



HAL
open science

Real time change-point detection in a nonlinear quantile model

Gabriela Ciuperca

► **To cite this version:**

Gabriela Ciuperca. Real time change-point detection in a nonlinear quantile model. *Sequential Analysis*, 2017, 36 (1), pp.87-110. 10.1080/07474946.2016.1275482 . hal-02072133

HAL Id: hal-02072133

<https://hal.science/hal-02072133v1>

Submitted on 8 Feb 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Real time change-point detection in a nonlinear quantile model

Gabriela Ciuperca *

Université Lyon 1, UMR 5208, Institut Camille Jordan, France

Abstract: Most studies in real time change-point detection either focus on the linear model or use the CUSUM method under classical assumptions on model errors. This paper considers the sequential change-point detection in a nonlinear quantile model. A test statistic based on the CUSUM of the quantile process subgradient is proposed and studied. Under null hypothesis that the model does not change, the asymptotic distribution of the test statistic is determined. Under alternative hypothesis that at some unknown observation there is a change in model, the proposed test statistic converges in probability to ∞ . These results allow to build the critical regions on open-end and on closed-end procedures. Simulation results, using Monte Carlo technique, investigate the performance of the test statistic, specially for heavy-tailed error distributions. We also compare it with the classical CUSUM test statistic.

Keywords: Asymptotic behaviour; Nonlinear quantile regression; Sequential detection; Test statistic.

Subject Classifications: 62F03 ; 62F05.

1 Introduction

Models with heavy-tailed errors are commonly encountered in applications. To address this problem, a very interesting possibility is the quantile method, which has as a particular case, the least absolute deviation (median) method. Quantile framework is also useful for regressions when the model errors don't meet the classical conditions: zero mean and bounded variance. On the other hand, in applications, it is possible that the model changes to unknown observations. We so obtain a change-point model. There are two types of change-point problem: a posteriori and a priori (sequential).

A posteriori change-point problem arises when the data are completely known at the end of the experiment to process.

In the sequential change-point problem, which will be presented here, the change detection is performed in real time. In order to detect a possible change in model under

*Université de Lyon, Université Lyon 1, CNRS, UMR 5208, Institut Camille Jordan, Bat. Braconnier, 43, blvd du 11 novembre 1918, F - 69622 Villeurbanne Cedex, France.
Email: Gabriela.Ciuperca@univ-lyon1.fr

classical suppositions on the errors, the most used technique is the CUSUM method.

For the two types of change-point models, the number of publications in the last years is very extensive. Given the contribution of the present paper, we prefer to only mention references concerning either the change-point detection in the quantile models or the sequential detection of a change-point in a model. Oka and Qu (2011) consider the estimation of multiple structural changes in a linear regression quantile, while in Su and Xiao (2008) a test statistic is proposed. Several tests are also considered by Qu (2008), Furno (2012) for structural change in linear regression quantiles. Recently, new contributions to the change-point estimation in quantile linear models were made by Aue et al. (2014), Zhang et al. (2016). The CUSUM method can also be used in a posteriori change-point model. So, Zhang et al. (2014) proposed a test based on the CUSUM of the subgradient of the quantile objective function for testing a posteriori presence of a change-point due to a covariate threshold. Wang and He (2007) give a possible application of a quantile framework for detecting the differential expressions in GeneChip microarray studies by proposing a rank test in a posteriori change-point linear model. Another application of a posteriori change-point quantile regression for longitudinal data is given in Li et al. (2015) for a study of cognitive changes in Alzheimer's disease.

The sequential detection of a change-point in a linear model based on CUSUM of the least squares (LS) residuals was considered by Horváth et al. (2004). Their results are improved later by bootstrapping in Hušková and Kirch (2012). We find the same method in Xia et al. (2009) for a generalized linear model. The CUSUM with adaptive LASSO residuals is used by Ciuperca (2015) for real time change-point detection in a linear model with a large number of explanatory variables. Zhou et al. (2015) propose and study a test statistic in a linear model based on subgradient of the quantile process.

While most of the contributions to change-point model have been focused on the linear models, Ciuperca (2013) considers the sequential detection of a change-point in a nonlinear model based on CUSUM of the least squares residuals. For a posteriori change-point nonlinear model, Boldea and Hall (2013) consider the least square method, Ciuperca and Salloum (2015) consider the empirical likelihood test and Ciuperca (2016) the quantile method.

In this paper, the sequential change-point detection in a nonlinear model is studied, when the errors don't satisfy the classical conditions.

The rest of this article is organized as follows. In Section 2, we introduce the model assumptions and the nonlinear model under the null and alternative hypotheses. We propose and study the asymptotic behavior of the test statistic. The test statistic is the CUSUM of the subgradient of the quantile objective function. For detecting the change location in model, we propose a stopping time from which the null hypothesis is rejected. In Section 3, simulation results illustrate the performance of the proposed test statistic. Two lemmas and the proofs of the main results are given in Section 4.

2 Models and main results

In this section, we propose and study a test statistic for detecting in real time a change in a nonlinear quantile model. The asymptotic distribution of the test statistic under null hypothesis will allow to build the asymptotic critical region. Notations and assumptions are also given.

We begin by some general notations. Throughout the paper, C denotes a positive generic constant not dependent on m , which may take different values in different formulas or even in different parts of the same formula. All vectors and matrices denoted by bold symbols and all vectors written column-wise. For a r -vector $\mathbf{v} = (v_1, \dots, v_r)$, $\|\mathbf{v}\|$ is its euclidean norm, $\|\mathbf{v}\|_1$ is its L_1 -norm and $\|\mathbf{v}\|_\infty = \max(|v_1|, \dots, |v_r|)$. For a matrix \mathbf{M} , we denote by $\|\mathbf{M}\|_1$ the subordinate norm to the vector norm $\|\cdot\|_1$.

Let us consider the following nonlinear parametric model with independent observations

$$Y_i = g(\mathbf{X}_i; \beta_i) + \varepsilon_i, \quad i = 1, \dots, m, \dots, m + T_m.$$

For observation i , Y_i denotes the response variable, \mathbf{X}_i is a q -vector of explicative variables and ε_i is (unobserved) model error. For simplicity, we suppose that the regressors \mathbf{X}_i are non random, although the results will typically hold for random \mathbf{X}_i 's independent of the ε_i 's and if \mathbf{X}_i independent of \mathbf{X}_j for $i \neq j$. If \mathbf{X}_i is random, then, the conditional distribution with respect to \mathbf{X}_i must be considered.

The regression function $g : \Upsilon \times \mathcal{B} \rightarrow \mathbb{R}$ is known up to the parameters β_i , with $\beta_i \in \mathcal{B} \subseteq \mathbb{R}^p$, $\Upsilon \subseteq \mathbb{R}^q$. We suppose that the set \mathcal{B} is compact (see Koenker (2005)).

We suppose that on the first m observations, no change in the model parameter has occur:

$$\beta_i = \beta^0, \quad \text{for } i = 1, \dots, m,$$

with β^0 the true value of the parameter on the observations $i = 1, \dots, m$. The observations $1, \dots, m$ are called historical observations and $(Y_i, \mathbf{X}_i)_{1 \leq i \leq m}$ are historical data. We test null hypothesis that, at each observation $i \geq m + 1$ we have the same model as the m first observations:

$$H_0 : \beta_i = \beta^0, \quad m + 1 \leq i \leq m + T_m, \quad (2.1)$$

against alternative hypothesis that, at some (unknown) observation k_m^0 there is a change, called also change-point, in model:

$$H_1 : \exists k_m^0 \geq 1 \quad \text{such that} \quad \begin{cases} \beta_i = \beta^0, & \text{for } m + 1 \leq i \leq m + k_m^0 \\ \beta_i = \beta^1 \neq \beta^0, & \text{for } m + k_m^0 + 1 < i \leq m + T_m. \end{cases} \quad (2.2)$$

The values of β^0, β^1 are unknown and the parameter β^1 can depend on m .

For the sample size, two cases are possible, which will give different results, for the test statistics under the null hypothesis:

- *the open-end procedure:* $T_m = \infty$;

- *the closed-end procedure*: $T_m < \infty$, $\lim_{m \rightarrow \infty} T_m = \infty$, with $\lim_{m \rightarrow \infty} \frac{T_m}{m} = T > 0$, with the possibility $T = \infty$.

We introduce now the quantile model. For a fixed quantile index $\tau \in (0, 1)$, the τ th conditional quantile regression of Y , given \mathbf{x} , is $g(\mathbf{x}, \boldsymbol{\beta}) + F^{-1}(\tau)$, with $F^{-1}(\tau)$ the τ th quantile (F^{-1} is the inverse of the distribution function F) of error ε . We suppose that $F(0) = \tau$. The density of ε is denoted by f .

For a fixed quantile index $\tau \in (0, 1)$, consider the check function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\rho_\tau(u) = u\psi_\tau(u),$$

with

$$\psi_\tau(u) = \tau - \mathbb{1}_{u \leq 0}.$$

The quantile estimator of parameter $\boldsymbol{\beta}$, calculated on the historical observations $1, \dots, m$, is defined as the minimizer of the corresponding quantile process:

$$\hat{\boldsymbol{\beta}}_m \equiv \arg \min_{\boldsymbol{\beta} \in \mathcal{B}} \sum_{i=1}^m \rho_\tau(Y_i - g(\mathbf{X}_i, \boldsymbol{\beta})).$$

We now state the assumptions on the errors, the design and on the regression function.

For the errors ε_i we suppose that:

(A1) The errors $(\varepsilon_i)_{1 \leq i \leq n}$ are supposed independent identically distributed (i.i.d.) random variables. We denote by f the density and by F the distribution function of ε .

(A2) f is continuous, uniformly bounded away from zero and infinity and has a bounded first derivative in the neighbourhood of $\pm \left(g(\mathbf{x}; \boldsymbol{\beta}^0 + m^{-1/2} \|\mathbf{u}\|) - g(\mathbf{x}; \boldsymbol{\beta}^0) \right)$, for all $\mathbf{x} \in \Upsilon$ and any bounded p -vector \mathbf{u} .

For a bounded p -vector $\mathbf{u} \in \mathbb{R}^p$, let $\mathcal{V}_m(\mathbf{u}) \equiv \{\boldsymbol{\beta} \in \mathcal{B}; \|\boldsymbol{\beta} - \boldsymbol{\beta}^0\| \leq m^{-1/2} \|\mathbf{u}\|\}$ be the neighbourhood of $\boldsymbol{\beta}^0$.

