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Empirical likelihood test for high-dimensional two-sample model

Gabriela Ciuperca · Zahraa Salloum

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Abstract A non parametric method based on the empirical likelihood is proposed for detecting the change in the coefficients of high-dimensional linear model where the number of model variables may increase as the sample size increases. This amounts to testing the null hypothesis of no change against the alternative of one change in the regression coefficients. Based on the theoretical asymptotic behaviour of the empirical likelihood ratio statistic, we propose, for a fixed design, a simpler test statistic, easier to use in practice. The asymptotic normality of the proposed test statistic under the null hypothesis is proved, a result which is different from the χ^2 law for a model with a fixed variable number. Under alternative hypothesis, the test statistic diverges. We can then find the asymptotic confidence region for the difference of parameters of the two phases. Some Monte-Carlo simulations study the behaviour of the proposed test statistic.

Keywords Two-sample · high-dimension · linear model · empirical likelihood test.

1 Introduction

The technology development and fast numerical techniques make possible to consider and study statistical models with a large number of variables. High-dimensional model refers to a model whose the number p of explanatory variables increases to infinity as the number n of observations converges to infinity. When p diverges, traditional statistical methods may not work with this kind of growth dimensionality.

Most of the literature works on high-dimensional model utilize the LASSO (Least Absolute Shrinkage and Selection Operator) type methods, in order to automatically select the significant variables. The principle of these methods, introduced by Tibshirani (1996), is to optimize a penalized process, more precisely, a process with a L_1 -type penalty. If the model contains outliers, the parameter estimators by the least squares method with LASSO penalty have a large error. An alternative method is then the penalized quantile method. Thereby, Dicker et al. (2013) consider a quantile model with seamless- L_0 penalty when the number p of explanatory variables is such that $p \rightarrow \infty$, $p/n \rightarrow 0$ as $n \rightarrow \infty$. For a general quantile regression, Wu and Liu (2009) propose the SCAD penalty, while, in Zou and Yuan (2008), a composite quantile regression is considered with an adaptive LASSO penalty. The case $p \rightarrow \infty$ is also considered in Fan and Peng (2004) for a non-concave penalized likelihood method, when $p^5/n \rightarrow \infty$. Concerning the group selection methods for high-dimensional models, the readers find in Huang et al. (2012) a review of methods. All these methods are based first on the principle of selecting (automatically) the significant variables. Then, the dependent variable is modeled only as a function of the significant variables, in order to have more accurate parameter estimators and a better adjustment for the dependent variable.

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If the goal is to have the most accurate prediction and also robust, in the case of a model with outliers, one possibility is to consider the empirical likelihood (EL) method. But, for this type of method, in literature, most papers are devoted to the case of fixed p . For a high-dimensional linear regression model, we can refer first to the paper of Guo et al (2013), when the design is deterministic. High-dimensional data are also studied by Liu et al. (2013), where EL method is considered for a sequence of i.i.d. random vectors with dimension p , when $p \rightarrow \infty$ as $n \rightarrow \infty$.

In this paper, we are interested by a change-point model, that is, a model which changes at some moment. The number p of explanatory variables varies with the number n of observations and p can converge to infinity if $n \rightarrow \infty$.

Since statistical techniques in high-dimension are fairly recent, there are not many papers in literature that address the change-point problem in a high-dimensional model. Lung-Yut-Fong et al. (2013) propose an approach for detection of a change-point in high-volume network traffic. The asymptotic distribution of the test statistic proposed in Lung-Yut-Fong et al. (2013), under the null hypothesis that there is no change-point, is the *argsup* of a Brownian Bridge. There are some papers where LASSO type methods are used. Lee et al. (2015) consider a possible change-point in a high-dimensional regression with Gaussian errors. The main result of the article is to show that the sparsity property is maintained, even if there is a change in the model. There is no hypothesis test to decide the presence or absence of change in model. In Ciuperca (2014), LASSO-type and adaptive LASSO estimators are studied, while in Ciuperca (2013) quantile model with SCAD penalty is considered. These last two papers consider models with p fixed. In order to choose the change-point number, a model selection criterion is also proposed by Ciuperca (2014).

To the authors' knowledge, the EL technique has not yet been addressed in a high-dimensional two-sample model, that makes the interest of this work. We study the asymptotic behaviour of the empirical likelihood ratio test statistic when the design is deterministic.

We consider a first linear model:

$$Y_i = \mathbf{X}_i^t \boldsymbol{\beta} + \varepsilon_i, \quad i = 1, \dots, n. \quad (1.1)$$

Consider now a second linear model which changes at observation k . It is called two-phase model, or model with one change-point:

$$Y_i = \begin{cases} \mathbf{X}_i^t \boldsymbol{\beta} + \varepsilon_i, & 1 \leq i \leq k, \\ \mathbf{X}_i^t \boldsymbol{\beta}_2 + \varepsilon_i, & k < i \leq n, \end{cases} \quad (1.2)$$

where \mathbf{X}_i is a $p \times 1$ vector of p explanatory variables, $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_2$ are $p \times 1$ vectors of unknown parameters and ε_i designates the model error. The parameter $\boldsymbol{\beta}$ of the first phase of (1.2) coincides with that of (1.1). For models (1.1) and (1.2), Y_i is observation i of the response variable. The errors ε_i are supposed independent identically distributed (i.i.d), with mean zero and finite variance σ^2 .

We assume that the number p of explanatory variables \mathbf{X}_i depends on the sample size n : $p = p_n$, such that $p_n \rightarrow \infty$ as $n \rightarrow \infty$. The change-point k of (1.2) also depends on n . The change in model (1.2) takes place far enough from the first observation and sufficiently previous to the last observation. So, we suppose that $\lim_{n \rightarrow \infty} k/n \in (0, 1)$.

In this paper, for given k , we use the empirical likelihood method to construct the confidence region for $\boldsymbol{\beta} - \boldsymbol{\beta}_2$, or equivalent to test the null hypothesis of no change in model (1.2). Under null hypothesis, the model has the form (1.1), that is

$$H_0 : \boldsymbol{\beta}_2 = \boldsymbol{\beta}. \quad (1.3)$$

The alternative hypothesis assumes that one change occurs in the regression parameters, that is

$$H_1 : \boldsymbol{\beta}_2 \neq \boldsymbol{\beta}. \quad (1.4)$$

The paper is organized as follows. In Section 2 we first present the EL method for the two-sample model. Some notations used throughout the paper are defined and needed assumptions for the theoretical study are also announced. In Section 3, we construct an empirical likelihood ratio test statistic and we study its asymptotic behaviour. The asymptotic distribution under H_0 of the test statistic is obtained, while, under H_1 , this statistic diverges. Next, in Section 4, we analyse the coverage accuracy and the empirical power by

means of simulations, which confirm the performance of proposed test. A new critical value is also proposed in order to improve the coverage rate. The proofs of the main results are given in Appendix (Section 5) followed by some Lemmas and their proofs.

2 Preliminaires

In this section, we introduce the EL method for the two-sample model. Notations and assumptions are also given.

Under null hypothesis H_0 , that is model (1.1), let β^0 denote the true value of the parameter β . Under alternative hypothesis H_1 , that is model (1.2), the true values of β , β_2 , respectively, are β^0 , β_2^0 . In order to define the profile empirical likelihood (under H_0 and under H_1), we introduce the following random p -vector, for all $\beta \in \mathbb{R}^p$ and $i = 1, \dots, n$:

$$\mathbf{z}_i(\beta) \equiv \mathbf{X}_i(Y_i - \mathbf{X}_i^t \beta).$$

Consider also the vector

$$\mathbf{z}_i^0 \equiv \mathbf{X}_i \varepsilon_i.$$

We remark that, under the hypothesis H_0 , we have $\mathbf{z}_i^0 = \mathbf{z}_i(\beta^0)$, for all $i = 1, \dots, n$ and $\mathbb{E}[\mathbf{z}_i^0] = \mathbf{0}_p$. On the other hand, for fixed design $(\mathbf{X}_i)_{1 \leq i \leq n}$, the random variables \mathbf{z}_i^0 are independent but not identically distributed.

On the change-point, we consider the notation $\theta_{nk} = k/n$. Thus, in view of the remark made in Introduction, we assume that $\theta_{nk} \rightarrow \theta^0 \in (0, 1)$ as $n \rightarrow \infty$.

For the dependent variable Y_i of model (1.2), let us consider the probability to observe the value y_i (respectively y_j) : $q_i \equiv \mathbb{P}[Y_i = y_i]$, for $i = 1, \dots, k$ and $q_j \equiv \mathbb{P}[Y_j = y_j]$, for $j = k+1, \dots, n$. Obviously, these probabilities satisfy the relations $\sum_{i=1}^k q_i = 1$ and $\sum_{j=k+1}^n q_j = 1$. Corresponding to these probabilities, we define the probability vectors (q_1, \dots, q_k) and (q_{k+1}, \dots, q_n) .

Under hypothesis H_0 given by (1.3), the profile empirical likelihood for β is

$$\mathcal{R}_{nk}(\beta) \equiv \sup_{(q_1, \dots, q_k)} \sup_{(q_{k+1}, \dots, q_n)} \left\{ \prod_{i=1}^k q_i \prod_{j=k+1}^n q_j; \sum_{i=1}^k q_i = 1, \sum_{j=k+1}^n q_j = 1, \sum_{i=1}^k q_i \mathbf{z}_i(\beta) = \sum_{j=k+1}^n q_j \mathbf{z}_j(\beta) = \mathbf{0}_p \right\},$$

with $\mathbf{0}_p$ the p -vector with all components zero.

Similarly, under hypothesis H_1 given by (1.4), the profile empirical likelihood is

$$\mathcal{R}_{nk}(\beta, \beta_2) \equiv \sup_{(q_1, \dots, q_k)} \sup_{(q_{k+1}, \dots, q_n)} \left\{ \prod_{i=1}^k q_i \prod_{j=k+1}^n q_j; \sum_{i=1}^k q_i = 1, \sum_{j=k+1}^n q_j = 1, \sum_{i=1}^k q_i \mathbf{z}_i(\beta) = \mathbf{0}_p, \sum_{j=k+1}^n q_j \mathbf{z}_j(\beta_2) = \mathbf{0}_p \right\}.$$

Then, using an idea similar to the maximum likelihood test for testing H_0 against H_1 , we consider the profile empirical likelihood ratio $\mathcal{R}_{nk}(\beta)/\mathcal{R}_{nk}(\beta, \beta_2)$.

Since $\mathcal{R}_{nk}(\beta, \beta_2) = k^{-k}(n-k)^{-(n-k)}$, we have that the corresponding empirical log-likelihood ratio is

$$-2 \sup_{(q_1, \dots, q_k)} \sup_{(q_{k+1}, \dots, q_n)} \left\{ \sum_{i=1}^k \log(kq_i) + \sum_{j=k+1}^n \log((n-k)q_j); \sum_{i=1}^k q_i = \sum_{j=k+1}^n q_j = 1, \sum_{i=1}^k q_i \mathbf{z}_i(\beta) = \sum_{j=k+1}^n q_j \mathbf{z}_j(\beta) = \mathbf{0}_p \right\}.$$

Applying the Lagrange multiplier method, the optimal probabilities q_i and q_j are

$$q_i = \frac{1}{k + n\lambda_1^t \mathbf{z}_i(\beta)}, \quad q_j = \frac{1}{n - k - n\lambda_2^t \mathbf{z}_j(\beta)}, \quad (2.1)$$

where $\lambda_1, \lambda_2 \in \mathbb{R}^p$ are the Lagrange multipliers. Consequently, the corresponding empirical log-likelihood function can be written as

$$2 \sum_{i=1}^k \log \left(1 + \frac{n}{k} \lambda_1^t \mathbf{z}_i(\beta) \right) + 2 \sum_{j=k+1}^n \log \left(1 - \frac{n}{n-k} \lambda_2^t \mathbf{z}_j(\beta) \right). \quad (2.2)$$

Taking into account relation (2.1), the derivative with respect to β of (2.2) is $\sum_{i=1}^k q_i \mathbf{X}_i \mathbf{X}_i^t \lambda_1 - \sum_{j=k+1}^n q_j \mathbf{X}_j \mathbf{X}_j^t \lambda_2 = \mathbf{0}_p$. We can apply Lemma 4 of Guo et al (2013) on each phase of model, that implies that $\|\lambda_1\| = O_{\mathbb{P}}(p^{1/2}k^{-1/2})$ and $\|\lambda_2\| = O_{\mathbb{P}}(p^{1/2}(n-k)^{-1/2})$. Then, the probabilities q_i and q_j of (2.1) are approximately k^{-1} and $(n-k)^{-1}$, respectively. Thus, we can restrict λ_1 and λ_2 such that

$$k^{-1} \sum_{i=1}^k \mathbf{X}_i \mathbf{X}_i^t \lambda_1 = (n-k)^{-1} \sum_{j=k+1}^n \mathbf{X}_j \mathbf{X}_j^t \lambda_2. \quad (2.3)$$

If the symmetric matrices $k^{-1} \sum_{i=1}^k \mathbf{X}_i \mathbf{X}_i^t$ and $(n-k)^{-1} \sum_{j=k+1}^n \mathbf{X}_j \mathbf{X}_j^t$ converge, as $n \rightarrow \infty$, to two strictly positive definite matrices, then the relation (2.3) can be written $\lambda_1 = (k^{-1} \sum_{i=1}^k \mathbf{X}_i \mathbf{X}_i^t)^{-1} ((n-k)^{-1} \sum_{j=k+1}^n \mathbf{X}_j \mathbf{X}_j^t) \lambda_2$. Noting by $\tilde{\lambda}_2 \equiv (k^{-1} \sum_{i=1}^k \mathbf{X}_i \mathbf{X}_i^t)^{-1} ((n-k)^{-1} \sum_{j=k+1}^n \mathbf{X}_j \mathbf{X}_j^t) \lambda_2$, we have the new Lagrange multipliers such that $\lambda_1 = \tilde{\lambda}_2$. For the sake of readability, we denote $\tilde{\lambda}_2$ by λ_2 .

With this remark, we will restrict the study to a particular case, when $\lambda_1 = \lambda_2 = \lambda$. Considering this constraint, instead of statistic (2.2) we consider the following particular empirical likelihood ratio (ELR) statistic

$$\text{EL}_{nk}(\beta) \equiv 2 \sum_{i=1}^k \log \left(1 + \frac{n}{k} \lambda^t \mathbf{z}_i(\beta) \right) + 2 \sum_{j=k+1}^n \log \left(1 - \frac{n}{n-k} \lambda^t \mathbf{z}_j(\beta) \right), \quad (2.4)$$

where the Lagrange multiplier $\lambda \in \mathbb{R}^p$ satisfies

$$\sum_{i=1}^k \frac{\mathbf{z}_i(\beta)}{k/n + \lambda^t \mathbf{z}_i(\beta)} - \sum_{j=k+1}^n \frac{\mathbf{z}_j(\beta)}{1 - k/n - \lambda^t \mathbf{z}_j(\beta)} = \mathbf{0}_p. \quad (2.5)$$

2.1 Notations

We provide a brief summary of notations used in the paper.

For exposition convenience, we define some general notation. All vectors are column and \mathbf{v}^t denotes the transposed of \mathbf{v} . All vectors and matrices are in bold. For a vector \mathbf{v} , by $\|\mathbf{v}\|$ we denote its Euclidean norm and by $\|\mathbf{v}\|_1$ its L_1 -norm. For a symmetric p -square matrix $\mathbf{A} = (a_{ij})$, let us denote by $\gamma_1(\mathbf{A}) \geq \gamma_2(\mathbf{A}) \geq \dots \geq \gamma_p(\mathbf{A})$ the eigenvalues and $\text{tr}(\mathbf{A})$ as the trace operator of the matrix \mathbf{A} . Consider also the following notation $\text{Mx}(\mathbf{A}) = \max_{1 \leq i, j \leq p} |a_{ij}|$. We denote by $\|\mathbf{A}\|_1 = \max_{j=1, \dots, p} (\sum_{i=1}^p |a_{ij}|)$, the subordinate norm to the vector norm $\|\cdot\|_1$.

All throughout the paper, C denotes a generic constant which may be different from line to line and even from formula to formula and whose value is not of interest.

Moreover, $\mathbf{0}_p$ denote the p -vector with all components zero.

At the beginning of this section, the notation $\theta_{nk} \equiv k/n$ was introduced. To simplify notations, we will use the notation θ instead θ_{nk} .

