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Adaptive group LASSO selection in quantile models

GABRIELA CIUPERCA

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Abstract The paper considers a linear model with grouped explanatory variables. If the model errors are not with zero mean and bounded variance or if model contains outliers, then the least squares framework is not appropriate. Thus, the quantile regression is an interesting alternative. In order to automatically select the relevant variable groups, we propose and study here the adaptive group LASSO quantile estimator. We establish the sparsity and asymptotic normality of the proposed estimator in two cases: fixed number and divergent number of variable groups. Numerical study by Monte Carlo simulations confirms the theoretical results and illustrates the performance of the proposed estimator.

Keywords group selection · quantile model · adaptive LASSO · selection consistency · oracle properties.

Mathematics Subject Classification (2010) 62J05 · 62J07

1 Introduction

Classically, for the regression model, the errors are assumed to be independent, of mean zero and bounded variance. Then, the model is estimated by the least squares (LS) method, eventually with a penalty of LASSO type when automatic detection of significant variables is performed. If the assumptions on the first two moments of the model error are not satisfied, then the LS framework breaks down. In this case, an alternative is to consider the quantile regression with a LASSO type penalty. This is one of the interests of this paper. The quantile regression is robust and allows relaxation of the two first moment conditions of the model error.

Often enough in practice, for example in the variance analysis case, are considered the regression linear models with grouped variables. For models with grouped explanatory variables it is more meaningful to identify relevant variable groups instead of individual variables. If the errors have

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Normal distribution, then for detecting the relevant variable groups, the F-statistic test is used. If the errors are not Gaussian and if more the number of groups is large, then the F-statistic test is inappropriate. From where, another interest of this paper: we consider the quantile process with LASSO type penalty in order to automatically detect the irrelevant variable groups.

The automatic selection method of the grouped variables using the LASSO penalties was introduced by Yuan and Lin (2008) for gaussian errors, by proposing the LASSO group penalty for the process of the error squares sum. Several recent papers have considered group selection using LASSO type penalties. For fixed parameter space and mean zero, finite second moment i.i.d. model errors, Nardi and Rinaldo (2008) established the model selection consistency and asymptotic normality of nonzero group LASSO estimator. The same estimator is studied by Nardi and Rinaldo (2008) when number of covariates is larger, for particular case of normal errors. For gaussian errors, Xu and Ghosh (2015) realize a Bayesian variable selection by penalization of the error squares sum with Bayesian group LASSO. For this estimation method, the posterior median estimator satisfies the sparsity property. The adaptive group LASSO estimators, when the number p of groups is fixed, was studied by Wang and Leng (2008). For high-dimensional model, Wei and Huang (2010) studied the selection and estimation properties of the adaptive group LASSO, but under assumption that the errors are gaussian. Still for the error squares sum penalized with adaptive LASSO penalty, Zhang and Xiang (2015) consider the case of the number of groups p_n converges to infinity when $n \rightarrow \infty$, for i.i.d. errors ε such that $\mathbb{E}[\varepsilon] = 0$ and $Var[\varepsilon] < \infty$. The consistency and asymptotic normality of the parameter estimator are established. A paper that doesn't consider the LS penalized process, but a process associated to a twice differentiable convex function, with LASSO penalty, for the case p large and small n was considered by Wang et al. (2015). When the number of groups can grow at a certain polynomial rate, the automatic selection property of variable groups for a LS process with SCAD penalty has been proven in Guo et al. (2015). Automatic selection of the relevant variable groups, when p converges to infinity, has also considered by Zou and Zhang (2009) penalizing the LS process with adaptive elastic-net penalty. For a review of group selection methods and several applications of these methods the reader can see Huang et al. (2012).

In this paper we consider the model selection problem and the estimation in a linear model with p groups of explanatory variables. We propose and study the asymptotic properties of the adaptive group LASSO quantile estimator in two cases: p fixed and $p \rightarrow \infty$ as $n \rightarrow \infty$. This estimator is the minimizer of the quantile process penalized by an adaptive group LASSO penalty. The oracle properties, i.e. the automatic selection of significant variables groups and their asymptotic distribution, are proved.

The remainder of the paper is organized as follows. In Section 2 we present the model and introduce some notations used throughout in this paper. Oracle properties for the adaptive group LASSO quantile estimator are proved for p fixed in Section 3 and for $p \rightarrow \infty$ as $n \rightarrow \infty$ in Section 4. Section 5 reports some simulation results which illustrate the method interest. We compare the adaptive group LASSO quantile estimation performance with the adaptive group LASSO least squares estimations, proposed by Zhang and Xiang (2015). All proofs are given in Section 6.

2 Model and notations

In this section, we present the statistical model and we also introduce some notations used throughout in the paper.

We begin by introducing some general notations. All vectors and matrices are denoted by bold symbols and all vectors are written as column vectors. For a vector \mathbf{v} , we denote by \mathbf{v}^t its transposed

and by $\|\mathbf{v}\|$ its Euclidean norm. Notations $\xrightarrow[n \rightarrow \infty]{\mathcal{L}}$, $\xrightarrow[n \rightarrow \infty]{\mathbb{P}}$ represent the convergence in distribution and in probability, respectively, as $n \rightarrow \infty$. For a positive definite matrix \mathbf{M} , we denote by $\lambda_{\min}(\mathbf{M})$ and $\lambda_{\max}(\mathbf{M})$ its the smallest and largest eigenvalues, respectively.

We will also use the following notations: if V_n and U_n are random variable sequences, $V_n = o_{\mathbb{P}}(U_n)$ means that $\lim_{n \rightarrow \infty} \mathbb{P}[|U_n/V_n| > e] = 0$ for any $e > 0$, $V_n = O_{\mathbb{P}}(U_n)$ means that there exists a finite $C > 0$ such that $\mathbb{P}[|U_n/V_n| > C] < e$ for any n and e . If V_n and U_n are deterministic sequences, $V_n = o(U_n)$ means that the sequence $V_n/U_n \rightarrow 0$ for $n \rightarrow \infty$, $V_n = O(U_n)$ means that the sequence V_n/U_n is bounded for sufficiently large n .

Throughout this paper, C will denote generic constant; not depending on size n which may take different values in different formula or even in different parts of the same formula. The value of C is not of interest. We will also use the notation $\mathbf{0}_k$ for the zero k -vector.

We consider the following linear model with p groups of explanatory variables:

$$Y_i = \sum_{j=1}^p \mathbf{X}_{ij}^t \boldsymbol{\beta}_j + \varepsilon_i = \mathbb{X}_i^t \boldsymbol{\beta} + \varepsilon_i, \quad i = 1, \dots, n, \quad (1)$$

with Y_i, ε_i random variables. For each group $j = 1, \dots, p$, the vector of the parameters is $\boldsymbol{\beta}_j \equiv (\beta_{j1}, \dots, \beta_{jd_j}) \in \mathbb{R}^{d_j}$ and the design for observation i is \mathbf{X}_{ij} , a column vector of size d_j . The vector with all coefficients is $\boldsymbol{\beta} \equiv (\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_p)$ and for observation i , the vector with all explanatory variables is $\mathbb{X}_i = (\mathbf{X}_{i1}, \dots, \mathbf{X}_{ip})$. Denote by $\boldsymbol{\beta}_j^0 = (\beta_{j1}^0, \dots, \beta_{jd_j}^0)$ the true value (unknown) of the parameter $\boldsymbol{\beta}_j$. For observation i , we denote by $X_{i,j,k}$ the k th variable of the j th group.

We emphasize that for the i th sample, we observe (Y_i, \mathbb{X}_i) , $i = 1, \dots, n$.

The relevant groups of explanatory variables correspond to the nonzero vectors. Without loss of generality, on suppose that the first p_0 ($p_0 \leq p$) groups of explanatory variables are relevant:

$$\|\boldsymbol{\beta}_j^0\| \neq 0, \quad \text{for all } j \leq p_0 \quad \text{and} \quad \|\boldsymbol{\beta}_j^0\| = 0, \quad \text{for all } j > p_0,$$

where $\|\cdot\|$ is the Euclidean norm. Let r be the total number of explanatory variables, so $r = \sum_{j=1}^p d_j$. We denote by $r^0 = \sum_{j=1}^{p_0} d_j$. So, p_0 is the number of nonzero true parameter vectors and r^0 is the total number of parameters in these nonzero true vectors.

The multi-factor ANOVA model is an example of this model.

We introduce now the quantile framework. For a fixed quantile index $\tau \in (0, 1)$, the check function $\rho_{\tau}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+$ is defined by $\rho_{\tau}(u) = u(\tau - \mathbb{1}_{u < 0})$.

The quantile estimator of $\boldsymbol{\beta}$, is the minimizer of the quantile process associated to model (1):

$$\tilde{\boldsymbol{\beta}}_n \equiv \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^r} \sum_{i=1}^n \rho_{\tau}(Y_i - \mathbb{X}_i^t \boldsymbol{\beta}). \quad (2)$$

For the particular case $\tau = 1/2$ we obtain the median regression and (2) becomes the least absolute deviations estimator. A great advantage of the quantile framework is that, compared to classical estimation methods that are sensitive to outliers, the quantile method provides more robust estimators. Moreover, the required assumptions to the error moments are relaxed.

The estimator $\tilde{\boldsymbol{\beta}}_n = (\tilde{\boldsymbol{\beta}}_{n;1}, \tilde{\boldsymbol{\beta}}_{n;2}, \dots, \tilde{\boldsymbol{\beta}}_{n;p})$ has as d_j -subvector $\tilde{\boldsymbol{\beta}}_{n;j}$ for each group $j = 1, \dots, p$. The quantile estimation method doesn't perform automatic variable selection. For finding the zero

vectors, i.e. the irrelevant groups of variables, hypothesis tests are required. However when model (1) has a large group number p , it is useful to estimate simultaneously the parameter groups and to eliminate the irrelevant groups without crossing every time by a hypothesis test. The adaptive LASSO penalties have the advantage of automatic selection and of parameter estimation (see for example Zhang and Xiang (2015), Wei and Huang (2010), Wang and Leng (2008)).

In order to introduce and study the adaptive LASSO estimator, we consider the following index set

$$\mathcal{A} \equiv \{j; \|\beta_j^0\| \neq 0\} = \{1, \dots, p_0\}$$

and $\mathcal{A}^c \equiv \{j; \|\beta_j^0\| = 0\} = \{p_0 + 1, \dots, p\}$ its complementary set. The set \mathcal{A} contains the index set corresponding to groups with nonzero true parameters.

For β a r -vector of parameters, we denote by $\beta_{\mathcal{A}}$ the r^0 -subvector of β which contains β_j , for $j = 1, \dots, p_0$. Similarly, the $(r - r^0)$ -vector $\beta_{\mathcal{A}^c}$ contains β_j for $j = p_0 + 1, \dots, p$.

In practice, the set \mathcal{A} is unknown. Then, we must find the set \mathcal{A} and estimate the corresponding parameters.

In Sections 3 and 4 we will introduce an estimator, denoted $\widehat{\beta}_n^*$, which minimizes the quantile process penalized with an adaptive group LASSO penalty, for two cases: p fixed and $p \rightarrow \infty$ as $n \rightarrow \infty$. We generalize the adaptive LASSO quantile estimator proposed by Ciuperca (2016) for individual variable selection to the case of group selection. We call this estimator, adaptive group LASSO quantile (*ag-LASSO-Q*) estimator.

