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# Vanishing of the anchored isoperimetric profile in bond percolation at $p_c$

Raphaël Cerf<sup>‡</sup>, Barbara Dembin<sup>§</sup>

## Abstract

We consider the anchored isoperimetric profile of the infinite open cluster, defined for  $p > p_c$ , whose existence has been recently proved in [3]. We extend adequately the definition for  $p = p_c$ , in finite boxes. We prove a partial result which implies that, if the limit defining the anchored isoperimetric profile at  $p_c$  exists, it has to vanish.

## 1 Introduction

The most well-known open question in percolation theory is to prove that the percolation probability vanishes at  $p_c$  in dimension three. In fact, the interesting quantities associated to the model are very difficult to study at the critical point or in its vicinity. We study here a very modest intermediate question. We consider the anchored isoperimetric profile of the infinite open cluster, defined for  $p > p_c$ , whose existence has been recently proved in [3]. We extend adequately the definition for  $p = p_c$ , in finite boxes. We prove a partial result which implies that, if the limit defining the anchored isoperimetric profile at  $p_c$  exists, it has to vanish.

**The Cheeger constant.** For a graph  $\mathcal{G}$  with vertex set  $V$  and edge set  $E$ , we define the edge boundary  $\partial_{\mathcal{G}}A$  of a subset  $A$  of  $V$  as

$$\partial_{\mathcal{G}}A = \left\{ e = \langle x, y \rangle \in E : x \in A, y \notin A \right\}.$$

We denote by  $|B|$  the cardinal of the finite set  $B$ . The Cheeger constant of the graph  $\mathcal{G}$  is defined as

$$\varphi_{\mathcal{G}} = \min \left\{ \frac{|\partial_{\mathcal{G}}A|}{|A|} : A \subset V, 0 < |A| \leq \frac{|V|}{2} \right\}.$$

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This constant was introduced by Cheeger in his thesis [2] in order to obtain a lower bound for the smallest eigenvalue of the Laplacian.

**The anchored isoperimetric profile  $\varphi_n(p)$ .** Let  $d \geq 2$ . We consider an i.i.d. supercritical bond percolation on  $\mathbb{Z}^d$ , every edge is open with a probability  $p > p_c(d)$ , where  $p_c(d)$  denotes the critical parameter for this percolation. We know that there exists almost surely a unique infinite open cluster  $\mathcal{C}_\infty$  [5]. We say that  $H$  is a valid subgraph of  $\mathcal{C}_\infty$  if  $H$  is connected and  $0 \in H \subset \mathcal{C}_\infty$ . We define the anchored isoperimetric profile  $\varphi_n(p)$  of  $\mathcal{C}_\infty$  as follows. We condition on the event  $\{0 \in \mathcal{C}_\infty\}$  and we set

$$\varphi_n(p) = \min \left\{ \frac{|\partial_{\mathcal{C}_\infty} H|}{|H|} : H \text{ valid subgraph of } \mathcal{C}_\infty, 0 < |H| \leq n^d \right\}.$$

The following theorem from [3] asserts the existence of the limit of  $n\varphi_n(p)$  when  $p > p_c(d)$ .

**Theorem 1.1.** *Let  $d \geq 2$  and  $p > p_c(d)$ . There exists a positive real number  $\varphi(p)$  such that, conditionally on  $\{0 \in \mathcal{C}_\infty\}$ ,*

$$\lim_{n \rightarrow \infty} n\varphi_n(p) = \varphi(p) \text{ almost surely.}$$

We wish to study how this limit behaves when  $p$  is getting closer to  $p_c$ . To do so, we need to extend the definition of the anchored isoperimetric profile so that it is well defined at  $p_c(d)$ . We say that  $H$  is a valid subgraph of  $\mathcal{C}(0)$ , the open cluster of 0, if  $H$  is connected and  $0 \in H \subset \mathcal{C}(0)$ . We define  $\widehat{\varphi}_n(p)$  for every  $p \in [0, 1]$  as

$$\widehat{\varphi}_n(p) = \min \left\{ \frac{|\partial_{\mathcal{C}(0)} H|}{|H|} : H \text{ valid subgraph of } \mathcal{C}(0), 0 < |H| \leq n^d \right\}.$$