For β^0 , the true value of the parameter β on the phase $1, \dots, k$, and the test value under H_0 , we define the following p -square matrix

$$\mathbf{S}_n(\beta^0) \equiv \frac{1}{n\theta^2} \sum_{i=1}^k \mathbf{z}_i(\beta^0) \mathbf{z}_i^t(\beta^0) + \frac{1}{n(1-\theta)^2} \sum_{j=k+1}^n \mathbf{z}_j(\beta^0) \mathbf{z}_j^t(\beta^0) \quad (2.6)$$

and the following p -vector

$$\psi_n(\beta^0) \equiv \frac{1}{n\theta} \sum_{i=1}^k \mathbf{z}_i(\beta^0) - \frac{1}{n(1-\theta)} \sum_{j=k+1}^n \mathbf{z}_j(\beta^0). \quad (2.7)$$

Under null hypothesis, for the true value β^0 of β , the mean of the random matrix $\mathbf{S}_n(\beta^0)$ is the following p -square matrix

$$\mathbf{V}_n^0 \equiv \frac{1}{n\theta^2} \sum_{i=1}^k \mathbf{V}_{(i)}^0 + \frac{1}{n(1-\theta)^2} \sum_{j=k+1}^n \mathbf{V}_{(j)}^0, \quad (2.8)$$

where, for $i = 1, \dots, n$

$$\mathbf{V}_{(i)}^0 \equiv \text{Var}(\mathbf{z}_i^0) = \sigma^2 \mathbf{X}_i \mathbf{X}_i^t. \quad (2.9)$$

For $i = 1, \dots, n$, let us also consider the following random vector

$$\mathbf{w}_i^0 \equiv (\mathbf{V}_n^0)^{-1/2} \mathbf{z}_i(\beta^0).$$

Corresponding to the components of $\mathbf{w}_i^0 = (w_{i,1}^0, \dots, w_{i,p}^0)$, we consider for $i = 1, \dots, n$, for $r \in \mathbb{N}^*$, $t_1, \dots, t_r \in \{1, \dots, p\}$, the following scalar

$$\alpha^{t_1 t_2 \dots t_r} \equiv \frac{1}{n\theta^r} \sum_{i=1}^k \mathbb{E}[w_{i,t_1}^0 w_{i,t_2}^0 \dots w_{i,t_r}^0] + \frac{1}{n(\theta-1)^r} \sum_{j=k+1}^n \mathbb{E}[w_{j,t_1}^0 w_{j,t_2}^0 \dots w_{j,t_r}^0] \quad (2.10)$$

and the following random variable

$$\omega^{t_1 t_2 \dots t_r} \equiv \frac{1}{n\theta^r} \sum_{i=1}^k w_{i,t_1}^0 w_{i,t_2}^0 \dots w_{i,t_r}^0 + \frac{1}{n(\theta-1)^r} \sum_{j=k+1}^n w_{j,t_1}^0 w_{j,t_2}^0 \dots w_{j,t_r}^0 - \alpha^{t_1 t_2 \dots t_r}, \quad (2.11)$$

where w_{i,t_r}^0 is the r -th component of \mathbf{w}_i^0 . In particular, for all $t_1, t_2 \in \{1, \dots, p\}$, $\alpha^{t_1} = 0$, $\alpha^{t_1 t_2} = \delta^{t_1 t_2}$ is the Kronecker delta, that is $\alpha^{t_1 t_2} = 1$ if $t_1 = t_2$, and 0 otherwise.

2.2 Assumptions

We now state the assumptions on the design, on the errors, on the number p of the explanatory variables and on the change-point location. These assumptions are needed in order to keep the properties obtained for EL statistic in a high-dimensional model, without change-point.

For assumptions (A3)-(A6) the constant q is such that $q \geq 4$.

- (A1) There exist positive constants $C_0, C_1 > 0$, such that $0 < C_0 < \inf_n \gamma_1(\mathbf{V}_n^0) \leq \sup_n \gamma_1(\mathbf{V}_n^0) < C_1 < \infty$.
- (A2) $\mathbb{E}(\varepsilon_1^4) < C_2$ for some $C_2 > 0$ and for all n .
- (A3) $p^{-1} \sum_{s=1}^p |X_{i,s}|^q < C_3, 1 \leq i \leq n$, for some $C_3 > 0$, and $q \geq 4$;
- (A4) $\mathbb{E}|\varepsilon_1|^{2q} < C_4$, for some $C_4 > 0$.
- (A5) $p k^{(2-q)/(2q)} \rightarrow 0$ and $p(n-k)^{(2-q)/(2q)} \rightarrow 0$, as $n \rightarrow \infty$.
- (A6) $p^{2+4/q} k^{-1} \rightarrow 0$ and $p^{2+4/q} (n-k)^{-1} \rightarrow 0$, as $n \rightarrow \infty$.
- (A7) $\sum_{r,s=1}^p \alpha^{rss} = O(p^2)$.
- (A8) $\sum_{r,s,u=1}^p \alpha^{rsu} \alpha^{rsu} = O(p^{5/2})$ and $\sum_{r,s,u=1}^p \alpha^{rss} \alpha^{suu} = O(p^{5/2})$.
- (A9) For all $i = 1, \dots, n$, for $l \in \mathbb{N}^*$, $j_1, \dots, j_l \in \{1, \dots, p\}$, and whenever $\sum_{i=1}^l d_i \leq 6$, there exists a positive absolute constant $C_5 < \infty$, then $\mathbb{E}(w_{i,j_1}^{d_1} \dots w_{i,j_l}^{d_l}) \leq C_5$.

Assumptions (A3) and (A6) guarantee that the eigenvalues of \mathbf{S}_n^0 are close to those of \mathbf{V}_n^0 (see Lemma 2). Assumption (A1) implies that \mathbf{V}_n^0 is uniformly nonsingular and bounded, for large n . Then, for n large enough, with probability close to one, \mathbf{S}_n^0 is nonsingular and $0 < C_0 < \gamma_p(\mathbf{S}_n^0) \leq \gamma_1(\mathbf{S}_n^0) < C_1 < \infty$. Assumption (A3) is also assumed by Guo et al (2013), Hjort et al. (2009), Liu et al. (2013) for high-dimensional model without change-point. Assumption (A4) together with (A3) and (A6) imply $\sup_{1 \leq i \leq n} |\boldsymbol{\lambda}^t \mathbf{z}_i^0| = o_p(1)$, which leads to Taylor expansions of (2.4) and (2.5) (see Lemma 6). Assumptions (A1), (A2), (A4) are also used by Guo et al (2013) for linear models without change-point with random design. Same assumption (A1) is requested in Zi et al. (2012) for a two-sample model with fixed p . Assumptions (A5)-(A9) are also assumed by Guo et al (2013), Liu et al. (2013), in order to have for the asymptotic normality of the ELR statistic.

3 Main Results

In this section, we present the main results of this paper. The asymptotic distribution of ELR test statistic under hypothesis H_0 will allow to build the asymptotic confidence region for the difference of the parameters of the two phases of model. We can also test if the models changes after observation k . In comparison to the obtained results for fixed p (see Liu et al. (2008), Zi et al. (2012) for linear model, Ciuperca and Salloum (2015) for nonlinear model) where the asymptotic law is the χ^2 distribution with p degrees of freedom, in the case presented here, the test statistic is different and it has a standard normal asymptotic distribution. In order to find this asymptotic distribution, we first need some intermediate results for studying the asymptotic behaviour of the ELR statistic.

We emphasize that the presence of the break point k complicates the study and leads to a different approach in respect to a model without change-point.

Note that under the hypothesis H_0 , we have: $\mathbf{z}_i(\boldsymbol{\beta}^0) = \mathbf{z}_i^0 = \mathbf{X}_i \varepsilon_i$, while under H_1 , the vector $\mathbf{z}_i(\boldsymbol{\beta}^0)$, for $i = k + 1, \dots, n$ becomes

$$\mathbf{z}_i(\boldsymbol{\beta}^0) = \mathbf{X}_i \mathbf{X}_i^t (\boldsymbol{\beta}_2^0 - \boldsymbol{\beta}^0) - \mathbf{z}_i^0. \quad (3.1)$$

When H_0 is true, we denote by \mathbf{S}_n^0 the matrix $\mathbf{S}_n(\boldsymbol{\beta}^0)$:

$$\mathbf{S}_n^0 \equiv \frac{1}{n\theta^2} \sum_{i=1}^k \mathbf{z}_i^0 \mathbf{z}_i^{0t} + \frac{1}{n(1-\theta)^2} \sum_{j=k+1}^n \mathbf{z}_j^0 \mathbf{z}_j^{0t} \quad (3.2)$$

and by $\boldsymbol{\psi}_n^0$ the vector $\boldsymbol{\psi}_n(\boldsymbol{\beta}^0)$:

$$\boldsymbol{\psi}_n^0 \equiv \frac{1}{n\theta} \sum_{i=1}^k \mathbf{z}_i^0 - \frac{1}{n(1-\theta)} \sum_{j=k+1}^n \mathbf{z}_j^0. \quad (3.3)$$

The Lagrange multiplier $\boldsymbol{\lambda}$ is a key element in any empirical likelihood formulation. The first result concerns the convergence rate to zero of $\boldsymbol{\lambda}$ defined in (2.5). When p is fixed, Zi et al. (2012) showed that $\|\boldsymbol{\lambda}\| = O_{\mathcal{P}}(n^{-1/2})$. When p is growing along with n , the above rate for $\|\boldsymbol{\lambda}\|$ is no longer valid as shown by the following proposition. In the proof we use Lemma 2, Lemma 3 and Lemma 4.

Proposition 1 *Suppose that assumptions (A1), (A3)-(A6) are satisfied. Then, under hypothesis H_0 , the Lagrange multiplier $\boldsymbol{\lambda}$ satisfies $\|\boldsymbol{\lambda}\| = O_{\mathcal{P}}(p^{1/2}n^{-1/2})$.*

Accordingly to this Proposition, by assumption (A6), we have that $\|\boldsymbol{\lambda}\| \xrightarrow{\mathcal{P}} 0$, as $n \rightarrow \infty$. More precisely, the Lagrange multiplier $\boldsymbol{\lambda}$ has the following approximate form given by Proposition 2. The proof, given in Appendix, is obtained by combining Lemma 4, Lemma 6 and Lemma 7. The p -square matrix \mathbf{V}_n^0 is defined by (2.8) and the p -vector $\boldsymbol{\psi}_n^0$ by (3.3).

Proposition 2 *If assumptions (A1), (A3)-(A6) are satisfied, then, under the null hypothesis H_0 , we have $\boldsymbol{\lambda} = (\mathbf{V}_n^0)^{-1} \boldsymbol{\psi}_n^0 (1 + o_{\mathcal{P}}(1))$.*

We prove now the following two propositions, all satisfied under hypothesis H_0 . They give two approximations for the ELR statistic $\text{EL}_{nk}(\boldsymbol{\beta}^0)$, defined by (2.4), approximations which will allow to find its asymptotic distribution.

In the proof of the following Proposition are used Lemma 5, Lemma 6, Proposition 2 and Lemma 7.

Proposition 3 *Suppose that assumptions (A1)-(A8) are satisfied. Then, under the null hypothesis H_0 , we have*

$$\text{EL}_{nk}(\boldsymbol{\beta}^0) = n\boldsymbol{\psi}_n^{0t} (\mathbf{S}_n^0)^{-1} \boldsymbol{\psi}_n^0 + o_{\mathcal{P}}(p^{1/2}).$$

For the proof of Proposition 4, given in Appendix, we use Lemma 4, Lemma 8, Proposition 2 and Proposition 3.

Proposition 4 *Suppose that assumptions (A1), (A3), (A4), (A6) and (A7) are fulfilled. If the hypothesis H_0 is true, then we have*

$$\text{EL}_{nk}(\boldsymbol{\beta}^0) = n\boldsymbol{\psi}_n^{0t} (\mathbf{V}_n^0)^{-1} \boldsymbol{\psi}_n^0 + o_{\mathcal{P}}(p^{1/2}).$$

The following theorem establishes the asymptotic normality of the ELR test statistic, when dimension p of the explanatory variables increases to infinity as $n \rightarrow \infty$. Its proof, given in Appendix, is very technical and moreover the change-point presence in the model occurs in an essential way. Proposition 3 and Proposition 4 are used in the proof. We note that the variance of standardization Δ_n^2 depends localisation of the change in the interval $[1 : n]$.

Theorem 1 *Under null hypothesis H_0 , if assumptions (A1)-(A9) are satisfied and $p = o(n^{1/3})$, then*

$$(i) \quad \frac{n\psi_n^{0t}(\mathbf{V}_n^0)^{-1}\psi_n^0 - p}{\Delta_n/n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1), \quad (3.4)$$

$$(ii) \quad \frac{\text{EL}_{nk}(\beta^0) - p}{\Delta_n/n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1),$$

where $\Delta_n^2 = \sum_{i=1}^n \sigma_i^2$, with $\sigma_1^2 = \theta^{-4}((\mathbf{X}_1^t(\mathbf{V}_n^0)^{-1}\mathbf{X}_1)^2 \mathbb{E}[\varepsilon_1^4] - [\text{tr}((\mathbf{V}_n^0)^{-1}\mathbf{V}_{(1)}^0)]^2)$ and:
- for $i = 2, \dots, k+1$,

$$\sigma_i^2 = \frac{4}{\theta^4} \sum_{l=1}^{i-1} \text{tr}((\mathbf{V}_n^0)^{-1}\mathbf{V}_{(i)}^0(\mathbf{V}_n^0)^{-1}\mathbf{V}_{(l)}^0) + \frac{(\mathbf{X}_i^t(\mathbf{V}_n^0)^{-1}\mathbf{X}_i)^2 \mathbb{E}[\varepsilon_1^4] - [\text{tr}((\mathbf{V}_n^0)^{-1}\mathbf{V}_{(i)}^0)]^2}{\theta^4},$$

- for $i = k+2, \dots, n$,

$$\sigma_i^2 = \frac{4}{\theta^2(1-\theta)^2} \sum_{l=1}^k \text{tr}((\mathbf{V}_n^0)^{-1}\mathbf{V}_{(i)}^0(\mathbf{V}_n^0)^{-1}\mathbf{V}_{(l)}^0) + \frac{4}{(1-\theta)^4} \sum_{l=k+1}^{i-1} \text{tr}((\mathbf{V}_n^0)^{-1}\mathbf{V}_{(i)}^0(\mathbf{V}_n^0)^{-1}\mathbf{V}_{(l)}^0) + \frac{(\mathbf{X}_i^t(\mathbf{V}_n^0)^{-1}\mathbf{X}_i)^2 \mathbb{E}[\varepsilon_1^4] - [\text{tr}((\mathbf{V}_n^0)^{-1}\mathbf{V}_{(i)}^0)]^2}{(1-\theta)^4}.$$

The following result is an immediate corollary of Theorem 1.

Corollary 1 *Testing the null hypothesis $H_0 : \beta = \beta_2 = \beta^0$ against the alternative hypothesis $H_1 : \beta = \beta^0, \beta_2 \neq \beta^0$, is equivalent to constructing the confidence regions for $\gamma = \beta^0 - \beta_2^0$, or to testing the null hypothesis $H'_0 : \gamma = \mathbf{0}_p$. Then, based to Theorem 1, in order to test H_0 against H_1 , we consider the following asymptotic test statistic*

$$\mathcal{Z}(\beta^0) \equiv \frac{n\psi_n^t(\beta^0)(\mathbf{V}_n^0)^{-1}\psi_n(\beta^0) - p}{\Delta_n/n}. \quad (3.5)$$

Note that $\psi_n(\beta^0)$ through $\mathbf{z}_i(\beta^0)$, given by relation (3.1), for $i = k+1, \dots, n$, depends of $\gamma = \beta^0 - \beta_2^0$.

The asymptotic behaviour under hypothesis H_1 of the test statistic $\mathcal{Z}(\beta^0)$ is given by the following theorem. We show that $\mathcal{Z}(\beta^0)$ diverges under alternative hypothesis.

Theorem 2 *Under alternative hypothesis H_1 , if assumptions (A1)-(A9) are satisfied and $p = o(n^{1/3})$, then $|\mathcal{Z}(\beta^0)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \infty$.*

Theorem 1 and Theorem 2 allow to build the asymptotic confidence region for the parameter $\gamma = \beta^0 - \beta_2$.

Corollary 2 *The α -level asymptotic confidence region for γ is*

$$\mathcal{R}_{1-\alpha/2} = \{\gamma : |\mathcal{Z}(\beta^0)| < c_{1-\alpha/2}\}, \quad (3.6)$$

where $c_{1-\alpha/2}$ is the quantile of the standard normal distribution.

For simulations, in order to calculate $\mathcal{R}_{1-\alpha/2}$, the matrix \mathbf{V}_n^0 is firstly calculated by relation (2.8). Once the model has been generated, we calculate \mathbf{z}_i^0 and then $\mathbf{z}_i(\beta^0)$ by relation (3.1). Finally, we calculate $\psi_n(\beta^0)$ by (2.7), Δ_n by Theorem 1 and the test statistic $\mathcal{Z}(\beta^0)$ by (3.5). For M Monte Carlo replications of the model, the coverage rate (CR), is the number of times when $|\mathcal{Z}(\beta^0)|$ is less than $c_{1-\alpha/2}$, divided by M .

For applications on real data, we will test model (1.1) against model (1.2). For these models, we know n values for the response variable Y and for the $p - 1$ explanatory variables X_2, \dots, X_p . The point k , where we want to test if there is a change, is known, while the values of β^0 on the first phase can be unknown.