We say that $\widehat{\beta}_n^*$ satisfies the *oracle properties* if:

- (i) *asymptotic normality*: $\sqrt{n}(\widehat{\beta}_n^* - \beta^0)_{\mathcal{A}}$ converges in law to a centred Normal distribution.
- (ii) *sparsity property*: $\lim_{n \rightarrow \infty} \mathbb{P}[\mathcal{A} = \{j = 1, \dots, p; \|\widehat{\beta}_{n;j}^*\| \neq \mathbf{0}_{d_j}\}] = 1$.

3 Fixed p case

In this section we propose and study the asymptotic properties of the *ag-LASSO-Q* estimator for the parameter β of model (1) when the group number p is fixed.

We define the *ag-LASSO-Q* estimator by:

$$\widehat{\beta}_n^* \equiv \arg \min_{\beta \in \mathbb{R}^r} Q(\beta),$$

where $Q(\beta)$ is the penalized quantile process with the adaptive group LASSO penalty:

$$Q(\beta) \equiv \sum_{i=1}^n \rho_{\tau}(Y_i - \sum_{j=1}^p \mathbf{X}_{ij}^t \beta_j) + \mu_n \sum_{j=1}^p \widehat{\omega}_{n;j} \|\beta_j\|, \quad (3)$$

with the weight $\widehat{\omega}_{n;j} \equiv \|\widetilde{\beta}_{n;j}\|^{-\gamma}$, $\gamma > 0$. The estimator $\widehat{\beta}_n^*$ is written as $\widehat{\beta}_n^* = (\widehat{\beta}_{n;1}^*, \dots, \widehat{\beta}_{n;p}^*)$ and $\widehat{\beta}_{n;j}^*$ is a subvector of size d_j , for $j = 1, \dots, p$.

For a particular case of a quantile model with non-grouped variables, $d_j = 1$ for all $j = 1, \dots, p$, we obtain the adaptive LASSO quantile estimator proposed and studied by Ciuperca (2016).

Before presenting the main results for $\widehat{\boldsymbol{\beta}}_n^*$ in the fixed p case, we give the required assumptions. The tuning parameter μ_n and the constant γ are such that, for $n \rightarrow \infty$,

$$\mu_n \rightarrow \infty, \quad \frac{\mu_n}{\sqrt{n}} \rightarrow 0, \quad n^{(\gamma-1)/2} \mu_n \rightarrow \infty. \quad (4)$$

For the design $(\mathbb{X}_i)_{1 \leq i \leq n}$ we consider the following assumption:

(A1) $n^{-1} \max_{1 \leq i \leq n} \mathbb{X}_i^t \mathbb{X}_i \xrightarrow[n \rightarrow \infty]{} 0$ and $n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t \xrightarrow[n \rightarrow \infty]{} \boldsymbol{\Upsilon}$, with $\boldsymbol{\Upsilon}$ a $r \times r$ positive definite matrix.

For the errors ε_i we suppose that:

(A2) $(\varepsilon_i)_{1 \leq i \leq n}$ are independent, identically distributed, with $F : \mathcal{B} \rightarrow [0, 1]$ the distribution function and a continuous positive density f in a neighborhood of 0. The τ th quantile of ε_i is zero: $\tau = F(0)$. Moreover, for every $e \in \text{int}(\mathcal{B})$, $\mathbf{1}_r \in \mathbb{R}^r$ we have

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n \int_0^{\mathbf{x}_i^t \mathbf{1}_r} \sqrt{n} [F(e + v/\sqrt{n}) - F(e)] dv = \frac{1}{2} f(e) \mathbf{1}_r^t \boldsymbol{\Upsilon} \mathbf{1}_r, \quad (5)$$

where $\mathbf{1}_r$ is the r -vector with all components 1. The set \mathcal{B} is a real set, with $0 \in \mathcal{B}$.

Assumption (A1) is standard for LASSO methods and (A2) is classic for quantile regression (see Ciuperca (2016), Koenker (2005), Zou and Yuan (2008), Wu and Liu (2009)). Assumption (A1) requests that the design matrix has a reasonable good behaviour. For the tuning parameter μ_n , the same conditions on (4) are required in Ciuperca (2016) for adaptive LASSO quantile model but with ungrouped explanatory variables.

We make the remark that for ANOVA model, since in the analysis of variance there is a constraint for each level of a factor, we consider as constraint that the effect of this level is zero. Then this zero level is not considered in the model in order that assumption (A1) is satisfied.

In order to study the asymptotic properties of the estimator $\widehat{\boldsymbol{\beta}}_n^*$, let us consider the index set of the groups selected by the adaptive group LASSO quantile method:

$$\widehat{\mathcal{A}}_n^* \equiv \{j \in \{1, \dots, p\}; \|\widehat{\boldsymbol{\beta}}_{n;j}^*\| \neq 0\}$$

and $\widehat{\mathcal{A}}_n^{*c}$ its complementary set.

The following Theorem shows that the *ag-LASSO-Q* estimators with the index in the set \mathcal{A} are asymptotically Gaussian. Then, the estimators of the nonzero parameter vectors have the same asymptotic distribution they would have if the zero parameter vectors were known.

Theorem 1 *Under assumptions (A1), (A2) and condition (4), we have $\sqrt{n}(\widehat{\boldsymbol{\beta}}_n^* - \boldsymbol{\beta}^0)_{\mathcal{A}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_{r,0}, \tau(1-\tau)f^{-2}(0)\boldsymbol{\Upsilon}_{\mathcal{A}}^{-1})$, with $\boldsymbol{\Upsilon}_{\mathcal{A}}$ the submatrix of $\boldsymbol{\Upsilon}$ with the row and column indices in $\{1, \dots, d_1, d_1 + 1, \dots, d_1 + d_2, \dots, \sum_{j=1}^{p_0} d_j\}$.*

We give now the Karush-Kuhn-Tucker(KKT) optimality conditions, needed to prove the sparsity property for $\widehat{\boldsymbol{\beta}}_n^*$.

For all $j \in \widehat{\mathcal{A}}_n^*$, we have, with probability one, the following d_j equalities

$$\tau \sum_{i=1}^n \mathbf{X}_{ij} - \sum_{i=1}^n \mathbf{X}_{ij} \mathbb{1}_{Y_i < \mathbb{X}_i^t \widehat{\boldsymbol{\beta}}_n^*} = \frac{\mu_n \widehat{\omega}_{n;j} \widehat{\boldsymbol{\beta}}_{n;j}^*}{\|\widehat{\boldsymbol{\beta}}_{n;j}^*\|}. \quad (6)$$

For all $j \notin \widehat{\mathcal{A}}_n^*$, for all $k = 1, \dots, d_j$ we have, with probability one, the following inequality

$$\left| \tau \sum_{i=1}^n X_{ij,k} - \sum_{i=1}^n X_{ij,k} \mathbb{1}_{Y_i < \mathbb{X}_i^t \widehat{\boldsymbol{\beta}}_n^*} \right| \leq \mu_n \widehat{\omega}_{n;j}. \quad (7)$$

The following theorem shows the sparsity property of the *ag-LASSO-Q* estimator. This result states that the adaptive group LASSO quantile estimators of the nonzero parameter vectors are exactly nonzero with a probability converging to one when n diverges to infinity.

Theorem 2 *Under the assumptions of Theorem 1 and under the condition $n^{\gamma/2-1} \lambda_n \rightarrow \infty$, as $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} \mathbb{P}[\widehat{\mathcal{A}}_n^* = \mathcal{A}] = 1$.*

Theorem 1 and Theorem 2 establish the asymptotic normality and the sparsity of the *ag-LASSO-Q* estimator, which means that this estimator still share the oracle properties in the case of fixed p .

Remark 1 For the weight $\widehat{\omega}_{n;j}$ associated to the j th group, we considered the quantile estimator norm to the power $-\gamma$. In view of the proofs of Theorem 1 and Theorem 2, these two theorems remain true also when $\widetilde{\boldsymbol{\beta}}_{n;j}$ is replaced by any estimator of $\boldsymbol{\beta}_j$, with convergence rate $n^{-1/2}$, under assumptions (A1), (A2).

4 The case of p depending on n

Consider now same model (1) with grouped variables, but with the number p of groups depending on n : $p = p_n$ and $p_n \rightarrow \infty$ as $n \rightarrow \infty$. More precisely, we consider $p_n = O(n^c)$, with the constant $c \in (0, 1)$. For readability, we keep the notation p instead of p_n . Similarly, we have $r = \sum_{j=1}^p d_j$, with r depending on n . Always for simplicity of notation, for the design \mathbb{X}_i , for the parameter $\boldsymbol{\beta}$, ever if their dimension depends on n , we do not put subscript n .

We will first find the convergence rate of the quantile estimator $\widetilde{\boldsymbol{\beta}}_n$ of $\boldsymbol{\beta}$. Afterwards, we will propose for $\boldsymbol{\beta}$ an adaptive group LASSO quantile estimator. Even though the number p diverges as $n \rightarrow \infty$, this estimator keeps the oracle properties.

Since the design size depends on n , we need reconsider the assumptions on \mathbb{X}_i . Then, let us consider the following assumptions for the errors (ε_i) , design (\mathbb{X}_i) and for the number p of groups: **(A3)** $(\varepsilon_i)_{1 \leq i \leq n}$ are i.i.d. Let F be the distribution function and f be the density function of (ε_i) . The density function f is continuously, strictly positive in a neighbourhood of zero and has a bounded first derivative in the neighbourhood of 0. The τ th quantile of ε_i is zero: $\tau = F(0)$.

(A4) There exist two constants $0 < m_0 \leq M_0 < \infty$, such that $m_0 \leq \lambda_{\min}(n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t) \leq \lambda_{\max}(n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t) \leq M_0$.

(A5) $(p/n)^{1/2} \max_{1 \leq i \leq n} \|\mathbb{X}_i\| \rightarrow 0$, as $n \rightarrow \infty$.

(A6) p is such that $p = O(n^c)$, with $0 < c < 1$.

Since $p \rightarrow \infty$, condition (5) of assumption (A2) for the case p fixed is now replaced by f' bounded in the neighborhood of 0. This assumption also been considered for always high-dimensional quantile model, with seamless L_0 penalty by Ciuperca (2015). In Ciuperca (2015), assumptions (A4) and (A5) are also required. Assumption (A6) was considered by Zhang and Xiang (2015) for an high-dimensional linear model where the objective function is the error squares sum, penalized with

an adaptive group LASSO penalty. Assumptions (A4), (A5), (A6) are also required for an high-dimensional linear model by Zou and Zhang (2009), which penalize the LS process with adaptive elastic-net penalty. In respect to the case p fixed, assumptions (A4) and (A5) are the similar of (A1).

We will start by finding the convergence rate of quantile estimator (2) in the case $p \rightarrow \infty$ as $n \rightarrow \infty$. For this, consider the quantile process:

$$G_n(\boldsymbol{\beta}) \equiv \sum_{i=1}^n \rho_\tau(Y_i - \mathbb{X}_i^t \boldsymbol{\beta}).$$

For the quantile estimator existence, we assume that the total number r of parameters is strictly less than n .

