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Vanishing of the anchored isoperimetric profile in bond percolation at $p_c$

Raphaël Cerf, Barbara Dembin

Abstract

We consider the anchored isoperimetric profile of the infinite open cluster, defined for $p > p_c$, whose existence has been recently proved in [3]. We extend adequately the definition for $p = p_c$, in finite boxes. We prove a partial result which implies that, if the limit defining the anchored isoperimetric profile at $p_c$ exists, it has to vanish.

1 Introduction

The most well-known open question in percolation theory is to prove that the percolation probability vanishes at $p_c$ in dimension three. In fact, the interesting quantities associated to the model are very difficult to study at the critical point or in its vicinity. We study here a very modest intermediate question. We consider the anchored isoperimetric profile of the infinite open cluster, defined for $p > p_c$, whose existence has been recently proved in [3]. We extend adequately the definition for $p = p_c$, in finite boxes. We prove a partial result which implies that, if the limit defining the anchored isoperimetric profile at $p_c$ exists, it has to vanish.

The Cheeger constant. For a graph $G$ with vertex set $V$ and edge set $E$, we define the edge boundary $\partial_G A$ of a subset $A$ of $V$ as

$$\partial_G A = \left\{ e = (x, y) \in E : x \in A, y \notin A \right\}.$$

We denote by $|B|$ the cardinal of the finite set $B$. The Cheeger constant of the graph $G$ is defined as

$$\varphi_G = \min \left\{ \frac{|\partial_G A|}{|A|} : A \subset V, 0 < |A| \leq \frac{|V|}{2} \right\}.$$

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This constant was introduced by Cheeger in his thesis [2] in order to obtain a lower bound for the smallest eigenvalue of the Laplacian.

The anchored isoperimetric profile $\varphi_n(p)$. Let $d \geq 2$. We consider an i.i.d. supercritical bond percolation on $\mathbb{Z}^d$, every edge is open with a probability $p > p_c(d)$, where $p_c(d)$ denotes the critical parameter for this percolation. We know that there exists almost surely a unique infinite open cluster $C_\infty$ [5]. We say that $H$ is a valid subgraph of $C_\infty$ if $H$ is connected and $0 \in H \subset C_\infty$. We define the anchored isoperimetric profile $\varphi_n(p)$ of $C_\infty$ as follows. We condition on the event \{0 $\in C_\infty$\} and we set

$$\varphi_n(p) = \min \left\{ \frac{|\partial C_\infty H|}{|H|} : H \text{ valid subgraph of } C_\infty, 0 < |H| \leq n^d \right\}.$$ 

The following theorem from [3] asserts the existence of the limit of $n\varphi_n(p)$ when $p > p_c(d)$.

**Theorem 1.1.** Let $d \geq 2$ and $p > p_c(d)$. There exists a positive real number $\varphi(p)$ such that, conditionally on \{0 $\in C_\infty$\},

$$\lim_{n \to \infty} n\varphi_n(p) = \varphi(p) \text{ almost surely.}$$

We wish to study how this limit behaves when $p$ is getting closer to $p_c$. To do so, we need to extend the definition of the anchored isoperimetric profile so that it is well defined at $p_c(d)$. We say that $H$ is a valid subgraph of $C(0)$, the open cluster of 0, if $H$ is connected and $0 \in H \subset C(0)$. We define $\widehat{\varphi}_n(p)$ for every $p \in [0, 1]$ as

$$\widehat{\varphi}_n(p) = \min \left\{ \frac{|\partial C(0) H|}{|H|} : H \text{ valid subgraph of } C(0), 0 < |H| \leq n^d \right\}.$$ 

In particular, if 0 is not connected to $\partial[-n/2, n/2]^d$ by a $p$-open path, then $|C(0)| < n^d$ and taking $H = C(0)$, we see that $\widehat{\varphi}_n(p)$ is equal to 0. Thanks to theorem 1.1, we have

$$\forall p > p_c \lim_{n \to \infty} n\widehat{\varphi}_n(p) = \theta(p)\delta_\varphi(p) + (1 - \theta(p))\delta_0,$$

where $\theta(p)$ is the probability that 0 belongs to an infinite open cluster. The techniques of [3] to prove the existence of this limit rely on coarse-graining estimates which can be employed only in the supercritical regime. Therefore we are not able so far to extend the above convergence at the critical point $p_c$. Naturally, we expect that $n\widehat{\varphi}_n(p_c)$ converges towards 0 as $n$ goes to infinity, unfortunately we are only able to prove a weaker statement.

**Theorem 1.2.** With probability one, we have

$$\liminf_{n \to \infty} n\widehat{\varphi}_n(p_c) = 0.$$

We shall prove this theorem by contradiction. We first define an exploration process of the cluster of 0 that remains inside the box $[-n, n]^d$. If the statement of the theorem does not hold, then the cluster of 0 satisfies a $d$-dimensional anchored isoperimetric inequality. It follows that the number of sites that are
revealed in the exploration of the cluster of 0 will grow fast enough of order $n^{d-1}$. Then, we can prove that the intersection of the cluster that we have explored with the boundary of the box $[-n, n]^d$ is of order $n^{d-1}$. Using the fact that there is no percolation in a half-space, we obtain a contradiction. Before starting the precise proof, we recall some results from [3] on the meaning of the limiting value $\varphi(p)$.

