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Cancellation of the Anchored isoperimetric profile in bond percolation at p_c

Raphaël Cerf[‡], Barbara Dembin[§]

Abstract

We consider the anchored isoperimetric profile of the infinite open cluster, defined for $p > p_c$, whose existence has been recently proved in [3]. We extend adequately the definition for $p = p_c$, in finite boxes. We prove a partial result which implies that, if the limit defining the anchored isoperimetric profile at p_c exists, it has to vanish.

The most well-known open question in percolation theory is to prove that the percolation probability vanishes at p_c in dimension three. In fact, the interesting quantities associated to the model are very difficult to study at the critical point or in its vicinity. We study here a very modest intermediate question. We consider the anchored isoperimetric profile of the infinite open cluster, defined for $p > p_c$, whose existence has been recently proved in [3]. We extend adequately the definition for $p = p_c$, in finite boxes. We prove a partial result which implies that, if the limit defining the anchored isoperimetric profile at p_c exists, it has to vanish.

The Cheeger constant. For a graph \mathcal{G} with vertex set V and edge set E , we define the edge boundary $\partial_{\mathcal{G}}A$ of a subset A of V as

$$\partial_{\mathcal{G}}A = \left\{ e = \langle x, y \rangle \in E : x \in A, y \notin A \right\}.$$

We denote by $|B|$ the cardinal of the finite set B . The Cheeger constant of the graph \mathcal{G} is defined as

$$\varphi_{\mathcal{G}} = \min \left\{ \frac{|\partial_{\mathcal{G}}A|}{|A|} : A \subset V, 0 < |A| \leq \frac{|V|}{2} \right\}.$$

This constant was introduced by Cheeger in his thesis [2] in order to obtain a lower bound for the smallest eigenvalue of the Laplacian.

The anchored isoperimetric profile $\varphi_n(p)$. Let $d \geq 2$. We consider an i.i.d. supercritical bond percolation on \mathbb{Z}^d , every edge is open with a probability

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$p > p_c(d)$, where $p_c(d)$ denotes the critical parameter for this percolation. We know that there exists almost surely a unique infinite open cluster \mathcal{C}_∞ [5]. We say that H is a valid subgraph of \mathcal{C}_∞ if H is connected and $0 \in H \subset \mathcal{C}_\infty$. We define the anchored isoperimetric profile $\varphi_n(p)$ of \mathcal{C}_∞ as follows. We condition on the event $\{0 \in \mathcal{C}_\infty\}$ and we set

$$\varphi_n(p) = \min \left\{ \frac{|\partial_{\mathcal{C}_\infty} H|}{|H|} : H \text{ valid subgraph of } \mathcal{C}_\infty, 0 < |H| \leq n^d \right\}.$$

The following theorem from [3] asserts the existence of the limit of $n\varphi_n(p)$ when $p > p_c(d)$.

Theorem 1. *Let $d \geq 2$ and $p > p_c(d)$. There exists a positive real number $\varphi(p)$ such that, conditionally on $\{0 \in \mathcal{C}_\infty\}$,*

$$\lim_{n \rightarrow \infty} n\varphi_n(p) = \varphi(p) \text{ almost surely.}$$

We wish to study how this limit behaves when p is getting closer to p_c . To do so, we need to extend the definition of the anchored isoperimetric profile so that it is well defined at $p_c(d)$. We say that H is a valid subgraph of $\mathcal{C}(0)$, the open cluster of 0, if H is connected and $0 \in H \subset \mathcal{C}(0)$. We define $\widehat{\varphi}_n(p)$ for every $p \in [0, 1]$ as

$$\widehat{\varphi}_n(p) = \min \left\{ \frac{|\partial_{\mathcal{C}(0)} H|}{|H|} : H \text{ valid subgraph of } \mathcal{C}(0), 0 < |H| \leq n^d \right\}.$$