We recall that in the case p fixed, the convergence rate of the quantile estimator $\tilde{\boldsymbol{\beta}}_n$ is of order $n^{-1/2}$ (see for example Koenker (2005)). We will show that, the quantile estimator has the convergence of order $(p/n)^{1/2}$, when the explanatory group variable number diverges with the sample size. In view of the proof of Lemma 1, the convergence rate of $\tilde{\boldsymbol{\beta}}_n$ depends only of p and not of total number r of parameters, thanks to assumption (A5). One needs the convergence rate of the quantile estimator is necessary for studying the asymptotic behaviour of the penalty which intervenes in adaptive group LASSO quantile process.

Lemma 1 *Under assumptions (A3)-(A6), we have $\|\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}^0\| = O_{\mathbb{P}}(\sqrt{\frac{p}{n}})$.*

Consider now the following adaptive group LASSO quantile (*ag-LASSO-Q*) estimator:

$$\hat{\boldsymbol{\beta}}_n^* \equiv \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^d} \left(\frac{1}{n} G_n(\boldsymbol{\beta}) + \lambda_n \sum_{j=1}^p \hat{\omega}_{n;j} \|\boldsymbol{\beta}_j\| \right),$$

where λ_n is a tuning parameter (positive) and the weights of the LASSO penalty are $\hat{\omega}_{n;j} \equiv \|\tilde{\boldsymbol{\beta}}_{n;j}\|^{-\gamma}$, with $\gamma > 0$. The relation between the tuning parameter μ_n of relation (3) for the case p fixed and λ_n for the case p depending on n is $\lambda_n = \mu_n/n$. We prefer to consider these forms as tuning parameter and as objective process, for having a similarity with the adaptive group LASSO LS (*ag-LASSO-LS*) case considered by Zhang and Xiang (2015).

In order to study the asymptotic normality of $\hat{\boldsymbol{\beta}}_n^*$ we need to impose an additional condition on the total number of nonzero parameters. More precisely, r^0 it is assumed to be the same order as p_0 . This is for controlling the penalty, so that it is smaller than the quantile process.

Concerning the size of the nonzero parameter vectors, we take the following assumption:

(A7) $r^0 = O(p_0)$.

For the smallest nonzero vector norm and on constant c of assumption (A6) we assume:

(A8) Let us denote $h_0 \equiv \min_{1 \leq j \leq p_0} \|\boldsymbol{\beta}_j^0\|$. There exists a constant $M > 0$ such that $M \leq n^{-\alpha} h_0$ and $\alpha > (c-1)/2$.

These two assumptions were also found in the paper Zhang and Xiang (2015), for *ag-LASSO-LS* method in high-dimensional linear model, but with a supplementary condition for r : $r = O(p)$. Here, we do not need this requirement, since assumption (A5) is imposed. On the other hand, in

Zhang and Xiang (2015), instead of assumption (A5) the condition $n^{-1/2} \max_{1 \leq i \leq n} \|\mathbb{X}_{i,\mathcal{A}}\|^2 \rightarrow 0$, as $n \rightarrow \infty$, is required.

The following theorem gives the convergence rate of the *ag-LASSO-Q* estimator when $p \rightarrow \infty$. We obtain the same convergence rate that of quantile estimator when group number diverges. This convergence rate is also obtained by Zhang and Xiang (2015) for the *ag-LASSO-LS* estimator, but for errors $(\varepsilon)_{1 \leq i \leq n}$ with mean zero and bounded variance.

Theorem 3 *Under assumptions (A3)-(A6), (A8) and the tuning parameter $(\lambda_n)_{n \in \mathbb{N}}$ satisfying $\lambda_n n^{(1+c)/2-\alpha\gamma} \rightarrow 0$, as $n \rightarrow \infty$, we have $\|\widehat{\boldsymbol{\beta}}_n^* - \boldsymbol{\beta}^0\| = O_{\mathbb{P}}(\sqrt{\frac{p}{n}})$.*

The following theorem shows the oracle properties for *ag-LASSO-Q* estimator when the number p of groups diverges. We denote by $\mathbb{X}_{i,\mathcal{A}}$ a r^0 -vector which contains the sub-vectors $\mathbf{X}_{i,j}$, for $j \in \{1, \dots, p_0\}$.

Theorem 4 *Suppose that assumptions (A3)-(A6), (A8) are satisfied and also that the tuning parameter satisfies $\lambda_n n^{(1-c)(1+\gamma)/2} \rightarrow \infty$, $\lambda_n n^{(c+1)/2-\alpha\gamma} \rightarrow 0$, as $n \rightarrow \infty$. Then:*

(i) $\mathbb{P}[\widehat{\mathcal{A}}_n^* = \mathcal{A}] \rightarrow 1$, for $n \rightarrow \infty$.

(ii) *If moreover assumption (A7) holds, then, for any vector \mathbf{u} of size r^0 such that $\|\mathbf{u}\| = 1$, with notation $\boldsymbol{\Upsilon}_{n,\mathcal{A}} \equiv n^{-1} \sum_{i=1}^n \mathbb{X}_{i,\mathcal{A}} \mathbb{X}_{i,\mathcal{A}}^t$, we have $\sqrt{n}(\mathbf{u}^t \boldsymbol{\Upsilon}_{n,\mathcal{A}}^{-1} \mathbf{u})^{-1/2} \mathbf{u}^t (\widehat{\boldsymbol{\beta}}_n^* - \boldsymbol{\beta}^0)_{\mathcal{A}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \tau(1 - \tau)f^{-2}(0))$.*

For the tuning parameter λ_n , the same conditions are required in Zhang and Xiang (2015) such that, the *ag-LASSO-LS* estimator in an high-dimensional linear model satisfies the oracle properties.

Remark 2 As for the case p fixed, we considered the weight $\widehat{\omega}_{n;j} = \|\widetilde{\boldsymbol{\beta}}_{n;j}\|^{-\gamma}$, with $\widetilde{\boldsymbol{\beta}}_{n;j}$ the quantile estimator of the d_j -vector $\boldsymbol{\beta}_j$, for any $j = 1, \dots, p$. In view of the proof of Theorem 4, the oracle properties for *ag-LASSO-Q* estimator remain true also when $\widetilde{\boldsymbol{\beta}}_{n;j}$ is replaced by any $(p/n)^{1/2}$ -estimator of $\boldsymbol{\beta}_j$, under assumptions (A3)-(A6).

Remark 3 If h_0 , defined in assumption (A8), doesn't depend on n , then $\alpha = 0$. In this case, the conditions required on $(\lambda_n)_{n \in \mathbb{N}}$ in Theorem 3 imply $\gamma > 2c/(1-c)$, and then γ can take values in the interval $(0, \infty)$. The value of γ increase with that of $c \in (0, 1)$. For example, if $c = 1/2$ then $\gamma > 2$.

5 Simulations

In order to evaluate the performance of the proposed estimation method, Monte Carlo simulations are realized in this section. To assess this performance we compare the *ag-LASSO-Q* and *ag-LASSO-LS* estimation methods.

The design \mathbb{X}_i is generated in the same way as in paper Wei and Huang (2010): $\mathbb{X} = (\mathbf{X}_1, \dots, \mathbf{X}_p)$, with the group explanatory variables $\mathbf{X}_j = (X_{5(j-1)+1}, \dots, X_{5j})$, for all $j = 1, \dots, p$. We first generate $r = 5p$ independent random variables R_1, \dots, R_r of standard normal distribution. We

also generate the variables Z_j of multivariate normal distribution with mean zero and covariance $Cov(Z_{j_1}, Z_{j_2}) = 0.9^{|j_1 - j_2|}$. Finally, the variables X_1, \dots, X_r are generated as:

$$X_{5(j-1)+k} = \frac{Z_j + R_{5(j-1)+k}}{\sqrt{2}}, \quad 1 \leq j \leq p, \quad 1 \leq k \leq 5.$$

Two model errors are considered: Normal $\mathcal{N}(0, 3^2)$ and Cauchy $\mathcal{C}(0, 3^2)$. For the parameters we take: $\beta_1^0 = (0.5, 1, 1.5, 1, 0.5)$, $\beta_2^0 = (1, 1, 1, 1, 1)$, $\beta_3^0 = (-1, 0, 1, 2, 1.5)$, $\beta_4^0 = (-1.5, 1, 0.5, 0.5, 0.5)$ and all other parameters are zero vectors. The nonzero vectors were also considered in Example 2 of Wei and Huang (2010) for errors $\mathcal{N}(0, 3^2)$, $p = 10$, when the parameters were estimated by LS method with adaptive group LASSO penalty.

The constant c of assumption (A6) is $c = 0.43$. Then, we will consider the following value couples for n and p : (30, 5), (60, 5), (60, 10), (100, 10), (200, 10), (400, 15), (1000, 25) and (1000, 100). On the other hand, p^0 will always be equal to 4. The response variable Y is generated as: $Y_i = \sum_{j=1}^p \mathbf{X}_{ij}^t \beta_j^0 + \varepsilon_i$, for $i = 1, \dots, n$.

We will compare the obtained results by the adaptive group LASSO quantile method, proposed in this paper, with those obtained by the adaptive group LASSO LS method, proposed by Wei and Huang (2010), Zhang and Xiang (2015).

For simulations, we used the R language. After a scale transformation, we can use the group LASSO methods instead of the adaptive LASSO group methods. Then, in order to calculate the adaptive group LASSO LS estimations we have used the function *grpreg* of package *grpreg*, the tuning parameter being chosen on a value grid, using the AIC criterion. In order to calculate the adaptive group LASSO quantile estimations, we have used the function *groupQICD* of package *rqPen* and the tuning parameter varies on a value grid.

For each considered case, 1000 Monte Carlo replications was made.

In Table 1 we give how the two estimation methods identify the parameter vectors (zero or nonzero), for the part that contains the four nonzero parameter vectors β_j^0 , $j = 1, \dots, 4$, and for the part with $p - 4$ zero vectors. We present the minimum, three quartiles (Q1, median, Q3), the mean and the maximum of the number of nonzero vectors ($j = 1, \dots, 4$), respectively, zero ($j = 5, \dots, p$), found by the two estimation methods.

For n large (equal to 100, 200, 400, 1000), we observe that for errors of $\mathcal{N}(0, 3^2)$ law, the two estimation methods well identify the zero and nonzero parameter vectors. However, for Cauchy errors, the *ag-LASSO-LS* method poorly identifies nonzero vectors (the group of the four significant variables). The zero vectors are very well identified by the two methods.

For n small (equal to 30 or 60), the two estimation methods well identify the four relevant variable groups, that errors are Normal or Cauchy (except for *ag-LASSO-Q*, in the case $n = 60$, $p = 5$, $\varepsilon \sim \mathcal{C}(0, 3^2)$). However, the $(p - 4)$ irrelevant variable groups are not well identified by the *ag-LASSO-LS* method.

Conclusion

For gaussian errors, the *ag-LASSO-LS* method identifies well the two (relevant and irrelevant) variable groups for n large. For n small, the irrelevant variable groups are not well identified. For Cauchy errors, this method, either does not identify the relevant variable groups or irrelevant variable groups, regardless of the value n . Then, for Cauchy errors, the *ag-LASSO-LS* estimations do not have the sparsity property.