The regression function $g(\mathbf{x}, \boldsymbol{\beta})$ is supposed continuous on Υ , twice differentiable in $\boldsymbol{\beta}$, continuous twice differentiable for $\boldsymbol{\beta} \in \mathcal{V}_m(\mathbf{u})$, for any \mathbf{u} bounded. In the following, for $\mathbf{x} \in \Upsilon$ and $\boldsymbol{\beta} \in \Gamma$ we use notation $\dot{\mathbf{g}}(\mathbf{x}, \boldsymbol{\beta}) \equiv \partial g(\mathbf{x}, \boldsymbol{\beta}) / \partial \boldsymbol{\beta}$ and $\ddot{\mathbf{g}}(\mathbf{x}, \boldsymbol{\beta}) \equiv \partial^2 g(\mathbf{x}, \boldsymbol{\beta}) / \partial \boldsymbol{\beta}^2$. Moreover, for the function g , following assumptions are considered:

(A3) For all $\mathbf{x} \in \Upsilon$, function $\dot{\mathbf{g}}(\mathbf{x}, \boldsymbol{\beta})$ is bounded for any $\boldsymbol{\beta}$ in every neighbourhood $\mathcal{V}_m(\mathbf{u})$, for bounded \mathbf{u} .

The design $(\mathbf{X}_i)_{1 \leq i \leq T_m}$ and the function g satisfy the following assumptions:

(A4) There exist two positive constants $C_1, C_2 > 0$ and natural m_0 , such that, for all $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \Gamma$ and $m \geq m_0$: $C_1 \|\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2\|_2 \leq \left(m^{-1} \sum_{i=1}^m [g(\mathbf{X}_i, \boldsymbol{\beta}_1) - g(\mathbf{X}_i, \boldsymbol{\beta}_2)]^2 \right)^{1/2} \leq C_2 \|\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2\|_2$. Moreover, we have that $m^{-1} \sum_{i=1}^m \dot{\mathbf{g}}(\mathbf{X}_i, \boldsymbol{\beta}^0) \dot{\mathbf{g}}^t(\mathbf{X}_i, \boldsymbol{\beta}^0)$ converges, as $m \rightarrow \infty$, to a positive definite matrix.

(A5) For any $j = 1, \dots, p$, all p -vector \mathbf{u} bounded, we have,

$\lim_{m \rightarrow \infty} m^{-1} \sum_{i=1}^m \sup_{\beta \in \mathcal{V}_m(\mathbf{u})} \|\ddot{\mathbf{g}}_j(\mathbf{X}_i; \beta)\|^2 < \infty$, with $\ddot{\mathbf{g}}_j(\mathbf{X}_i; \beta) \equiv \left(\frac{\partial^2 g(\mathbf{X}_i; \beta)}{\partial \beta_j \partial \beta_l} \right)_{1 \leq l \leq p}$ and $\beta = (\beta_1, \dots, \beta_p)$.

(A6) For any \mathbf{u} bounded, we have, $\lim_{m \rightarrow \infty} m^{-1} \sum_{i=1}^m \sup_{\beta \in \mathcal{V}_m(\mathbf{u})} \|\dot{\mathbf{g}}(\mathbf{X}_i; \beta)\|^3 < \infty$.

Assumptions (A1), (A4) and the fact that the density f is bounded are considered also in Koenker (2005). They are necessary to have the consistency and asymptotic normality of $\widehat{\beta}_m$.

In the linear case, assumption (A6) becomes $\sum_{i=1}^m \|\mathbf{X}_i\|^3 = O(m)$, assumption also considered by Koenker and Portnoy (1987) and by Zhou et al. (2015) for linear quantile regression. Assumption (A2) is considered by Zhou et al. (2015) for real time detection of a change in a linear quantile model, when f' is bounded in the neighbourhood of $\mathbf{X}_i^t \beta^0$. Obviously, assumption (A5) is needed only in nonlinear models, so, it is for example in multiple structural change-point nonlinear quantile model, considered by Ciuperca (2016), but in a stronger assumption: for all $\beta \in \mathcal{B}$, $\mathbf{x} \in \Upsilon$, it is imposed that $\|\dot{\mathbf{g}}(\mathbf{x}; \beta)\|_1$ is bounded.

For a vector $\beta \in \mathcal{B}$, let be the following p -square matrix:

$$\mathbf{J}_m(\beta) \equiv \tau(1 - \tau) \frac{1}{m} \sum_{i=1}^m \dot{\mathbf{g}}(\mathbf{X}_i; \beta) \dot{\mathbf{g}}^t(\mathbf{X}_i; \beta)$$

and for the quantile estimator $\widehat{\beta}_m$, let be the following random p -vector:

$$\mathbf{S}(m, k) \equiv \mathbf{J}_m^{-1/2}(\widehat{\beta}_m) \sum_{i=m+1}^{m+k} \dot{\mathbf{g}}(\mathbf{X}_i; \widehat{\beta}_m) \psi_\tau(Y_i - g(\mathbf{X}_i; \widehat{\beta}_m)). \quad (2.3)$$

The statistic $\mathbf{S}(m, k)$ is the similar to the cumulative sum (CUSUM) of quantile residuals $\widehat{\varepsilon}_i \equiv Y_i - g(\mathbf{X}_i; \widehat{\beta}_m)$, defined by $\sum_{i=m+1}^{m+k} \widehat{\varepsilon}_i$. More precisely, statistic (2.3) is the CUSUM of the subgradient of the quantile objective function. The common principle of the CUSUM method and of the test statistic considered in present paper, is to assume that there is the same model as for historical data, in each observation $m + k$, for $k \geq 1$.

In order to construct a test statistic for H_0 against H_1 , we consider, for a given constant $\gamma \in [0, 1/2)$, the normalisation function $z(m, k, \gamma) \equiv m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{k+m}\right)^\gamma$ and the following random variable:

$$\Gamma(m, k, \gamma) \equiv \frac{\|\mathbf{S}(m, k)\|_\infty}{z(m, k, \gamma)}. \quad (2.4)$$

The function $z(m, k, \gamma)$, proposed by Horváth et al. (2004), is used as a boundary function.

For testing H_0 , given by (2.1), against H_1 , given by (2.2), we will consider the following test statistic:

$$Z_m(\gamma) \equiv \sup_{1 \leq k \leq T_m} \Gamma(m, k, \gamma). \quad (2.5)$$

Remark 2.1 For linear function $g(\mathbf{x}; \boldsymbol{\beta}) = \mathbf{x}^t \boldsymbol{\beta}$, statistic $\mathbf{S}(m, k)$ becomes:

$(\tau(1-\tau)m^{-1} \sum_{i=1}^m \mathbf{X}_i \mathbf{X}_i^t)^{-1/2} \sum_{i=m+1}^{m+k} \mathbf{X}_i \psi_\tau(Y_i - g(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m))$ and then $Z_m(\gamma)$ is the test statistic studied by Zhou et al. (2015) for sequential change-point detection in a linear quantile model.

In order to study the asymptotic behaviour of statistic $Z_m(\gamma)$, that depends on quantile estimator $\widehat{\boldsymbol{\beta}}_m$, one needs a similar expression to the Bahadur representation for the linear quantile model. For this, by assumption (A4), the following matrix can be considered:

$$\boldsymbol{\Omega} \equiv f(0) \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \dot{\mathbf{g}}^t(\mathbf{X}_i; \boldsymbol{\beta}^0). \quad (2.6)$$

On the other hand, assumption (A3) implies that $\max_{1 \leq i \leq m} m^{-1/2} \|\dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0)\| \rightarrow 0$ as $m \rightarrow \infty$. This implies the uniqueness of $\widehat{\boldsymbol{\beta}}_m$ and the following representation (see Koenker (2005), page 100) for $\widehat{\boldsymbol{\beta}}_m$:

$$\widehat{\boldsymbol{\beta}}_m = \boldsymbol{\beta}^0 + \frac{1}{m} \boldsymbol{\Omega}^{-1} \sum_{l=1}^m \dot{\mathbf{g}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \psi_\tau(Y_l - g(\mathbf{X}_l; \boldsymbol{\beta}^0)) + o_{\mathbb{P}}(m^{-1/2}). \quad (2.7)$$

This representation implies that, in order to show results where $\widehat{\boldsymbol{\beta}}_m$ appears, we can use instead of estimator $\widehat{\boldsymbol{\beta}}_m$, some parameter $\boldsymbol{\beta}^0 + m^{-1/2} \mathbf{u}$, with \mathbf{u} a bounded vector of size p .

Remark 2.2 For real time detection in a nonlinear model by CUSUM method with LS residuals, considered by Ciuperca (2013), we must calculate the LS estimators of the model parameters. Numerically, it can cause problems, since the algorithm should start with a point in a neighbourhood of the unknown true value. If the algorithm starts with a bad starting point, than it can not converge. For the test statistic proposed in the present paper, the quantile estimator and then the test statistic $Z_m(\gamma)$ can always be calculated.

The following theorem establishes the limiting distribution of the test statistic under null hypothesis. This distribution depends on the value of $\gamma \in [0, 1/2)$. Will be studied by numerical simulations, in Section 3, the most appropriate values of γ . The proof of Theorem 2.1 is given in Subsection 4.2.

Theorem 2.1 If hypothesis H_0 holds, under assumptions (A1)-(A6), for all constant $\gamma \in [0, 1/2)$,

(i) if $T_m = \infty$ or ($T_m < \infty$ and $\lim_{m \rightarrow \infty} T_m/m = \infty$), then,

$$Z_m(\gamma) \xrightarrow{\mathcal{L}} \sup_{0 < t < 1} \frac{\|\mathbf{W}(t)\|_\infty}{t^\gamma}.$$

(ii) if $T_m < \infty$ and $\lim_{m \rightarrow \infty} T_m/m = T < \infty$, then $Z_m(\gamma)$ converges, as $m \rightarrow \infty$, to

$$\sup_{0 < t < T/(1+T)} \frac{\|\mathbf{W}(t)\|_\infty}{t^\gamma}.$$

For the two statements, $\{\mathbf{W}(t), t \in [0, \infty)\}$ is a Wiener process of dimension p .

Remark 2.3 Beside the linear quantile model considered by Zhou et al. (2015), in our case, in order to obtain relations (4.14) and (4.15) in the proof of Theorem 2.1, one must use the K-M-T approximation and not relation (3) of Leisch and Hornik (2000) which gives a version of FCLT and not uniform approximations for the two sums of relations (4.11) and (4.12).

In order to have a test statistic, thus, to build a critical region, it is necessary to study the behaviour of $Z_m(\gamma)$ under alternative hypothesis H_1 . For this, we suppose that, under hypothesis H_1 , the (unknown) change-point k_m^0 is not very far from the last observation of historical observations. Obviously, this supposition poses no problem for practical applications, since if hypothesis H_0 was not rejected until an observation k_m of order m^s , we reconsider as historical periode, all observations of 1 to k_m . Consider then the following assumption on k_m^0 of (2.2).

(A7) $k_m^0 = O(m^s)$, with the constant s : $\begin{cases} s \geq 0, & \text{for open-end procedure,} \\ 0 \leq s \leq 1, & \text{for closed-end procedure.} \end{cases}$

Obviously, an identifiability assumption is also necessary, that is, if the parameters before and after the change are different, then the model is also different. On the other hand, the jump in the parameters must be strictly greater than the convergence rate of the quantile estimator $\hat{\beta}_m$. Then, the following assumption is considered on the regression function g and on the parameter β^1 after the change-point k_m^0 .