In particular, if 0 is not connected to $\partial B(n^d)$ by a p -open path, then taking $H = \mathcal{C}(0)$, we see that $\widehat{\varphi}_n(p)$ is equal to 0. Thanks to Theorem 1, we have

$$\forall p > p_c \quad \lim_{n \rightarrow \infty} n\widehat{\varphi}_n(p) = \theta(p)\delta_{\varphi(p)} + (1 - \theta(p))\delta_0,$$

where $\theta(p)$ is the probability that 0 belongs to an infinite open cluster. The techniques of [3] to prove the existence of this limit rely on coarse-graining estimates which can be employed only in the supercritical regime. Therefore we are not able so far to extend the above convergence at the critical point p_c . Naturally, we expect that $n\widehat{\varphi}_n(p_c)$ converges towards 0 as n goes to infinity, unfortunately we are only able to prove a weaker statement.

Theorem 2. *With probability one, we have*

$$\liminf_{n \rightarrow \infty} n\widehat{\varphi}_n(p_c) = 0.$$

We shall prove this theorem by contradiction. We first define an exploration process of the cluster of 0 that remains inside the box $[-n, n]^d$. If the statement of the theorem does not hold, then it turns out that the intersection of the cluster that we have explored with the boundary of the box $[-n, n]^d$ is of order n^{d-1} . Using the fact that there is no percolation in a half-space, we obtain a contradiction. Before starting the precise proof, we recall some results from [3] on the meaning of the limiting value $\varphi(p)$.

The Wulff theorem. We denote by \mathcal{L}^d the d -dimensional Lebesgue measure and by \mathcal{H}^{d-1} denotes the $(d-1)$ -Hausdorff measure in dimension d . Given a

norm τ on \mathbb{R}^d and a subset E of \mathbb{R}^d having a regular boundary, we define $\mathcal{I}_\tau(E)$, the surface tension of E for the norm τ , as

$$\mathcal{I}_\tau(E) = \int_{\partial E} \tau(n_E(x)) \mathcal{H}^{d-1}(dx).$$

We consider the anisotropic isoperimetric problem associated with the norm τ :

$$\text{minimize } \frac{\mathcal{I}_\tau(E)}{\mathcal{L}^d(E)} \text{ subject to } \mathcal{L}^d(E) \leq 1. \quad (1)$$

The famous Wulff construction provides a minimizer for this anisotropic isoperimetric problem. We define the set \widehat{W}_τ as

$$\widehat{W}_\tau = \bigcap_{v \in \mathbb{S}^{d-1}} \{x \in \mathbb{R}^d : x \cdot v \leq \tau(v)\},$$

where \cdot denotes the standard scalar product and \mathbb{S}^{d-1} is the unit sphere of \mathbb{R}^d . Up to translation and Lebesgue negligible sets, the set

$$\frac{1}{\mathcal{L}^d(\widehat{W}_\tau)^{1/d}} \widehat{W}_\tau$$

is the unique solution to the problem (1).

Representation of $\varphi(p)$. In [3], we build an appropriate norm β_p for our problem that is directly related to the open edge boundary ratio. We define the Wulff crystal W_p as the dilate of \widehat{W}_{β_p} such that $\mathcal{L}^d(W_p) = 1/\theta(p)$, where $\theta(p) = \mathbb{P}(0 \in \mathcal{C}_\infty)$. We denote by \mathcal{I}_p the surface tension associated with the norm β_p . In [3], we prove that

$$\forall p > p_c(d) \quad \varphi(p) = \mathcal{I}_p(W_p).$$

We prove next the following lemma, which is based on two important results due to Zhang [9] and Rossignol and Th  ret [6]. To alleviate the notation, the critical point $p_c(d)$ is denoted simply by p_c .