The *ag-LASSO-Q* method, for the two types of errors, identifies the two variable groups (significant and irrelevant), the precision increasing with n . Then, the *ag-LASSO-Q* estimations have the sparsity property.

We conclude then that the simulations confirm the theoretical results for the *ag-LASSO-Q* estimators.

6 Proofs

In this section we provide the proofs of all results presented in Sections 3 and 4.

6.1 Proofs for results of Section 3

Proof of Theorem 1. The proof is similar to that of Theorem 4.1 of Zou and Yuan (2008).

We denote $\sqrt{n}(\hat{\beta}_n^* - \beta^0) \equiv \hat{\mathbf{u}}_n$, and in general $\sqrt{n}(\beta - \beta^0) \equiv \mathbf{u} \equiv (\mathbf{u}_1, \dots, \mathbf{u}_p)$, with $\mathbf{u}_j = (u_{j,1}, \dots, u_{j,d_j})$, for $j = 1, \dots, p$.

Since $Y_i = \mathbb{X}_i^t \beta^0 + \varepsilon_i$, then $Y_i - \mathbb{X}_i^t \beta = \frac{\mathbb{X}_i^t \mathbf{u}}{\sqrt{n}} + \varepsilon_i$. Let us consider the following random variables

$$\begin{aligned} \mathcal{D}_i &\equiv (1 - \tau)\mathbb{1}_{\varepsilon_i < 0} - \tau\mathbb{1}_{\varepsilon_i \geq 0}, \\ v_n &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathcal{D}_i, \\ B_n(\mathbf{u}) &\equiv \sum_{i=1}^n \int_0^{\mathbb{X}_i^t \mathbf{u} / \sqrt{n}} [\mathbb{1}_{\varepsilon_i < t} - \mathbb{1}_{\varepsilon_i < 0}] dt \end{aligned} \quad (8)$$

and the random vector

$$\mathbf{z}_n \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{X}_i \mathcal{D}_i.$$

Obviously, $E[\mathcal{D}_i] = 0$ et $E[\mathbf{z}_n] = \mathbf{0}_r$. By the CLT, using assumptions (A1) and (A2), we have

$$\mathbf{z}_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_r, \tau(1 - \tau)\mathbf{\Upsilon}), \quad v_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \tau(1 - \tau)). \quad (9)$$

The vector $\hat{\mathbf{u}}_n$ is the minimizer of the following random process:

$$L_n(\mathbf{u}) \equiv \sum_{i=1}^n \left[\rho_\tau\left(\varepsilon_i - \frac{\mathbb{X}_i^t \mathbf{u}}{\sqrt{n}}\right) - \rho_\tau(\varepsilon_i) \right] + \mu_n \sum_{j=1}^p \hat{\omega}_{n;j} \left[\left\| \beta_j^0 + \frac{\mathbf{u}_j}{\sqrt{n}} \right\| - \|\beta_j^0\| \right],$$

which can be written under the following form:

$$L_n(\mathbf{u}) = [\mathbf{z}_n^t \mathbf{u} + B_n(\mathbf{u})] + \mu_n \sum_{j=1}^p \hat{\omega}_{n;j} \left[\left\| \beta_j^0 + \frac{\mathbf{u}_j}{\sqrt{n}} \right\| - \|\beta_j^0\| \right] \frac{\sqrt{n}}{\sqrt{n}}. \quad (10)$$

We first study the last sum of the right hand side of (10).

For all $j \leq p_0$ (thus $\|\beta_j^0\| \neq 0$) we have, since the quantile estimators are consistent:

$$\hat{\omega}_{n;j} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \|\beta_j^0\|^{-\gamma} \neq 0 \quad (11)$$

Table 1 Model selection results by ag_LASSO_Q and ag_LASSO_LS methods for $p^0 = 4$, $c = 0.43$, errors $\mathcal{N}(0, 3^2)$ or $\mathcal{C}(0, 3^2)$.

n	p	ϵ	nonzero true parameter vectors												zero true parameter vectors												
			min		Q1		Q2		mean		Q3		max		true	min		Q1		Q2		mean		Q3		max	
			LS	Q	LS	Q	LS	Q	LS	Q	LS	Q	LS	Q	LS	Q	LS	Q	LS	Q	LS	Q	LS	Q	LS	Q	LS
30	5	C	0	0	4	3	4	3	3.5	3.3	4	4	4	4	1	0	0	0	0	0	0	0.1	0.7	0	1	1	1
		N	4	3	4	4	4	4	4	4	3.8	4	4	4	4	1	0	0	0	0	0	0.01	0.5	0	1	1	1
60	5	C	0	1	0	3	1	4	1.3	3.6	4	4	4	4	1	0	0	0	1	1	1	0.7	0.8	1	1	1	1
		N	4	3	4	4	4	4	4	4	3.9	4	4	4	4	1	0	0	0	0	0	0.1	0.7	0	1	1	1
60	10	C	0	0	4	3	4	4	3.4	3.5	4	4	4	4	6	0	2	0	5	0	5	0.8	5.2	0	6	6	6
		N	4	3	4	4	4	4	4	3.9	4	4	4	4	6	0	1	0	4	0	4	0.06	4.3	0	5	6	6
100	10	C	0	3	0	4	0	4	0.9	3.9	1	4	4	4	6	0	2	5	4	6	5	5	5	6	6	6	6
		N	4	3	4	4	4	4	4	3.9	4	4	4	4	6	0	1	0	4	0	5	2	4.4	5	5	6	6
200	10	C	0	3	0	3	0	4	0.5	3.7	1	4	4	4	6	0	2	6	6	6	6	5.7	5.7	6	6	6	6
		N	4	3	4	4	4	4	4	3.9	4	4	4	4	6	0	3	0	5	4	6	3.1	5.5	6	6	6	6
400	15	C	0	3	0	3	0	4	0.4	3.6	11	0	4	4	4	8	10	11	11	11	11	10.8	10.9	11	11	11	11
		N	4	3	4	4	4	4	4	3.99	4	4	4	4	11	0	9	9	11	10	11	9.3	10.8	11	11	11	11
1000	25	C	0	4	0	4	0	4	0.5	4	1	4	4	4	21	19	19	21	21	21	21	20.8	20.7	21	21	21	21
		N	4	4	4	4	4	4	4	4	4	4	4	4	21	17	20	20	21	21	21	20.3	20.9	21	21	21	21
	100	C	0	3	0	3	0	4	0.3	3.5	0	4	3	4	96	94	96	96	96	96	96	95.8	96	96	96	96	96
		N	4	3	4	4	4	4	4	3.98	4	4	4	4	96	92	95	95	96	96	96	95.4	96	96	96	96	96

and by elementary calculus

$$\sqrt{n}[\|\beta_j^0 + n^{-1/2}\mathbf{u}_j\| - \|\beta_j^0\|] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{\mathbf{u}_j^t \beta_j^0}{\|\beta_j^0\|}. \quad (12)$$

Then, using condition $\mu_n n^{-1/2} \rightarrow 0$, when $n \rightarrow \infty$, taking into account relations (11) and (12), by Slutsky's Lemma, we have:

$$\mu_n \sum_{j=1}^{p_0} \widehat{\omega}_{n;j} [\|\beta_j^0 + n^{-1/2}\mathbf{u}_j\| - \|\beta_j^0\|] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (13)$$

For $j > p_0$, we have $\beta_j^0 = \mathbf{0}_{d_j}$. Then: $\sqrt{n}[\|\beta_j^0 + n^{-1/2}\mathbf{u}_j\| - \|\beta_j^0\|] = \|\mathbf{u}_j\|$. Since $\widehat{\omega}_{n;j} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \infty$, by assumption $n^{(\gamma-1)/2} \mu_n \rightarrow \infty$, we have $n^{-1/2} \mu_n \widehat{\omega}_{n;j} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \infty$. Thus

$$\mu_n \widehat{\omega}_{n;j} \frac{\sqrt{n}}{\sqrt{n}} [\|\beta_j^0 + n^{-1/2}\mathbf{u}_j\| - \|\beta_j^0\|] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \begin{cases} 0, & \text{if } \mathbf{u}_j = \mathbf{0}_{d_j} \\ \infty, & \text{if } \|\mathbf{u}_j\| \neq 0. \end{cases} \quad (14)$$

Then, taking into account relations (13) and (14), we have the following result for the third term of the right hand side of (10):

$$\mu_n \sum_{j=1}^p \widehat{\omega}_{n;j} \left[\left\| \beta_j^0 + \frac{\mathbf{u}_j}{\sqrt{n}} \right\| - \|\beta_j^0\| \right] \frac{\sqrt{n}}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \sum_{j=1}^p W(\beta_j^0, \mathbf{u}), \quad (15)$$

with

$$W(\beta_j^0, \mathbf{u}) \equiv \begin{cases} 0, & \text{if } \beta_j^0 \neq \mathbf{0}_{d_j} \\ 0, & \text{if } \beta_j^0 = \mathbf{0}_{d_j} \text{ and } \mathbf{u}_j = \mathbf{0}_{d_j} \\ \infty, & \text{if } \beta_j^0 = \mathbf{0}_{d_j} \text{ and } \mathbf{u}_j \neq \mathbf{0}_{d_j}. \end{cases}$$

On the other hand, by the two results of (9), we have for the first two terms of the right hand side of (10), with \mathbf{z} a random d -vector of law $\mathcal{N}(\mathbf{0}_d, \tau(1-\tau)\Upsilon)$, that

$$\mathbf{z}^t \mathbf{u} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathbf{z}^t \mathbf{u}, \quad B_n(\mathbf{u}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{1}{2} f(0) \mathbf{u}^t \Upsilon \mathbf{u}.$$

Taking into account these last two results and relation (15), then, $L_n(\mathbf{u})$ of relation (10) has an asymptotic distribution:

$$L_n(\mathbf{u}) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathbf{z}^t \mathbf{u} + \frac{1}{2} f(0) \mathbf{u}^t \Upsilon \mathbf{u} + \sum_{j=1}^p W(\beta_j^0, \mathbf{u}).$$

Let us denote $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ with \mathbf{u}_1 of size r^0 , \mathbf{u}_2 of size $r - r^0$ and $\widehat{\mathbf{u}}_n = (\widehat{\mathbf{u}}_{1n}, \widehat{\mathbf{u}}_{2n})$, where $\widehat{\mathbf{u}}_{1n}$ contains the first $\sum_{j=1}^{p_0} d_j = r^0$ elements of \mathbf{u} . Since $\widehat{\mathbf{u}}_n = \arg \min_{\mathbf{u} \in \mathbb{R}^r} L_n(\mathbf{u})$, we obtain that $\widehat{\mathbf{u}}_{2n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \mathbf{0}_{r-r^0}$ et $\widehat{\mathbf{u}}_{1n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_{r^0}, \tau(1-\tau)f^{-2}(0)\Upsilon_{\mathcal{A}}^{-1})$. \blacksquare