(A8) (a) If $\|\beta^1 - \beta^0\| > c_1$, with c_1 a positive constant not depending on m , there exists a positive constant c_2 such that

$$\frac{1}{a_m} \left\| \sum_{i=m+k_m^0+1}^{m+k_m^0+a_m} \dot{\mathbf{g}}(\mathbf{X}_i; \beta^0) [g(\mathbf{X}_i; \beta^1) - g(\mathbf{X}_i; \beta^0)] \right\|_{\infty} > c_2 > 0,$$

for any sequence $(a_m)_{m \in \mathbb{N}}$ converging to infinity as $m \rightarrow \infty$.

(b) If $\beta^1 - \beta^0$ converges to $\mathbf{0}$ with the rate b_m , furthermore $m^{1/2}b_m \rightarrow \infty$, then we suppose that $\dot{\mathbf{g}}(\mathbf{x}, \beta)$ is bounded for any $\mathbf{x} \in \Upsilon$ and β such that $\|\beta - \beta^0\| \leq b_m$.

Assumption (A7) is very natural and typically used in sequential change-point detection works (see Ciuperca (2013), Hušková and Kirch (2012)). Concerning assumption (A8) for linear models, assumption (b) becomes assumption (A3), while assumption (a) is true for all β^1 such that $\|\beta^1 - \beta^0\| > c_1 > 0$ by assumption (A4). For nonlinear random models under classical assumptions on errors, in papers of Ciuperca (2013) and of Boldea and Hall (2013), where sequential and a posteriori change-point detection is considered, respectively, the imposed identifiability assumption is $\mathbb{E}[g(\mathbf{X}; \beta^0)] \neq \mathbb{E}[g(\mathbf{X}, \beta^1)]$, for $\beta^0 \neq \beta^1$.

By the following theorem, we prove that under alternative hypothesis H_1 , the test statistic $Z_m(\gamma)$ converges in probability to infinity, as $m \rightarrow \infty$, for all $\gamma \in [0, 1/2)$. Its proof is given in Subsection 4.2.

Theorem 2.2 *Suppose that alternative hypothesis H_1 holds, assumptions (A1)-(A6) are satisfied. Then, we have that:*

$$Z_m(\gamma) = \sup_{1 \leq k \leq T_m} \Gamma(m, k, \gamma) \xrightarrow[m \rightarrow \infty]{\mathbb{P}} \infty, \quad \text{for all } \gamma \in [0, 1/2).$$

Consequence of Theorem 2.1 and Theorem 2.2, we deduce that the statistic $Z_m(\gamma)$ can be considered for testing H_0 against H_1 . So, the asymptotic critical region is $\{Z_m(\gamma) > c_\alpha(\gamma)\}$, where $c_\alpha(\gamma)$ is the $(1 - \alpha)$ quantile of the distribution of $\sup_{0 < t < 1} \frac{\|\mathbf{W}(t)\|_\infty}{t^\gamma}$, if $T_m = \infty$ or $(T_m < \infty$ and $\lim_{m \rightarrow \infty} T_m/m = \infty)$, and of $\sup_{0 < t < T/(1+T)} \frac{\|\mathbf{W}(t)\|_\infty}{t^\gamma}$, if $T_m < \infty$ and $\lim_{m \rightarrow \infty} T_m/m = T < \infty$. On the other hand, for a fixed $\alpha \in (0, 1)$, the proposed test statistic has the asymptotic type I error probability (size) α and the asymptotic power 1.

Remark 2.4 *The sequential detector statistic, built as the cumulative sum of the LS residuals, considered by Ciuperca (2013), always in a nonlinear model, has an asymptotic distribution depending on regression function g and on true value β^0 of the parameter before the change-point. Then, for this test statistic, one must calculate the value of critical value for each function g and value β^0 . In the present work case, the asymptotic distribution of the proposed test statistic, and then the asymptotic critical value, don't depend on g or β^0 . In contrast, them must be calculated for each value of p .*

The stopping time for the proposed test statistic is the first observation k for which $\Gamma(m, k, \gamma)$ is greater than $c_\alpha(\gamma)$:

$$\hat{k}_m \equiv \begin{cases} \inf \{1 \leq k \leq T_m, \Gamma(m, k, \gamma) \geq c_\alpha(\gamma)\}, \\ \infty, & \text{if } \Gamma(m, k, \gamma) < c_\alpha(\gamma), \text{ for every } 1 \leq k \leq T_m. \end{cases} \quad (2.8)$$

Then, hypothesis H_0 is rejected in \hat{k}_m .

3 Simulations

In this section, Monte Carlo simulation studies are carried out to assess the performance of the proposed test statistic and to compare it with another real time detection method for nonlinear models. All simulations were performed using the R language. The program codes can be requested from the author. The Brownian motions are generated using the *BM* function in R package *'sde'*. The following R packages were also used: package *"quantreg"* for functions *nlrq* (nonlinear quantile model), *rq* (linear quantile model), package *"MASS"* for function *ginv* which calculates the generalized inverse of a matrix, package *"expm"* for function *sqrtn* which computes the matrix square root of a square matrix.

Firstly, the critical values for a model with two parameters are given, in open-end and in closed-end procedure. In Table 1 are given the empirical quantile (critical values) for the open-end procedure, for six values of γ : 0, 0.15, 0.25, 0.35, 0.45, 0.49 and five values for the nominal size $\alpha \in \{0.01, 0.025, 0.05, 0.10, 0.25\}$. For the same values of γ

Table 1: The empirical $(1 - \alpha)$ quantiles (critical values) $c_\alpha(\gamma)$ of the random variable $\sup_{0 < t < 1} \|\mathbf{W}(t)\|_\infty / t^\gamma$, for $p = 2$, calculated on 50000 Monte-Carlo replications. Open-end procedure.

$\gamma \downarrow; \alpha \rightarrow$	0.01	0.025	0.05	0.10	0.25
0	2.9963	2.7121	2.4806	2.2257	1.8277
0.15	3.0538	2.7921	2.7684	2.5406	1.9059
0.25	3.1121	2.8333	2.6103	2.3678	1.9865
0.35	3.2161	2.9422	2.7273	2.4868	2.1214
0.45	3.4625	3.2067	2.9943	2.7675	2.4226
0.49	3.7316	3.4731	3.2634	3.0303	2.6801

Table 2: The empirical $(1 - \alpha)$ quantiles (critical values) $c_\alpha(\gamma)$ of the random variable $\sup_{0 < t < T/(1+T)} \|\mathbf{W}(t)\|_\infty / t^\gamma$, for $p = 2$, calculated on 50000 Monte-Carlo replications. Closed-end procedure for $T = 5/2$.

$\gamma \downarrow; \alpha \rightarrow$	0.01	0.025	0.05	0.10	0.25
0	2.5478	2.3058	2.1026	1.8830	1.5430
0.15	2.7267	2.4791	2.2665	2.0377	1.6925
0.25	2.8697	2.6244	2.4098	2.1743	1.8263
0.35	3.0702	2.8154	2.5979	2.3646	2.0190
0.45	3.4099	3.1626	2.9533	2.7271	2.383
0.49	3.7263	3.4631	3.2561	3.0311	2.6740

and α , we give the critical values in Table 2, for the case $T = \lim_{m \rightarrow \infty} T_m/m = 5/2$. Two models are considered: a linear model in subsection 3.1 and the growth model in Subsection 3.2. Everywhere we take $m = 200$, $T_m = 500$ and $\beta^0 = (1, 1)$. The design is of dimension 1 ($q = 1$) and it is obtained by considering $X_i \sim \mathcal{N}(0, 1)$, for any $i = 1, \dots, m, \dots, m + T_m$.

In view of the asymptotic distribution of the test statistic under hypothesis H_0 , we deduce that for a fixed nominal size α , we have the following relation between the empirical powers, sizes for the open-end and closed-end procedures: $\hat{\pi}_{open} \leq \hat{\pi}_{closed}$.

3.1 Linear model

In this subsection, we present the case already considered by Zhou et al. (2015), where only standardized Gaussian errors were considered, for the simple linear regression function $g(x; \beta) = a + bx$, with $\beta = (a, b)$. We consider in addition of $\mathcal{N}(0, 1)$ law for the errors, two Cauchy distributions: $\mathcal{C}(0, 2)$ and $\mathcal{C}(0, 1)$. The results are given in Table 3 for the open-end procedure and in Table 4 for the closed-end procedure. For the historical period, we consider $\beta^0 = (1, 1)$. For the model after the change k_m^0 , we consider two possible values for β^1 : the first possible value has been considered in the paper Zhou et al. (2015): $\beta^1 = (1, 2)$, the second considered value is $\beta^1 = (2, 3)$. Compared to Zhou et al. (2015), we give empirical powers when the change occurs in $k_m^0 = 5$. Concerning the empirical sizes, for $\varepsilon \sim \mathcal{N}(0, 1)$, we find the results given in Zhou et al. (2015).

Comparing the results of Tables 3 and 4, for the two Cauchy laws $\mathcal{C}(0, 1)$ and $\mathcal{C}(0, 2)$,

Table 3: Empirical test sizes ($\hat{\alpha}$) and powers ($\hat{\pi}$), calculated on 5000 Monte-Carlo replications. Open-end procedure, linear quantile model, $k_m^0 = 5$, $\varepsilon \sim \mathcal{N}(0, 1)$, $\varepsilon \sim \mathcal{C}(0, 1)$ or $\varepsilon \sim \mathcal{C}(0, 2)$.

$\hat{\alpha}$ or $\hat{\pi}$	β^1	α	ε	γ					
				0	0.15	0.25	0.35	0.45	0.49
$\hat{\alpha}$	β^0	0.025	$\mathcal{N}(0, 1)$	0.007	0.012	0.017	0.020	0.024	0.022
			$\mathcal{C}(0, 2)$	0.006	0.011	0.013	0.017	0.021	0.016
			$\mathcal{C}(0, 1)$	0.008	0.015	0.019	0.022	0.022	0.17
		0.05	$\mathcal{N}(0, 1)$	0.017	0.025	0.032	0.040	0.041	0.036
			$\mathcal{C}(0, 2)$	0.016	0.024	0.030	0.038	0.039	0.034
			$\mathcal{C}(0, 1)$	0.020	0.026	0.031	0.038	0.039	0.031
$\hat{\pi}$	(1, 2)	0.025	$\mathcal{N}(0, 1)$	1	1	1	1	1	1
			$\mathcal{C}(0, 2)$	0.552	0.614	0.641	0.650	0.587	0.488
			$\mathcal{C}(0, 1)$	0.997	0.998	0.999	0.999	0.997	0.995
		0.05	$\mathcal{N}(0, 1)$	1	1	1	1	1	1
			$\mathcal{C}(0, 2)$	0.695	0.738	0.756	0.755	0.695	0.593
			$\mathcal{C}(0, 1)$	1	1	0.1	1	0.999	0.997
$\hat{\pi}$	(2, 3)	0.025	$\mathcal{N}(0, 1)$	1	1	1	1	1	1
			$\mathcal{C}(0, 2)$	0.994	0.997	0.997	0.997	0.995	0.985
			$\mathcal{C}(0, 1)$	1	1	1	1	1	1
		0.05	$\mathcal{N}(0, 1)$	1	1	1	1	1	1
			$\mathcal{C}(0, 2)$	0.998	0.999	1	0.999	0.998	0.995
			$\mathcal{C}(0, 1)$	1	1	1	1	1	1

Table 4: Empirical test sizes ($\hat{\alpha}$) and powers ($\hat{\pi}$), calculated on 5000 Monte-Carlo replications. Closed-end procedure, linear quantile model, $k_m^0 = 5$, $\varepsilon \sim \mathcal{N}(0, 1)$, $\varepsilon \sim \mathcal{C}(0, 1)$ or $\varepsilon \sim \mathcal{C}(0, 2)$.