If β^0 is unknown, then it is estimated by a convergent estimator on the observations $i = 1, \dots, k$, for example by LS method or quantile method, depending on the distribution of Y . Once we dispose of an estimator $\hat{\beta}_k$ for β^0 , the variance σ^2 of ε is estimated afterwards by a convergent estimator, for instance $\hat{\sigma}_k^2 = (k - p)^{-1} \sum_{i=1}^k (Y_i - \mathbf{X}_i^t \hat{\beta}_k)^2$. We calculate thereby $\mathbf{V}_{(i)}^0 = \hat{\sigma}_k^2 \mathbf{X}_i \mathbf{X}_i^t$, for any $i = 1, \dots, n$ and then \mathbf{V}_n^0 by (2.8). For any $i = 1, \dots, n$ we calculate $\mathbf{z}_i(\hat{\beta}_k) = \mathbf{X}_i(Y_i - \mathbf{X}_i^t \hat{\beta}_k)$, which will allow us to calculate the vector $\psi_n(\hat{\beta}_k)$ of relation (2.7). With all of these elements in place, we can calculate the value of the statistic $\mathcal{Z}(\hat{\beta}_k) = (n\psi_n^t(\hat{\beta}_k)(\mathbf{V}_n^0)^{-1}\psi_n(\hat{\beta}_k) - p)(\Delta_n/n)^{-1}$, using for Δ_n the relation given in Theorem 1. For a given size $\alpha \in (0, 1)$, if the value of $|\mathcal{Z}(\hat{\beta}_k)|$ is less than $c_{1-\alpha/2}$, then hypothesis H_0 is accepted, that is to say that the model does not change after observation k , otherwise hypothesis H_1 is accepted.

If β^0 is known, we can consider as an estimator for σ^2 : $\hat{\sigma}_k^2 = (k-p)^{-1} \sum_{i=1}^k (Y_i - \mathbf{X}_i^t \beta^0)^2$. For any $i = 1, \dots, n$ we calculate $\mathbf{z}_i(\beta^0) = \mathbf{X}_i(Y_i - \mathbf{X}_i^t \beta^0)$ and afterwards $\psi_n(\beta^0)$ by relation (2.7). Finally, the absolute value of $\mathcal{Z}(\beta^0)$ will be compared with $c_{1-\alpha/2}$.

Remark 1 Compared to Liu et al. (2008), where, for fixed p , a test statistic is proposed for testing the presence of the change-point, by maximizing ELR in respect to β and λ , in the present work we fix the parameter on the first phase and we test whether the parameter of the second phase is the same. In Liu et al. (2008), the system of equations in λ and β of the score functions must be solved, which can be numerically quite tedious. In this paper, apart from the fact that we consider $p \rightarrow \infty$, using theoretical properties for the Lagrange multiplier λ , we propose a simpler form for ELR statistic, easier to use in practice. Parameter β^0 , if it is unknown, can be estimated on the observations $1, \dots, k$ by a simpler computational method, in order to obtain $\hat{\beta}_k$ a convergent estimator, i.e. $\|\hat{\beta}_k - \beta^0\| = o_{\mathcal{P}}(1)$.

4 Simulation study

We now conduct simulation studies to evaluate, in terms of coverage accuracy and empirical power, the test statistic specified by Theorem 1(ii), with $\text{EL}_{nk}(\beta^0)$ approximated by Proposition 4, ie $\mathcal{Z}(\beta^0)$ given by relation (3.5). For these studies, we use Monte Carlo simulations. Throughout, we consider the size $\alpha = 0.05$.

The p explanatory variables are generated as follows: $X_1 = 1$ and $(X_2, \dots, X_p) \sim \mathcal{N}_{p-1}(\mathbf{0}_{p-1}, \Sigma)$, with the covariance matrix $\Sigma = (\varsigma_{hl})$, $\varsigma_{hl} = 2^{-|h-l|}$, $1 \leq h, l \leq p-1$, the same matrix considered by Guo et al (2013), for a model without change-point. In order to be in a fixed design, we consider the same realization for $(\mathbf{X}_i)_{1 \leq i \leq n}$ for each Monte Carlo replication.

Concerning the coefficients β^0 of the model, under H_0 , we take $\beta^0 = (\beta_1^0, \beta_2^0, \dots, \beta_p^0) = (1, 2, \dots, p)$.

For model (1.2), under hypothesis $H_0 : \beta = \beta_2 = \beta^0$, we first calculate the coverage rate (CR) based on Corollary 2, for a given change-point k .

We consider different values for n and k and two different distributions for the errors (ε_i) : standard normal distribution $\mathcal{N}(0, 1)$ and $\varepsilon \sim \mathcal{Exp}(1) - 1$, where $\mathcal{Exp}(1)$ is the exponential distribution with mean 1.

4.1 Importance of assumptions (A5), (A6)

In this subsection we realise throughout 2000 Monte Carlo replications for studying the behaviour of the test statistic behaviour, under null hypothesis and afterwards, when model has a change-point. Coverage rate and empirical power are investigated. Values of n and k are $n \in \{20, 100, 200, 400, 600, 1000\}$, $k \in \{5, 25, 75, 280, 350\}$.

Analyse of coverage rate. The results are summarized in Table 1 where we give $CR = 1 - \text{empirical size}$, based on Corollary 2, relation (3.5). For n and k fixed, the CRs decrease when p increases, this decreasing trend being more pronounced in the exponential error case. We observe that whether for exponential errors or for gaussian errors, if assumptions (A5), (A6) are not satisfied, then the CRs are well below 0.95. These results are in accordance with those obtained by Guo et al (2013), for models without change-point, with fixed design.

In order to confirm this supposition, in Table 2, the values of n and k are varied such that $\theta = k/n = 1/2$

Table 1 Coverage rates (CR) for Exponential and Gaussian errors.

n	k	p	Exponential errors CR	Gaussian errors CR
20	5	2	0.94	0.96
		3	0.92	0.95
		5	0.91	0.92
		7	0.77	0.87
100	25	2	0.94	0.95
		10	0.88	0.94
		20	0.71	0.84
200	75	2	0.93	0.94
		10	0.89	0.93
		20	0.77	0.94
		50	0.73	0.85
400	280	2	0.92	0.91
		10	0.87	0.88
		20	0.79	0.86
		50	0.75	0.84
		100	0.60	0.83
600	350	2	0.94	0.92
		10	0.92	0.93
		20	0.86	0.87
		50	0.85	0.88
		100	0.78	0.84
		200	0.66	0.84
1000	350	2	0.95	0.94
		50	0.86	0.92
		100	0.84	0.89
		200	0.78	0.80
		300	0.52	0.77

and p satisfies (A5), (A6). We obtain then that the CRs are larger than 0.90.

Analyse of power. Under H_1 , we consider $\beta_2^0 = 1 - \beta^0$. In the all considered cases, for n , k , p and ε in Tables 1 and 2, we obtain that the empirical powers are equal to 1.

Table 2 Coverage rates (CR), by 2000 Monte Carlo replications, for Exponential and Gaussian errors, $\theta = 1/2$.

n	k	p	Exponential errors CR	Gaussian errors CR
20	10	2	0.92	0.96
100	50	4	0.93	0.94
200	100	5	0.96	0.97
400	200	10	0.90	0.94
600	300	20	0.90	0.92
800	400	20	0.93	0.93
800	400	30	0.90	0.91
2000	1000	30	0.91	0.91

4.2 CR's improvement

In order to obtain more precise false probabilities, for fixed size α , we will calculate, by 10000 Monte Carlo replications, the $(1 - \alpha/2)$ and $\alpha/2$ quantiles, denoted \hat{c}_1 , \hat{c}_2 , respectively, for test statistic $\mathcal{Z}(\beta^0)$. We consider the new critical value $\hat{c}_{1-\alpha/2} \equiv \max(\hat{c}_1, |\hat{c}_2|)$. These new critical values, for $p = 50$, for different values of n and k , for $\mathcal{N}(0, 1)$ and $\mathcal{Exp}(1) - 1$ distribution errors, are given in Table 3. These values are not

influenced by value of k , for fixed n . This is observed by calculating $\hat{c}_{1-\alpha/2}$ for k such that $\theta_1 = k/n = 3/8$ and afterwards we calculate the CRs, denoted \widehat{CR} , for another k such that $\theta_2 = k/n = 5/8$. We observe that the values of $\hat{c}_{1-\alpha/2}$ are larger than the quantile of the standard normal distribution and $\hat{c}_{1-\alpha/2}$ are larger for exponential errors than those for normal errors. On the other hand, the values of $\hat{c}_{1-\alpha/2}$ decrease when n (and k) increases and they approach to quantile of $\mathcal{N}(0, 1)$.

In the same Table, are given empirical powers, denoted $\hat{\pi}$, calculated for $\beta_2^0 = 1 - \beta^0$ under H_1 , considering $\hat{c}_{1-\alpha/2}$ as critical value. We obtain that all $\hat{\pi}$ are equal to 1.

If under H_1 , only two components of β^0 change: β_2^0 is such that $\beta_{2,j}^0 = \beta_j^0$ for all $j \in \{1, \dots, p\} \setminus \{3, 30\}$, $\beta_{2,3}^0 = \beta_3^0 + 1$, $\beta_{2,30}^0 = \beta_{30}^0 + 1$, we always get $\hat{\pi} = 1$. Here we have denoted by $\beta_{2,j}^0$ the j th component of β_2^0 . Hence, even if there is a small change in the coefficients, most coefficients remaining unchanged, the test statistic detects this change.

Table 3 Empirical critical value $\hat{c}_{1-\alpha/2}$ and corresponding coverage rates (\widehat{CR}), empirical powers ($\hat{\pi}$), for Exponential and Gaussian errors, $p = 50$, $\beta_2^0 = 1 - \beta^0$.

n	k	Exponential errors			Gaussian errors		
		$\hat{c}_{1-\alpha/2}$	\widehat{CR}	$\hat{\pi}$	$\hat{c}_{1-\alpha}$	\widehat{CR}	$\hat{\pi}$
200	75	4.01	0.97	1	3.24	0.97	1
	125		0.97	1		0.93	1
400	150	3.41	0.97	1	2.96	0.97	1
	250		0.97	1		0.98	1
600	225	3.40	0.97	1	2.85	0.97	1
	375		0.97	1		0.98	1
800	300	2.68	0.95	1	2.43	0.97	1
	500		0.95	1		0.95	1
2000	750	2.48	0.97	1	2.30	0.97	1
	1250		0.95	1		0.93	1

4.3 Conclusion of simulations

Proposed test statistic (3.5), with $\mathcal{N}(0, 1)$ the asymptotic distribution under H_0 , involves the construction of a confidence region for the parameters of the second phase of the model (on observations $k + 1, \dots, n$). If assumptions (A5), (A6) are satisfied, then the coverage rates are close to the nominal coverage level. Contrariwise, if the coefficients change on the second phase, the test always detects this change. For improving the coverage rate in the case $n - k \not\gg p^2$ or $k \not\gg p^2$, we proposed to calculate new critical values. With these critical values, the rate of false changes is generally smaller than the size α . If there are changes in the coefficients of the second phase of the model, the test statistic based on the new confidence region always detects this change. For fixed p , if n and k increase, such that $k/n = \text{constant}$, then these new critical values decrease and approach the $(1 - \alpha)$ quantile of $\mathcal{N}(0, 1)$ distribution.

5 Appendix

This section is divided into two subsections. In the first we give the proofs of the Propositions and of the Theorems. In the second subsection, we present Lemmas and their proofs.

We recall that under the hypothesis H_0 , the vector $\mathbf{z}_i(\beta^0)$ is $\mathbf{z}_i^0 = \mathbf{X}_i \varepsilon_i$. Then, in all proofs, if hypothesis H_0 is true, we will use \mathbf{z}_i^0 instead of $\mathbf{z}_i(\beta^0)$.

5.1 Proposition and Theorem proofs

Proof of Proposition 1. Let us write $\boldsymbol{\lambda}$ as $\boldsymbol{\lambda} = \|\boldsymbol{\lambda}\|\mathbf{u}$, where \mathbf{u} is a p -vector with norm one. Using Lemma 3, for $\theta = k/n$, we have, for any $i = 1, \dots, k$, with probability one

$$0 < 1 + \frac{\|\boldsymbol{\lambda}\|}{\theta} \mathbf{u}^t \mathbf{z}_i^0 \leq 1 + \|\boldsymbol{\lambda}\| T_n^0, \quad (5.1)$$

where $T_n^0 \equiv \max_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq n}} \{(k/n)^{-1} \|\mathbf{z}_i^0\|, (1 - k/n)^{-1} \|\mathbf{z}_j^0\|\}$. For $j = k+1, \dots, n$ we have, with probability one,

$$0 < 1 - \frac{\|\boldsymbol{\lambda}\|}{1 - \theta} \mathbf{u}^t \mathbf{z}_j^0 \leq 1 + \|\boldsymbol{\lambda}\| T_n^0. \quad (5.2)$$

Using relations (5.1) and (5.2), then we get from (2.5) that

$$\begin{aligned} 0 &= \frac{1}{n} \sum_{i=1}^k \frac{\mathbf{u}^t \mathbf{z}_i^0}{\theta + \boldsymbol{\lambda}^t \mathbf{z}_i^0} - \frac{1}{n} \sum_{j=k+1}^n \frac{\mathbf{u}^t \mathbf{z}_j^0}{1 - \theta - \boldsymbol{\lambda}^t \mathbf{z}_j^0} \\ &= \frac{1}{n\theta} \sum_{i=1}^k \mathbf{u}^t \mathbf{z}_i^0 - \frac{1}{n\theta^2} \|\boldsymbol{\lambda}\| \sum_{i=1}^k \frac{\mathbf{u}^t \mathbf{z}_i^0 \mathbf{z}_i^{0t} \mathbf{u}}{1 + \theta^{-1} \|\boldsymbol{\lambda}\| \mathbf{u}^t \mathbf{z}_i^0} - \frac{1}{n(1-\theta)} \sum_{j=k+1}^n \mathbf{u}^t \mathbf{z}_j^0 - \frac{1}{n(1-\theta)^2} \|\boldsymbol{\lambda}\| \sum_{j=k+1}^n \frac{\mathbf{u}^t \mathbf{z}_j^0 \mathbf{z}_j^{0t} \mathbf{u}}{1 - (1-\theta)^{-1} \|\boldsymbol{\lambda}\| \mathbf{u}^t \mathbf{z}_j^0}. \end{aligned}$$

By the last equality, using also notations given by (3.2) and (3.3), it follows that

$$0 \leq \mathbf{u}^t \boldsymbol{\psi}_n^0 - \frac{\|\boldsymbol{\lambda}\|}{1 + \|\boldsymbol{\lambda}\| T_n^0} \mathbf{u}^t \mathbf{S}_n^0 \mathbf{u}.$$

Then, we have with probability one, that $\mathbf{u}^t \boldsymbol{\psi}_n^0 (1 + \|\boldsymbol{\lambda}\| T_n^0) \geq \|\boldsymbol{\lambda}\| \mathbf{u}^t \mathbf{S}_n^0 \mathbf{u}$. Therefore

$$\|\boldsymbol{\lambda}\| \leq \frac{\mathbf{u}^t \boldsymbol{\psi}_n^0}{\mathbf{u}^t \mathbf{S}_n^0 \mathbf{u} - \mathbf{u}^t \boldsymbol{\psi}_n^0 T_n^0}. \quad (5.3)$$

On the other hand, we have $|\mathbf{u}^t \boldsymbol{\psi}_n^0| \leq \|\boldsymbol{\psi}_n^0\|$. Then, using Lemma 4, we obtain that $\|\boldsymbol{\psi}_n^0\| = O_{\mathbb{P}}(n^{-1/2} p^{1/2})$, which gives

$$\mathbf{u}^t \boldsymbol{\psi}_n^0 = O_{\mathbb{P}}(n^{-1/2} p^{1/2}). \quad (5.4)$$

Using Lemma 3 and relation (5.4), we have that $\mathbf{u}^t \boldsymbol{\psi}_n^0 T_n^0 = O_{\mathbb{P}}(n^{-1/2} p^{1/2}) o_{\mathbb{P}}(n^{1/q} p^{1/2}) = o_{\mathbb{P}}(n^{-(q+2)/2q})$. Then, by assumption (A5), we obtain that $\mathbf{u}^t \boldsymbol{\psi}_n^0 T_n^0 = o_{\mathbb{P}}(1)$.

On the other hand, according to Lemma 2, $\mathbf{u}^t \mathbf{S}_n^0 \mathbf{u} \geq \gamma_p(\mathbf{S}_n^0) > C_0 > 0$ holds with a probability tending to 1 as $n \rightarrow \infty$. Then, for relation (5.3), we obtain that

$$\|\boldsymbol{\lambda}\| = O_{\mathbb{P}}(|\mathbf{u}^t \boldsymbol{\psi}_n^0|/C_0) = O_{\mathbb{P}}(\|\boldsymbol{\psi}_n^0\|) = O_{\mathbb{P}}(p^{1/2} n^{-1/2}).$$

■

Proof of Proposition 2. By Lemma 6 we have that $\boldsymbol{\lambda} = (\mathbf{S}_n^0)^{-1} (\mathbf{R}_n^0 + \boldsymbol{\psi}_n^0) (1 + o_{\mathbb{P}}(1))$.