Proof of Theorem 2. By Theorem 1, for all $j \in \mathcal{A}$ we have that $\sqrt{n}(\widehat{\beta}_{n;j}^* - \beta_j^0) \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}_{d_j}, \tau(1 - \tau)f^{-2}(0)\mathbf{\Upsilon}_{\mathcal{A}_j})$ as $n \rightarrow \infty$. The square matrix $\mathbf{\Upsilon}_{\mathcal{A}_j}$ of size $d_j \times d_j$ is the submatrix of $\mathbf{\Upsilon}$ with the row and column indices in $\{d_{j-1} + 1, d_{j-1} + 2, \dots, d_j\}$, with $d_0 = 0$. Since $\beta_j^0 \neq \mathbf{0}_{d_j}$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}[\mathcal{A} \subseteq \widehat{\mathcal{A}}_n^*] = 1. \quad (16)$$

To finish the proof we show that for all $j \notin \mathcal{A}$ we have $\mathbb{P}[j \in \widehat{\mathcal{A}}_n^*] \rightarrow 0$ as $n \rightarrow \infty$. Since $j \notin \mathcal{A}$, then $\beta_j^0 = \mathbf{0}_{d_j}$. Considering the Euclidean norm for equalities (6) we have with probability one, since we suppose $j \in \widehat{\mathcal{A}}_n^*$, that:

$$\mu_n \widehat{\omega}_{n;j} < 2 \left\| \sum_{i=1}^n \mathbf{X}_{ij} \right\| \leq 2 \sum_{i=1}^n \|\mathbf{X}_{ij}\| = 2 \sum_{i=1}^n \left(\sum_{k=1}^{d_j} X_{ij,k}^2 \right)^{1/2}.$$

By the Cauchy-Schwarz inequality, we have that,

$$\sum_{i=1}^n \frac{1}{n} \left(\sum_{k=1}^{d_j} X_{ij,k}^2 \right)^{1/2} \leq \left(\frac{1}{n} \sum_{i=1}^n \left(\sum_{k=1}^{d_j} X_{ij,k}^2 \right) \right)^{1/2} = \left(\frac{1}{n} \sum_{i=1}^n \|\mathbf{X}_{ij}\|^2 \right)^{1/2}.$$

Then, taking into account assumption (A1), there exists a bounded constant $C_1 > 0$ such that

$$\frac{1}{n} \mu_n \widehat{\omega}_{n;j} < 2 \left(\frac{1}{n} \sum_{i=1}^n \|\mathbf{X}_{ij}\|^2 \right)^{1/2} \leq C_1 < \infty. \quad (17)$$

On the other hand, left-hand side of inequality (17), can be written:

$$\frac{\mu_n \widehat{\omega}_{n;j}}{n} = \frac{\mu_n}{n^{\gamma/2} \|\widetilde{\beta}_{n;j}\|^\gamma} \frac{n^{\gamma/2}}{n}.$$

Since we have supposed that $j \in \widehat{\mathcal{A}}_n^*$ and $j \notin \mathcal{A}$, we have that for all $\epsilon > 0$, there exists $\eta_\epsilon > 0$ such that $\mathbb{P}[n^{-1/2} \|\widetilde{\beta}_{n;j}\|^{-1} > \eta_\epsilon] > 1 - \epsilon$. The last two relations, together with the supposition $n^{\gamma/2-1} \mu_n \rightarrow \infty$, imply, for all constant $A > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{\mu_n \widehat{\omega}_{n;j}}{n} > A \right] = 1. \quad (18)$$

Then, relations (17) and (18) are in contradiction. Thus

$$\lim_{n \rightarrow \infty} \mathbb{P}[j \in \mathcal{A}^c \cap \widehat{\mathcal{A}}_n^*] = 0. \quad (19)$$

The theorem follows from relations (16) and (19). ■

6.2 Proofs for results of Section 4

Proof of Lemma 1. We show that for all $\epsilon > 0$, there exists a constant $B_\epsilon > 0$ (without loss of generality, we consider $B_\epsilon > 0$, otherwise we take $|B_\epsilon|$) large enough such that for n large enough:

$$\mathbb{P} \left[\inf_{\|\mathbf{u}\|=1} G_n \left(\boldsymbol{\beta}^0 + B_\epsilon \sqrt{\frac{p}{n}} \mathbf{u} \right) > G_n(\boldsymbol{\beta}^0) \right] \geq 1 - \epsilon. \quad (20)$$

For this, we consider for some constant $C > 0$, the expectation of the difference:

$$\begin{aligned} \mathbb{E} \left[G_n \left(\boldsymbol{\beta}^0 + C \sqrt{\frac{p}{n}} \mathbf{u} \right) - G_n(\boldsymbol{\beta}^0) \right] &= \sum_{i=1}^n \mathbb{E} \left[\rho_\tau \left(\varepsilon_i - C \sqrt{\frac{p}{n}} \mathbb{X}_i^t \mathbf{u} \right) - \rho_\tau(\varepsilon_i) \right] \\ &= \sum_{i=1}^n \mathbb{E} \left[\int_0^{C \sqrt{\frac{p}{n}} \mathbb{X}_i^t \mathbf{u}} \mathbb{1}_{0 < \varepsilon_i < t} dt \right] = \sum_{i=1}^n \int_0^{C \sqrt{\frac{p}{n}} \mathbb{X}_i^t \mathbf{u}} [F(t) - F(0)] dt. \end{aligned} \quad (21)$$

By assumption (A6) we have $p/n \rightarrow 0$. Moreover, by assumption (A5), we have that $\sqrt{\frac{p}{n}} \mathbb{X}_i^t \mathbf{u} = o(1)$, for $\|\mathbf{u}\| = 1$. Thus, by mean value theorem and since the density f has a bounded first derivative in the neighbourhood of 0, relation (21) becomes:

$$= \frac{f(0)}{2} C^2 \frac{p}{n} \sum_{i=1}^n (\mathbb{X}_i^t \mathbf{u})^2 + o \left(\frac{p}{n} \sum_{i=1}^n \mathbf{u}^t (\mathbb{X}_i^t \mathbb{X}_i^t) \mathbf{u} \right).$$

Then, taking into account assumption (A4),

$$\frac{1}{n} \mathbb{E} \left[G_n \left(\boldsymbol{\beta}^0 + C \sqrt{\frac{p}{n}} \mathbf{u} \right) - G_n(\boldsymbol{\beta}^0) \right] = C^2 \frac{f(0)}{2} \frac{p}{n} \frac{1}{n} \sum_{i=1}^n (\mathbb{X}_i^t \mathbf{u})^2 (1 + o(1)). \quad (22)$$

Let be the following random variable

$$\mathcal{R}_i \equiv \rho_\tau \left(\varepsilon_i - C \sqrt{\frac{p}{n}} \mathbb{X}_i^t \mathbf{u} \right) - \rho_\tau(\varepsilon_i) - C \sqrt{\frac{p}{n}} \mathcal{D}_i \mathbb{X}_i^t \mathbf{u}$$

and the following random vector

$$\mathbf{W}_n \equiv C \sqrt{\frac{p}{n}} \sum_{i=1}^n \mathcal{D}_i \mathbb{X}_i^t,$$

with the random variable \mathcal{D}_i defined by (8). The vector \mathbf{W}_n is the similar of the vector \mathbf{z}_n when p was fixed. Then, the process G_n can be written:

$$G_n \left(\boldsymbol{\beta}^0 + C \sqrt{\frac{p}{n}} \mathbf{u} \right) - G_n(\boldsymbol{\beta}^0) = \mathbb{E} \left[G_n \left(\boldsymbol{\beta}^0 + C \sqrt{\frac{p}{n}} \mathbf{u} \right) - G_n(\boldsymbol{\beta}^0) \right] + \mathbf{W}_n \mathbf{u} + \sum_{i=1}^n [\mathcal{R}_i - \mathbb{E}[\mathcal{R}_i]]. \quad (23)$$

First all, remark that

$$\|\mathbf{u}\|^2 \lambda_{\min} \left(\frac{1}{n} \mathbb{X}_i \mathbb{X}_i^t \right) \leq \frac{1}{n} \sum_{i=1}^n \mathbf{u}^t \mathbb{X}_i \mathbb{X}_i^t \mathbf{u} \leq \|\mathbf{u}\|^2 \lambda_{\max} \left(\frac{1}{n} \mathbb{X}_i \mathbb{X}_i^t \right). \quad (24)$$

Since the errors $(\varepsilon_i)_{1 \leq i \leq n}$ are independent, using also $|\mathcal{R}_i| < C\sqrt{\frac{p}{n}} \|\mathbb{X}_i^t \mathbf{u}\| \mathbb{1}_{|\varepsilon_i| < C\sqrt{\frac{p}{n}} \|\mathbb{X}_i^t \mathbf{u}\|}$, together with assumption (A5), we obtain

$$\mathbb{E} \left[\mathbb{1}_{|\varepsilon_i| < C\sqrt{\frac{p}{n}} \|\mathbb{X}_i^t \mathbf{u}\|} \right] \leq C\sqrt{\frac{p}{n}} \|\mathbb{X}_i\| \leq C\sqrt{\frac{p}{n}} \max_{1 \leq i \leq n} \|\mathbb{X}_i\| = o(1),$$

which imply, since $(\varepsilon_i)_{1 \leq i \leq n}$ are i.i.d.,

$$\mathbb{E} \left[\sum_{i=1}^n [\mathcal{R}_i - \mathbb{E}[\mathcal{R}_i]] \right]^2 = \sum_{i=1}^n \mathbb{E}[\mathcal{R}_i - \mathbb{E}[\mathcal{R}_i]]^2 \leq \sum_{i=1}^n \mathbb{E}[\mathcal{R}_i^2]$$

and by assumptions (A3), (A5) together with relation (24), we have

$$\leq C^2 \frac{p}{n} \sum_{i=1}^n \|\mathbb{X}_i^t \mathbf{u}\|^2 \mathbb{E} \left[\mathbb{1}_{|\varepsilon_i| < C\sqrt{\frac{p}{n}} \|\mathbb{X}_i^t \mathbf{u}\|} \right] = o \left(\frac{p}{n} \sum_{i=1}^n \mathbf{u}^t \mathbb{X}_i \mathbb{X}_i^t \mathbf{u} \right) = o(p). \quad (25)$$

For the last relation we have used assumption (A4).