$\hat{\alpha}$ or $\hat{\pi}$	β^1	α	ε	γ					
				0	0.15	0.25	0.35	0.45	0.49
$\hat{\alpha}$	β^0	0.025	$\mathcal{N}(0, 1)$	0.058	0.030	0.032	0.029	0.026	0.022
			$\mathcal{C}(0, 2)$	0.029	0.029	0.029	0.027	0.024	0.017
			$\mathcal{C}(0, 1)$	0.030	0.030	0.031	0.030	0.024	0.018
		0.05	$\mathcal{N}(0, 1)$	0.058	0.057	0.058	0.056	0.046	0.037
			$\mathcal{C}(0, 2)$	0.054	0.054	0.053	0.054	0.045	0.035
			$\mathcal{C}(0, 1)$	0.055	0.056	0.051	0.051	0.045	0.032
$\hat{\pi}$	(1, 2)	0.025	$\mathcal{N}(0, 1)$	1	1	1	1	1	1
			$\mathcal{C}(0, 2)$	0.782	0.767	0.748	0.715	0.609	0.493
			$\mathcal{C}(0, 1)$	1	1	1	1	1	0.995
		0.05	$\mathcal{N}(0, 1)$	1	1	1	1	1	1
			$\mathcal{C}(0, 2)$	0.860	0.849	0.833	0.808	0.715	0.598
			$\mathcal{C}(0, 1)$	1	1	1	1	1	0.997
$\hat{\pi}$	(2, 3)	0.025	$\mathcal{N}(0, 1)$	1	1	1	1	1	1
			$\mathcal{C}(0, 2)$	0.999	0.999	0.999	0.999	0.996	0.985
			$\mathcal{C}(0, 1)$	1	1	1	1	0.998	0.996
		0.05	$\mathcal{N}(0, 1)$	1	1	1	1	1	1
			$\mathcal{C}(0, 2)$	1	1	1	0.999	0.999	0.995
			$\mathcal{C}(0, 1)$	1	1	1	1	1	1

we deduct that for $\mathcal{C}(0, 2)$, if $\beta^1 = (1, 2)$, "only" 495 observations were not sufficient to detect the change, when the distribution of the errors ε has very heavy tails. If $\beta^1 = (2, 3)$, thus the difference between β^0 and β^1 is bigger, then the empirical powers are very close to 1. As argument that $T_m = 500$ is not large enough to detect change when $\varepsilon \sim \mathcal{C}(0, 2)$, we see improved results when $T_m = 1500$ observations are considered after $m = 200$. We got, for the *open-end procedure*, the following empirical powers:

	$\gamma = 0$	$\gamma = 0.15$	$\gamma = 0.25$	$\gamma = 0.35$	$\gamma = 0.45$	$\gamma = 0.49$
$\alpha = 0.025 :$	0.856	0.863	0.859	0.844	0.774	0.667
$\alpha = 0.05 :$	0.921	0.922	0.918	0.903	0.849	0.767

and for the *closed-end procedure*:

	$\gamma = 0$	$\gamma = 0.15$	$\gamma = 0.25$	$\gamma = 0.35$	$\gamma = 0.45$	$\gamma = 0.49$
$\alpha = 0.025 :$	0.952	0.934	0.915	0.882	0.796	0.671
$\alpha = 0.05 :$	0.973	0.963	0.954	0.932	0.862	0.776

For $\varepsilon \sim \mathcal{N}(0, 1)$ or $\mathcal{C}(0, 1)$, the empirical powers $\hat{\pi}$ are either 1 or very close to 1.

In the open-end procedure (see Table 3), for all cases, the empirical sizes $\hat{\alpha}$ are inferior to the corresponding nominal sizes α . In the closed-end procedure (see Table 4), for $\gamma \in \{0, 0.15, 0.25, 0.35\}$ the empirical sizes are slightly higher than the nominal sizes for the three laws. For $\gamma \in \{0.45, 0.49\}$ the empirical sizes $\hat{\alpha}$ are lower to α .

3.2 Nonlinear model: growth model

Let us consider now the growth function $g(x; \beta) = b_1 - \exp(-b_2x)$, which models many phenomena, with the parameters $\beta = (b_1, b_2) \in \mathbb{R}^2$ and $x \in \mathbb{R}$. We consider the following values for the true parameters: $\beta^0 = (1, 1)$ and $\beta^1 = (1, 2)$. The results are given in Table 5 for the closed-end procedure and in Table 6 for the open-end procedure.

For the open-end procedure, we compared our results with the results obtained by Ciuperca (2013) using CUSUM method for the least squares (LS) residuals. Recall that for the CUSUM method for the LS residuals, in addition to γ and α , the critical values must be calculated for each regression function $g(\mathbf{x}, \beta)$, each value of β^0 and each law of the design $(\mathbf{X}_i)_{1 \leq i \leq m}$. The proposed test statistic $Z_m(\gamma)$ given by (2.5), have asymptotic distribution under hypothesis H_0 and then the critical values, only depend on the value of $\gamma \in (0, 1/2)$ and on the nominal size $\alpha \in (0, 1)$. We will study by simulations if there are more appropriate values of γ than other. The influence of error distribution on the results is also studied. We will consider the standard normal distribution and the Cauchy distribution, a law which does not have finite moments, with heavy tails.

For the open-end procedure, the empirical sizes are inferior to α , for any considered value of γ , α and for the two error distributions $\mathcal{N}(0, 1)$ and $\mathcal{C}(0, 1)$.

For the closed-end procedure, we have considered three distributions for the errors: $\mathcal{N}(0, 1)$, $\mathcal{C}(0, 1)$ and $\mathcal{C}(0, 2)$. For errors $\mathcal{C}(0, 2)$, we have that $\hat{\alpha}$ are slightly superior to α . If $\varepsilon \sim \mathcal{N}(0, 1)$ or $\mathcal{C}(0, 1)$, then $\hat{\alpha} \leq \alpha$ for $\gamma \in \{0.45, 0.49\}$.

On empirical powers $\hat{\pi}$, in the all cases, for both types of procedure, they are either

Table 5: Empirical test sizes ($\hat{\alpha}$) and powers ($\hat{\pi}$), calculated on 5000 Monte-Carlo replications. Closed-end procedure, growth model by quantile method, $k_m^0 = 5$, $\varepsilon \sim \mathcal{N}(0, 1)$, $\varepsilon \sim \mathcal{C}(0, 1)$, $\varepsilon \sim \mathcal{C}(0, 2)$.

$\hat{\alpha}$ or $\hat{\pi}$	β^1	α	ε	γ					
				0	0.15	0.25	0.35	0.45	0.49
$\hat{\alpha}$	β^0	0.025	$\mathcal{N}(0, 1)$	0.028	0.028	0.027	0.026	0.025	0.022
			$\mathcal{C}(0, 2)$	0.031	0.032	0.032	0.035	0.038	0.030
			$\mathcal{C}(0, 1)$	0.025	0.026	0.026	0.023	0.022	0.017
		0.05	$\mathcal{N}(0, 1)$	0.050	0.051	0.053	0.051	0.045	0.035
			$\mathcal{C}(0, 2)$	0.057	0.059	0.059	0.063	0.057	0.048
			$\mathcal{C}(0, 1)$	0.050	0.049	0.047	0.047	0.041	0.036
$\hat{\pi}$	(1, 2)	0.025	$\mathcal{N}(0, 1)$	1	1	1	1	1	1
			$\mathcal{C}(0, 2)$	0.999	0.998	0.998	0.997	0.994	0.988
			$\mathcal{C}(0, 1)$	0.999	0.999	0.999	0.999	0.999	0.999
		0.05	$\mathcal{N}(0, 1)$	1	1	1	1	1	1
			$\mathcal{C}(0, 2)$	0.999	0.999	0.999	0.999	0.997	0.994
			$\mathcal{C}(0, 1)$	0.999	0.999	0.999	0.999	0.999	0.999

1 or very close to 1. Compared to the linear case considered in Subsection 3.1, the changes in model are detected almost always ($\hat{\pi} \geq 0.988$) for the errors $\mathcal{C}(0, 2)$. A possible explanation of the result improvement is the model shape. More precisely, in the nonlinear case, the derivatives $\mathbf{g}(x, \beta)$ depend on β and therefore changes appear earlier in the test statistic. In the linear case, changes occur only in ψ_τ , which lengthens change-point detection. In Table 7 are given descriptive statistics of the detected stopping times \hat{k}_m , given by (2.8), for two values of k_m^0 : 5 and 50. The value "Inf" for "max" means that there has been at least one Monte Carlo realization for which no changes were detected under hypothesis H_1 (corroborating with the empirical power results for error law $\mathcal{C}(0, 1)$ in Table 6). We observe that the shortest delay time is achieved for $\gamma = 0.45$. This delay time is longer than that obtained by CUSUM method for LS normal errors (see Ciuperca (2013)).

3.3 Conclusion on simulations

For linear model, in the open-end procedure case, the empirical sizes $\hat{\alpha}$ are inferior to the nominal sizes α . In the closed-end procedure case, for $\gamma \in \{0, 0.15, 0.25, 0.35\}$, the empirical sizes are slightly than the nominal size and for $\gamma \in \{0.45, 0.49\}$ we have $\hat{\alpha} \leq \alpha$. For very heavy-tailed errors, if T_m is not large enough and $\|\beta^1 - \beta^0\|$ is small, then the proposed test statistic does not detect the change.

In contrast, for models with derivatives dependent on parameter β , the empirical powers are either 1 or very close to one. All values of γ between 0 and 1/2 can be considered, with a slight advantage for the values close to 0.49, since there are fewer false alarms for heavy-tailed errors. The preference for value $\gamma = 0.45$ is given by the shortest delay for detecting the change.

Not least, the proposed test statistic gives very good results in comparison to the CUSUM method for the LS residuals.

Table 6: Empirical test sizes ($\hat{\alpha}$) and powers ($\hat{\pi}$), calculated on 5000 Monte-Carlo replications. Open-end procedure, growth model by quantile(Q) and LS methods, $k_m^0 = 5$, $\varepsilon \sim \mathcal{N}(0, 1)$, $\varepsilon \sim \mathcal{C}(0, 1)$.