In the other hand, by Lemma 4, we have that $\|\boldsymbol{\psi}_n^0\| = O_{\mathbb{P}}(p^{1/2} n^{-1/2})$. Using this fact and relation (5.64), we obtain

$$\frac{\|\mathbf{R}_n^0\|}{\|\boldsymbol{\psi}_n^0\|} = o_{\mathbb{P}}(n^{1/q} p^{1/2} n^{-1/2} p^{1/2}) = o_{\mathbb{P}}(pn^{(2-q)/2q}).$$

Therefore, by assumption (A5) we obtain that $\mathbf{R}_n^0 = \boldsymbol{\psi}_n^0 o_{\mathbb{P}}(1)$. Then $\boldsymbol{\lambda} = (\mathbf{S}_n^0)^{-1} \boldsymbol{\psi}_n^0 (1 + o_{\mathbb{P}}(1))$. In the other hand, by Lemma 7, we have that $((\mathbf{S}_n^0)^{-1} - (\mathbf{V}_n^0)^{-1}) \boldsymbol{\psi}_n^0 = (\mathbf{V}_n^0)^{-1} \boldsymbol{\psi}_n^0 o_{\mathbb{P}}(1)$. Then, $(\mathbf{S}_n^0)^{-1} \boldsymbol{\psi}_n^0 = (\mathbf{V}_n^0)^{-1} \boldsymbol{\psi}_n^0 (1 + o_{\mathbb{P}}(1))$. Therefore, for $\boldsymbol{\lambda} = (\mathbf{S}_n^0)^{-1} \boldsymbol{\psi}_n^0 (1 + o_{\mathbb{P}}(1))$, we obtain that

$$\boldsymbol{\lambda} = (\mathbf{V}_n^0)^{-1} \boldsymbol{\psi}_n^0 (1 + o_{\mathbb{P}}(1)). \quad (5.5)$$

■

Proof of Proposition 3. By Lemma 5 we have that

$$\mathbf{E}L_{nk}(\beta^0) = 2n\lambda^t \psi_n^0 - n\lambda^t \mathbf{S}_n^0 \lambda + \mathcal{E}_3 + o_{\mathcal{P}}(1), \quad (5.6)$$

$$\text{with } \mathcal{E}_3 \equiv \frac{2}{3} \left(\frac{1}{\theta^3} \sum_{i=1}^k (\mathbf{z}_i^{0t} \lambda)^3 - \frac{1}{(1-\theta)^3} \sum_{j=k+1}^n (\mathbf{z}_j^{0t} \lambda)^3 \right).$$

Consider now, the following p -vector

$$\mathbf{R}_n^0 \equiv \frac{1}{n\theta^3} \sum_{i=1}^k \mathbf{z}_i^0 (\lambda^t \mathbf{z}_i^0)^2 - \frac{1}{n(1-\theta)^3} \sum_{j=k+1}^n \mathbf{z}_j^0 (\lambda^t \mathbf{z}_j^0)^2. \quad (5.7)$$

By Lemma 6 we have that $\lambda = (\mathbf{S}_n^0)^{-1}(\mathbf{R}_n^0 + \psi_n^0)(1 + o_{\mathcal{P}}(1))$. Then, we have for (5.6) that

$$\mathbf{E}L_{nk}(\beta^0) = n\psi_n^{0t} (\mathbf{S}_n^0)^{-1} \psi_n^0 - n\mathbf{R}_n^{0t} (\mathbf{S}_n^0)^{-1} \mathbf{R}_n^0 + \mathcal{E}_3 + o_{\mathcal{P}}(1). \quad (5.8)$$

We now study \mathcal{E}_3 and $n(\mathbf{R}_n^0)^t (\mathbf{S}_n^0)^{-1} \mathbf{R}_n^0$ in parallel. By Proposition 2, we have that $\lambda = (\mathbf{V}_n^0)^{-1} \psi_n^0 (1 + o_{\mathcal{P}}(1))$, which implies

$$\mathbf{z}_i^{0t} \lambda = \mathbf{z}_i^{0t} (\mathbf{V}_n^0)^{-1} \psi_n^0 (1 + o_{\mathcal{P}}(1)). \quad (5.9)$$

Then, \mathcal{E}_3 becomes

$$\begin{aligned} \mathcal{E}_3 &= \frac{2}{3\theta^3} \sum_{i=1}^k \left(\mathbf{z}_i^{0t} (\mathbf{V}_n^0)^{-1/2} (\mathbf{V}_n^0)^{-1/2} \left(\frac{1}{n\theta} \sum_{i=1}^k \mathbf{z}_i^0 + \frac{1}{n(\theta-1)} \sum_{j=k+1}^n \mathbf{z}_j^0 \right) \right)^3 (1 + o_{\mathcal{P}}(1)) \\ &\quad + \frac{2}{3(\theta-1)^3} \sum_{j=k+1}^n \left(\mathbf{z}_j^{0t} (\mathbf{V}_n^0)^{-1/2} (\mathbf{V}_n^0)^{-1/2} \left(\frac{1}{n\theta} \sum_{i=1}^k \mathbf{z}_i^0 + \frac{1}{n(\theta-1)} \sum_{j=k+1}^n \mathbf{z}_j^0 \right) \right)^3 (1 + o_{\mathcal{P}}(1)). \end{aligned}$$

Using notations given by (2.10), (2.11) and the strong law of large numbers (Markov's Theorem), we obtain

$$\begin{aligned} \mathcal{E}_3 &= \frac{2}{\theta^3} \sum_{l=1}^k (\mathbf{w}_l^{0t} \left(\frac{1}{n\theta} \sum_{i=1}^k \mathbf{w}_i^0 + \frac{1}{n(\theta-1)} \sum_{j=k+1}^n \mathbf{w}_j^0 \right))^3 + \frac{2}{(1-\theta)^3} \sum_{l=k+1}^n (\mathbf{w}_l^{0t} \left(\frac{1}{n\theta} \sum_{i=1}^k \mathbf{w}_i^0 + \frac{1}{n(\theta-1)} \sum_{j=k+1}^n \mathbf{w}_j^0 \right))^3 (1 + o_{\mathcal{P}}(1)) \\ &= \frac{2n}{3} \sum_{r,s,u=1}^p \omega^r \omega^s \omega^u \left(\frac{1}{n\theta^3} \sum_{i=1}^k w_{i,r}^0 w_{i,s}^0 w_{i,u}^0 + \frac{1}{n(\theta-1)^3} \sum_{j=k+1}^n w_{j,r}^0 w_{j,s}^0 w_{j,u}^0 \right) (1 + o_{\mathcal{P}}(1)) \\ &= \frac{2n}{3} \sum_{r,s,u=1}^p \omega^r \omega^s \omega^u \alpha^{rsu} (1 + o_{\mathcal{P}}(1)). \end{aligned} \quad (5.10)$$

In the other hand, replacing λ in relation (5.7) we obtain

$$\mathbf{R}_n^0 = \left(\frac{1}{n\theta^3} \sum_{i=1}^k \mathbf{z}_i^0 (\mathbf{z}_i^{0t} \psi_n^{0t} (\mathbf{V}_n^0)^{-1})^2 - \frac{1}{n(1-\theta)^3} \sum_{j=k+1}^n \mathbf{z}_j^0 (\mathbf{z}_j^{0t} \psi_n^{0t} (\mathbf{V}_n^0)^{-1})^2 \right) (1 + o_{\mathcal{P}}(1)). \quad (5.11)$$

Consider now for $n(\mathbf{R}_n^0)^t (\mathbf{S}_n^0)^{-1} \mathbf{R}_n^0$ the following decomposition

$$\mathbf{R}_n^{0t} (\mathbf{S}_n^0)^{-1} \mathbf{R}_n^0 = \mathbf{R}_n^{0t} ((\mathbf{S}_n^0)^{-1} - (\mathbf{V}_n^0)^{-1}) \mathbf{R}_n^0 + \mathbf{R}_n^{0t} (\mathbf{V}_n^0)^{-1} \mathbf{R}_n^0. \quad (5.12)$$

By Lemma 7(ii), we have that $((\mathbf{S}_n^0)^{-1} - (\mathbf{V}_n^0)^{-1}) \mathbf{R}_n^0 = (\mathbf{V}_n^0)^{-1} \mathbf{R}_n^0 o_{\mathcal{P}}(1)$. Then, relation (5.12) becomes $\mathbf{R}_n^{0t} (\mathbf{S}_n^0)^{-1} \mathbf{R}_n^0 = \mathbf{R}_n^{0t} (\mathbf{V}_n^0)^{-1} \mathbf{R}_n^0 (1 + o_{\mathcal{P}}(1))$. Using relations (3.3), (5.7), (5.11) and the fact that $\mathbf{w}_i^0 =$

$(\mathbf{V}_n^0)^{-1/2}\mathbf{z}_i^0$, for $i = 1, \dots, n$, we have that $n\mathbf{R}_n^{0t}(\mathbf{S}_n^0)^{-1}\mathbf{R}_n^0$ can be written

$$\begin{aligned} n\mathbf{R}_n^{0t}(\mathbf{S}_n^0)^{-1}\mathbf{R}_n^0 &= \left\{ \frac{1}{\theta^3} \sum_{l=1}^k \mathbf{w}_l^0 \left[\left(\frac{1}{n\theta} \sum_{i=1}^k \mathbf{w}_i^0 - \frac{1}{n(1-\theta)} \sum_{j=k+1}^n \mathbf{w}_j^0 \right)^t \mathbf{w}_l^0 \right]^2 \right. \\ &\quad - \frac{1}{(1-\theta)^3} \sum_{l=k+1}^n \mathbf{w}_l^0 \left[\left(\frac{1}{n\theta} \sum_{i=1}^k \mathbf{w}_i^0 - \frac{1}{n(1-\theta)} \sum_{j=k+1}^n \mathbf{w}_j^0 \right)^t \mathbf{w}_l^0 \right]^2 \\ &\quad \cdot \left(\frac{1}{n\theta^3} \sum_{l=1}^k \mathbf{w}_l^0 \left[\left(\frac{1}{n\theta} \sum_{i=1}^k \mathbf{w}_i^0 - \frac{1}{n(1-\theta)} \sum_{j=k+1}^n \mathbf{w}_j^0 \right)^t \mathbf{w}_l^0 \right]^2 \right. \\ &\quad \left. \left. - \frac{1}{n(1-\theta)^3} \sum_{l=k+1}^n \mathbf{w}_l^0 \left[\left(\frac{1}{n\theta} \sum_{i=1}^k \mathbf{w}_i^0 - \frac{1}{n(1-\theta)} \sum_{j=k+1}^n \mathbf{w}_j^0 \right)^t \mathbf{w}_l^0 \right]^2 \right) \right\} (1 + o_{\mathcal{P}}(1)). \end{aligned}$$

Thus, using notations given by (2.10) and (2.11), we obtain

$$\begin{aligned} n\mathbf{R}_n^{0t}(\mathbf{S}_n^0)^{-1}\mathbf{R}_n^0 &= n \sum_{r,s,l,u,v=1}^p \omega^r \omega^s \left[\frac{1}{n\theta^3} \sum_{i=1}^k w_{i,r}^0 w_{i,s}^0 w_{i,l}^0 - \frac{1}{n(1-\theta)^3} \sum_{j=k+1}^n w_{j,r}^0 w_{j,s}^0 w_{j,l}^0 \right] \\ &\quad \cdot \omega^u \omega^v \left[\frac{1}{n\theta^3} \sum_{i=1}^k w_{i,u}^0 w_{i,v}^0 w_{i,l}^0 - \frac{1}{n(1-\theta)^3} \sum_{j=k+1}^n w_{j,u}^0 w_{j,v}^0 w_{j,l}^0 \right] (1 + o_{\mathcal{P}}(1)) \\ &= n \sum_{r,s,l,u,v=1}^p \alpha^{rsl} \alpha^{uvl} \omega^r \omega^s \omega^u \omega^v (1 + o_{\mathcal{P}}(1)). \end{aligned}$$

In conclusion, for \mathcal{E}_3 of (5.10) and for $n\mathbf{R}_n^{0t}(\mathbf{S}_n^0)^{-1}\mathbf{R}_n^0$, using assumptions (A6), (A7) and (A8), together with the proof of Proposition 1 of Guo et al (2013), we obtain: $\mathcal{E}_3 = o_{\mathcal{P}}(p^{1/2})$ and $n\mathbf{R}_n^{0t}(\mathbf{S}_n^0)^{-1}\mathbf{R}_n^0 = o_{\mathcal{P}}(p^{1/2})$. Combining the last two relations together relation (5.8), we obtain that

$$\mathbf{EL}_{nk}(\beta^0) = n\psi_n^0(\mathbf{S}_n^0)^{-1}\psi_n^0 + o_{\mathcal{P}}(p^{1/2}).$$

■

Proof of Proposition 4. We first prove

$$n\psi_n^{0t}((\mathbf{V}_n^0)^{-1} - (\mathbf{S}_n^0)^{-1})\psi_n^0 = o_{\mathcal{P}}(p^{1/2}). \quad (5.13)$$

For this, we introduce the following two p -square matrices

$$\mathbf{B}_n^0 \equiv (\mathbf{V}_n^0)^{-1/2} \mathbf{S}_n^0 (\mathbf{V}_n^0)^{-1/2}, \quad \mathbf{K}_n^0 \equiv \mathbf{I}_p - \mathbf{B}_n^0$$

and the following p -vector

$$\boldsymbol{\eta}_n^0 \equiv (\mathbf{V}_n^0)^{-1/2} \boldsymbol{\psi}_n^0.$$

With this notations, the left hand side of relation (5.13), can be written

$$n\psi_n^{0t}((\mathbf{V}_n^0)^{-1} - (\mathbf{S}_n^0)^{-1})\psi_n^0 = n\boldsymbol{\eta}_n^{0t}(\mathbf{I}_p - (\mathbf{V}_n^0)^{1/2}(\mathbf{S}_n^0)^{-1}(\mathbf{V}_n^0)^{1/2})\boldsymbol{\eta}_n^0 = n\boldsymbol{\eta}_n^{0t}(\mathbf{K}_n^0)^{-1}\boldsymbol{\eta}_n^0.$$

We consider the following decomposition for $n\boldsymbol{\eta}_n^{0t}(\mathbf{K}_n^0)^{-1}\boldsymbol{\eta}_n^0$

$$n\boldsymbol{\eta}_n^{0t}(\mathbf{K}_n^0)^{-1}\boldsymbol{\eta}_n^0 = \left(n\boldsymbol{\eta}_n^{0t}\mathbf{K}_n^0\boldsymbol{\eta}_n^0 - n\boldsymbol{\eta}_n^{0t}(\mathbf{K}_n^0)^2\boldsymbol{\eta}_n^0 - \dots - (-1)^b n\boldsymbol{\eta}_n^{0t}(\mathbf{K}_n^0)^b\boldsymbol{\eta}_n^0 \right) + (-1)^b n\boldsymbol{\eta}_n^{0t}(\mathbf{K}_n^0)^b(\mathbf{I}_p - (\mathbf{B}_n^0)^{-1})\boldsymbol{\eta}_n^0, \quad (5.14)$$

for any $b \in \mathbb{N}^*$.