Let be the following random variable $U_n \equiv p^{-1/2} \sum_{i=1}^n [\mathcal{R}_i - \mathbb{E}[\mathcal{R}_i]]$. Then, relation (25) implies $\mathbb{E}[U_n^2] = o(1)$. This, together with $\mathbb{E}[U_n] = 0$, imply, by the Bienaymé-Tchebychev inequality, that $U_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$. Thus $\sum_{i=1}^n [\mathcal{R}_i - \mathbb{E}[\mathcal{R}_i]] = o_{\mathbb{P}}(p^{1/2})$. Then, relation (23) becomes

$$G_n \left(\boldsymbol{\beta}^0 + C\sqrt{\frac{p}{n}} \mathbf{u} \right) - G_n(\boldsymbol{\beta}^0) = \mathbb{E} \left[G_n \left(\boldsymbol{\beta}^0 + C\sqrt{\frac{p}{n}} \mathbf{u} \right) - G_n(\boldsymbol{\beta}^0) \right] + \mathbf{W}_n \mathbf{u} + o_{\mathbb{P}}(p^{1/2}) \quad (26)$$

which is equal to, using (22):

$$= \left[C^2 \frac{f(0)}{2} p \left(\frac{1}{n} \sum_{i=1}^n \mathbf{u}^t \mathbb{X}_i \mathbb{X}_i^t \mathbf{u} \right) + C\sqrt{p} \left(\sum_{i=1}^n \frac{\mathcal{D}_i \mathbb{X}_i^t}{\sqrt{n}} \right) \mathbf{u} \right] (1 + o_{\mathbb{P}}(1)) + o_{\mathbb{P}}(p^{1/2}).$$

Since $\mathbb{E}[\mathcal{D}_i] = 0$, $\text{Var}[\mathcal{D}_i \mathbb{X}_i^t \mathbf{u}] = \tau(1 - \tau) \mathbf{u}^t \mathbb{X}_i \mathbb{X}_i^t \mathbf{u}$ and $\|\mathbf{u}\| = 1$, then, using (24), we have that $n^{-1/2} \sum_{i=1}^n \mathcal{D}_i \mathbb{X}_i^t \mathbf{u}$ converges in law to a centred Gaussian distribution. Taking into account assumptions (A4) and (A6), for B large enough, we obtain

$$G_n \left(\boldsymbol{\beta}^0 + B\sqrt{\frac{p}{n}} \mathbf{u} \right) - G_n(\boldsymbol{\beta}^0) = B^2 f(0) p \left(\frac{1}{n} \sum_{i=1}^n \mathbf{u}^t \mathbb{X}_i \mathbb{X}_i^t \mathbf{u} \right) (1 + o_{\mathbb{P}}(1)) > 0, \quad (27)$$

for n large enough. Thus, relation (20) follows taking into account assumptions (A3) and (A4). ■

Proof of Theorem 3. We have the following inequality, with probability 1, for the quantile estimator $\tilde{\boldsymbol{\beta}}_{n;j}$:

$$\min_{j \in \mathcal{A}} \|\boldsymbol{\beta}_j^0\| \leq \max_{j \in \mathcal{A}} \|\tilde{\boldsymbol{\beta}}_{n;j} - \boldsymbol{\beta}_j^0\| + \min_{j \in \mathcal{A}} \|\tilde{\boldsymbol{\beta}}_{n;j}\|.$$

By Lemma 1, we have that $\max_{j \in \mathcal{A}} \|\tilde{\boldsymbol{\beta}}_{n;j} - \boldsymbol{\beta}_j^0\| = O_{\mathbb{P}}((p/n)^{1/2}) = O_{\mathbb{P}}(n^{(c-1)/2})$. On the other hand, we denoted in assumption (A8), $h_0 = \min_{j \in \mathcal{A}} \|\boldsymbol{\beta}_j^0\|$. Then, we have with probability one,

$h_0 n^{-\alpha} \leq O_{\mathbb{P}}(n^{(c-1)/2-\alpha}) + n^{-\alpha} \min_{j \in \mathcal{A}} \|\tilde{\boldsymbol{\beta}}_{n;j}\|$ and taking into account assumption (A8): $M \leq o_{\mathbb{P}}(1) + n^{-\alpha} \min_{j \in \mathcal{A}} \|\tilde{\boldsymbol{\beta}}_{n;j}\|$. Hence:

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\min_{j \in \mathcal{A}} \|\tilde{\boldsymbol{\beta}}_{n;j}\| > \frac{Mn^\alpha}{2} \right] = 1. \quad (28)$$

We consider the r -vector \mathbf{u} such that $\|\mathbf{u}\| = 1$. Similarly to Theorem 2.1 of Zhang and Xiang (2015), we have, for a constant B ,

$$\sum_{j=1}^p \|\tilde{\boldsymbol{\beta}}_{n;j}\|^{-\gamma} \left[\|\boldsymbol{\beta}_j^0 + B\sqrt{\frac{p}{n}}\mathbf{u}_j\| - \|\boldsymbol{\beta}_j^0\| \right] \geq \sum_{j=1}^{p_0} \|\tilde{\boldsymbol{\beta}}_{n;j}\|^{-\gamma} \left[\|\boldsymbol{\beta}_j^0 + B\sqrt{\frac{p}{n}}\mathbf{u}_j\| - \|\boldsymbol{\beta}_j^0\| \right] \geq -B\sqrt{\frac{p}{n}} \sum_{j=1}^{p_0} \|\tilde{\boldsymbol{\beta}}_{n;j}\|^{-\gamma} \|\mathbf{u}_j\|.$$

Since $\|\mathbf{u}\| = 1$, taking into account relation (28), applying the Cauchy-Schwarz inequality, we obtain

$$-\sqrt{\frac{p}{n}} \lambda_n \sum_{j=1}^{p_0} \|\tilde{\boldsymbol{\beta}}_{n;j}\|^{-\gamma} \|\mathbf{u}_j\| \geq -\sqrt{\frac{p}{n}} \lambda_n \left(\sum_{j=1}^{p_0} \|\tilde{\boldsymbol{\beta}}_{n;j}\|^{-2\gamma} \right)^{1/2} \|\mathbf{u}\| \geq -\sqrt{\frac{p}{n}} \frac{\lambda_n \sqrt{p_0}}{\left(\min_{j \in \mathcal{A}} \|\tilde{\boldsymbol{\beta}}_{n;j}\| \right)^\gamma} \geq -\sqrt{\frac{p}{n}} \frac{\lambda_n \sqrt{p_0}}{\left(\frac{Mn^\alpha}{2} \right)^\gamma}$$

and by assumption $\lambda_n n^{(1+c)/2-\alpha\gamma} \xrightarrow[n \rightarrow \infty]{} 0$, we obtain:

$$= O_{\mathbb{P}} \left(\frac{p}{n} \right). \quad (29)$$

The Theorem is proved if we have the similar of inequality (20) for

$$\frac{1}{n} G_n \left(\boldsymbol{\beta}^0 + B\sqrt{\frac{p}{n}}\mathbf{u} \right) - \frac{1}{n} G_n(\boldsymbol{\beta}^0) + \sum_{j=1}^p \lambda_n \hat{\omega}_{n;j} \left[\|\boldsymbol{\beta}_j^0 + \sqrt{\frac{p}{n}} B \mathbf{u}_j\| - \|\boldsymbol{\beta}_j^0\| \right] \equiv Q_n \left(\boldsymbol{\beta}^0 + B\sqrt{\frac{p}{n}}\mathbf{u} \right) - Q_n(\boldsymbol{\beta}^0).$$

We show that for all $\epsilon > 0$ there exists B_ϵ large enough, such that, for any n large enough

$$\mathbb{P} \left[\inf_{\|\mathbf{u}\|=1} Q_n \left(\boldsymbol{\beta}^0 + B_\epsilon \sqrt{\frac{p}{n}}\mathbf{u} \right) > Q_n(\boldsymbol{\beta}^0) \right] > 1 - \epsilon. \quad (30)$$

By the definition of Q_n , we have for all constant $B > 0$, that

$$Q_n \left(\boldsymbol{\beta}^0 + B\sqrt{\frac{p}{n}}\mathbf{u} \right) - Q_n(\boldsymbol{\beta}^0) = \frac{1}{n} G_n \left(\boldsymbol{\beta}^0 + B\sqrt{\frac{p}{n}}\mathbf{u} \right) - \frac{1}{n} G_n(\boldsymbol{\beta}^0) + \sum_{j=1}^p \lambda_n \hat{\omega}_{n;j} \left[\|\boldsymbol{\beta}_j^0 + \sqrt{\frac{p}{n}} B \mathbf{u}_j\| - \|\boldsymbol{\beta}_j^0\| \right]$$

and using relations (27) and (29)

$$> B^2 f(0) \frac{p}{n} \left(\frac{1}{n} \sum_{i=1}^n \mathbf{u}^t \mathbb{X}_i \mathbb{X}_i^t \mathbf{u} \right) (1 + o_{\mathbb{P}}(1)) - B O_{\mathbb{P}} \left(\frac{p}{n} \right).$$

Relation (30) follows from the last relation, for n and B large enough and using assumption (A4). ■

Proof of Theorem 4. (i) By Theorem 3, we have that $\hat{\boldsymbol{\beta}}_n^*$ belongs, with a probability converging to one, to the set: $\mathcal{V}_p(\boldsymbol{\beta}^0) \equiv \{\boldsymbol{\beta}; \|\boldsymbol{\beta} - \boldsymbol{\beta}^0\| \leq B\sqrt{\frac{p}{n}}\}$, with $B > 0$ large enough as in relation (30).

For $p > p_0$, we show that for all $\boldsymbol{\beta} = (\boldsymbol{\beta}_{\mathcal{A}}, \boldsymbol{\beta}_{\mathcal{A}^c}) \in \mathcal{V}_p(\boldsymbol{\beta}^0)$ such that $\|\boldsymbol{\beta}_{\mathcal{A}} - \boldsymbol{\beta}_{\mathcal{A}}^0\| = O(\sqrt{\frac{p}{n}})$ and for all constant $C \in (0, B)$, we have

$$Q_n(\boldsymbol{\beta}_{\mathcal{A}}, \mathbf{0}_{p-p_0}) = \min_{\|\boldsymbol{\beta}_{\mathcal{A}^c}\| \leq C\sqrt{\frac{p}{n}}} Q_n(\boldsymbol{\beta}_{\mathcal{A}}, \boldsymbol{\beta}_{\mathcal{A}^c}), \quad (31)$$

with a probability tending to one, as sample size $n \rightarrow \infty$.

Let us consider the parameter set $\mathcal{W}_n \equiv \{\boldsymbol{\beta} \in \mathcal{V}_p(\boldsymbol{\beta}^0); \|\boldsymbol{\beta}_{\mathcal{A}^c}\| > 0\}$.

We show that $\mathbb{P}[\widehat{\boldsymbol{\beta}}_n^* \in \mathcal{W}_n] \rightarrow 0$, as $n \rightarrow \infty$. For this, we firstly consider two parameter vectors $\boldsymbol{\beta} = (\boldsymbol{\beta}_{\mathcal{A}}, \boldsymbol{\beta}_{\mathcal{A}^c}) \in \mathcal{W}_n$ and $\boldsymbol{\beta}^{(1)} = (\boldsymbol{\beta}_{\mathcal{A}}^{(1)}, \boldsymbol{\beta}_{\mathcal{A}^c}^{(1)}) \in \mathcal{V}_p(\boldsymbol{\beta}^0)$, such that $\boldsymbol{\beta}_{\mathcal{A}}^{(1)} = \boldsymbol{\beta}_{\mathcal{A}}$ and $\boldsymbol{\beta}_{\mathcal{A}^c}^{(1)} = \mathbf{0}_{d-d^0}$.