In particular, if 0 is not connected to  $\partial[-n/2, n/2]^d$  by a  $p$ -open path, then  $|\mathcal{C}(0)| < n^d$  and taking  $H = \mathcal{C}(0)$ , we see that  $\widehat{\varphi}_n(p)$  is equal to 0. Thanks to theorem 1.1, we have

$$\forall p > p_c \quad \lim_{n \rightarrow \infty} n\widehat{\varphi}_n(p) = \theta(p)\delta_{\varphi(p)} + (1 - \theta(p))\delta_0,$$

where  $\theta(p)$  is the probability that 0 belongs to an infinite open cluster. The techniques of [3] to prove the existence of this limit rely on coarse-graining estimates which can be employed only in the supercritical regime. Therefore we are not able so far to extend the above convergence at the critical point  $p_c$ . Naturally, we expect that  $n\widehat{\varphi}_n(p_c)$  converges towards 0 as  $n$  goes to infinity, unfortunately we are only able to prove a weaker statement.

**Theorem 1.2.** *With probability one, we have*

$$\liminf_{n \rightarrow \infty} n\widehat{\varphi}_n(p_c) = 0.$$

We shall prove this theorem by contradiction. We first define an exploration process of the cluster of 0 that remains inside the box  $[-n, n]^d$ . If the statement of the theorem does not hold, then the cluster of 0 satisfies a  $d$ -dimensional anchored isoperimetric inequality. It follows that the number of sites that are

revealed in the exploration of the cluster of 0 will grow fast enough of order  $n^{d-1}$ . Then, we can prove that the intersection of the cluster that we have explored with the boundary of the box  $[-n, n]^d$  is of order  $n^{d-1}$ . Using the fact that there is no percolation in a half-space, we obtain a contradiction. Before starting the precise proof, we recall some results from [3] on the meaning of the limiting value  $\varphi(p)$ .

**The Wulff theorem.** We denote by  $\mathcal{L}^d$  the  $d$ -dimensional Lebesgue measure and by  $\mathcal{H}^{d-1}$  denotes the  $(d-1)$ -Hausdorff measure in dimension  $d$ . Given a norm  $\tau$  on  $\mathbb{R}^d$  and a subset  $E$  of  $\mathbb{R}^d$  having a regular enough boundary, we define  $\mathcal{I}_\tau(E)$ , the surface tension of  $E$  for the norm  $\tau$ , as

$$\mathcal{I}_\tau(E) = \int_{\partial E} \tau(n_E(x)) \mathcal{H}^{d-1}(dx).$$

We consider the anisotropic isoperimetric problem associated with the norm  $\tau$ :

$$\text{minimize } \frac{\mathcal{I}_\tau(E)}{\mathcal{L}^d(E)} \text{ subject to } \mathcal{L}^d(E) \leq 1. \quad (1)$$

The famous Wulff construction provides a minimizer for this anisotropic isoperimetric problem. We define the set  $\widehat{W}_\tau$  as

$$\widehat{W}_\tau = \bigcap_{v \in \mathbb{S}^{d-1}} \{x \in \mathbb{R}^d : x \cdot v \leq \tau(v)\},$$

where  $\cdot$  denotes the standard scalar product and  $\mathbb{S}^{d-1}$  is the unit sphere of  $\mathbb{R}^d$ . Up to translation and Lebesgue negligible sets, the set

$$\frac{1}{\mathcal{L}^d(\widehat{W}_\tau)^{1/d}} \widehat{W}_\tau$$

is the unique solution to the problem (1).

**Representation of  $\varphi(p)$ .** In [3], we build an appropriate norm  $\beta_p$  for our problem that is directly related to the open edge boundary. We define the Wulff crystal  $W_p$  as the dilate of  $\widehat{W}_{\beta_p}$  such that  $\mathcal{L}^d(W_p) = 1/\theta(p)$ , where  $\theta(p) = \mathbb{P}(0 \in \mathcal{C}_\infty)$ . We denote by  $\mathcal{I}_p$  the surface tension associated with the norm  $\beta_p$ . In [3], we prove that

$$\forall p > p_c(d) \quad \varphi(p) = \mathcal{I}_p(W_p).$$

## 2 Proofs

We prove next the following lemma, which is based on two important results due to Zhang [9] and Rossignol and Th  ret [6]. To alleviate the notation, the critical point  $p_c(d)$  is denoted simply by  $p_c$ .