We will study the convergence of the expansion given by (5.14). By Lemma 6 of Chen et al. (2009), we have the inequality $\boldsymbol{\eta}_n^{0t}\mathbf{A}\boldsymbol{\eta}_n^0 \leq \|\boldsymbol{\eta}_n^0\|^2(\text{tr}(\mathbf{A}^2))^{1/2}$, for any symmetric matrix \mathbf{A} . Then, for the first term of the right-hand side of relation (5.14) we have that, with probability one:

$$n\boldsymbol{\eta}_n^{0t}\mathbf{K}_n^0\boldsymbol{\eta}_n^0 \leq n \|\boldsymbol{\eta}_n^0\|^2(\text{tr}(\mathbf{K}_n^0)^2)^{1/2}. \quad (5.15)$$

Using assumption (A1) and Lemma 4 of Liu et al. (2013), we obtain that, with probability one:

$$\|\boldsymbol{\eta}_n^0\|^2 = \boldsymbol{\psi}_n^{0t} (\mathbf{V}_n^0)^{-1} \boldsymbol{\psi}_n^0 \leq \frac{1}{\gamma_1(\mathbf{V}_n^0)} \|\boldsymbol{\psi}_n^0\|^2 \leq \frac{1}{C_0} \|\boldsymbol{\psi}_n^0\|^2.$$

By Lemma 4, we have that $\|\boldsymbol{\psi}_n^0\| = O_{\mathbb{P}}(p^{1/2}n^{-1/2})$ and thus

$$\|\boldsymbol{\eta}_n^0\|^2 = O_{\mathbb{P}}(pn^{-1}). \quad (5.16)$$

By Lemma 8, for $1 \leq r \leq p$, we have

$$\gamma_r(\mathbf{K}_n^0) \leq (\text{tr}(\mathbf{K}_n^0)^2)^{1/2} = O_{\mathbb{P}}(pn^{-1/2}). \quad (5.17)$$

Using relations (5.16), (5.17) and condition $p = o(n^{1/2})$ obtained by assumption (A6), we have for (5.15) that

$$n\boldsymbol{\eta}_n^{0t} \mathbf{K}_n^0 \boldsymbol{\eta}_n^0 = n O_{\mathbb{P}}(n^{-1}p) O_{\mathbb{P}}(pn^{-1/2}) = O_{\mathbb{P}}(p^2n^{-1/2}) = o_{\mathbb{P}}(p^{1/2}). \quad (5.18)$$

On the other hand, using relations (5.16) and (5.17), we obtain

$$|\boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b \boldsymbol{\eta}_n^0| \leq \|\boldsymbol{\eta}_n^0\|^2 \max_{1 \leq r \leq p} |\gamma_r(\mathbf{K}_n^0)^b| \leq \|\boldsymbol{\eta}_n^0\|^2 (\text{tr}(\mathbf{K}_n^0)^2)^{b/2} = O_{\mathbb{P}}(pn^{-1}) O_{\mathbb{P}}(p^b n^{-b/2}),$$

which gives

$$\boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b \boldsymbol{\eta}_n^0 = O_{\mathbb{P}}(p^{b+1}n^{-(b+2)/2}). \quad (5.19)$$

The last equation means that the series $n \sum_{b=1}^{\infty} (-1)^{b-1} \boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b \boldsymbol{\eta}_n^0$ is convergent for fixed n when $p = o(n^{1/2})$. Then, taking also into account relation (5.18), we can conclude that

$$n \sum_{b=1}^{\infty} (-1)^{b-1} \boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b \boldsymbol{\eta}_n^0 = o_{\mathbb{P}}(p^{1/2}). \quad (5.20)$$

The remaining task is to prove that the term $n\boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b (\mathbf{I}_p - (\mathbf{B}_n^0)^{-1}) \boldsymbol{\eta}_n^0$ in (5.14) is negligible as $b \rightarrow \infty$. For the last term of (5.14), we have that

$$|n\boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b (\mathbf{I}_p - (\mathbf{B}_n^0)^{-1}) \boldsymbol{\eta}_n^0| \leq |n\boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b \boldsymbol{\eta}_n^0| + |n(\boldsymbol{\eta}_n^0)^t (\mathbf{K}_n^0)^b (\mathbf{B}_n^0)^{-1} \boldsymbol{\eta}_n^0|. \quad (5.21)$$

For the first term of the right hand side of (5.21), by relation (5.19), we have, with probability one, that $\boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b \boldsymbol{\eta}_n^0 = O_{\mathbb{P}}(p^{b+1}n^{-(b+2)/2})$. Then

$$n\boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b \boldsymbol{\eta}_n^0 = O_{\mathbb{P}}(p^{b+1}n^{-b/2}). \quad (5.22)$$

For the second term of the right hand side of (5.21), we have, with probability one, that

$$|n\boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b (\mathbf{B}_n^0)^{-1} \boldsymbol{\eta}_n^0| \leq n \|\boldsymbol{\eta}_n^0\|^2 \text{Mx}((\mathbf{K}_n^0)^b (\mathbf{B}_n^0)^{-1}). \quad (5.23)$$

Furthermore, according to Lemma 4 of Liu et al. (2013), for any $p \times p$ symmetric matrix $\mathbf{A} = (a_{ij})$, we have $\text{Mx}(\mathbf{A}) \leq \max_{1 \leq i \leq p} |\gamma_i(\mathbf{A})|$. Then, with probability one,

$$\text{Mx}((\mathbf{K}_n^0)^b (\mathbf{B}_n^0)^{-1}) \leq p \cdot \text{Mx}(\mathbf{K}_n^0)^b \cdot \text{Mx}(\mathbf{B}_n^0)^{-1} \leq p \cdot \max_{1 \leq r \leq p} |\gamma_r(\mathbf{K}_n^0)^b| \cdot \max_{1 \leq r \leq p} |\gamma_r(\mathbf{B}_n^0)^{-1}|.$$

By Lemma 8, we know that $|\gamma_r(\mathbf{K}_n^0)| \leq (\text{tr}(\mathbf{K}_n^0)^2)^{1/2} = O_{\mathbb{P}}(pn^{-1/2})$ and then, $|\gamma_r(\mathbf{K}_n^0)^b| \leq (\text{tr}(\mathbf{K}_n^0)^2)^{b/2} = O_{\mathbb{P}}(p^b n^{-b/2})$. On the other hand, by Proposition 2 it is clear that $\gamma_1(\mathbf{B}_n^0)^{-1} = \gamma_1((\mathbf{S}_n^0)^{-1} \mathbf{V}_n^0) \leq \gamma_1(\mathbf{V}_n^0)/C_0$, with probability tending to one. All these imply $\text{Mx}((\mathbf{K}_n^0)^b (\mathbf{B}_n^0)^{-1}) \leq p O_{\mathbb{P}}(p^b n^{-b/2})$. Then, we obtain

$$\text{Mx}((\mathbf{K}_n^0)^b (\mathbf{B}_n^0)^{-1}) = O_{\mathbb{P}}(p^{b+1}n^{-b/2}). \quad (5.24)$$

Combining relations (5.16), (5.23) and (5.24), for the second term of the right hand side of (5.21), we obtain that

$$|n\boldsymbol{\eta}_n^{0t} (\mathbf{K}_n^0)^b (\mathbf{B}_n^0)^{-1} \boldsymbol{\eta}_n^0| \leq np \|\boldsymbol{\eta}_n^0\|^2 \text{Mx}((\mathbf{K}_n^0)^b (\mathbf{B}_n^0)^{-1}) = O_{\mathbb{P}}(p^{b+3}n^{-b/2}). \quad (5.25)$$

By relations (5.21), (5.22), (5.25) and assumption (A6), it follows that

$$n\boldsymbol{\eta}_n^{0t}(\mathbf{I}_p - \mathbf{V}_n^0)\boldsymbol{\eta}_n^0 = O_{\mathcal{P}}(p^{b+1}n^{-b/2}) + O_{\mathcal{P}}(p^{b+3}n^{-b/2}) = O_{\mathcal{P}}((p^{2+4/b}n^{-1})^b) = o_{\mathcal{P}}(1). \quad (5.26)$$

Combining the results obtained in relations (5.20) and (5.26), we obtain (5.13).

The Proposition follows combining relation (5.13) and Proposition 3. \blacksquare

Proof of Theorem 1. Since $\theta = k/n \rightarrow \theta^0 \in (0, 1)$ and the point k , where the test is realised, is known, we suppose that θ is θ^0 , then it is fixed.

(i) We prove relation (3.4), by constructing a martingale and applying the martingale central limit theorem (see Chow and Teicher (1997)). We will prove this relation in four steps. In Step 1 we construct a martingale, in Steps 2 and 3 we propose two sufficient conditions for applying a central limit theorem and finally, in Step 4 we prove relation (3.4).

Step 1. In this step, we will construct a martingale.

For $i = 1, \dots, n$, let us define the following random vector sequence:

$$\mathbf{G}_i^0 \equiv \begin{cases} \frac{1}{\theta} \sum_{j=1}^i \mathbf{w}_j^0, & \text{if } i \leq k, \\ \frac{1}{\theta} \sum_{j=1}^k \mathbf{w}_j^0 - \frac{1}{1-\theta} \sum_{j=k+1}^i \mathbf{w}_j^0, & \text{if } i > k, \end{cases}$$

and also the random variable $H_i^0 \equiv \|\mathbf{G}_i^0\|^2 - ip$.

Then, the left hand side of (3.4) can be written

$$\frac{n\psi_n^{0t}(\mathbf{V}_n^0)^{-1}\psi_n^0 - p}{\Delta_n/n} = \frac{H_n^0}{\Delta_n}.$$

The relation between \mathbf{G}_i^0 and \mathbf{G}_{i-1}^0 is:

$$\mathbf{G}_i^0 = \begin{cases} \mathbf{G}_{i-1}^0 + \frac{\mathbf{w}_i^0}{\theta}, & \text{if } i \leq k, \\ \mathbf{G}_{i-1}^0 - \frac{\mathbf{w}_i^0}{1-\theta}, & \text{if } i > k, \end{cases}$$

with $\mathbf{G}_0^0 \equiv \mathbf{0}$.

Consider now the following filtration $\mathcal{F}_i = \sigma(\mathbf{w}_1^0, \dots, \mathbf{w}_i^0) = \sigma(\mathbf{G}_1^0, \dots, \mathbf{G}_i^0)$ for $i = 1, \dots, n$ the σ -field generated by $\mathbf{w}_1^0, \dots, \mathbf{w}_i^0$ or by $\mathbf{G}_1^0, \dots, \mathbf{G}_i^0$. Firstly, we study if $\{H_i^0, \mathcal{F}_i\}_{i \geq 1}$ is a martingale. For this, consider for example i such that $i > k$. Then

$$\mathbb{E}[H_i^0 | \mathcal{F}_{i-1}] = \|\mathbf{G}_{i-1}^0\|^2 + \frac{\mathbb{E}[\|\mathbf{w}_i^0\|^2]}{(1-\theta)^2} - ip \neq H_{i-1}^0.$$

Consequently, $\{H_i^0, \mathcal{F}_i\}_{i \geq 1}$ is not a martingale. We will now construct a martingale based on \mathbf{G}_n^0 with respect to the filtration $\{\mathcal{F}_i\}_{i \geq 1}$. For this, we define the following random variable sequence $U_i \equiv H_i^0 - H_{i-1}^0$, for $i = 1, \dots, n$, with $H_0^0 = 0$. Then

$$U_i^0 = \begin{cases} \frac{2}{\theta} \mathbf{G}_{i-1}^{0t} \mathbf{w}_i^0 + \frac{\|\mathbf{w}_i^0\|^2}{\theta^2} - p, & \text{if } i \leq k, \\ -\frac{2}{1-\theta} \mathbf{G}_{i-1}^{0t} \mathbf{w}_i^0 + \frac{\|\mathbf{w}_i^0\|^2}{(1-\theta)^2} - p, & \text{if } i > k. \end{cases}$$

We consider the following two random variable sequences:

$$\tau_i^0 \equiv U_i^0 - \mathbb{E}[U_i^0],$$

and

$$\varphi_i^0 \equiv \sum_{j=1}^i \tau_j^0 = H_i^0 - \sum_{j=1}^i \mathbb{E}[U_j^0].$$

For all i such that $i \leq k$ we have that the condition expectation of φ_i^0 given the σ -field \mathcal{F}_{i-1} is:

$$\mathbb{E}[\varphi_i^0 | \mathcal{F}_{i-1}] = \|\mathbf{G}_{i-1}^0\|^2 + \frac{\mathbb{E}[\|\mathbf{w}_i^0\|^2]}{\theta^2} - ip - \sum_{j=1}^i \mathbb{E}[U_j^0] = \|\mathbf{G}_{i-1}^0\|^2 - (i-1)p - \sum_{j=1}^{i-1} \mathbb{E}[U_j^0] = \varphi_{i-1}^0$$

and for all $i > k$:

$$\mathbb{E}[\varphi_i^0 | \mathcal{F}_{i-1}] = \|\mathbf{G}_{i-1}^0\|^2 + \frac{\mathbb{E}[\|\mathbf{w}_i^0\|^2]}{(1-\theta)^2} - ip - \sum_{j=1}^i \mathbb{E}[U_j^0] = \varphi_{i-1}^0.$$

Thus, $\{\varphi_i^0\}_{i \geq 1}$ is a martingale with respect to $\{\mathcal{F}_i\}_{i \geq 1}$.

To apply the martingale central limit theorem of Chow and Teicher (1997) for $\{\varphi_i^0, \mathcal{F}_i, i \geq 1\}$, it suffices to show that

$$\sum_{i=1}^n \mathbb{E}[|\tau_i^0|^3] = o(\Delta_n^3) \quad (5.27)$$

and

$$\sum_{i=1}^n \mathbb{E}[|\mathbb{E}[(\tau_i^0)^2 | \mathcal{F}_{i-1}] - \sigma_i^2|] = o(\Delta_n^2), \quad (5.28)$$

Step 2. In this step, we will prove relation (5.27).

In order to facilitate writing, for $i = 1, \dots, n$, we denote

$$\mathbf{N}_i^0 \equiv \frac{\mathbf{w}_i^0}{\theta} \mathbb{1}_{i \leq k} - \frac{\mathbf{w}_i^0}{1-\theta} \mathbb{1}_{i > k}. \quad (5.29)$$

Then, the random variable τ_i^0 can be written

$$\tau_i^0 = 2\mathbf{G}_{i-1}^{0t} \mathbf{N}_i^0 + \|\mathbf{N}_i^0\|^2 - \mathbb{E}[\|\mathbf{N}_i^0\|^2]. \quad (5.30)$$

By assumption (A9), for some positive absolute constant $C_7 < \infty$, for all $i = 1, \dots, n$ and all $j_1, \dots, j_l = 1, \dots, p$, $l \in \mathbb{N}$, whenever $\sum_{i=1}^l d_i \leq 6$, we have

$$\mathbb{E}[N_{i,j_1}^{0d_1} \dots N_{i,j_l}^{0d_l}] \leq C_7, \quad (5.31)$$

with N_{i,j_1}^0 is the j_1 -th components of the vector \mathbf{N}_i^0 defined in (5.29). By the Holder inequality, for any $b \leq 3$ we have

$$\mathbb{E}[\|\mathbf{N}_i^0\|^{2b}] \leq C \mathbb{E}[(\sum_{j=1}^p w_{ij}^2)^b] \leq Cp^{b-1} \sum_{j=1}^p \mathbb{E}[w_{ij}^{2b}] \leq Cp^b. \quad (5.32)$$

By the Cauchy Schwartz's inequality, we have that

$$\mathbb{E}[\mathbf{N}_i^{0t} \mathbf{G}_{i-1}^0]^3 \leq [\mathbb{E}[\mathbf{N}_i^{0t} \mathbf{G}_{i-1}^0]^6]^{1/2}. \quad (5.33)$$

On the other hand, by Lemma 7 of Guo et al (2013), we have

$$\mathbb{E}[\mathbf{N}_i^{0t} \mathbf{G}_{i-1}^0]^6 \leq Cp^6(i^3 + i^2 + i). \quad (5.34)$$

Then, by (5.33) and (5.34), we obtain that $\mathbb{E}[\mathbf{N}_i^{0t} \mathbf{G}_{i-1}^0]^3 \leq [Cp^6(i^3 + i^2 + i)]^{1/2}$. This implies that

$$\mathbb{E}[\mathbf{N}_i^{0t} \mathbf{G}_{i-1}^0]^3 \leq Cp^3(i^{3/2} + i + i^{1/2}). \quad (5.35)$$

On the other hand, we have

$$\mathbb{E}[\|\mathbf{N}_i^0\|^2 - \mathbb{E}[\|\mathbf{N}_i^0\|^2]]^3 = \mathbb{E}[\|\mathbf{N}_i^0\|^6 - 3\|\mathbf{N}_i^0\|^4 \mathbb{E}[\|\mathbf{N}_i^0\|^2] + 3\|\mathbf{N}_i^0\|^2 (\mathbb{E}[\|\mathbf{N}_i^0\|^2])^2 - (\mathbb{E}[\|\mathbf{N}_i^0\|^2])^3].$$

Using (5.32), we obtain that

$$\left| \mathbb{E}[\|\mathbf{N}_i^0\|^2 - \mathbb{E}[\|\mathbf{N}_i^0\|^2]]^3 \right| \leq Cp^3. \quad (5.36)$$

Using relations (5.35) and (5.36), we can write $\mathbb{E}[|\tau_i^0|^3] \leq Cp^3(1 + i^{3/2} + i + i^{1/2})$. Then $\sum_{i=1}^n \mathbb{E}[|\tau_i^0|^3] \leq Cnp^3(1 + n^{3/2} + n + n^{1/2})$, which gives

$$\sum_{i=1}^n \mathbb{E}[|\tau_i^0|^3] \leq Cp^3(n^{5/2} + n^2 + n^{3/2} + n). \quad (5.37)$$

On the other hand, using assumption (A1), by similar arguments as for relation (16) of Guo et al (2013), we have that $\Delta_n^2 \geq Cn^2p + O_p(p^2n)$. Then

$$\Delta_n^3 \geq Cn^3p^{3/2} + O_p(p^3n^{3/2}). \quad (5.38)$$

From relations (5.37) and (5.38), we obtain

$$\frac{\sum_{i=1}^n \mathbb{E}[|\tau_i^0|^3]}{\Delta_n^3} \leq C \frac{p^3 n^{5/2}}{n^3 p^{3/2}} = O_p((p^3/n)^{1/2}).$$

Since $p = o(n^{1/3})$, relation (5.27) follows.

Step 3. Now, in this step we prove relation (5.28).

