\omega^\nu(\xi_1,-\theta_1) = \overline{J_\omega^\nu(\xi_1,\theta_1)} \tag{246}$$

$$R_\omega^\nu(-\theta_1) = -R_\omega^\nu(\theta_1) \tag{247}$$

$$c_{0,\omega}\big(\xi_1,\eta_\omega(-\theta_1),-\theta_1\big) = c_{0,\omega}\big(\xi_1,\eta_\omega(\theta_1),\theta_1\big) \tag{248}$$

There follows

$$a_\omega^j(\xi_1,-\theta_1) = \overline{a_\omega^j(\xi_1,\theta_1)} \tag{249}$$

$$J_\omega^{-\nu-1}(\xi_1,-\theta_1) = \overline{J_\omega^{-\nu-1}(\xi_1,\theta_1)} \tag{250}$$

Second group of symmetries involve the reflexions in ω and ε :

$$\widehat{\Theta}_{-\varepsilon,-\omega}\big(\theta_{-\omega}(-\xi_1)\big) = -\widehat{\Theta}_{\varepsilon,\omega}\big(\theta_\omega(\xi_1)\big) \tag{251}$$

$$\check{\Theta}_{-\varepsilon,-\omega}\big(\theta_{-\omega}(-\xi_1)\big) = -\check{\Theta}_{\varepsilon,\omega}\big(\theta_\omega(\xi_1)\big) \tag{252}$$

$$T^{\nu}_{-\varepsilon,-\omega}\big(-\xi_1,\theta_{-\omega}(-\xi_1)\big) = -T^{\nu}_{\varepsilon,\omega}\big(\xi_1,\theta_\omega(\xi_1)\big) \tag{253}$$

$$T^{-\nu-1}_{\varepsilon,-\omega}\big(-\xi_1,\theta_{-\omega}(-\xi_1)\big) = -T^{-\nu-1}_{\varepsilon,\omega}\big(\xi_1,\theta_\omega(\xi_1)\big) \tag{254}$$

$$R^j_{-\omega}\big(\theta_{-\omega}(-\xi_1)\big) = -R^j_{\omega}\big(\theta_\omega(\xi_1)\big), \quad j=\nu,-\nu-1 \tag{255}$$

We also get:

$$h_{-\omega}(-\xi_1,\theta_1) = -h_\omega(\xi_1,\theta_1) \tag{256}$$

$$\theta_{-\omega}(-\xi_1) = \theta_\omega(\xi_1) \tag{257}$$

$$c_{0,-\omega}\big(-\xi_1,\eta_{-\omega}\big(\theta_{-\omega}(-\xi_1)\big),\theta_{-\omega}(-\xi_1)\big) = c_{0,\omega}\big(\xi_1,\eta_\omega\big(\theta_\omega(\xi_1)\big),\theta_\omega(\xi_1)\big) \tag{258}$$

$$J^j_{-\omega}\big(-\xi_1,\theta_{-\omega}(-\xi_1)\big) = \overline{J^j_\omega\big(\xi_1,\theta_\omega(\xi_1)\big)} \tag{259}$$

$$a^j_{-\omega}\big(-\xi_1,\theta_{-\omega}(-\xi_1)\big) = \overline{a^j_\omega\big(\xi_1,\theta_\omega(\xi_1)\big)}, \quad j=\nu,-\nu-1 \tag{260}$$

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