**Lemma 2.1.** *We have*

$$\lim_{\substack{p \rightarrow p_c \\ p > p_c}} \left( \theta(p) \delta_{\mathcal{I}_p(W_p)} + (1 - \theta(p)) \delta_0 \right) = \delta_0.$$

*Proof.* If  $\lim_{p \rightarrow p_c} \theta(p) = 0$ , then the result is clear. Otherwise, let us assume that

$$\lim_{\substack{p \rightarrow p_c \\ p > p_c}} \theta(p) = \delta > 0.$$

Let  $B$  be a subset of  $\mathbb{R}^d$  having a regular boundary and such that  $\mathcal{L}^d(B) = 1/\delta$ . As the map  $p \mapsto \theta(p)$  is non-decreasing and  $\mathcal{L}^d(W_p) = 1/\theta(p)$ , we have

$$\forall p > p_c \quad \mathcal{L}^d(W_p) \leq \mathcal{L}^d(B).$$

Moreover as  $W_p$  is the dilate of the minimizer associated to the isoperimetric problem (1), we have

$$\forall p > p_c \quad \mathcal{I}_p(W_p) \leq \mathcal{I}_p(B).$$

In [9], Zhang proved that  $\beta_{p_c} = 0$ . In [6], Rossignol and Th  ret proved the continuity of the flow constant. Combining these two results, we get that

$$\lim_{\substack{p \rightarrow p_c \\ p > p_c}} \beta_p = \beta_{p_c} = 0 \quad \text{and so} \quad \lim_{\substack{p \rightarrow p_c \\ p > p_c}} \mathcal{I}_p(B) = 0.$$

Finally, we obtain

$$\lim_{\substack{p \rightarrow p_c \\ p > p_c}} \mathcal{I}_p(W_p) = 0.$$

This yields the result.  $\square$

*Proof of theorem 1.2.* We assume by contradiction that

$$\mathbb{P} \left( \liminf_{n \rightarrow \infty} n \widehat{\varphi}_n(p_c) = 0 \right) < 1.$$

Therefore there exist positive constants  $c$  and  $\delta$  such that

$$\mathbb{P} \left( \liminf_{n \rightarrow \infty} n \widehat{\varphi}_n(p_c) > c \right) = \lim_{n \rightarrow \infty} \mathbb{P} \left( \inf_{k \geq n} k \widehat{\varphi}_k(p_c) > c \right) = \delta. \quad (2)$$

Therefore, there exists a positive integer  $n_0$  such that

$$\mathbb{P} \left( \inf_{k \geq n_0} k \widehat{\varphi}_k(p_c) > c \right) \geq \frac{\delta}{2}. \quad (3)$$

In what follows, we condition on the event

$$\left\{ \inf_{k \geq n_0} k \widehat{\varphi}_k(p_c) > c \right\}.$$

Note that on this event, 0 is connected to infinity by a  $p_c$ -open path. For  $H$  a subgraph of  $\mathbb{Z}^d$ , we define

$$\partial^\circ H = \left\{ e \in \partial H, e \text{ is open} \right\}.$$

Note that if  $H \subset \mathcal{C}_\infty$ , then  $\partial_{\mathcal{C}_\infty} H = \partial^\circ H$ . Moreover, if  $H$  is equal to  $\mathcal{C}(0)$ , the open cluster of 0, then  $\partial_{\mathcal{C}(0)} H = \partial^\circ H = \emptyset$ . We define next an exploration

process of the cluster of 0. We set  $\mathcal{C}_0 = \{0\}$ ,  $\mathcal{A}_0 = \emptyset$ . Let us assume that  $\mathcal{C}_0, \dots, \mathcal{C}_l$  and  $\mathcal{A}_0, \dots, \mathcal{A}_l$  are already constructed. We define

$$\mathcal{A}_{l+1} = \{x \in \mathbb{Z}^d : \exists y \in \mathcal{C}_l \quad \langle x, y \rangle \in \partial^\circ \mathcal{C}_l\}$$