By elementary calculations and using relation (5.32), we obtain $(\tau_i^0)^2 = 4\mathbf{G}_{i-1}^{0t}\mathbf{N}_i^0\mathbf{N}_i^{0t}\mathbf{G}_{i-1}^0 + 4\mathbf{G}_{i-1}^{0t}\mathbf{N}_i^0(\|\mathbf{N}_i^0\|^2 - \mathbb{E}[\|\mathbf{N}_i^0\|^2]) + O_p(p^2)$. We observe also that $\sigma_i^2 = \mathbb{E}[(\tau_i^0)^2]$. Then

$$\mathbb{E}[(\tau_i^0)^2 | \mathcal{F}_{i-1}] = 4\mathbf{G}_{i-1}^{0t}\mathbb{E}[\mathbf{N}_i^0\mathbf{N}_i^{0t}] \mathbf{G}_{i-1}^0 + 4\mathbf{G}_{i-1}^{0t}\mathbb{E}[\mathbf{N}_i^0(\|\mathbf{N}_i^0\|^2 - \mathbb{E}[\|\mathbf{N}_i^0\|^2])] + O_p(p^2) \quad (5.39)$$

and

$$\sigma_i^2 = 4\mathbb{E}[\mathbf{G}_{i-1}^{0t}\mathbb{E}[\mathbf{N}_i^0\mathbf{N}_i^{0t}] \mathbf{G}_{i-1}^0] + \mathbb{E}[(\|\mathbf{N}_i^0\|^2 - \mathbb{E}[\|\mathbf{N}_i^0\|^2])^2] + O(p^2).$$

By inequality (5.32) we have $\mathbb{E}[(\|\mathbf{N}_i^0\|^2 - \mathbb{E}[\|\mathbf{N}_i^0\|^2])^2] = O(p^2)$. Then

$$\sigma_i^2 = 4\mathbb{E}[\mathbf{G}_{i-1}^{0t}\mathbb{E}[\mathbf{N}_i^0\mathbf{N}_i^{0t}] \mathbf{G}_{i-1}^0] + O(p^2). \quad (5.40)$$

Using relations (5.39) and (5.40), we obtain that, for any $i = 1, \dots, n$,

$$\begin{aligned} \mathbb{E}[|\mathbb{E}[\tau_i^2 | \mathcal{F}_{i-1}] - \sigma_i^2|^2] &\leq 16 \left(\mathbb{E}[\mathbf{G}_{i-1}^{0t}\mathbb{E}[\mathbf{N}_i^0\mathbf{N}_i^{0t}] \mathbf{G}_{i-1}^0 - \mathbb{E}[\mathbf{G}_{i-1}^{0t}\mathbf{N}_i^0\mathbf{N}_i^{0t}\mathbf{G}_{i-1}^0]]^2 \right. \\ &\quad \left. + \mathbb{E}[\mathbf{N}_i^{0t}(\|\mathbf{N}_i^0\|^2 - \mathbb{E}[\|\mathbf{N}_i^0\|^2])] \mathbb{E}[\mathbf{G}_{i-1}^0\mathbf{G}_{i-1}^{0t}] \mathbb{E}[\mathbf{N}_i^0(\|\mathbf{N}_i^0\|^2 - \mathbb{E}[\|\mathbf{N}_i^0\|^2])] + O(p^4) \right) \\ &\equiv 16(A + B + O(p^4)) \end{aligned} \quad (5.41)$$

For the term A of (5.41), we have the decomposition

$$A = \mathbb{E}[\mathbf{G}_{i-1}^{0t}\mathbb{E}[\mathbf{N}_i^0\mathbf{N}_i^{0t}] \mathbf{G}_{i-1}^0]^2 - (\mathbb{E}[\mathbf{G}_{i-1}^{0t}\mathbf{N}_i^0\mathbf{N}_i^{0t}\mathbf{G}_{i-1}^0])^2 \equiv A_1 - A_2. \quad (5.42)$$

Before analysing the terms A_1 and A_2 , we note that

$$\mathbb{E}(\mathbf{N}_i^0\mathbf{N}_i^{0t}) = (\mathbf{V}_n^0)^{-1/2} \left[\frac{1}{\theta^2} \mathbb{1}_{i \leq k} + \frac{1}{(1-\theta)^2} \mathbb{1}_{i > k} \right] \mathbf{V}_{(i)}^0 (\mathbf{V}_n^0)^{-1/2}. \quad (5.43)$$

In order to facilitate writing, we consider the following matrix

$$\mathbf{M}_i^0 \equiv \left(\frac{1}{\theta^2} \mathbb{1}_{i \leq k} + \frac{1}{(1-\theta)^2} \mathbb{1}_{i > k} \right) \mathbf{V}_{(i)}^0.$$

Then, $\mathbb{E}[\mathbf{N}_i^0\mathbf{N}_i^{0t}]$ can be expressed as:

$$\mathbb{E}[\mathbf{N}_i^0\mathbf{N}_i^{0t}] = (\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2}. \quad (5.44)$$

Hence, the term A_1 of (5.42), can be written

$$A_1 = \mathbb{E}[\mathbf{G}_{i-1}^{0t}(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{G}_{i-1}^0]^2.$$

Taking into account that the random vectors \mathbf{w}_i^0 are independent, with mean zero, for all $i = 1, \dots, n$, we can decompose A_1 as

$$A_1 \equiv A_{11} + A_{12} + A_{13}, \quad (5.45)$$

with

- if $i - 1 \leq k$:

$$A_{11} \equiv \frac{1}{\theta^4} \sum_{h \neq l=1}^{i-1} \mathbb{E}[\mathbf{w}_h^0(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_h^0] \mathbb{E}[\mathbf{w}_l^0(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_l^0],$$

$$A_{12} \equiv \frac{2}{\theta^4} \sum_{h \neq l=1}^{i-1} \mathbb{E}[\mathbf{w}_h^0(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_l^0\mathbf{w}_h^{0t}(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_l^0]$$

$$A_{13} \equiv \frac{1}{\theta^4} \sum_{h=1}^{i-1} \mathbb{E}[\mathbf{w}_h^{0t}(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_h^0]^2.$$

- if $i - 1 > k$:

$$A_{11} \equiv \frac{1}{\theta^4} \sum_{h \neq l=1}^k \mathbb{E}[\mathbf{w}_h^0(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_h^0] \mathbb{E}[\mathbf{w}_l^0(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_l^0]$$

$$+ \frac{1}{(1-\theta)^4} \sum_{h' \neq l'=k+1}^{i-1} \mathbb{E}[\mathbf{w}_{h'}^0(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_{h'}^0] \mathbb{E}[\mathbf{w}_{l'}^0(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_{l'}^0],$$

$$A_{12} \equiv \frac{2}{\theta^4} \sum_{h \neq l=1}^k \mathbb{E}[\mathbf{w}_h^0(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_l^0\mathbf{w}_h^{0t}(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_l^0]$$

$$+ \frac{2}{(1-\theta)^4} \sum_{h' \neq l'=k+1}^{i-1} \mathbb{E}[\mathbf{w}_{h'}^0(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_{l'}^0\mathbf{w}_{h'}^{0t}(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_{l'}^0].$$

$$A_{13} \equiv \frac{1}{\theta^4} \sum_{h=1}^k \mathbb{E}[\mathbf{w}_h^{0t}(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_h^0]^2 + \frac{1}{(1-\theta)^4} \sum_{h'=k+1}^{i-1} \mathbb{E}[\mathbf{w}_{h'}^{0t}(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_{h'}^0]^2.$$

We study A_{11} , A_{12} , A_{13} . For this, we consider the case $i - 1 > k$, the other is similar.

For A_{11} , applying Cauchy-Schwarz inequality, we have

$$A_{11} \leq \frac{1}{\theta^4} \left(\sum_{h=1}^k \mathbb{E}[\mathbf{w}_h^{0t}(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_h^0] \right)^2 + \frac{1}{(1-\theta)^4} \left(\sum_{h'=k+1}^{i-1} \mathbb{E}[\mathbf{w}_{h'}^{0t}(\mathbf{V}_n^0)^{-1/2}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1/2}\mathbf{w}_{h'}^0] \right)^2$$

$$= \frac{1}{\theta^4} \left(\sum_{h=1}^k \mathbb{E}[\mathbf{z}_h^{0t}(\mathbf{V}_n^0)^{-1}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1}\mathbf{z}_h^0] \right)^2 + \frac{1}{(1-\theta)^4} \left(\sum_{h'=k+1}^{i-1} \mathbb{E}[\mathbf{z}_{h'}^{0t}(\mathbf{V}_n^0)^{-1}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1}\mathbf{z}_{h'}^0] \right)^2.$$

Then, since $(\varepsilon_i)_{1 \leq i \leq n}$ are independent, we have:

$$A_{11} = \frac{1}{\theta^4} \left(\sum_{h=1}^k \text{tr}(\mathbf{V}_{(h)}^0(\mathbf{V}_n^0)^{-1}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1}) \right)^2 + \frac{1}{(1-\theta)^4} \left(\sum_{h'=k+1}^{i-1} \text{tr}(\mathbf{V}_{(h')}^0(\mathbf{V}_n^0)^{-1}\mathbf{M}_i^0(\mathbf{V}_n^0)^{-1}) \right)^2. \quad (5.46)$$

Similarly, we have for A_{12} :

$$\begin{aligned}
A_{12} &= \frac{2}{\theta^4} \mathbb{E} \left[\sum_{h \neq l=1}^k \mathbf{w}_h^0 (\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \mathbf{w}_l^0 \mathbf{w}_l^{0t} (\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \mathbf{w}_h^0 \right] \\
&\quad + \frac{2}{(1-\theta)^4} \mathbb{E} \left[\sum_{h' \neq l'=k+1}^{i-1} \mathbf{w}_{h'}^0 (\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \mathbf{w}_{l'}^0 \mathbf{w}_{l'}^{0t} (\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \mathbf{w}_{h'}^0 \right] \\
&= \frac{2}{\theta^4} \mathbb{E} \left[\sum_{h \neq l=1}^k \mathbf{w}_h^0 (\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \mathbb{E}[\mathbf{w}_l^0 \mathbf{w}_l^{0t}] (\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \mathbf{w}_h^0 \right] \\
&\quad + \frac{2}{(1-\theta)^4} \mathbb{E} \left[\sum_{h' \neq l'=k+1}^{i-1} \mathbf{w}_{h'}^0 (\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \mathbb{E}[\mathbf{w}_{l'}^0 \mathbf{w}_{l'}^{0t}] (\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \mathbf{w}_{h'}^0 \right] \\
&\leq \frac{2}{\theta^4} p k^2 \gamma_1^4 (\mathbf{V}_n^0)^{-1} \sup_{h, l \in \{1, \dots, k\}} (\gamma_1(\mathbf{V}_n^0) \gamma_1(\mathbf{V}_n^0)) \cdot \gamma_1^2(\mathbf{M}_i^0) \\
&\quad + \frac{2}{(1-\theta)^4} p (i-1-k)^2 \gamma_1^4 (\mathbf{V}_n^0)^{-1} \sup_{h', l' \in \{k+1, \dots, i-1\}} (\gamma_1(\mathbf{V}_n^0) \gamma_1(\mathbf{V}_n^0)) \cdot \gamma_1^2(\mathbf{M}_i^0).
\end{aligned}$$

Taking into account assumption (A1), we obtain that

$$A_{12} \leq Cp(i-1)^2. \quad (5.47)$$

For the term A_{13} of (5.45), we have

$$\begin{aligned}
A_{13} &= \frac{1}{\theta^4} \sum_{h=1}^k \sum_{j, l, s, t=1}^p \left((\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \right)_{jl} \left((\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \right)_{st} \mathbb{E}[w_{h,j}^0 w_{h,l}^0 w_{h,s}^0 w_{h,t}^0] \\
&\quad + \frac{1}{(1-\theta)^4} \sum_{h'=k+1}^{i-1} \sum_{j, l, s, t=1}^p \left((\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \right)_{jl} \left((\mathbf{V}_n^0)^{-1/2} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1/2} \right)_{st} \mathbb{E}[w_{h',j}^0 w_{h',l}^0 w_{h',s}^0 w_{h',t}^0].
\end{aligned}$$

Taking into account assumption (A9), we obtain that

$$A_{13} \leq \frac{1}{\theta^4} Cp^4 k \gamma_1^4 (\mathbf{V}_n^0)^{-1} \gamma_1^2(\mathbf{M}_i^0) + \frac{1}{(1-\theta)^4} Cp^4 (i-k) \gamma_1^4 (\mathbf{V}_n^0)^{-1} \gamma_1^2(\mathbf{M}_i^0).$$

Using also assumption (A1), we obtain get

$$A_{13} \leq Cp^4 (i-1). \quad (5.48)$$

For the term A_2 of (5.42), by similar calculations, we obtain that

- if $i-1 \leq k$:

$$A_2 = \frac{1}{\theta^4} \left(\sum_{h=1}^{i-1} \text{tr}(\mathbf{V}_n^0 (\mathbf{V}_n^0)^{-1} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1}) \right)^2$$

- if $i-1 > k$:

$$A_2 = \frac{1}{\theta^4} \left(\sum_{h=1}^k \text{tr}(\mathbf{V}_n^0 (\mathbf{V}_n^0)^{-1} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1}) \right)^2 + \frac{1}{(1-\theta)^4} \left(\sum_{h'=k+1}^{i-1} \text{tr}(\mathbf{V}_n^0 (\mathbf{V}_n^0)^{-1} \mathbf{M}_i^0 (\mathbf{V}_n^0)^{-1}) \right)^2. \quad (5.49)$$

For $i-1 > k$, by relations (5.46) and (5.49), we obtain that

$$A_{11} - A_2 \leq 0. \quad (5.50)$$

Inequality (5.50) is also true for $i-1 \leq k$.

Then, since the term A of (5.42) can be written $A = (A_{11} + A_{12} + A_{13}) - A_2$, combining relations (5.47), (5.48) and (5.50), we obtain that

$$A \leq Cp(i-1)^2 + Cp^4(i-1). \quad (5.51)$$

For the term B of (5.41), in the case $i - 1 > k$, taking into account the fact that $\mathbb{E}(\mathbf{N}_h^0) = \mathbf{0}_p$, $\mathbb{E}(\mathbf{w}_h^0) = \mathbf{0}_p$ for all $h = 1, \dots, n$ and the fact \mathbf{w}_h^0 is independent of $\mathbf{w}_{h'}^0$ for $h \neq h'$, we have that

$$\begin{aligned} B &= \mathbb{E}[\mathbf{N}_i^{0t} \|\mathbf{N}_i^0\|^2] \mathbb{E}\left[\left(\frac{1}{\theta} \sum_{h=1}^k \mathbf{w}_h^0 - \frac{1}{1-\theta} \sum_{h'=k+1}^{i-1} \mathbf{w}_{h'}^0\right) \cdot \left(\frac{1}{\theta} \sum_{h=1}^k \mathbf{w}_h - \frac{1}{1-\theta} \sum_{h'=k+1}^{i-1} \mathbf{w}_{h'}\right)^t\right] \mathbb{E}[\mathbf{N}_i^0 \|\mathbf{N}_i^0\|^2] \\ &= \frac{1}{\theta^2} \mathbb{E}[\mathbf{N}_i^{0t} \|\mathbf{N}_i^0\|^2] \sum_{h,l=1}^k \mathbb{E}[\mathbf{w}_h^0 \mathbf{w}_l^{0t}] \mathbb{E}[\mathbf{N}_i^{0t} \|\mathbf{N}_i^0\|^2] + \frac{1}{(1-\theta)^2} \mathbb{E}[\mathbf{N}_i^{0t} \|\mathbf{N}_i^0\|^2] \sum_{h',l'=k+1}^{i-1} \mathbb{E}[\mathbf{w}_{h'}^0 \mathbf{w}_{l'}^{0t}] \mathbb{E}[\mathbf{N}_i^0 \|\mathbf{N}_i^0\|^2] \\ &= \frac{1}{\theta^2} \sum_{u,s=1}^p \sum_{l,r=1}^p \left(\sum_{h=1}^k \mathbf{V}_n^{-1/2} \mathbf{V}_{(h)}^0 \mathbf{V}_n^{-1/2} \right)_{us} \mathbb{E}[N_{i,u}^0 (N_{i,l}^0)^2] \mathbb{E}[N_{i,s}^0 (N_{i,r}^0)^2] \\ &\quad + \frac{1}{(1-\theta)^2} \sum_{u,s=1}^p \sum_{l,r=1}^p \left(\sum_{h'=k+1}^{i-1} (\mathbf{V}_n^0)^{-1/2} \mathbf{V}_{(h')}^0 (\mathbf{V}_n^0)^{-1/2} \right)_{us} \mathbb{E}[N_{i,u}^0 (N_{i,l}^0)^2] \mathbb{E}[N_{i,s}^0 (N_{i,r}^0)^2], \end{aligned}$$

where $\left(\sum_{h=1}^k \mathbf{V}_n^{-1/2} \mathbf{V}_{(h)}^0 \mathbf{V}_n^{-1/2} \right)_{us}$ is the (u, s) -th element of the matrix $\sum_{h=1}^k \mathbf{V}_n^{-1/2} \mathbf{V}_{(h)}^0 \mathbf{V}_n^{-1/2}$. Using Lemma 4 of Liu et al. (2013) and relation (5.31), we get

$$\begin{aligned} B &\leq \frac{1}{\theta^2} \sum_{u,s=1}^p \sum_{l,r=1}^p \text{Mx} \left(\sum_{h=1}^k \mathbf{V}_n^{-1/2} \mathbf{V}_{(h)}^0 \mathbf{V}_n^{-1/2} \right)_{us} \mathbb{E}[N_{i,u}^0 (N_{i,l}^0)^2] \mathbb{E}[N_{i,s}^0 (N_{i,r}^0)^2] \\ &\quad + \frac{1}{(1-\theta)^2} \sum_{u,s=1}^p \sum_{l,r=1}^p \text{Mx} \left(\sum_{h'=k+1}^{i-1} (\mathbf{V}_n^0)^{-1/2} \mathbf{V}_{(h')}^0 (\mathbf{V}_n^0)^{-1/2} \right)_{us} \mathbb{E}[N_{i,u}^0 (N_{i,l}^0)^2] \mathbb{E}[N_{i,s}^0 (N_{i,r}^0)^2] \\ &\leq \frac{C}{\theta^2} p^4 (k \gamma_1^2 (\mathbf{V}_n^{-1/2})) \sup_{h \in \{1, \dots, k\}} (\gamma_1 (\mathbf{V}_{(h)}^0)) + \frac{C}{(1-\theta)^2} p^4 (i-1-k) \gamma_1^2 (\mathbf{V}_n^{-1/2}) \sup_{h \in \{k+1, \dots, i-1\}} (\gamma_1 (\mathbf{V}_{(h')}^0)). \end{aligned}$$

Then, by assumption (A1), we obtain that

$$B \leq Cp^4(i-1). \quad (5.52)$$

Similarly, we can prove that inequality (5.52) is also true for $i - 1 \leq k$.