$\hat{\alpha}$ or $\hat{\pi}$	β^1	α	ε	method	γ					
					0	0.15	0.25	0.35	0.45	0.49
$\hat{\alpha}$	β^0	0.025	$\mathcal{N}(0, 1)$	Q	0.007	0.01	0.013	0.017	0.020	0.019
				LS	0.002	0.002	0.002	0.003	0	0
			$\mathcal{C}(0, 1)$	Q	0.006	0.009	0.013	0.017	0.019	0.017
				LS	0.3	0.317	0.35	0.372	0.395	0.286
		0.05	$\mathcal{N}(0, 1)$	Q	0.015	0.022	0.028	0.036	0.039	0.034
				LS	0.005	0.003	0.002	0.004	0	0.001
			$\mathcal{C}(0, 1)$	Q	0.013	0.021	0.027	0.032	0.037	0.033
				LS	0.327	0.352	0.374	0.404	0.42	0.30
$\hat{\pi}$	(1, 2)	0.025	$\mathcal{N}(0, 1)$	Q	1	1	1	1	1	1
				LS	0.672	0.704	0.685	0.738	0.719	0.675
			$\mathcal{C}(0, 1)$	Q	999	0.999	0.999	0.999	0.999	0.999
				LS	0.672	0.704	0.685	0.738	0.719	0.675
		0.05	$\mathcal{N}(0, 1)$	Q	1	1	1	1	1	1
				LS	1	1	1	1	1	1
			$\mathcal{C}(0, 1)$	Q	999	0.999	0.999	0.999	0.999	0.999
				LS	0.701	0.731	0.714	0.754	0.74	0.703

Table 7: Descriptive statistics for the stopping times, calculated on 5000 Monte-Carlo replications. Open-end procedure, growth model by quantile method, $k_m^0 = 5$ or $k_m^0 = 50$, $\varepsilon \sim \mathcal{N}(0, 1)$, $\varepsilon \sim \mathcal{C}(0, 1)$.

k_m^0	α	ε	stat	γ					
				0	0.15	0.25	0.35	0.45	0.49
5	0.025	$\mathcal{N}(0, 1)$	median	97	77	65	54	47	48
			min	43	26	19	11	1	1
			max	230	214	190	184	187	200
		$\mathcal{C}(0, 1)$	median	124	100	85	71	63	66
			min	51	31	25	1	1	1
			max	437	421	413	413	437	Inf
	0.05	$\mathcal{N}(0, 1)$	median	86	68	58	48	42	43
			min	36	24	15	2	1	1
			max	200	184	172	171	171	184
		$\mathcal{C}(0, 1)$	median	109	87	74	62	54	56
			min	46	28	19	1	1	1
			max	405	398	317	308	411	437
50	0.025	$\mathcal{N}(0, 1)$	median	162	146	136	129	128	134
			min	85	66	61	37	1	1
			max	316	296	293	290	296	330
		$\mathcal{C}(0, 1)$	median	194	174	163	154	154	164
			min	82	45	37	1	1	1
			max	Inf	Inf	469	469	Inf	Inf
	0.05	$\mathcal{N}(0, 1)$	median	149	134	127	120	119	125
			min	75	63	45	1	1	1
			max	293	279	277	2464	283	293
		$\mathcal{C}(0, 1)$	median	175	158	148	141	141	150
			min	63	57	35	1	1	1
			max	456	443	442	442	467	Inf

4 Lemmas and proofs of Theorems

This section contains two lemmas and the proofs of Theorems 2.1 and 2.2.

4.1 Lemmas

In order to prove the main results, we need two lemmas.

By Hoeffding's inequality (see Hoeffding (1963), Wang and He (2007)) we have the following result:

Lemma 4.1 *Let $(Z_{in})_{1 \leq i \leq n}$ be a sequence of bounded independent random variables. Denote by $S_n \equiv \sum_{i=1}^n Z_{in}$. Let $(\delta_n)_{n \in \mathbb{N}}$ be a given sequence and suppose that*

$$\sum_{i=1}^n \text{Var}[Z_{in}] \leq \delta_n, \quad \max_{1 \leq i \leq n} |Z_{in}| \leq \frac{\log 2}{2} \frac{\sqrt{\delta_n}}{\sqrt{\log n}}.$$

Then, for all $L > 0$, there exists $N_0 \in \mathbb{N}$, such that for any $n \geq N_0$, we have:

$$\mathbb{P}[|S_n - \mathbb{E}[S_n]| \geq (1 + L)\sqrt{\delta_n}\sqrt{\log n}] \leq 2\exp(-L \log n) = 2n^{-L}.$$

For a p -vector \mathbf{u} , observations $i = m + 1, \dots, m + T_m$, let us consider the following random p -vectors, which will be used in the following Lemma and in the proofs of Theorems 2.1 and 2.2.

$$\mathbf{R}_i(\mathbf{u}) \equiv \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u})\psi_\tau(Y_i - g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u})) - \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0)\psi_\tau(\varepsilon_i) \quad (4.1)$$

and, for $k = 1, \dots, T_m$,

$$\mathbf{r}_{m,k}(\mathbf{u}) \equiv \sum_{i=m+1}^{m+k} \mathbf{R}_i(\mathbf{u}). \quad (4.2)$$

Lemma 4.2 *Suppose that assumptions (A1), (A2), (A3), (A5) are satisfied. Let $C_1 > 0$ be some constant. For all constant $L > 0$ and any $k \in \mathbb{N}$ sufficiently large, under hypothesis H_0 , we have:*

$$\mathbb{P}\left[\sup_{\|\mathbf{u}\| \leq C_1} \|\mathbf{r}_{m,k}(\mathbf{u}) - \mathbb{E}[\mathbf{r}_{m,k}(\mathbf{u})]\| \geq \sqrt{p}(1 + L)m^{-1/4}k^{1/2}\sqrt{\log k}\right] \leq 2k^{-L}.$$

Proof of Lemma 4.2

In order to prove the lemma, we show that all conditions of Lemma 4.1 are satisfied for the sequence of independent random variables $(R_{ij}(\mathbf{u}))_{m+1 \leq i \leq m+k}$, for any $j = 1, \dots, p$, with $R_{ij}(\mathbf{u})$ the j th coordinate of $\mathbf{R}_i(\mathbf{u})$.

By assumption (A3), since the function g is continuous twice differentiable in any neighbourhood $\mathcal{V}_m(\mathbf{u})$ of β^0 , with \mathbf{u} a bounded p -vector, we have that the random vectors $(\mathbf{R}_i(\mathbf{u}))_{m+1 \leq i \leq m+k}$ are bounded with probability one:

$$\max_{m+1 \leq i \leq m+k} \left(\max_{\|\mathbf{u}\| \leq C_1} \|\mathbf{R}_i(\mathbf{u})\| \right) = O_{\mathbb{P}}(1).$$

We study now $\sum_{i=m+1}^{m+k} \text{Var}[R_{ij}(\mathbf{u})]$. For this, we write $\mathbf{R}_i(\mathbf{u})$, defined by (4.1), under the form: $\mathbf{R}_i(\mathbf{u}) = \dot{\mathbf{g}}(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) [\mathbb{1}_{\varepsilon_i \leq 0} - \mathbb{1}_{\varepsilon_i \leq g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \beta^0)}] + [\dot{\mathbf{g}}(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) - \dot{\mathbf{g}}(\mathbf{X}_i; \beta^0)] [\tau - \mathbb{1}_{\varepsilon_i \leq 0}]$.

For all $j \in \{1, \dots, p\}$ and $i = m+1, \dots, m+k$, we have:

$$\begin{aligned} R_{ij}(\mathbf{u}) &= \frac{\partial g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j} \left(\mathbb{1}_{\varepsilon_i \leq 0} - \mathbb{1}_{\varepsilon_i \leq g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \beta^0)} \right) \\ &+ \left(\frac{\partial g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j} - \frac{\partial g(\mathbf{X}_i; \beta^0)}{\partial \beta_j} \right) (\tau - \mathbb{1}_{\varepsilon_i \leq 0}) \equiv A_{ij}(\mathbf{u}) + B_{ij}(\mathbf{u}). \end{aligned} \quad (4.3)$$

Thus, we have for its variance,

$$\text{Var}[R_{ij}(\mathbf{u})] \leq \mathbb{E}[(A_{ij}(\mathbf{u}) + B_{ij}(\mathbf{u}))^2] \leq 2(\mathbb{E}[A_{ij}^2(\mathbf{u})] + \mathbb{E}[B_{ij}^2(\mathbf{u})]). \quad (4.4)$$

By the Taylor's expansion at β^0 of $\frac{\partial g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j}$, with $\mathbf{u} = (u_1, \dots, u_p)$, we obtain, applying also the Cauchy-Schwarz inequality:

$$\mathbb{E}[B_{ij}^2(\mathbf{u})] \leq \left[m^{-1/2} \sum_{l=1}^p u_l \frac{\partial^2 g(\beta^0 + m^{-1/2}\tilde{\mathbf{u}})}{\partial \beta_j \partial \beta_l} \right]^2 \leq m^{-1} \|\mathbf{u}\|^2 \left\| \ddot{\mathbf{g}}_j(\beta^0 + m^{-1/2}\tilde{\mathbf{u}}) \right\|^2,$$

with the vector $\ddot{\mathbf{g}}_j(\beta^0 + m^{-1/2}\tilde{\mathbf{u}}) \equiv \left(\frac{\partial^2 g(\beta^0 + m^{-1/2}\tilde{\mathbf{u}})}{\partial \beta_j \partial \beta_l} \right)_{1 \leq l \leq p}$ and the vector $\tilde{\mathbf{u}} = (\tilde{u}_1, \dots, \tilde{u}_p)$ such that $0 \leq |\tilde{u}_j - u_j| \leq 1$, for any $j = 1, \dots, p$.