and

$$\mathcal{C}_{l+1} = \mathcal{C}_l \cup \mathcal{A}_{l+1}.$$

We have

$$\partial^\circ \mathcal{C}_l \subset \{\langle x, y \rangle \in \mathbb{E}^d : x \in \mathcal{A}_{l+1}\}$$

so that  $|\partial^\circ \mathcal{C}_l| \leq 2d|\mathcal{A}_{l+1}|$ . Since  $\mathcal{A}_{l+1}$  and  $\mathcal{C}_l$  are disjoint, we have

$$|\mathcal{C}_{l+1}| = |\mathcal{C}_l| + |\mathcal{A}_{l+1}| \geq |\mathcal{C}_l| + \frac{|\partial^\circ \mathcal{C}_l|}{2d}. \quad (4)$$

Let us set  $\alpha = 1/n_0^d$  so that  $|\mathcal{C}_0| = \alpha n_0^d$ . Let  $k$  be the smallest integer greater than  $2^{d+1}d/c$ . We recall that  $c$  and  $n_0$  were defined in (2) and (3). Let us prove by induction on  $n$  that

$$\forall n \geq n_0 \quad |\mathcal{C}_{(n-n_0)k}| \geq \alpha n^d. \quad (5)$$

This is true for  $n = n_0$ . Let us assume that this inequality is true for some integer  $n \geq n_0$ . If  $|\mathcal{C}_{(n+1-n_0)k}| \geq n^d$ , then we are done. Suppose that  $|\mathcal{C}_{(n+1-n_0)k}| < n^d$ . In this case, for any integer  $l \leq k$ , we have also  $|\mathcal{C}_{(n-n_0)k+l}| < n^d$ , and since  $\mathcal{C}_{(n-n_0)k+l}$  is a valid subgraph of  $\mathcal{C}(0)$  and  $\widehat{\varphi}_n(p_c) > c/n$ , we conclude that

$$\frac{|\partial^\circ \mathcal{C}_{(n-n_0)k+l}|}{|\mathcal{C}_{(n-n_0)k+l}|} \geq \frac{c}{n}$$

and so  $|\partial^\circ \mathcal{C}_{(n-n_0)k+l}| \geq \alpha c n^{d-1}$ . Thanks to inequality (4) applied  $k$  times, we have

$$|\mathcal{C}_{(n+1-n_0)k}| \geq \alpha \left( n^d + \frac{ck}{2d} n^{d-1} \right).$$

As  $k \geq 2^{d+1}d/c$ , we get

$$|\mathcal{C}_{(n+1-n_0)k}| \geq \alpha(n^d + 2^d n^{d-1}) \geq \alpha(n+1)^d.$$

This concludes the induction.

Let  $\eta > 0$  be a constant that we will choose later. In [1], Barsky, Grimmett and Newman proved that there is no percolation in a half-space at criticality. An important consequence of the result of Grimmett and Marstrand [4] is that the critical value for bond percolation in a half-space equals to the critical parameter  $p_c(d)$  of bond percolation in the whole space, *i.e.*, we have

$$\mathbb{P}(0 \text{ is connected to infinity by a } p_c\text{-open path in } \mathbb{N} \times \mathbb{Z}^{d-1}) = 0,$$

so that for  $n$  large enough,

$$\mathbb{P}(\exists \gamma \text{ a } p_c\text{-open path starting from } 0 \text{ in } \mathbb{N} \times \mathbb{Z}^{d-1} \text{ such that } |\gamma| \geq n) \leq \eta.$$

In what follows, we will consider an integer  $n$  such that the above inequality holds. By construction the set  $\mathcal{C}_n$  is inside the box  $[-n, n]^d$ . Starting from this

cluster, we are going to resume our exploration but with the constraint that we do not explore anything outside the box  $[-n, n]^d$ . We set  $\mathcal{C}'_0 = \mathcal{C}_n$  and  $\mathcal{A}'_0 = \emptyset$ . Let us assume  $\mathcal{C}'_0, \dots, \mathcal{C}'_l$  and  $\mathcal{A}'_0, \dots, \mathcal{A}'_l$  are already constructed. We define

$$\mathcal{A}'_{l+1} = \{x \in [-n, n]^d : \exists y \in \mathcal{C}'_l \quad \langle x, y \rangle \in \partial^\circ \mathcal{C}'_l\}$$