In conclusion, combining relations (5.41), (5.51) and (5.52), we get that

$$\mathbb{E}[|\mathbb{E}[(\tau_i^0)^2 | \mathcal{F}_{i-1}] - \sigma_i^2|^2] \leq Cp(i-1)^2 + Cp^4(i-1) + O(p^4).$$

Then, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{\sum_{i=1}^n \mathbb{E}[|\mathbb{E}[(\tau_i^0)^2 | \mathcal{F}_{i-1}] - \sigma_i^2|]}{\Delta_n^2} &\leq \frac{\sum_{i=1}^n \left(\mathbb{E}[|\mathbb{E}[(\tau_i^0)^2 | \mathcal{F}_{i-1}] - \sigma_i^2|^2] \right)^{1/2}}{\Delta_n^2} \\ &\leq \frac{Cn(n^{1/2}p^2 + p^{1/2}n + O(p^2))}{Cn^2p + O_{\mathbb{P}}(p^2n)} \\ &\leq C \frac{n^{3/2}p^2}{n^2p}. \end{aligned}$$

and hence the relation (5.28) follows.

Step 4. In this step, on the basis of the central limit theorem for martingales, we will complete the proof of relation (3.4). On the basis of relations (5.27) and (5.28) proved in Step 2 and Step 3, applying the martingale central limit Theorem of Chow and Teicher (1997) (Theorem 1, page 336), for $\{\varphi_n^0, \mathcal{F}_n\}_{n \geq 1}$ we get:

$$\frac{\varphi_n^0}{\Delta_n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1). \quad (5.53)$$

Using notations given in *Step 1*, we have $n(n\psi_n^{0t}(\mathbf{V}_n^0)^{-1}\psi_n^0 - p) = H_n^0 = \varphi_n^0 + \sum_{i=1}^n \mathbb{E}[U_i^0]$. In the other hand, since $H_n^0 = 0$, we have $\sum_{i=1}^n \mathbb{E}[U_i^0] = \mathbb{E}[H_n^0] = \mathbb{E}[\|\mathbf{G}_n^0\|^2] - np$. But, taking into account relations (2.8) and (2.9), we get:

$$\begin{aligned} \mathbb{E}[\|\mathbf{G}_n^0\|^2] &= \frac{1}{\theta^2} \sum_{i=1}^k \mathbb{E}[\|\mathbf{w}_i^0\|^2] + \frac{1}{(1-\theta)^2} \sum_{j=k+1}^n \mathbb{E}[\|\mathbf{w}_j^0\|^2] \\ &= \frac{\sigma^2}{\theta^2} \sum_{i=1}^k (\mathbf{X}_i^t (\mathbf{V}_n^0)^{-1/2}) ((\mathbf{V}_n^0)^{-1/2} \mathbf{X}_i) + \frac{\sigma^2}{(1-\theta)^2} \sum_{j=k+1}^n (\mathbf{X}_j^t (\mathbf{V}_n^0)^{-1/2}) ((\mathbf{V}_n^0)^{-1/2} \mathbf{X}_j) \\ &= \frac{\sigma^2}{\theta^2} \sum_{i=1}^k \text{tr} \left((\mathbf{X}_i^t (\mathbf{V}_n^0)^{-1/2}) ((\mathbf{V}_n^0)^{-1/2} \mathbf{X}_i) \right) + \frac{\sigma^2}{(1-\theta)^2} \sum_{j=k+1}^n \text{tr} \left((\mathbf{X}_j^t (\mathbf{V}_n^0)^{-1/2}) ((\mathbf{V}_n^0)^{-1/2} \mathbf{X}_j) \right) \\ &= \sigma^2 \text{tr} \left((\mathbf{V}_n^0)^{-1/2} \left(\frac{1}{\theta^2} \sum_{i=1}^k \mathbf{X}_i \mathbf{X}_i^t + \frac{1}{(1-\theta)^2} \sum_{j=k+1}^n \mathbf{X}_j \mathbf{X}_j^t \right) (\mathbf{V}_n^0)^{-1/2} \right) \\ &= n \text{tr} \left((\mathbf{V}_n^0)^{-1/2} \mathbf{V}_n^0 (\mathbf{V}_n^0)^{-1/2} \right) \\ &= np. \end{aligned}$$

Thus $\mathbb{E}[H_n^0] = 0$. Then, taking into account relation (5.53), we obtain claim (3.4).

(ii) The assertion results from (i) combined with Proposition 3 and Proposition 4. \blacksquare

Proof of Theorem 2. By elementary calculations, we have under hypothesis H_1 , with probability one:

$$\psi_n(\beta^0) = \psi_n^0 - \frac{1}{n(1-\theta)} \left(\sum_{j=k+1}^n \mathbf{X}_j \mathbf{X}_j^t \right) (\beta_2^0 - \beta^0).$$

Also, under H_1 , since $\mathbb{E}[\mathbf{z}_i^0] = \mathbf{0}_p$, using assumption (A1), we have

$$\psi_n(\beta^0) = -\frac{1}{n(1-\theta)} \left(\sum_{j=k+1}^n \mathbf{X}_j \mathbf{X}_j^t \right) (\beta_2^0 - \beta^0) (1 + o_{\mathbb{P}}(1)).$$

Then,

$$n\psi_n^t(\beta^0)(\mathbf{V}_n^0)^{-1}\psi_n(\beta^0) = \left(n\psi_n^{0t}(\mathbf{V}_n^0)^{-1}\psi_n^0 + n|(\beta_2^0 - \beta^0)^t \frac{1}{n(1-\theta)} \sum_{j=k+1}^n \mathbf{X}_j \mathbf{X}_j^t \right) (1 + o_{\mathbb{P}}(1)).$$

The test statistic becomes

$$\mathcal{Z}(\beta^0) = \frac{n\psi_n^{0t}(\mathbf{V}_n^0)^{-1}\psi_n^0 - p}{\Delta_n/n} + n \frac{|(\beta_2^0 - \beta^0)^t \frac{1}{n(1-\theta)} \sum_{j=k+1}^n \mathbf{X}_j \mathbf{X}_j^t|^2}{\Delta_n/n} (1 + o_{\mathbb{P}}(1)).$$

Since $\beta_2^0 \neq \beta^0$, using with Theorem 1, together assumption (A1), we have that $|\mathcal{Z}(\beta^0)| \rightarrow \infty$, in probability, as $n \rightarrow \infty$. \blacksquare

5.2 Lemmas

In order to prove Propositions 1, 2, 3, 4 and Theorems 1 and 2, we need the following lemmas.

The first four lemmas establish equivalent results obtained by Guo et al (2013) for a linear model without change-point.

Let us consider the following $p \times p$ matrix $\mathbf{L}_n \equiv \mathbf{S}_n^0 - \mathbf{V}_n^0$. For $1 \leq u, v \leq p$ let us consider $L_{n;u,v}$ the (u, v) element of the matrix \mathbf{L}_n . Let also consider the largest absolute element of \mathbf{L}_n : $\max_{1 \leq u, v \leq p} |S_{n,(u,v)}^0 - V_{n,(u,v)}^0|$, with $S_{n,(u,v)}^0$ and $V_{n,(u,v)}^0$ denote the (u, v) components of \mathbf{S}_n^0 and \mathbf{V}_n^0 , respectively.

Lemma 1 *Under null hypothesis H_0 , suppose that assumptions (A3) and (A4) are satisfied. For any $\epsilon > 0$, there exists a positive constant C_q that depending only on $q \geq 4$, such that*

$$\mathbb{P}\left[\max_{1 \leq u, v \leq p} |S_{n,(u,v)}^0 - V_{n,(u,v)}^0| \geq \epsilon\right] \leq C_q \frac{p^2}{n^{q/2} \epsilon^q}.$$

Proof. The matrix \mathbf{L}_n can be written

$$\mathbf{L}_n = \frac{1}{n\theta^2} \sum_{i=1}^k \mathbf{X}_i \mathbf{X}_i^t (\epsilon_i^2 - \sigma^2) + \frac{1}{n(1-\theta)^2} \sum_{j=k+1}^n \mathbf{X}_j \mathbf{X}_j^t (\epsilon_j^2 - \sigma^2).$$

The (u, v) -th element of the matrix \mathbf{L}_n , for $1 \leq u, v \leq p$, is

$$L_{n,(u,v)} = \frac{1}{n\theta^2} \sum_{i=1}^k X_{i,u} X_{i,v} (\epsilon_i^2 - \sigma^2) + \frac{1}{n(1-\theta)^2} \sum_{j=k+1}^n X_{j,u} X_{j,v} (\epsilon_j^2 - \sigma^2) \equiv L_{n,(u,v)}^{(1)} + L_{n,(u,v)}^{(2)}. \quad (5.54)$$

Since $\theta = k/n \rightarrow \theta^0 \in (0, 1)$ as $n \rightarrow \infty$, we can apply Lemma 1 of Guo et al (2013) for $L_{n,(u,v)}^{(1)}$ and $L_{n,(u,v)}^{(2)}$. Then, for fixed $\theta \in (0, 1)$, for all $\epsilon > 0$ there exists two positive constants $C_q^{(1)}$ and $C_q^{(2)}$ such that:

$$\mathbb{P}[|L_{n,(u,v)}^{(1)}| \geq \epsilon] \leq C_q^{(1)} \frac{p^2}{n^{q/2} \epsilon^q}. \quad (5.55)$$

and

$$\mathbb{P}[|L_{n,(u,v)}^{(2)}| \geq \epsilon] \leq C_q^{(2)} \frac{p^2}{n^{q/2} \epsilon^q}. \quad (5.56)$$

Then the Lemma follows from relations (5.54), (5.55) and (5.56), considering $C_q = \max(C_q^{(1)}, C_q^{(2)})$. ■

By the next lemma we prove first that all eigenvalues of \mathbf{S}_n^0 converge to those of \mathbf{V}_n^0 uniformly with the rate $O_p(p \max_{1 \leq u, v \leq p} |S_{n,(u,v)}^0 - V_{n,(u,v)}^0|)$ and then that all eigenvalues of \mathbf{S}_n^0 are bounded and strictly positive for n enough large.

Lemma 2 *Under assumptions (A1), (A3), (A4) and (A6), there exists two constants $C_0, C_1 > 0$ such that the inequality $\gamma_p(\mathbf{S}_n^0) > C_0$ and $\gamma_1(\mathbf{S}_n^0) < C_1$ hold with probability tending to one as $n \rightarrow \infty$.*

Proof. Arguments based on the properties of the trace and of the eigenvalues of symmetric square matrices, together with Lemma 2 of Guo et al (2013) imply that

$$\max_{1 \leq r \leq p} |\gamma_r(\mathbf{V}_n^0) - \gamma_r(\mathbf{S}_n^0)| \leq p \max_{1 \leq u, v \leq p} |S_{n,(u,v)}^0 - V_{n,(u,v)}^0|. \quad (5.57)$$

Lemma follows by combining this last relation with Lemma 1 and assumption (A1). ■

Let us consider the following random variable $T_n^0 \equiv \max_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq n}} \left\{ \theta^{-1} \|\mathbf{z}_i^0\|, (1-\theta)^{-1} \|\mathbf{z}_j^0\| \right\}$.

Lemma 3 *Under the null hypothesis H_0 , suppose that assumptions (A3) and (A4) are satisfied. Then, for $q \geq 4$, we have $T_n^0 = o_{\mathbb{P}}(p^{1/2} n^{1/q})$.*

Proof. Since $\theta = k/n \rightarrow \theta^0 \in (0, 1)$ as $n \rightarrow \infty$, we have with probability 1, for enough large n : $T_n^0 \leq C \max_{1 \leq i \leq n} \|\mathbf{z}_i^0\|$, with $C > 0$. By Lemma 3 of Guo et al (2013), we have that: $\max_{1 \leq i \leq n} \|\mathbf{z}_i^0\| = o_{\mathbb{P}}(p^{1/2} n^{1/q})$ and the lemma follows. ■

By the following Lemma we give an asymptotic approximation for the L_2 -norm of the vector $\boldsymbol{\psi}_n(\boldsymbol{\beta}^0)$, given by (2.7), under hypothesis H_0 .

Lemma 4 *Under the null hypothesis H_0 , if assumption (A1) holds, we have $\|\boldsymbol{\psi}_n^0\| = O_{\mathbb{P}}(p^{1/2} n^{-1/2})$.*

Proof. Let $\mathbf{X}^{(1)}$ is the $p \times k$ design matrix whose k columns are \mathbf{X}_i , for $i = 1, \dots, k$ and $\mathbf{X}^{(2)}$ is the $p \times (n - k)$ design matrix whose $(n - k)$ columns are \mathbf{X}_j , for $j = k + 1, \dots, n$. Since (ε_i) are independent, we have that

$$\begin{aligned} \mathbb{E}[\psi_n^{0t} \psi_n^0] &= \frac{1}{(n\theta)^2} \mathbb{E}\left[\sum_{i,i'=1}^k \mathbf{z}_i^{0t} \mathbf{z}_{i'}^0\right] + \frac{1}{(n(1-\theta))^2} \mathbb{E}\left[\sum_{j,j'=k+1}^n \mathbf{z}_j^{0t} \mathbf{z}_{j'}^0\right] \\ &= \frac{\sigma^2}{n} \text{tr}\left(\frac{1}{n\theta^2} \mathbf{X}^{(1)} (\mathbf{X}^{(1)})^t + \frac{1}{n(1-\theta)^2} \mathbf{X}^{(2)} (\mathbf{X}^{(2)})^t\right). \end{aligned}$$

Using assumption (A1), we obtain that

$$\mathbb{E}[\psi_n^{0t} \psi_n^0] \leq \sigma^2 \frac{p}{n} \left(\gamma_1 \left(\frac{1}{n\theta} \mathbf{X}^{(1)} (\mathbf{X}^{(1)})^t + \frac{1}{n(1-\theta)} \mathbf{X}^{(2)} (\mathbf{X}^{(2)})^t \right) \right) = O(pn^{-1}).$$

Then, $\|\psi_n^0\| = O_{\mathbb{P}}(p^{1/2}n^{-1/2})$. ■

The following Lemma gives a first approximation for the EL statistic, under hypothesis H_0 .

Lemma 5 *Under the same assumptions as in Proposition 3, if hypothesis H_0 is true, we have:*

$$\text{EL}_{nk}(\beta^0) = 2n\lambda^t \psi_n^0 - n\lambda^t \mathbf{S}_n^0 \lambda + \frac{2}{3} \left(\frac{1}{\theta^3} \sum_{i=1}^k (\mathbf{z}_i^{0t} \lambda)^3 - \frac{1}{(1-\theta)^3} \sum_{j=k+1}^n (\mathbf{z}_j^{0t} \lambda)^3 \right) + o_{\mathbb{P}}(1),$$

with $\theta = k/n$.

Proof. The limited development of the statistic $\text{EL}_{nk}(\beta^0)$ specified by relation (2.4), in the neighbourhood of $\lambda = \mathbf{0}_p$, up to order 3 can be written

$$\begin{aligned} \text{EL}_{nk}(\beta^0) &= \lambda^t \left(\frac{2}{\theta} \sum_{i=1}^k \mathbf{z}_i^0 - \frac{2}{1-\theta} \sum_{j=k+1}^n \mathbf{z}_j^0 \right) - \lambda^t \left(\frac{1}{\theta^2} \sum_{i=1}^k \mathbf{z}_i^0 \mathbf{z}_i^{0t} + \frac{1}{(1-\theta)^2} \sum_{j=k+1}^n \mathbf{z}_j^0 \mathbf{z}_j^{0t} \right) \lambda \\ &\quad + \frac{2}{3} \left(\frac{1}{\theta^3} \sum_{i=1}^k (\mathbf{z}_i^{0t} \lambda)^3 - \frac{1}{(1-\theta)^3} \sum_{j=k+1}^n (\mathbf{z}_j^{0t} \lambda)^3 \right) + \frac{1}{4!} \sum_{u,v,r,s=1}^p \frac{\partial^4 \text{EL}_{nk}(\beta^0)(\tilde{\lambda}_{uvrs})}{\partial \lambda_u \partial \lambda_v \partial \lambda_r \partial \lambda_s} (\lambda_u)(\lambda_v)(\lambda_r)(\lambda_s) \\ &\equiv \mathcal{E}_1 - \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4, \end{aligned} \tag{5.58}$$

where for all $1 \leq u, v, r, s \leq p$, λ_u is the u -th component of λ and $\tilde{\lambda}_{uvrs} = a_{uvrs} \lambda$, with $a_{uvrs} \in [0, 1]$.