Let us take the difference of the objective random process for the two parameter vectors. We denote this difference $D_n(\boldsymbol{\beta}, \boldsymbol{\beta}^{(1)})$:

$$\begin{aligned} D_n(\boldsymbol{\beta}, \boldsymbol{\beta}^{(1)}) &\equiv Q_n(\boldsymbol{\beta}) - Q_n(\boldsymbol{\beta}^{(1)}) \\ &= n^{-1} \sum_{i=1}^n \left[\rho_{\tau}(Y_i - \mathbb{X}_i^t \boldsymbol{\beta}) - \rho_{\tau}(Y_i - \mathbb{X}_i^t \boldsymbol{\beta}^{(1)}) \right] + \sum_{j=p_0+1}^p \lambda_n \widehat{\omega}_{n;j} \|\boldsymbol{\beta}_j\|. \end{aligned} \quad (32)$$

From Knight (1998), we have the following identity, for any $x, y \in \mathbb{R}$:

$$\rho_{\tau}(x - y) - \rho_{\tau}(x) = y(\mathbb{1}_{x \leq 0} - \tau) + \int_0^y (\mathbb{1}_{x \leq t} - \mathbb{1}_{x \leq 0}) dt.$$

Using this relation for the first sum of (32), we obtain:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n [\rho_{\tau}(Y_i - \mathbb{X}_i^t \boldsymbol{\beta}) - \rho_{\tau}(Y_i - \mathbb{X}_i^t \boldsymbol{\beta}^{(1)})] &= \frac{1}{n} (\boldsymbol{\beta} - \boldsymbol{\beta}^{(1)})^t \sum_{i=1}^n \mathbb{X}_i [\mathbb{1}_{Y_i - \mathbb{X}_i^t \boldsymbol{\beta}^{(1)} \leq 0} - \tau] \\ &+ \frac{1}{n} \sum_{i=1}^n \int_0^{\mathbb{X}_i^t (\boldsymbol{\beta} - \boldsymbol{\beta}^{(1)})} [\mathbb{1}_{Y_i - \mathbb{X}_i^t \boldsymbol{\beta}^{(1)} \leq t} - \mathbb{1}_{Y_i - \mathbb{X}_i^t \boldsymbol{\beta}^{(1)} \leq 0}] dt \equiv T_{1n} + T_{2n}. \end{aligned} \quad (33)$$

For T_{1n} , since the density f is bounded in a neighbourhood of 0, we have, by assumption (A5), that:

$$\mathbf{E}[T_{1n}] = (\boldsymbol{\beta} - \boldsymbol{\beta}^{(1)})^t \frac{1}{n} \sum_{i=1}^n \mathbb{X}_i [F(\mathbb{X}_i^t (\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0)) - F(0)] = (\boldsymbol{\beta} - \boldsymbol{\beta}^{(1)})^t \frac{f(0)}{n} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t (\boldsymbol{\beta}^0 - \boldsymbol{\beta}^{(1)}) (1 + o(1)).$$

Then, $|\mathbf{E}[T_{1n}]| \leq \|\boldsymbol{\beta} - \boldsymbol{\beta}^{(1)}\| \cdot \|n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t\| \cdot \|\boldsymbol{\beta}^0 - \boldsymbol{\beta}^{(1)}\| f(0) (1 + o(1))$. Since the matrix $n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t$ is Hermitian, we have, taking into account assumption (A4), that $n^{-1} \|\sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t\| = \lambda_{\max}(n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t) \leq M_0$. Hence, we have $|\mathbf{E}[T_{1n}]| \leq M_0 f(0) \|\boldsymbol{\beta} - \boldsymbol{\beta}^{(1)}\| \cdot \|\boldsymbol{\beta}^0 - \boldsymbol{\beta}^{(1)}\|$. Therefore $\mathbf{E}[T_{1n}] = O(\|\boldsymbol{\beta} - \boldsymbol{\beta}^{(1)}\|^2)$. By calculations analogous to $\mathbf{E}[T_{1n}]$, using independence of $(\varepsilon_i)_{1 \leq i \leq n}$, we have that $\mathbf{E}[T_{1n}^2] = Cn^{-1} \|\boldsymbol{\beta} - \boldsymbol{\beta}^{(1)}\|^3 \rightarrow 0$, for $n \rightarrow \infty$. Since $\text{Var}[T_{1n}] \leq \mathbf{E}[T_{1n}^2]$, using the Bienaymé-Tchebychev inequality, we have:

$$T_{1n} = C \|\boldsymbol{\beta} - \widetilde{\boldsymbol{\beta}}\|^2 (1 + o_{\mathbb{P}}(1)). \quad (34)$$

Consider now T_{2n} of relation (33), which can be written as:

$$T_{2n} = n^{-1} \sum_{i=1}^n \int_0^{\mathbb{X}_i^t (\boldsymbol{\beta} - \boldsymbol{\beta}^{(1)})} [\mathbb{1}_{\varepsilon_i \leq t - \mathbb{X}_i^t (\boldsymbol{\beta}^0 - \boldsymbol{\beta}^{(1)})} - \mathbb{1}_{\varepsilon_i \leq -\mathbb{X}_i^t (\boldsymbol{\beta}^0 - \boldsymbol{\beta}^{(1)})}] dt.$$

Taking into account that $\beta \in \mathcal{V}_p(\beta_n^0)$, together with assumptions (A3), (A5), we get $\mathbf{E}[T_{2n}] = n^{-1} \sum_{i=1}^n \int_0^{\mathbb{X}_i^t(\beta - \beta^{(1)})} [F(t - \mathbb{X}_i^t(\beta^0 - \beta^{(1)})) - F(-\mathbb{X}_i^t(\beta^0 - \beta^{(1)}))] dt = n^{-1} \sum_{i=1}^n \int_0^{\mathbb{X}_i^t(\beta - \beta^{(1)})} [t f(\mathbb{X}_i^t(\beta^{(1)} - \beta^0)) + o(t)] dt$.

By the proof of Lemma 1, assumptions (A3), (A5), we obtain that $f(\mathbb{X}_i^t(\beta^{(1)} - \beta^0))$ is bounded by a constant $C > 0$. Thus, as for T_{1n} , using assumption (A4) and the fact that $n^{-1} \sum_{i=1}^n \|\mathbb{X}_i\|^2 - \text{tr}(n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t) \xrightarrow{n \rightarrow \infty} 0$, we have $|\mathbf{E}[T_{2n}]| \leq C n^{-1} \sum_{i=1}^n \|\mathbb{X}_i\|^2 \|\beta - \beta^{(1)}\| \cdot \|\beta^{(1)} - \beta^0\| + o(n^{-1} \sum_{i=1}^n \mathbb{X}_i^t(\beta - \beta^{(1)})) = C \|\beta - \beta^{(1)}\|^2$. We show similarly that $\mathbf{E}[T_{2n}^2] = C n^{-1} \|\beta - \beta^{(1)}\|^3$. Then, by the Bienaymé-Tchebychev inequality, we get:

$$T_{2n} = C \|\beta - \beta^{(1)}\|^2 (1 + o_{\mathbb{P}}(1)). \quad (35)$$

Hence, by relations (34), (35), we obtain

$$T_{1n} + T_{2n} = C \|\beta - \beta^{(1)}\|^2 (1 + o_{\mathbb{P}}(1)). \quad (36)$$

This last relation together with relations (33), (36), give for relation (32):

$$D_n(\beta, \beta^{(1)}) = C \|\beta - \beta^{(1)}\|^2 (1 + o_{\mathbb{P}}(1)) + \sum_{j=p_0+1}^p \lambda_n \hat{\omega}_{n;j} \|\beta_j\|.$$

On the other hand, by Lemma 1, we have

$$\hat{\omega}_{n;j} = \frac{1}{\|\tilde{\beta}_{n;j}\|^\gamma} = \frac{1}{\|\tilde{\beta}_{n;j} - \beta_j^0\|^\gamma} = O_{\mathbb{P}}\left(\left(\frac{p}{n}\right)^{-\gamma/2}\right)$$

and moreover for all $j \geq p_0 + 1$, since $\beta \in \mathcal{W}_n \subseteq \mathcal{V}_p(\beta^0)$ we have $0 < \|\beta_j\| = O(\sqrt{\frac{p}{n}})$. Then

$$\sum_{j=p_0+1}^p \lambda_n \hat{\omega}_{n;j} \|\beta_j\| = \sum_{j=p_0+1}^p \lambda_n O_{\mathbb{P}}\left(\left(\frac{p}{n}\right)^{(1-\gamma)/2}\right).$$

Thus

$$\frac{D_n(\beta, \beta^{(1)})}{\|\beta - \beta^{(1)}\|} \geq C \|\beta - \beta^{(1)}\| (1 + o_{\mathbb{P}}(1)) + \sum_{j=p_0+1}^p \lambda_n O_{\mathbb{P}}\left(\left(\frac{p}{n}\right)^{-\gamma/2}\right).$$

We have that $\|\beta - \beta^{(1)}\| = O\left(\left(\frac{p}{n}\right)^{1/2}\right)$. Since $p/n = O(n^{c-1})$, under the assumption that $\lambda_n n^{(1-c)(1+\gamma)/2} \rightarrow \infty$, as $n \rightarrow \infty$, and $p > p_0$, we have

$$\frac{D_n(\beta, \beta^{(1)})}{\|\beta - \beta^{(1)}\|} \geq \lambda_n O_{\mathbb{P}}\left(n^{\gamma(1-c)/2}\right). \quad (37)$$

To finish the proof of relation (31), consider now other two parameter vectors: β^0 the true value and $\beta^{(1)}$ a parameter such that $\beta^{(1)} \equiv (\beta_{\mathcal{A}}^{(1)}, \beta_{\mathcal{A}^c}^{(1)})$, $\beta_{\mathcal{A}}^{(1)} = \beta_{\mathcal{A}}$, $\beta_{\mathcal{A}^c}^{(1)} = \beta_{\mathcal{A}^c}^0 = \mathbf{0}_{d-d^0}$. We obtain as for (36) that:

$$D_n(\beta^0, \beta^{(1)}) = n^{-1} \sum_{i=1}^n \left[\rho_\tau(Y_i - \mathbb{X}_i^t \beta^0) - \rho_\tau(Y_i - \mathbb{X}_i^t \beta^{(1)}) \right] = C \|(\beta^0 - \beta^{(1)})_{\mathcal{A}}\|^2 (1 + o_{\mathbb{P}}(1)).$$

Then

$$\frac{D_n(\boldsymbol{\beta}^0, \boldsymbol{\beta}^{(1)})}{\|\boldsymbol{\beta}^0 - \boldsymbol{\beta}^{(1)}\|} = C\|(\boldsymbol{\beta}^0 - \boldsymbol{\beta}^{(1)})_{\mathcal{A}}\|(1 + o_{\mathbb{P}}(1)) = O_{\mathbb{P}}\left(\left(\frac{p}{n}\right)^{1/2}\right) = O_{\mathbb{P}}\left(n^{(c-1)/2}\right). \quad (38)$$

Since $\lambda_n n^{(1-c)(1+\gamma)/2} \rightarrow \infty$ as $n \rightarrow \infty$, we have that (37) is much bigger than (38), for n large enough. Then, relation (31) follows.