and

$$\mathcal{C}'_{l+1} = \mathcal{C}'_l \cup \mathcal{A}'_{l+1}.$$

We stop the process when  $\mathcal{A}'_{l+1} = \emptyset$ . As the number of vertices in the box  $[-n, n]^d$  is finite, this process of exploration will eventually stop for some integer  $l$ . We have that  $|\mathcal{C}'_l| \leq n^d$  and  $n\hat{\varphi}_k(p_c) > c$  so that

$$|\partial^\circ \mathcal{C}'_l| \geq \frac{c}{n} |\mathcal{C}'_l| \geq \frac{c}{n} |\mathcal{C}_n|.$$

Moreover, for  $n \geq kn_0$ , we have, thanks to inequality (5),

$$|\mathcal{C}_n| \geq |\mathcal{C}_{\lfloor \frac{n}{k} \rfloor k}| \geq |\mathcal{C}_{(\lfloor \frac{n}{k} \rfloor - n_0)k}| \geq \alpha \left( \left\lfloor \frac{n}{k} \right\rfloor \right)^d.$$

We suppose that  $n$  is large enough so that  $n \geq kn_0$  and  $\lfloor \frac{n}{k} \rfloor \geq n/2k$ . Combining the two previous display inequalities, we conclude that

$$|\partial^\circ \mathcal{C}'_l| \geq \frac{c\alpha}{2^d k^d} n^{d-1}.$$

Therefore, for  $n$  large enough, there exists one face of  $[-n, n]^d$  such that there are at least  $c\alpha n^{d-1}/(2^d k^d 2d)$  vertices that are connected to 0 by a  $p_c$ -open path that remains inside the box  $[-n, n]^d$  and so

$$\mathbb{P} \left( \begin{array}{l} \text{there exists one face of } [-n, n]^d \text{ with at least} \\ c\alpha n^{d-1}/(2^d k^d 2d) \text{ vertices that are connected to 0 by a} \\ p_c\text{-open path that remains inside the box } [-n, n]^d \end{array} \right) \geq \frac{\delta}{2}. \quad (6)$$

Let us denote by  $X_n$  the number of vertices in the face  $\{-n\} \times [-n, n]^{d-1}$  that are connected to 0 by a  $p_c$ -open path inside the box  $[-n, n]^d$ . We have

$$\begin{aligned} \mathbb{E}(X_n) &\leq |(\{-n\} \times [-n, n]^{d-1}) \cap \mathbb{Z}^d| \mathbb{P} \left( \begin{array}{l} \exists \gamma \text{ a } p_c\text{-open path starting} \\ \text{from 0 in } \mathbb{N} \times \mathbb{Z}^{d-1} \text{ such that} \\ |\gamma| \geq n \end{array} \right) \\ &\leq (2n+1)^{d-1} \eta. \end{aligned} \quad (7)$$

Moreover, we have

$$\mathbb{E}(X_n) \geq \frac{c\alpha}{2d2^d k^d} n^{d-1} \mathbb{P} \left( X_n > \frac{c\alpha}{2d2^d k^d} n^{d-1} \right). \quad (8)$$

Finally, combining inequalities (7) and (8), we get

$$\mathbb{P} \left( X_n > \frac{c\alpha}{2d2^d k^d} n^{d-1} \right) \leq \frac{2d\eta 3^{d-1} 2^d k^d}{c\alpha}.$$

Therefore, we can choose  $\eta$  small enough such that

$$\mathbb{P} \left( X_n > \frac{c\alpha}{2d2^d k^d} n^{d-1} \right) \leq \frac{\delta}{10d}$$

and so using the symmetry of the lattice

$$\mathbb{P} \left( \begin{array}{c} \text{there exists one face of } [-n, n]^d \text{ such there are at least} \\ c\alpha n^{d-1}/(2^d k^d 2d) \text{ vertices that are connected to } 0 \text{ by a } p_c\text{-open} \\ \text{path that remains inside the box } [-n, n]^d \end{array} \right) \\ \leq 2d \mathbb{P} \left( X_n > \frac{c\alpha}{2^d 2^d k^d} n^{d-1} \right) \leq \frac{\delta}{5}.$$

This contradicts inequality (6) and yields the result.  $\square$

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