We first study \mathcal{E}_4 , which can be written

$$\mathcal{E}_4 = -\frac{1}{4!} \left(\frac{12}{\theta^4} \sum_{i=1}^k \frac{(\mathbf{z}_i^{0t} \lambda)^4}{(1 + \theta^{-1} \tilde{\lambda} \mathbf{z}_i^0)^4} + \frac{12}{(1-\theta)^4} \sum_{j=k+1}^n \frac{(\mathbf{z}_j^{0t} \lambda)^4}{(1 - (1-\theta)^{-1} \tilde{\lambda} \mathbf{z}_j^0)^4} \right). \tag{5.59}$$

By Proposition 1 of Ciuperca and Salloum (2015), we have that, for all $\epsilon > 0$, there exists two positive absolute constants M_1 and M_2 such that

$$\mathbb{P} \left[\frac{1}{M_1 \theta^4} \sum_{i=1}^k (\mathbf{z}_i^{0t} \lambda)^4 \leq \sum_{i=1}^k \frac{(\mathbf{z}_i^{0t} \lambda)^4}{(1 + \theta^{-1} \tilde{\lambda} \mathbf{z}_i^0)^4} \leq \frac{1}{M_2 \theta^4} \sum_{i=1}^k (\mathbf{z}_i^{0t} \lambda)^4 \right] \geq 1 - \epsilon.$$

Then, for the first term of the right-hand side of (5.59), applying Cauchy-Schwartz's inequality, we obtain that

$$\sum_{i=1}^k \frac{(\mathbf{z}_i^{0t} \lambda)^4}{(1 + \theta^{-1} \tilde{\lambda} \mathbf{z}_i^0)^4} \leq \frac{1}{M_2 \theta^4} \sum_{i=1}^k (\mathbf{z}_i^{0t} \lambda)^4 \leq \frac{1}{M_2 \theta^4} \sum_{i=1}^k \|\mathbf{X}_i\|^4 \varepsilon_i^4 \|\lambda\|^4.$$

Using assumptions (A1) (A2), together with the fact that $\|\lambda\| = O_{\mathbb{P}}(p^{1/2}n^{-1/2})$ given by Proposition 1 and $p = o(n^{1/2})$, we obtain that the first term of (5.59) is $o_{\mathbb{P}}(1)$. In the same way we can demonstrate that

each term of (5.59) is $o_{\mathcal{P}}(1)$, which implies that $\mathcal{E}_4 = o_{\mathcal{P}}(1)$.

Using notations given by (3.2) and (3.3), we obtain that $\mathcal{E}_1 = 2n\lambda^t \psi_n^0$ and $\mathcal{E}_2 = n\lambda^t \mathbf{S}_n^0 \lambda$. Then, relation (5.58) becomes $\mathbb{E}L_{nk}(\beta^0) = 2n\lambda^t \psi_n^0 - n\lambda^t \mathbf{S}_n^0 \lambda + \mathcal{E}_3 + o_{\mathcal{P}}(1)$ and the lemma follows. \blacksquare

The following Lemma gives, under hypothesis H_0 , an approximation for the Lagrange multiplier λ and the asymptotic behaviour of $\sup_{1 \leq i \leq n} |\lambda^t \mathbf{z}_i^0| = o_{\mathcal{P}}(1)$.

Lemma 6 *Under the null hypothesis H_0 , suppose that assumptions (A1), (A3)-(A6) are satisfied. Then for $q \geq 4$ and fixed $\theta \in (0, 1)$, we have $\sup_{1 \leq i \leq n} |\lambda^t \mathbf{z}_i^0| = o_{\mathcal{P}}(1)$ and*

$$\lambda = (\mathbf{S}_n^0)^{-1}(\psi_n^0 + o_{\mathcal{P}}(n^{(1-q)/q} p^{3/2})). \quad (5.60)$$

Proof. By Proposition 1, we have that $\|\lambda\| = O_{\mathcal{P}}(n^{-1/2} p^{1/2})$. Note that, by Cauchy-Schwartz's inequality and by Lemma 3, we have

$$\theta^{-1} \sup_{1 \leq i \leq k} |\lambda^t \mathbf{z}_i^0| \leq \|\lambda\| T_n^0 = O_{\mathcal{P}}(n^{-1/2} p^{1/2}) o_{\mathcal{P}}(n^{1/q} p^{1/2}) = o_{\mathcal{P}}(n^{(-q+2)/2q} p)$$

and

$$(1-\theta)^{-1} \sup_{k+1 \leq j \leq n} |\lambda^t \mathbf{z}_j^0| \leq \|\lambda\| T_n^0 = O_{\mathcal{P}}(n^{-1/2} p^{1/2}) o_{\mathcal{P}}(n^{1/q} p^{1/2}) = o_{\mathcal{P}}(n^{(-q+2)/2q} p),$$

with $\theta = k/n$. These two relations together assumption (A5) involve $\sup_{1 \leq i \leq n} |\lambda^t \mathbf{z}_i^0| = o_{\mathcal{P}}(1)$.

We prove now relation (5.60). The limited development of (2.5), in the neighbourhood of $\lambda = \mathbf{0}_p$, up to order 3 can be written

$$\begin{aligned} \mathbf{0}_p &= \left(\frac{1}{\theta} \sum_{i=1}^k \mathbf{z}_i^0 - \frac{1}{1-\theta} \sum_{j=k+1}^n \mathbf{z}_j^0 \right) - \left(\frac{1}{\theta^2} \sum_{i=1}^k \mathbf{z}_i^0 \mathbf{z}_i^{0t} + \frac{1}{(1-\theta)^2} \sum_{j=k+1}^n \mathbf{z}_j^0 \mathbf{z}_j^{0t} \right) \lambda \\ &+ \left(\frac{1}{\theta^3} \sum_{i=1}^k \mathbf{z}_i^0 (\mathbf{z}_i^{0t} \lambda)^2 - \frac{1}{(1-\theta)^3} \sum_{j=k+1}^n \mathbf{z}_j^0 (\mathbf{z}_j^{0t} \lambda)^2 \right) \\ &+ \left(\frac{1}{(\theta + \tilde{\lambda}^t \mathbf{z}_i^0)^4} \sum_{i=1}^k \mathbf{z}_i^0 (\mathbf{z}_i^{0t} \lambda)^3 - \frac{1}{(1-\theta - \tilde{\lambda}^t \mathbf{z}_j^0)^4} \sum_{j=k+1}^n \mathbf{z}_j^0 (\mathbf{z}_j^{0t} \lambda)^3 \right), \end{aligned} \quad (5.61)$$

where $\tilde{\lambda} = u\lambda$, with $u \in (0, 1)$.

Using Proposition 1 of Ciuperca and Salloum (2015), similarly as for the term \mathcal{E}_4 of Proposition 3, we can demonstrate easily that the last term of the right hand side of relation (5.61) is $o_{\mathcal{P}}(1)$. We recall that

$$\mathbf{R}_n^0 \equiv \frac{1}{n\theta^3} \sum_{i=1}^k \mathbf{z}_i^0 (\lambda^t \mathbf{z}_i^0)^2 - \frac{1}{n(1-\theta)^3} \sum_{j=k+1}^n \mathbf{z}_j^0 (\lambda^t \mathbf{z}_j^0)^2. \quad (5.62)$$

Then, using notations given by (3.2), (3.3) and (5.62) we obtain that relation (5.61) becomes $\psi_n^0 - \mathbf{S}_n^0 \lambda + \mathbf{R}_n^0 = \mathbf{0}_p$. Thus

$$\lambda = (\mathbf{S}_n^0)^{-1}(\mathbf{R}_n^0 + \psi_n^0)(1 + o_{\mathcal{P}}(1)). \quad (5.63)$$

We recall that $T_n^0 \equiv \max_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq n}} \left\{ \theta^{-1} \|\mathbf{z}_i^0\|, (1-\theta)^{-1} \|\mathbf{z}_j^0\| \right\}$. Then, we have for \mathbf{R}_n^0 :

$$\|\mathbf{R}_n^0\| \leq T_n^0 \lambda^t \left(\frac{1}{n\theta^2} \sum_{i=1}^k \mathbf{z}_i^0 \mathbf{z}_i^{0t} + \frac{1}{n(1-\theta)^2} \sum_{j=k+1}^n \mathbf{z}_j^0 \mathbf{z}_j^{0t} \right) \lambda \leq T_n^0 \lambda^t \mathbf{S}_n^0 \lambda.$$

Using Lemma 4 of Liu et al. (2013) and Lemma 2 we obtain that $\|\mathbf{R}_n^0\| \leq T_n^0 \|\lambda\|^2 \gamma_1(\mathbf{S}_n^0) = O_{\mathcal{P}}(T_n^0 \|\lambda\|^2)$. On the other hand, by Proposition 1 we have that $\|\lambda\| = O_{\mathcal{P}}(n^{-1/2} p^{1/2})$ and by Lemma 3 that $T_n^0 = o_{\mathcal{P}}(n^{1/q} p^{1/2})$. Then for $\|\mathbf{R}_n^0\|$, we obtain that

$$\|\mathbf{R}_n^0\| = o_{\mathcal{P}}(n^{(1-q)/q} p^{3/2}). \quad (5.64)$$

Relation (5.60) follows from (5.63) and (5.64). \blacksquare

The following result gives an asymptotic approximation for $[(\mathbf{S}_n^0)^{-1} - (\mathbf{V}_n^0)^{-1}]\boldsymbol{\psi}_n^0$ and $[(\mathbf{S}_n^0)^{-1} - (\mathbf{V}_n^0)^{-1}]\mathbf{R}_n^0$.

Lemma 7 *Under the null hypothesis (H_0), if assumptions (A3), (A4) and (A6) hold, we have*

- (i) $((\mathbf{S}_n^0)^{-1} - (\mathbf{V}_n^0)^{-1})\boldsymbol{\psi}_n^0 = ((\mathbf{V}_n^0)^{-1}\boldsymbol{\psi}_n^0)_{o_{\mathcal{P}}}(1)$.
- (ii) $((\mathbf{S}_n^0)^{-1} - (\mathbf{V}_n^0)^{-1})\mathbf{R}_n^0 = ((\mathbf{V}_n^0)^{-1}\mathbf{R}_n^0)_{o_{\mathcal{P}}}(1)$.

Proof. (i) By Lemma 1 we have that, under assumptions (A3) and (A4), for all $\epsilon > 0$ there exist $C_q > 0$, for $q \geq 4$, such that

$$\mathbb{P}\left[p \max_{1 \leq u, v \leq p} |S_{n,(u,v)}^0 - V_{n,(u,v)}^0| \geq \epsilon\right] \leq C_q \frac{p^{2+q}}{nq/2\epsilon^q}.$$

Furthermore, under assumption (A6) we have

$$\max_{1 \leq u, v \leq p} |S_{n,(u,v)}^0 - V_{n,(u,v)}^0| = o_{\mathcal{P}}(1). \quad (5.65)$$

We recall that, for a matrix \mathbf{A} , $\|\mathbf{A}\|_1$ is the subordinate norm to the vector norm $\|\cdot\|_1$. Using Lemma 1(iii) of Ciuperca and Salloum (2015), Lemma 2, relations (5.57), (5.65) and the identity $\mathbf{V}_n^0((\mathbf{S}_n^0)^{-1} - (\mathbf{V}_n^0)^{-1})\boldsymbol{\psi}_n^0 = (\mathbf{V}_n^0(\mathbf{S}_n^0)^{-1} - \mathbf{I}_p)\boldsymbol{\psi}_n^0$, we have

$$\begin{aligned} \|\mathbf{V}_n^0((\mathbf{S}_n^0)^{-1} - (\mathbf{V}_n^0)^{-1})\boldsymbol{\psi}_n^0\| &= \|(\mathbf{V}_n^0 - \mathbf{S}_n^0)(\mathbf{S}_n^0)^{-1}\boldsymbol{\psi}_n^0\| \leq \|\mathbf{V}_n^0 - \mathbf{S}_n^0\|_1 \|(\mathbf{S}_n^0)^{-1}\|_1 \|\boldsymbol{\psi}_n^0\| \\ &\leq \max_{1 \leq r \leq p} |\gamma_r(\mathbf{V}_n^0 - \mathbf{S}_n^0)| \cdot |\gamma_1(\mathbf{S}_n^0)^{-1}| \cdot \|\boldsymbol{\psi}_n^0\| \leq Cp \max_{1 \leq u, v \leq p} |S_{n,(u,v)}^0 - V_{n,(u,v)}^0| \cdot \|\boldsymbol{\psi}_n^0\| \\ &= \|\boldsymbol{\psi}_n^0\|_{o_{\mathcal{P}}}(1), \end{aligned}$$

which implies that $((\mathbf{S}_n^0)^{-1} - (\mathbf{V}_n^0)^{-1})\boldsymbol{\psi}_n^0 = (\mathbf{V}_n^0)^{-1}\boldsymbol{\psi}_n^0_{o_{\mathcal{P}}}(1)$.

(ii) The proof of (ii) is similar to (i). \blacksquare

The following lemma is needed for proving Proposition 4. We recall that $\mathbf{K}_n^0 = \mathbf{I}_p - (\mathbf{V}_n^0)^{-1/2}\mathbf{S}_n^0(\mathbf{V}_n^0)^{-1/2}$.

Lemma 8 *Under null hypothesis H_0 , if assumption (A7) holds, then*

$$\text{tr}(\mathbf{K}_n^0)^2 = O_{\mathcal{P}}(p^2 n^{-1}).$$

Proof. We show that $\mathbb{E}[\text{tr}(\mathbf{K}_n^0)^2] = O(p^2 n^{-1})$. For this, we write the matrix \mathbf{K}_n^0 as

$$\mathbf{K}_n^0 = \mathbf{I}_p - \frac{1}{n\theta^2} \sum_{i=1}^k \mathbf{w}_i^0 \mathbf{w}_i^{0t} \frac{1}{n(1-\theta)^2} \sum_{j=k+1}^n \mathbf{w}_j^0 \mathbf{w}_j^{0t}.$$

Then

$$\begin{aligned} \mathbb{E}[\text{tr}(\mathbf{K}_n^0)^2] &= n^{-2} \sum_{r,s=1}^p \mathbb{E}\left[(\theta^{-2} \sum_{i=1}^k w_{i,r}^0 w_{i,s}^0 + \frac{1}{(1-\theta)^2} \sum_{j=k+1}^n w_{j,r}^0 w_{j,s}^0)^2\right] \\ &\quad - 2n^{-1} \sum_{r=1}^p (\theta^{-2} \sum_{i=1}^k \mathbb{E}[w_{i,r}^0 w_{i,r}^0] + (1-\theta)^{-2} \sum_{j=k+1}^n \mathbb{E}[w_{j,r}^0 w_{j,r}^0]) + p. \end{aligned} \quad (5.66)$$

For the first term of the right-hand side of (5.66), using the independence of \mathbf{w}_i^0 for all $i = 1, \dots, n$, we have that

$$\begin{aligned} &\frac{1}{n^2} \sum_{r,s=1}^p \mathbb{E}\left[\left(\frac{1}{\theta^2} \sum_{i=1}^k w_{i,r}^0 w_{i,s}^0 + \frac{1}{(1-\theta)^2} \sum_{j=k+1}^n w_{j,r}^0 w_{j,s}^0\right)^2\right] \\ &= \frac{1}{n^2} \sum_{r,s=1}^p \left(\frac{1}{\theta^4} \sum_{i=1}^k \mathbb{E}[w_{i,r}^0 w_{i,r}^0 w_{i,s}^0 w_{i,s}^0] + \frac{1}{(1-\theta)^4} \sum_{j=k+1}^n \mathbb{E}[w_{j,r}^0 w_{j,r}^0 w_{j,s}^0 w_{j,s}^0]\right) \\ &\quad + \frac{1}{n^2} \sum_{r,s=1}^p \left(\frac{1}{\theta^2} \sum_{i=1}^k \mathbb{E}[w_{i,r}^0 w_{i,s}^0] + \frac{1}{(1-\theta)^2} \sum_{j=k+1}^n \mathbb{E}[w_{j,r}^0 w_{j,s}^0]\right)^2 \end{aligned}$$

$$= \frac{1}{n} \sum_{r,s=1}^p \alpha^{rrss} + \sum_{r,s=1}^p (\alpha^{rs})^2 = O(p^2 n^{-1}). \quad (5.67)$$

The last equation is due to assumption (A7) and to the fact that $\alpha^{rs} = 0$, for $r \neq s$. For the second term of the right-hand side of (5.66), by the fact that $\alpha^{rr} = 1$, we have that

$$\frac{1}{n} \sum_{r=1}^p \left(\frac{1}{\theta^2} \sum_{i=1}^k \mathbb{E}[w_{i,r}^0 w_{i,r}^0] + \frac{1}{(1-\theta)^2} \sum_{j=k+1}^n \mathbb{E}[w_{j,r}^0 w_{j,r}^0] \right) = \sum_{r=1}^p \alpha^{rr} = p. \quad (5.68)$$

Then, using relations (5.66), (5.67) and (5.68), we obtain that

$$\mathbb{E}[\text{tr}(\mathbf{K}_n^2)] = O(p^2 n^{-1}).$$

By Markov's inequality, Lemma yields. ■

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