Using assumption (A5) and $\|\mathbf{u}\| \leq C_1$, we obtain, uniformly in \mathbf{u} and in j :

$$\sum_{i=m+1}^{m+k} \mathbb{E}[B_{ij}^2(\mathbf{u})] = O(m^{-1}k). \quad (4.5)$$

For the term $A_{ij}(\mathbf{u})$ of relation (4.3), using the identity that, for all real numbers a and b , we have with probability one, that, $\mathbb{1}_{\varepsilon_i \leq a} - \mathbb{1}_{\varepsilon_i \leq b} = \mathbb{1}_{\min(a,b) \leq \varepsilon_i \leq \max(a,b)} \text{sgn}(a-b)$, we obtain for $A_{ij}^2(\mathbf{u})$:

$$\begin{aligned} \mathbb{E}[A_{ij}^2(\mathbf{u})] &\leq \left(\frac{\partial g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j} \right)^2 \mathbb{E} \left[\left(\mathbb{1}_{\varepsilon_i \leq 0} - \mathbb{1}_{\varepsilon_i \leq g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \beta^0)} \right)^2 \right] \\ &= \left(\frac{\partial g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j} \right)^2 \mathbb{E} \left[\mathbb{1}_{\min(0, g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \beta^0)) \leq \varepsilon_i \leq \max(0, g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \beta^0))} \right]. \end{aligned}$$

We have used the notation $\text{sgn}(\cdot)$ for the sign function. We assume, without loss of generality that,

$$\max(0, g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \boldsymbol{\beta}^0)) = g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \boldsymbol{\beta}^0).$$

Then, $\mathbb{E} \left[\mathbb{1}_{\min(0, g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \boldsymbol{\beta}^0)) \leq \varepsilon_i \leq \max(0, g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \boldsymbol{\beta}^0))} \right] = F(g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \boldsymbol{\beta}^0)) - F(0)$. Thus, by two Taylor expansions, first for F and after for g , we have:

$$\begin{aligned} \mathbb{E}[A_{ij}^2(\mathbf{u})] &\leq \left(\frac{\partial g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j} \right)^2 [g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \boldsymbol{\beta}^0)] f(b_i) \\ &= \left(\frac{\partial g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j} \right)^2 \left| m^{-1/2}\mathbf{u}^t \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{w}_{i,m}) \right| f(b_i) \\ &\leq \left(\frac{\partial g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j} \right)^2 m^{-1/2} \|\mathbf{u}\| \|\dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{w}_{i,m})\| f(b_i), \end{aligned}$$

with the scalar b_i between $g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \boldsymbol{\beta}^0)$ and 0, the p -vector $\mathbf{w}_{i,m}$ between $\boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u}$ and $\boldsymbol{\beta}^0$. For the last inequality, we have used the Cauchy-Schwarz inequality. The vector $\mathbf{w}_{i,m}$ is such that there exists a constant C_4 , which does not depend on i, j, m , such that, $\|\mathbf{w}_{i,m}\| \leq C_4$ for any i and \mathbf{u} .

Then, taking into account assumption (A2), we get

$$\sum_{i=m+1}^{m+k} \mathbb{E}[A_{ij}^2(\mathbf{u})] \leq C m^{-1/2} \|\mathbf{u}\| \sum_{i=m+1}^{m+k} \left(\frac{\partial g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j} \right)^2 \|\dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{w}_{i,m})\|.$$

Thus, using assumption (A3) and the fact that \mathbf{u} is bounded, we obtain, uniformly in \mathbf{u} :

$$\sum_{i=m+1}^{m+k} \mathbb{E}[A_{ij}^2(\mathbf{u})] \leq O(m^{-1/2}k). \quad (4.6)$$

Taking into account relations (4.4), (4.5), (4.6) together with assumption (A1), we get:

$$\sum_{i=m+1}^{m+k} \text{Var}[R_{ij}(\mathbf{u})] = O(m^{-1/2}k).$$

Thus we can apply Lemma 4.1 for the random variable $R_{ij}(\mathbf{u})$, for any $j = 1, \dots, p$ and for $\delta_k = km^{-1/2}$. Therefore, for all $L > 0$, $\|\mathbf{u}\| \leq C_1$, for $k \in \mathbb{N}$ sufficiently large, we get:

$$\mathbb{P} \left[\left[R_{ij}(\mathbf{u}) - \mathbb{E}[R_{ij}(\mathbf{u})] \right]^2 \geq (1+L)^2 m^{-1/2} k \log k \right] \leq 2k^{-L}. \quad (4.7)$$

On the other hand, we have, the following inequalities, for all $L > 0$, any observation $i = m + 1, \dots, m + k$, all vector \mathbf{u} bounded and enough large k, m , that:

$$\begin{aligned} \mathbb{P} \left[\|\mathbf{R}_i(\mathbf{u}) - \mathbb{E}[\mathbf{R}_i(\mathbf{u})]\|^2 \geq p(1+L)^2 m^{-1/2} k \log k \right] &\leq \mathbb{P} \left[\max_{1 \leq j \leq p} |R_{ij}(\mathbf{u}) - \mathbb{E}[R_{ij}(\mathbf{u})]|^2 \geq (1+L)^2 m^{-1/2} k \log k \right] \\ &\leq 2k^{-L}. \end{aligned}$$

For the last inequality we have used relation (4.7). The lemma is proved. \blacksquare

4.2 Proofs of Theorems

Here we present the proofs of Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1

Statements (i) and (ii) are shown at the same time.

Case k small. Under assumptions (A1)-(A4), we have that the quantile estimator $\widehat{\beta}_m$ is consistent (see Koenker (2005)). By similar arguments as in the proof of Lemma 4.2 and by Bienaymé-Tchebychev inequality, we have that $\psi_\tau(Y_i - g(\mathbf{X}_i; \widehat{\beta}_m)) \xrightarrow[m \rightarrow \infty]{\mathbb{P}} \mathbf{0}$, for any $i \geq m + 1$, under hypothesis H_1 . Then, taking into account assumption (A3), we have that $\dot{\mathbf{g}}(\mathbf{X}_i; \widehat{\beta}_m) \psi_\tau(Y_i - g(\mathbf{X}_i; \widehat{\beta}_m)) \xrightarrow[m \rightarrow \infty]{\mathbb{P}} \mathbf{0}$ and thus, taking also into account relations (2.3) and (2.4), we have that $\Gamma(m, k, \gamma)$ converges in probability to 0, as $m \rightarrow \infty$. We deduct that, the sup of $\Gamma(m, k, \gamma)$ is not achieved for small values of k .

Case k large. From Lemma 4.2 we have for $\mathbf{r}_{m,k}(\mathbf{u})$, defined by (4.2), that:

$$\mathbf{r}_{m,k}(\mathbf{u}) = \mathbb{E}[\mathbf{r}_{m,k}(\mathbf{u})] + O_{\mathbb{P}}(m^{-1/4} k^{1/2} \sqrt{\log k}). \quad (4.8)$$

We will calculate $\mathbb{E}[\mathbf{r}_{m,k}(\mathbf{u})] = \sum_{i=m+1}^{m+k} \mathbb{E}[\mathbf{R}_i(\mathbf{u})]$. By Taylor expansions, we have:

$$\begin{aligned} \mathbb{E}[\mathbf{R}_i(\mathbf{u})] &= [\dot{\mathbf{g}}(\mathbf{X}_i; \beta^0) + m^{-1/2} \mathbf{u} \ddot{\mathbf{g}}(\mathbf{X}_i; \tilde{\beta})] [F(0) - F(g(\mathbf{X}_i; \beta^0 + m^{-1/2} \mathbf{u}) - g(\mathbf{X}_i; \beta^0))] \\ &= -[\dot{\mathbf{g}}(\mathbf{X}_i; \beta^0) + m^{-1/2} \mathbf{u} \ddot{\mathbf{g}}(\mathbf{X}_i; \tilde{\beta})] \left\{ [g(\mathbf{X}_i; \beta^0 + m^{-1/2} \mathbf{u}) - g(\mathbf{X}_i; \beta^0)] f(0) \right. \\ &\quad \left. + 2^{-1} [g(\mathbf{X}_i; \beta^0 + m^{-1/2} \mathbf{u}) - g(\mathbf{X}_i; \beta^0)]^2 f'(\tilde{b}_i) \right\} \\ &= -[\dot{\mathbf{g}}(\mathbf{X}_i; \beta^0) + m^{-1/2} \mathbf{u} \ddot{\mathbf{g}}(\mathbf{X}_i; \tilde{\beta})] \left\{ [m^{-1/2} \mathbf{u} \dot{\mathbf{g}}(\mathbf{X}_i; \beta^0) + 2^{-1} m^{-1} \mathbf{u} \ddot{\mathbf{g}}(\mathbf{X}_i; \tilde{\beta}) \mathbf{u}] f(0) \right. \\ &\quad \left. + 2^{-1} [g(\mathbf{X}_i; \beta^0 + m^{-1/2} \mathbf{u}) - g(\mathbf{X}_i; \beta^0)]^2 f'(\tilde{b}_i) \right\}, \end{aligned}$$

with the scalar \tilde{b}_i between $g(\mathbf{X}_i; \beta^0 + m^{-1/2} \mathbf{u}) - g(\mathbf{X}_i; \beta^0)$ and 0, the vector $\tilde{\beta}$ between β^0 and $\beta^0 + m^{-1/2} \mathbf{u}$. Thus, for k large enough, using assumptions (A1)-(A6), we have

$$\mathbb{E}[\mathbf{r}_{m,k}(\mathbf{u})] = -m^{-1/2} f(0) \mathbf{u} \sum_{i=m+1}^{m+k} \dot{\mathbf{g}}^t(\mathbf{X}_i; \beta^0) \dot{\mathbf{g}}(\mathbf{X}_i; \beta^0) + o(km^{-1/2}). \quad (4.9)$$

On the other hand, using relation (2.7), we consider for \mathbf{u} the following value:

$$\mathbf{u} = m^{1/2}(\widehat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}^0) = m^{-1/2}\boldsymbol{\Omega}^{-1} \sum_{l=1}^m \dot{\mathbf{g}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \psi_\tau(Y_l - g(\mathbf{X}_l; \boldsymbol{\beta}^0)) + o_{\mathbb{P}}(1).$$

Then, replacing in (4.8), taking into account of (4.9), we have:

$$\begin{aligned} \mathbf{r}_{m,k}(m^{1/2}(\widehat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}^0)) &= [-m^{-1}\boldsymbol{\Omega}^{-1} \sum_{l=1}^m \dot{\mathbf{g}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \psi_\tau(Y_l - g(\mathbf{X}_l; \boldsymbol{\beta}^0))] [f(0) \sum_{i=m+1}^{m+k} \dot{\mathbf{g}}^t(\mathbf{X}_i; \boldsymbol{\beta}^0) \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0)] \\ &\quad + O_{\mathbb{P}}(m^{-1/4}k^{1/2} \sqrt{\log k}) + o_{\mathbb{P}}(km^{-1/2}). \end{aligned}$$

On the other hand, by assumption (A4), for k large enough, we have:

$$f(0) \sum_{i=m+1}^{m+k} \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \dot{\mathbf{g}}^t(\mathbf{X}_i; \boldsymbol{\beta}^0) = k\boldsymbol{\Omega}(1 + O(1)),$$

with the matrix $\boldsymbol{\Omega}$ defined by (2.6). Hence

$$\mathbf{r}_{m,k}(m^{1/2}(\widehat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}^0)) = -km^{-1} \sum_{l=1}^m \dot{\mathbf{g}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \psi_\tau(Y_l - g(\mathbf{X}_l; \boldsymbol{\beta}^0)) + O_{\mathbb{P}}(m^{-1/4}k^{1/2} \sqrt{\log k}) + o_{\mathbb{P}}(km^{-1/2}),$$

which implies, taking into account (4.1) and (4.2), that

$$\begin{aligned} \mathbf{J}_m^{-1/2}(\boldsymbol{\beta}^0) \sum_{i=m+1}^{m+k} \dot{\mathbf{g}}(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m) \psi_\tau(Y_i - g(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m)) &= \mathbf{J}_m^{-1/2}(\boldsymbol{\beta}^0) \sum_{i=m+1}^{m+k} \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \psi_\tau(Y_i - g(\mathbf{X}_i; \boldsymbol{\beta}^0)) \\ &\quad - km^{-1} \mathbf{J}_m^{-1/2}(\boldsymbol{\beta}^0) \sum_{l=1}^m \dot{\mathbf{g}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \psi_\tau(Y_l - g(\mathbf{X}_l; \boldsymbol{\beta}^0)) + O_{\mathbb{P}}(m^{-1/4}k^{1/2} \sqrt{\log k}) + o_{\mathbb{P}}(km^{-1/2}). \end{aligned} \quad (4.10)$$