To finish the proof of (i), we will show that

$$\lim_{n \rightarrow \infty} \mathbb{P}\left[\min_{j \in \mathcal{A}} \|\widehat{\boldsymbol{\beta}}_{n;j}^*\| > 0\right] = 1. \quad (39)$$

With probability 1, we have that: $\min_{j \in \mathcal{A}} \|\widehat{\boldsymbol{\beta}}_{n;j}^*\| \geq \min_{j \in \mathcal{A}} \|\boldsymbol{\beta}_j^0\| - \max_{j \in \mathcal{A}} \|\widehat{\boldsymbol{\beta}}_{n;j}^* - \boldsymbol{\beta}_j^0\|$. By Theorem 3 and assumption (A8), we have,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left[n^{-\alpha} \min_{j \in \mathcal{A}} \|\widehat{\boldsymbol{\beta}}_{n;j}^*\| \geq \frac{M}{2}\right] = 1,$$

which implies

$$\lim_{n \rightarrow \infty} \mathbb{P}\left[n^{-\alpha} \min_{j \in \mathcal{A}} \|\widehat{\boldsymbol{\beta}}_{n;j}^*\| > 0\right] = 1,$$

from which relation (39) follows. Relations (39) and (31) imply (i).

(ii) Taking into account claim (i) and assumption (A7), the estimator $\widehat{\boldsymbol{\beta}}_n^*$ can be written with a probability converging to 1 as $\widehat{\boldsymbol{\beta}}_n^* = \boldsymbol{\beta}^0 + \sqrt{\frac{p}{n}}\boldsymbol{\delta}$, with $\boldsymbol{\delta} = (\boldsymbol{\delta}_{\mathcal{A}}, \boldsymbol{\delta}_{\mathcal{A}^c})$, $\boldsymbol{\delta}_{\mathcal{A}^c} = \mathbf{0}_{r-r^0}$, $\|\boldsymbol{\delta}_{\mathcal{A}}\| \leq C$. Then

$$Q_n\left(\boldsymbol{\beta}^0 + \sqrt{\frac{p}{n}}\boldsymbol{\delta}\right) - Q_n(\boldsymbol{\beta}^0) = \frac{1}{n} \sum_{i=1}^n \left[\rho_{\tau}\left(Y_i - \mathbb{X}_i^t(\boldsymbol{\beta}^0 + \sqrt{\frac{p}{n}}\boldsymbol{\delta})\right) - \rho_{\tau}(\varepsilon_i)\right] + \mathcal{P}, \quad (40)$$

with $\mathcal{P} \equiv \sum_{j=1}^{p_0} \lambda_n \widehat{\omega}_{n;j} [\|\boldsymbol{\beta}_j\| - \|\boldsymbol{\beta}_j^0\|]$.

For all $j \in \{1, \dots, p_0\}$ we have $\left|\|\widehat{\boldsymbol{\beta}}_{n;j}^*\| - \|\boldsymbol{\beta}_j^0\|\right| = C\sqrt{\frac{p}{n}}\boldsymbol{\delta}_j^t \boldsymbol{\beta}_j^0$, with a probability converging to 1. On the other hand, using assumption (A8) and Lemma 1 we have for the quantile estimators

$$\|\widetilde{\boldsymbol{\beta}}_{n;j}\| = \|\boldsymbol{\beta}_j^0\| + O_{\mathbb{P}}\left(\sqrt{\frac{p}{n}}\right) \geq h_0 + O_{\mathbb{P}}\left(n^{-(1-c)/2}\right) = O(n^{\alpha}) + O_{\mathbb{P}}\left(n^{-(1-c)/2}\right) = O(n^{\alpha}).$$

Thus, we have $\widehat{\omega}_{n;j} \leq O_{\mathbb{P}}(n^{-\alpha\gamma})$, for all $j \in \mathcal{A}$. Then, for the the second term on the right hand side of (40), we have with a probability converging to one:

$$\begin{aligned} |\mathcal{P}| &= \sum_{j=1}^{p_0} \lambda_n \widehat{\omega}_{n;j} \left| \|\boldsymbol{\beta}_j^0 + \sqrt{\frac{p}{n}}\boldsymbol{\delta}_j\| - \|\boldsymbol{\beta}_j^0\| \right| \leq Cn^{-\alpha\gamma} \sum_{j=1}^{p_0} \lambda_n \sqrt{\frac{p}{n}} |\boldsymbol{\delta}_j^t \boldsymbol{\beta}_j^0| \\ &\leq Cn^{-\alpha\gamma} \sqrt{\frac{p}{n}} \lambda_n \sum_{j=1}^{p_0} \|\boldsymbol{\delta}_j\| \cdot \|\boldsymbol{\beta}_j^0\| \leq C\sqrt{\frac{p}{n}} r^0 \lambda_n n^{-\alpha\gamma} = Cr^0 \lambda_n n^{(c-1)/2 - \alpha\gamma}. \end{aligned} \quad (41)$$

We study now the first term of the right hand side of (40), which can be written as:

$$\frac{1}{n} \sum_{i=1}^n \left[\rho_{\tau}\left(Y_i - \mathbb{X}_i^t\left(\boldsymbol{\beta}^0 + \sqrt{\frac{p}{n}}\boldsymbol{\delta}\right)\right) - \rho_{\tau}(\varepsilon_i) \right]$$

$$= \frac{1}{n} \sqrt{\frac{p}{n}} \sum_{i=1}^n \mathbb{X}_i^t \boldsymbol{\delta} [\mathbb{1}_{\varepsilon \leq 0} - \tau] + \frac{1}{n} \sum_{i=1}^n \int_0^{\sqrt{\frac{p}{n}} \mathbb{X}_i^t \boldsymbol{\delta}} [\mathbb{1}_{\varepsilon_i \leq t} - \mathbb{1}_{\varepsilon_i \leq 0}] dt \equiv J_1 + J_2.$$

Since $\mathbb{E}[J_1] = 0$, using assumption (A5) and the Cauchy-Schwarz inequality, we have:

$$\text{Var}[J_1] \leq \mathbb{E}[J_1^2] = \frac{1}{n^2} \frac{p}{n} \tau(1-\tau) \sum_{i=1}^n (\mathbb{X}_i^t \boldsymbol{\delta})^2 \leq \frac{p}{n^3} \tau(1-\tau) \sum_{i=1}^n \|\mathbb{X}_{i,\mathcal{A}}\|^2 \|\boldsymbol{\delta}_{\mathcal{A}}\|^2 \leq C \frac{p}{n^3} n \max_{1 \leq i \leq n} \|\mathbb{X}_{i,\mathcal{A}}\|^2 \xrightarrow{n \rightarrow \infty} 0.$$

Using assumption (A2), we have for the expectation of J_2 :

$$\mathbb{E}[J_2] = \frac{1}{n} \int_0^{\sqrt{\frac{p}{n}} \mathbb{X}_i^t \boldsymbol{\delta}} (tf(0) + o(t^2)) dt = \frac{f(0)}{2} \frac{p}{n} \boldsymbol{\delta}^t \frac{1}{n} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t \boldsymbol{\delta} (1 + o(1)). \quad (42)$$

Using assumption (A4) and relation (24), we have that

$$f(0)pn^{-1} \|\boldsymbol{\delta}\|_2^2 \lambda_{\min}(n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t) \leq \mathbb{E}[J_2] \leq f(0)pn^{-1} \|\boldsymbol{\delta}\|_2^2 \lambda_{\max}(n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t).$$

Taking into account the fact that $\|\boldsymbol{\delta}\|_2^2 = \|\boldsymbol{\delta}_{\mathcal{A}}\|_2^2 \leq C$, we have:

$$\mathbb{E}[J_2] = Cf(0) \frac{p}{n}. \quad (43)$$

Similarly we obtain $\text{Var}[J_2] = O\left(\frac{1}{n} \left(\frac{p}{n}\right)^{3/2}\right) \rightarrow 0$, as $n \rightarrow \infty$.

But $\mathbb{E}[J_2] = O(n^{c-1})$ and by assumption (A7), together with relation (41), we have $|\mathcal{P}| \leq O_{\mathbb{P}}(\lambda_n n^c n^{(c-1)/2-\alpha\gamma}) = O_{\mathbb{P}}(\lambda_n n^{(3c-1)/2-\alpha\gamma})$. Then $\frac{|\mathcal{P}|}{\mathbb{E}[J_2]} = O_{\mathbb{P}}(\lambda_n n^{(c+1)/2-\alpha\gamma}) \rightarrow 0$ and thus $\mathbb{E}[J_2] \gg |\mathcal{P}|$.

Hence, minimizing (40) amounts to minimizing $J_1 + J_2$, with respect to $\sqrt{\frac{p}{n}} \boldsymbol{\delta}$. Using relation (42), we obtain:

$$\frac{1}{n} \sum_{i=1}^n [\rho_{\tau}(Y_i - \mathbb{X}_i^t(\boldsymbol{\beta}_n^0 + \sqrt{\frac{p}{n}} \boldsymbol{\delta})) - \rho_{\tau}(\varepsilon_i)] = \frac{\sqrt{\frac{p}{n}}}{n} \sum_{i=1}^n \mathbb{X}_{i,\mathcal{A}}^t \boldsymbol{\delta}_{\mathcal{A}} [\mathbb{1}_{\varepsilon_i < 0} - \tau] + \frac{f(0)}{2} \frac{p}{n} \boldsymbol{\delta}_{\mathcal{A}}^t \boldsymbol{\Upsilon}_{n,\mathcal{A}} \boldsymbol{\delta}_{\mathcal{A}} (1 + o_{\mathbb{P}}(1)).$$

The minimizer of the right hand side of the last equation is:

$$\sqrt{\frac{p}{n}} \boldsymbol{\delta}_{\mathcal{A}} = -\frac{1}{n} \frac{1}{f(0)} \boldsymbol{\Upsilon}_{n,\mathcal{A}}^{-1} \left(\sum_{i=1}^n \mathbb{X}_{i,\mathcal{A}} (\mathbb{1}_{\varepsilon_i \leq 0} - \tau) \right). \quad (44)$$

For studying (44), let us consider the following independent random variable sequence, for $i = 1, \dots, n$,

$$W_i \equiv (f(0))^{-1} \mathbf{u}^t \boldsymbol{\Upsilon}_{n,\mathcal{A}}^{-1} \mathbb{X}_{i,\mathcal{A}} (\mathbb{1}_{\varepsilon_i \leq 0} - \tau),$$

with \mathbf{u} a vector of dimension r^0 , such that $\|\mathbf{u}\|_2 = 1$. We have that $\mathbb{E}[W_i] = 0$ and $\sum_{i=1}^n \text{Var}[W_i] = n\tau(1-\tau)(f(0))^{-2} \mathbf{u}^t \boldsymbol{\Upsilon}_{n,\mathcal{A}}^{-1} \mathbf{u}$. Then, by CLT for independent random variable sequences $(W_i)_{1 \leq i \leq n}$, we have

$$\sqrt{n} f(0) \frac{\mathbf{u}^t (\hat{\boldsymbol{\beta}}_{\mathcal{A}} - \boldsymbol{\beta}_{\mathcal{A}}^0)}{\sqrt{\tau(1-\tau)(\mathbf{u}^t \boldsymbol{\Upsilon}_{n,\mathcal{A}}^{-1} \mathbf{u})}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1). \quad (45)$$

Claim (ii) results by taking into account of the fact that $\hat{\boldsymbol{\beta}}_{\mathcal{A}} - \boldsymbol{\beta}_{\mathcal{A}}^0 = \sqrt{\frac{p}{n}} \boldsymbol{\delta}_{\mathcal{A}}$, together with relations (44), (45). ■

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