Lemma 3. *We have*

$$\lim_{\substack{p \rightarrow p_c \\ p > p_c}} \left(\theta(p) \delta_{\mathcal{I}_p(W_p)} + (1 - \theta(p)) \delta_0 \right) = \delta_0.$$

Proof. If $\lim_{p \rightarrow p_c} \theta(p) = 0$, then the result is clear. Otherwise, let us assume that

$$\lim_{\substack{p \rightarrow p_c \\ p > p_c}} \theta(p) = \delta > 0.$$

Let B be a subset of \mathbb{R}^d having a regular boundary and such that $\mathcal{L}^d(B) = 1/\delta$. As the map $p \mapsto \theta(p)$ is non-decreasing and $\mathcal{L}^d(W_p) = 1/\theta(p)$, we have

$$\forall p > p_c \quad \mathcal{L}^d(W_p) \leq \mathcal{L}^d(B).$$

Moreover as W_p is the dilate of the minimizer associated to the isoperimetric problem (1), we have

$$\forall p > p_c \quad \mathcal{I}_p(W_p) \leq \mathcal{I}_p(B).$$

In [9], Zhang proved that $\beta_{p_c} = 0$. In [6], Rossignol and Th eret proved the continuity of the flow constant. Combining these two results, we get that

$$\lim_{\substack{p \rightarrow p_c \\ p > p_c}} \beta_p = \beta_{p_c} = 0 \quad \text{and so} \quad \lim_{\substack{p \rightarrow p_c \\ p > p_c}} \mathcal{I}_p(B) = 0.$$

Finally, we obtain

$$\lim_{\substack{p \rightarrow p_c \\ p > p_c}} \mathcal{I}_p(W_p) = 0.$$

This yields the result. \square

Proof of Theorem 2. We assume by contradiction that

$$\mathbb{P} \left(\liminf_{n \rightarrow \infty} n \widehat{\varphi}_n(p_c) = 0 \right) < 1.$$

Therefore there exist positive constants c and δ such that

$$\mathbb{P} \left(\liminf_{n \rightarrow \infty} n \widehat{\varphi}_n(p_c) > c \right) = \lim_{n \rightarrow \infty} \mathbb{P} \left(\inf_{k \geq n} k \widehat{\varphi}_k(p_c) > c \right) = \delta.$$

Therefore, there exists a positive integer n_0 such that

$$\mathbb{P} \left(\inf_{k \geq n_0} k \widehat{\varphi}_k(p_c) > c \right) \geq \frac{\delta}{2}.$$

In what follows, we condition on the event

$$\left\{ \inf_{k \geq n_0} k \widehat{\varphi}_k(p_c) > c \right\}.$$

Note that on this event, 0 is connected to infinity by a p_c -open path. For H a subgraph of \mathbb{Z}^d , we define

$$\partial^\circ H = \left\{ e \in \partial H, e \text{ is open} \right\}.$$

Note that if $H \subset \mathcal{C}_\infty$, then $\partial_{\mathcal{C}_\infty} H = \partial^\circ H$. Moreover, if H is equal to $\mathcal{C}(0)$, the open cluster of 0, then $\partial_{\mathcal{C}(0)} H = \partial^\circ H = \emptyset$. We define next an exploration process of the cluster of 0. We set $\mathcal{C}_0 = \{0\}$, $\mathcal{A}_0 = \emptyset$. Let us assume that $\mathcal{C}_0, \dots, \mathcal{C}_l$ and $\mathcal{A}_0, \dots, \mathcal{A}_l$ are already constructed. We define

$$\mathcal{A}_{l+1} = \{x \in \mathbb{Z}^d : \exists y \in \mathcal{C}_l \quad \langle x, y \rangle \in \partial^\circ \mathcal{C}_l\}$$

and

$$\mathcal{C}_{l+1} = \mathcal{C}_l \cup \mathcal{A}_{l+1}.$$

We have

$$\partial^\circ \mathcal{C}_l \subset \{\langle x, y \rangle \in \mathbb{E}^d : x \in \mathcal{A}_{l+1}\}$$

so that $|\partial^\circ \mathcal{C}_l| \leq 2d|\mathcal{A}_{l+1}|$. We claim that \mathcal{A}_{l+1} and \mathcal{C}_l are disjoint. Let us assume that there exists $x \in \mathcal{A}_{l+1} \cap \mathcal{C}_l$. In this case, there exists $y \in \mathcal{C}_l$ such that $\langle x, y \rangle \in \partial^\circ \mathcal{C}_l$ but this is impossible as $x, y \in \mathcal{C}_l$. Thus, we have $\mathcal{A}_{l+1} \cap \mathcal{C}_l = \emptyset$ and

$$|\mathcal{C}_{l+1}| = |\mathcal{C}_l| + |\mathcal{A}_{l+1}| \geq |\mathcal{C}_l| + \frac{|\partial^\circ \mathcal{C}_l|}{2d}. \quad (2)$$