By K-M-T approximation (see Komlós et al. (1975), Komlós et al. (1976), for each m there exists two independent p -dimensional Wiener processes on $[0, \infty)$, $\{\mathbf{W}_{1,m}(t), t \in [0, \infty)\}$ and $\{\mathbf{W}_{2,m}(t), t \in [0, \infty)\}$ such that, as $m \rightarrow \infty$:

$$\sup_{1 \leq k < \infty} k^{-1/\nu} \left\| \mathbf{J}_m^{-1/2}(\boldsymbol{\beta}^0) \sum_{i=m+1}^{m+k} \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \psi_\tau(Y_i - g(\mathbf{X}_i; \boldsymbol{\beta}^0)) - \mathbf{W}_{1,m}(k/m) \right\|_{\infty} = O_{\mathbb{P}}(1) \quad (4.11)$$

and

$$\left\| \mathbf{J}_m^{-1/2}(\boldsymbol{\beta}^0) \sum_{l=1}^m \dot{\mathbf{g}}^t(\mathbf{X}_l; \boldsymbol{\beta}^0) \psi_\tau(Y_l - g(\mathbf{X}_l; \boldsymbol{\beta}^0)) - \mathbf{W}_{2,m}(1) \right\|_{\infty} = o_{\mathbb{P}}(m^{1/\nu}). \quad (4.12)$$

Then, by relations (4.10), (4.11), (4.12) and Lemma 5.3 of Horvath et al.(2004), we have:

$$\sup_{1 \leq k < \infty} \frac{\left\| \mathbf{J}_m^{-1/2}(\boldsymbol{\beta}^0) \sum_{i=m+1}^{m+k} \dot{\mathbf{g}}(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m) \psi_\tau(Y_i - g(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m)) - [\mathbf{W}_{1,m}(k/m) - k/m \mathbf{W}_{2,m}(1)] \right\|_{\infty}}{z(m, k, \gamma)} = o_{\mathbb{P}}(1). \quad (4.13)$$

By a similar way as in the proof of Theorem 2.1 of Horváth et al. (2004), we have that (see also the proof in the linear case, Theorem 1 of Zhou et al. (2015)):

(i) if $T_m = \infty$ or ($T_m < \infty$ and $\lim_{m \rightarrow \infty} T_m/m = \infty$), then,

$$\sup_{1 \leq k < T_m} \frac{\|\mathbf{W}_{1,m}(k/m) - k/m \mathbf{W}_{2,m}(1)\|_\infty}{z(m, k, \gamma)} \xrightarrow[m \rightarrow \infty]{\mathcal{L}} \sup_{0 \leq t < \infty} \frac{\|\mathbf{W}_1(t) - t \mathbf{W}_2(1)\|_\infty}{(1+t)(t/(1+t))^\gamma} \stackrel{\mathcal{L}}{=} \sup_{0 \leq t \leq 1} \frac{\|\mathbf{W}(t)\|_\infty}{t^\gamma}, \quad (4.14)$$

(ii) if $T_m < \infty$ and $\lim_{m \rightarrow \infty} T_m/m = T < \infty$, then,

$$\sup_{1 \leq k < T_m} \frac{\|\mathbf{W}_{1,m}(k/m) - k/m \mathbf{W}_{2,m}(1)\|_\infty}{z(m, k, \gamma)} \xrightarrow[m \rightarrow \infty]{\mathcal{L}} \sup_{0 \leq t \leq T} \frac{\|\mathbf{W}_1(t) - t \mathbf{W}_2(1)\|_\infty}{(1+t)(t/(1+t))^\gamma} \stackrel{\mathcal{L}}{=} \sup_{0 \leq t \leq T/(1+T)} \frac{\|\mathbf{W}(t)\|_\infty}{t^\gamma}, \quad (4.15)$$

with $\{\mathbf{W}_1(t), t \in [0, \infty)\}$ and $\{\mathbf{W}_2(t), t \in [0, \infty)\}$ two independent p -dimensional Wiener process on $[0, \infty)$.

On the other hand, by Lemma 2 of Zhou et al. (2015), we have that for all $\gamma \in [0, 1/2)$ and $\nu > 2$:

$$\lim_{m \rightarrow \infty} \frac{k^{1/\nu} + km^{1/\nu-1} + o(km^{-1/2}) + O(m^{-1/4}k^{1/2}\sqrt{\log k})}{z(m, k, \gamma)} = 0.$$

The theorem follows by taking into account relations (4.13), (4.14), (4.15) and by the fact that by assumptions (A4) and (A5) we have the following decomposition:

$$\mathbf{J}_m(\widehat{\boldsymbol{\beta}}_m) \equiv \tau(1-\tau)m^{-1} \sum_{l=1}^m \dot{\mathbf{g}}(\mathbf{X}_l; \widehat{\boldsymbol{\beta}}_m) \dot{\mathbf{g}}^t(\mathbf{X}_l; \widehat{\boldsymbol{\beta}}_m) = \mathbf{J}_m(\boldsymbol{\beta}^0) + O_{\mathbb{P}}(m^{-1/2}).$$

■

Proof of Theorem 2.2

The theorem is shown if we prove that there exists an observation k depending on m , denoted \tilde{k} , for which we have convergence in probability of $\Gamma(m, \tilde{k}, \gamma)$, as $m \rightarrow \infty$, to infinity.

(i) *Open-end procedure case.* We suppose, without loss of generality that $k_m^0 \leq m^s/2$, with $s > 1$. We consider $\tilde{k} \equiv k_m^0 + m^s$. Then, for the statistic defined by (2.3), we have:

$$\mathbf{S}(m, \tilde{k}) \equiv \mathbf{J}_m^{-1/2}(\widehat{\boldsymbol{\beta}}_m) \left[\sum_{i=m+1}^{m+k_m^0} \dot{\mathbf{g}}(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m) \psi_\tau(Y_i - g(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m)) + \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m) \psi_\tau(Y_i - g(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m)) \right]. \quad (4.16)$$

Similar as for Theorem 2.1, we have that there exists a constant $0 < C < \infty$, such that:

$$\frac{\|\mathbf{J}_m^{-1/2}(\widehat{\boldsymbol{\beta}}_m) \sum_{i=m+1}^{m+k_m^0} \dot{\mathbf{g}}(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m) \psi_\tau(Y_i - g(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m))\|_\infty}{z(m, k_m^0, \gamma)} \leq C,$$

with probability converging to 1, as $m \rightarrow \infty$.

Since the function $h(x) = (1+x)\left(\frac{x}{1+x}\right)^\gamma$, is increasing in $x > 0$, we have:

$$\frac{\|\mathbf{J}_m^{-1/2}(\widehat{\boldsymbol{\beta}}_m) \sum_{i=m+1}^{m+k_m^0} \dot{\mathbf{g}}(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m) \psi_\tau(Y_i - g(\mathbf{X}_i; \widehat{\boldsymbol{\beta}}_m))\|_\infty}{z(m, \tilde{k}, \gamma)} \leq C, \quad (4.17)$$

with probability converging to 1, as $m \rightarrow \infty$.

We will study $\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_m) \psi_\tau(Y_i - g(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_m))$ of relation (4.16). For this, we consider, for any p -vector $\mathbf{u} = O(1)$, the following random process:

$$\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \equiv \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \left[\dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2} \mathbf{u}) \psi_\tau(Y_i - g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2} \mathbf{u})) - \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \psi_\tau(\varepsilon_i) \right].$$

Since for $i = m + k_m^0 + 1, \dots, m + \tilde{k}$, hypothesis H_1 is true, we have that:

$$\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) = \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \left[\dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2} \mathbf{u}) \psi_\tau(Y_i - g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2} \mathbf{u})) - \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \psi_\tau(Y_i - g(\mathbf{X}_i; \boldsymbol{\beta}^1)) \right] \quad (4.18)$$

with $\mathbf{R}_i(\mathbf{u})$ defined by (4.1). Since $\mathbb{E}[\psi_\tau(Y_i - g(\mathbf{X}_i; \boldsymbol{\beta}^1))] = 0$ for any $i \geq m + k_m^0 + 1$, we have:

$$\mathbb{E} \left[\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \right] = \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2} \mathbf{u}) [F(0) - F(g(\mathbf{X}_i; \boldsymbol{\beta}^1) - g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2} \mathbf{u}))]. \quad (4.19)$$

For continuing the study of $\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u})$, we consider the two possible cases for difference $\boldsymbol{\beta}^1 - \boldsymbol{\beta}^0$.

(a) If $\|\boldsymbol{\beta}^1 - \boldsymbol{\beta}^0\| > c_1 > 0$, then by a Taylor expansion of the distribution function F in relation (4.19), we have:

$$\mathbb{E} \left[\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \right] = \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2} \mathbf{u}) [g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2} \mathbf{u}) - g(\mathbf{X}_i; \boldsymbol{\beta}^1)] f(\tilde{b}_i),$$

with the \tilde{b}_i between 0 and $g(\mathbf{X}_i; \boldsymbol{\beta}^1) - g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2} \mathbf{u})$.