Let us set $\alpha = 1/n_0^d$ so that $|\mathcal{C}_0| = \alpha n_0^d$. Let k be the smallest integer greater than $2^{d+1}d/c$. Let us prove by induction on n that

$$\forall n \geq n_0 \quad |\mathcal{C}_{(n-n_0)k}| \geq \alpha n^d. \quad (3)$$

This is true for $n = n_0$. Let us assume that this inequality is true for some integer $n \geq n_0$. If $|\mathcal{C}_{(n+1-n_0)k}| \geq n^d$, then we are done. Suppose that $|\mathcal{C}_{(n+1-n_0)k}| < n^d$. In this case, for any integer $l \leq k$, we have also $|\mathcal{C}_{(n-n_0)k+l}| < n^d$, and since $\mathcal{C}_{(n-n_0)k+l}$ is a valid subgraph of $\mathcal{C}(0)$ and $\widehat{\varphi}_n(p_c) > c/n$, we conclude that

$$\frac{|\partial^\circ \mathcal{C}_{(n-n_0)k+l}|}{|\mathcal{C}_{(n-n_0)k+l}|} \geq \frac{c}{n}$$

and so $|\partial^\circ \mathcal{C}_{(n-n_0)k+l}| \geq \alpha c n^{d-1}$. Thanks to inequality (2) applied k times, we have

$$|\mathcal{C}_{(n+1-n_0)k}| \geq \alpha \left(n^d + \frac{ck}{2d} n^{d-1} \right).$$

As $k \geq 2^{d+1}d/c$, we get

$$|\mathcal{C}_{(n+1-n_0)k}| \geq \alpha(n^d + 2^d n^{d-1}) \geq \alpha(n+1)^d.$$

This concludes the induction.

Let $\eta > 0$ be a constant that we will choose later. In [1], Barsky, Grimmett and Newman proved that there is no percolation in a half-space at criticality. An important consequence of the result of Grimmett and Marstrand [4] is that the critical value for bond percolation in a half-space corresponds to the critical parameter $p_c(d)$ of bond percolation in the whole space, *i.e.*, we have

$$\mathbb{P}(0 \text{ is connected to infinity by a } p_c\text{-open path in } \mathbb{N} \times \mathbb{Z}^{d-1}) = 0,$$

so that for n large enough,

$$\mathbb{P}(\exists \gamma \text{ a } p_c\text{-open path starting from } 0 \text{ in } \mathbb{N} \times \mathbb{Z}^{d-1} \text{ such that } |\gamma| \geq n) \leq \eta.$$

In what follows, we will consider an integer n such that the above inequality holds. By construction the set \mathcal{C}_n is inside the box $[-n, n]^d$. Starting from this cluster, we are going to resume our exploration but with the constraint that we do not explore anything outside the box $[-n, n]^d$. We set $\mathcal{C}'_0 = \mathcal{C}_n$ and $\mathcal{A}'_0 = \emptyset$. Let us assume $\mathcal{C}'_0, \dots, \mathcal{C}'_l$ and $\mathcal{A}'_0, \dots, \mathcal{A}'_l$ are already constructed. We define

$$\mathcal{A}'_{l+1} = \{x \in [-n, n]^d : \exists y \in \mathcal{C}'_l \quad \langle x, y \rangle \in \partial^\circ \mathcal{C}'_l\}$$

and

$$\mathcal{C}'_{l+1} = \mathcal{C}'_l \cup \mathcal{A}'_{l+1}.$$

We stop the process when $\mathcal{A}'_{l+1} = \emptyset$. As the number of vertices in the box $[-n, n]^d$ is finite, this process of exploration will eventually stop for some integer l . We have that $|\mathcal{C}'_l| \leq n^d$ and $n\widehat{\varphi}_k(p_c) > c$ so that

$$|\partial^\circ \mathcal{C}'_l| \geq \frac{c}{n} |\mathcal{C}'_l| \geq \frac{c}{n} |\mathcal{C}_n|.$$