Since there exists a constant $c_3 > 0$ such that $f(\tilde{b}_i) > c_3$ and since, by assumption (A8),

$$\frac{1}{\tilde{k} - k_m^0} \left\| \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0) [g(\mathbf{X}_i; \boldsymbol{\beta}^1) - g(\mathbf{X}_i; \boldsymbol{\beta}^0)] \right\|_{\infty} > c_2 > 0,$$

then, we have:

$$\mathbb{E} \left[\left\| \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \right\| \right] = O(\tilde{k} - k_m^0).$$

By assumption (A3) and the strong law of large numbers, together with the last relation, we have that

$$\left\| \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \right\| = O_{\mathbb{P}} \left(\mathbb{E} \left[\left\| \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \right\| \right] \right) = O_{\mathbb{P}}(\tilde{k} - k_m^0) = O_{\mathbb{P}}(m^{s-1/2} \|m^{1/2}(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^0)\|). \quad (4.20)$$

(b) Consider now the case $\beta^1 - \beta^0 \rightarrow \mathbf{0}$, such that $m^{1/2}\|\beta^1 - \beta^0\| \rightarrow \infty$, as $m \rightarrow \infty$. Since $\dot{\mathbf{g}}(\mathbf{x}, \beta)$ is bounded for any $\mathbf{x} \in \Upsilon$ and for any β such that $\|\beta - \beta^0\| \leq b_m$, we have:

$$\max_{m+k_m^0+1 \leq i \leq m+\tilde{k}} \left(\max_{\substack{\|\mathbf{u}\| \leq C_1 \\ \|\beta - \beta^0\| \leq \|\beta^1 - \beta^0\|}} \|\mathbf{R}_i(\mathbf{u})\| \right) = O_{\mathbb{P}}(1).$$

In this case, relation (4.3) becomes: $R_{ij}(\mathbf{u}) = A_{ij}(\mathbf{u}) + B_{ij}(\mathbf{u})$, with

$$A_{ij}(\mathbf{u}) = \frac{\partial g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j} [\mathbb{1}_{\varepsilon_i \leq 0} - \mathbb{1}_{\varepsilon_i \leq g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) - g(\mathbf{X}_i; \beta^1)}],$$

$$B_{ij}(\mathbf{u}) = \left[\frac{\partial g(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u})}{\partial \beta_j} - \frac{\partial g(\mathbf{X}_i; \beta^0)}{\partial \beta_j} \right] [\tau - \mathbb{1}_{\varepsilon_i \leq 0}].$$

Similarly as for relation (4.5), we have uniformly in \mathbf{u} : $\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbb{E}[B_{ij}^2(\mathbf{u})] = O((\tilde{k} - k_m^0)m^{-1})$.

For random variable $A_{ij}(\mathbf{u})$, we have similarly as in the proof of Lemma 4.2, taking into account that $m^{1/2}\|\beta^1 - \beta^0\| \rightarrow \infty$, and that $\dot{\mathbf{g}}$ is bounded in the neighbourhood of β^0 , that: $\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbb{E}[A_{ij}^2(\mathbf{u})] = O((\tilde{k} - k_m^0)\|\beta^1 - \beta^0\|)$.

Then

$$\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \text{Var}[R_{ij}(\mathbf{u})] \leq O((\tilde{k} - k_m^0)\|\beta^1 - \beta^0\|).$$

We take $\delta_m = (\tilde{k} - k_m^0)\|\beta^1 - \beta^0\|$ and applying Lemma 4.1, we get:

$$\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) = \mathbb{E} \left[\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \right] + O_{\mathbb{P}}(m^{s/2} \sqrt{s \log m} \sqrt{\|\beta^1 - \beta^0\|}). \quad (4.21)$$

We study now $\mathbb{E}[\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u})]$. Taking into account relation (4.18), by a Taylor expansion, we get

$$\begin{aligned} \mathbb{E} \left[\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \right] &= \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) \left[(-m^{-1/2}\mathbf{u} + \beta^1 - \beta^0)^t \dot{\mathbf{g}}(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) \right. \\ &\quad \left. + \frac{(\beta^1 - \beta^0)^t}{2} \ddot{\mathbf{g}}(\mathbf{X}_i; \tilde{\beta})(\beta^1 - \beta^0) \right] f(\tilde{b}_i). \end{aligned}$$

Since $m^{1/2}\|\beta^1 - \beta^0\| \rightarrow \infty$ as $m \rightarrow \infty$, taking into account assumptions (A2), (A4) and (A5), we have, uniformly in \mathbf{u} bounded:

$$\begin{aligned} \mathbb{E} \left[\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \right] &= (\beta^1 - \beta^0)^t \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) \left\{ \dot{\mathbf{g}}(\mathbf{X}_i; \beta^0 + m^{-1/2}\mathbf{u}) f(\tilde{b}_i) \right\} (1 + o(1)) \\ &= O(m^s (\beta^1 - \beta^0)). \end{aligned}$$

Hence, replacing this in (4.21):

$$\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) = O(m^s(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^0)) + O_{\mathbb{P}}(m^{s/2} \sqrt{s \log m} \sqrt{\|\boldsymbol{\beta}^1 - \boldsymbol{\beta}^0\|}) = O_{\mathbb{P}}(m^s(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^0)). \quad (4.22)$$

Thus, in the two cases of $\boldsymbol{\beta}^1 - \boldsymbol{\beta}^0$, taking into account (4.20) and (4.22), we have

$$\left\| \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \right\| = O_{\mathbb{P}}(m^{s-1/2} \|m^{1/2}(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^0)\|). \quad (4.23)$$

On the other hand, for (4.18), we have by functional central limit theorem:

$$\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \psi_{\tau}(\varepsilon_i) = O_{\mathbb{P}}(m^{s/2}). \quad (4.24)$$

Taking into account relations (4.18), (4.23), (4.24), and $s > 1$, we obtain:

$$\begin{aligned} & \left\| \mathbf{J}_m^{-1/2}(\hat{\boldsymbol{\beta}}_m) \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u}) \psi_{\tau}(Y_i - g(\mathbf{X}_i; \boldsymbol{\beta}^0 + m^{-1/2}\mathbf{u})) \right\|_{\infty} / z(m, \tilde{k}, \gamma) \\ &= \frac{O_{\mathbb{P}}(m^{s/2}) + O_{\mathbb{P}}(m^{s-1/2} \|m^{1/2}(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^0)\|_{\infty})}{\sqrt{m}(1 + \tilde{k}/m)(\tilde{k}/(m + \tilde{k}))^{\gamma}}. \end{aligned} \quad (4.25)$$

Since $\tilde{k} = k_m^0 + m^s$, we have $\lim_{m \rightarrow \infty} \left(\frac{\tilde{k}}{m+\tilde{k}}\right)^{\gamma} = 1$ and $\sqrt{m} \left(1 + \frac{\tilde{k}}{m}\right) = m^{s-1/2}$. Then, relation (4.25) converges to infinity as $m \rightarrow \infty$. Thus,

$$\frac{\left\| \mathbf{J}_m^{-1/2}(\hat{\boldsymbol{\beta}}_m) \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_m) \psi_{\tau}(Y_i - g(\mathbf{X}_i; \hat{\boldsymbol{\beta}}_m)) \right\|_{\infty}}{z(m, \tilde{k}, \gamma)} \xrightarrow[m \rightarrow \infty]{\mathbb{P}} \infty.$$

Theorem follows for the open-end procedure taking into account the last relation together with relations (4.16), (4.17).

(ii) *Closed-end procedure case.* The proof is similar to that of (i), taking $\tilde{k} = k_m^0 + m$. In this case, by analogous calculations, relation (4.23) becomes $\left\| \sum_{i=m+k_m^0+1}^{m+\tilde{k}} \mathbf{R}_i(\mathbf{u}) \right\| \geq O_{\mathbb{P}}(m^{1/2} \|m^{1/2}(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^0)\|)$ and relation (4.24) becomes $\sum_{i=m+k_m^0+1}^{m+\tilde{k}} \dot{\mathbf{g}}(\mathbf{X}_i; \boldsymbol{\beta}^0) \psi_{\tau}(\varepsilon_i) = O_{\mathbb{P}}(m^{1/2})$. Therefore, relation (4.25) converges to infinity as $m \rightarrow \infty$. \blacksquare

References

- Aue, A., Cheung, R.C.Y., Lee, T.C.M., Zhong, M. (2014). Segmented model selection in quantile regression using the minimum description length principle, *Journal of the American Statistical Association*, 109: 1241-1256.
- Boldea, O., Hall, A.R. (2013). Estimation and inference in unstable nonlinear least squares models. *Journal of Econometrics*, 172: 158-167.
- Ciuperca, G. (2013). Two tests for sequential detection of a change-point in a nonlinear model, *Journal of Statistical Planning and Inference*, 143: 1621-1834.
- Ciuperca, G. (2015). Real time change-point detection in a model by adaptive LASSO and CUSUM, *Journal de la Société Française de Statistique*, 156: 113-132.
- Ciuperca, G., Salloum, Z. (2015). Empirical likelihood test in a posteriori change-point nonlinear model, *Metrika*, 78: 919-952.
- Ciuperca, G. (2016). Estimation in a change-point nonlinear quantile model, *Communications in Statistics – Theory and Methods*, to appear.
- Furno, M. (2012). Tests for structural break in quantile regressions. *AStA Advances in Statistical Analysis*, 96: 493-515.
- Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables, *Journal of the American Statistical Association*, 58: 13-30.
- Horváth, L., Hušková, M., Kokoszka, P., Steinebach, J. (2004). Monitoring changes in linear models, *Journal of Statistical Planning and Inference*, 126: 225-251.
- Hušková, M., Kirch, C. (2012). Bootstrapping sequential change-point tests for linear regression, *Metrika*, 75: 673-708.
- Koenker, R., Portnoy, S. (1987). L-estimation for linear models, *Journal of the American Statistical Association*, 82: 851-857.
- Koenker, R. (2005). *Quantile Regression*, Econometric Society Monographs, 38, Cambridge University Press, Cambridge.
- Komlós, J., Major, P., Tusnády, G. (1975). An approximation of partial sums of independent RV's, and the sample DF. I. *Z. Wahrsch. Verw. Gebiete*, 32: 111-131.
- Komlós, J., Major, P., Tusnády, G. (1976). An approximation of partial sums of independent RV's, and the sample DF. II, *Z. Wahrsch. Verw. Gebiete*, 34: 33-58.
- Leisch, F., Hornik, K. (2000). Monitoring structural changes with the generalized fluctuation test, *Econometric Theory*, 16: 835-854.

- Li, C., Dowling, N.M., Chappell, R. (2015). Quantile regression with a change-point model for longitudinal data: an application to the study of cognitive changes in pre-clinical Alzheimer's disease, *Biometrics*, 71: 625–635.
- Oka, T., Qu, Z. (2011). Estimating structural changes in regression quantiles. *Journal of Econometrics*, 162: 248-267.
- Qu, Z. (2008). Testing for structural change in regression quantiles, *Journal of Econometrics*, 146: 170-184.
- Su, L., Xiao, Z. (2008). Testing for parameter stability in quantile regression models, *Statistics and Probability Letters*, 78: 2768–2775.
- Wang, H.J., He, X. (2007). Detecting differential expressions in GeneChip microarray studies: a quantile approach, *Journal of the American Statistical Association*, 102: 104–112.
- Xia, Z., Guo, P., Zhao, W. (2009). Monitoring structural changes in generalized linear models, *Communications in Statistics-Theory and Methods*, 38: 1927-1947.
- Zhang, L., Wang, H.J., Zhu, Z. (2014). Testing for change points due to a covariate threshold in quantile regression, *Statistica Sinica*, 24: 1859–1877.
- Zhang, L., Wang, H.J., Zhu, Z. (2016). Composite change point estimation for bent line quantile regression, *Annals of the Institute of Statistical Mathematics*, DOI: 10.1007/s10463-015-0538-5.
- Zhou, M., Wang, H.J., Tang, Y. (2015). Sequential change point detection in linear quantile regression models, *Statistics and Probability Letters*, 100: 98-103.