Moreover, for $n \geq kn_0$, we have, thanks to inequality (3),

$$|\mathcal{C}_n| \geq |\mathcal{C}_{\lfloor \frac{n}{k} \rfloor k}| \geq |\mathcal{C}_{(\lfloor \frac{n}{k} \rfloor - n_0)k}| \geq \alpha \left(\left\lfloor \frac{n}{k} \right\rfloor \right)^d.$$

We suppose that n is large enough so that $n \geq kn_0$ and $\lfloor \frac{n}{k} \rfloor \geq n/2k$. Combining the two previous display inequalities, we conclude that

$$|\partial^o \mathcal{C}'_i| \geq \frac{c\alpha}{2^d k^d} n^{d-1}.$$

Therefore, for n large enough, there exists one face of $[-n, n]^d$ such that there are at least $c\alpha n^{d-1}/(2^d k^d 2d)$ vertices that are connected to 0 by a p_c -open path that remains inside the box $[-n, n]^d$ and so

$$\mathbb{P} \left(\begin{array}{l} \text{there exists one face of } [-n, n]^d \text{ with at least} \\ c\alpha n^{d-1}/(2^d k^d 2d) \text{ vertices that are connected to 0 by a} \\ p_c\text{-open path that remains inside the box } [-n, n]^d \end{array} \right) \geq \frac{\delta}{2}. \quad (4)$$

Let us denote by X_n the number of vertices in the face $\{-n\} \times [-n, n]^{d-1}$ that are connected to 0 by a p_c -open path inside the box $[-n, n]^d$. We have

$$\begin{aligned} \mathbb{E}(X_n) &\leq |(\{-n\} \times [-n, n]^{d-1}) \cap \mathbb{Z}^d| \mathbb{P} \left(\begin{array}{l} \exists \gamma \text{ a } p_c\text{-open path starting} \\ \text{from 0 in } \mathbb{N} \times \mathbb{Z}^{d-1} \text{ such that} \\ |\gamma| \geq n \end{array} \right) \\ &\leq (2n+1)^{d-1} \eta. \end{aligned} \quad (5)$$

Moreover, we have

$$\mathbb{E}(X_n) \geq \frac{c\alpha}{2d2^d k^d} n^{d-1} \mathbb{P} \left(X_n > \frac{c\alpha}{2d2^d k^d} n^{d-1} \right). \quad (6)$$

Finally, combining inequalities (5) and (6), we get

$$\mathbb{P} \left(X_n > \frac{c\alpha}{2d2^d k^d} n^{d-1} \right) \leq \frac{2d\eta 3^{d-1} 2^d k^d}{c\alpha}.$$

Therefore, we can choose η small enough such that

$$\mathbb{P} \left(X_n > \frac{c\alpha}{2d2^d k^d} n^{d-1} \right) \leq \frac{\delta}{10d}$$

and so using the symmetry of the lattice

$$\begin{aligned} &\mathbb{P} \left(\begin{array}{l} \text{there exists one face of } [-n, n]^d \text{ such there are at least} \\ c\alpha n^{d-1}/(2^d k^d 2d) \text{ vertices that are connected to 0 by a } p_c\text{-open} \\ \text{path that remains inside the box } [-n, n]^d \end{array} \right) \\ &\leq 2d \mathbb{P} \left(X_n > \frac{c\alpha}{2d2^d k^d} n^{d-1} \right) \leq \frac{\delta}{5}. \end{aligned}$$

This contradicts inequality (4) and yields the result. \square

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