**The Wulff theorem.** We denote by $\mathcal{L}^d$ the $d$-dimensional Lebesgue measure and by $\mathcal{H}^{d-1}$ denotes the $(d-1)$--Hausdorff measure in dimension $d$. Given a norm $\tau$ on $\mathbb{R}^d$ and a subset $E$ of $\mathbb{R}^d$ having a regular enough boundary, we define $I_\tau(E)$, the surface tension of $E$ for the norm $\tau$, as

$$I_\tau(E) = \int_{\partial E} \tau(n_E(x)) \mathcal{H}^{d-1}(dx).$$

We consider the anisotropic isoperimetric problem associated with the norm $\tau$:

$$\min \frac{I_\tau(E)}{\mathcal{L}^d(E)} \text{ subject to } \mathcal{L}^d(E) \leq 1. \quad (1)$$

The famous Wulff construction provides a minimizer for this anisotropic isoperimetric problem. We define the set $\tilde{W}_\tau$ as

$$\tilde{W}_\tau = \bigcap_{v \in S^{d-1}} \{ x \in \mathbb{R}^d : x \cdot v \leq \tau(v) \},$$

where $\cdot$ denotes the standard scalar product and $S^{d-1}$ is the unit sphere of $\mathbb{R}^d$. Up to translation and Lebesgue negligible sets, the set

$$\frac{1}{\mathcal{L}^d(\tilde{W}_\tau)^{1/d}} \tilde{W}_\tau$$

is the unique solution to the problem (1).

**Representation of $\varphi(p)$.** In [3], we build an appropriate norm $\beta_p$ for our problem that is directly related to the open edge boundary. We define the Wulff crystal $W_p$ as the dilate of $\tilde{W}_{\beta_p}$ such that $\mathcal{L}^d(W_p) = 1/\theta(p)$, where $\theta(p) = P(0 \in C_{\infty})$. We denote by $I_p$ the surface tension associated with the norm $\beta_p$. In [3], we prove that

$$\forall p > p_c(d) \quad \varphi(p) = I_p(W_p).$$

2 Proofs

We prove next the following lemma, which is based on two important results due to Zhang [9] and Rossignol and Théret [6]. To alleviate the notation, the critical point $p_c(d)$ is denoted simply by $p_c$.

**Lemma 2.1.** We have

$$\lim_{p \to p_c} \left( \lim_{p' > p_c} \left( \frac{\theta(p)\delta_{I_p}(W_p) + (1 - \theta(p))\delta_0}{p'} \right) \right) = \delta_0.$$
Proof. If \( \lim_{p \to p_c} \theta(p) = 0 \), then the result is clear. Otherwise, let us assume that
\[
\lim_{p \to p_c} \theta(p) = \delta > 0.
\]
Let \( B \) be a subset of \( \mathbb{R}^d \) having a regular boundary and such that \( \mathcal{L}^d(B) = 1/\delta \). As the map \( p \mapsto \theta(p) \) is non-decreasing and \( \mathcal{L}^d(W_p) = 1/\theta(p) \), we have
\[
\forall p > p_c \quad \mathcal{L}^d(W_p) \leq \mathcal{L}^d(B).
\]
Moreover as \( W_p \) is the dilate of the minimizer associated to the isoperimetric problem (1), we have
\[
\forall p > p_c \quad \mathcal{I}^d(W_p) \leq \mathcal{I}^d(B).
\]
In [9], Zhang proved that \( \beta_{p_c} = 0 \). In [6], Rossignol and Théret proved the continuity of the flow constant. Combining these two results, we get that
\[
\lim_{p \to p_c} \beta_p = \beta_{p_c} = 0 \quad \text{and so} \quad \lim_{p \to p_c} \mathcal{I}^d(B) = 0.
\]
Finally, we obtain
\[
\lim_{p \to p_c} \mathcal{I}^d(W_p) = 0.
\]
This yields the result. \( \square \)

Proof of theorem 1.2. We assume by contradiction that
\[
P \left( \liminf_{n \to \infty} n \hat{\varphi}_n(p_c) = 0 \right) < 1.
\]
Therefore there exist positive constants \( c \) and \( \delta \) such that
\[
P \left( \liminf_{n \to \infty} n \hat{\varphi}_n(p_c) > c \right) = \lim_{n \to \infty} P \left( \inf_{k \geq n} k \hat{\varphi}_k(p_c) > c \right) = \delta. \tag{2}
\]
Therefore, there exists a positive integer \( n_0 \) such that
\[
P \left( \inf_{k \geq n_0} k \hat{\varphi}_k(p_c) > c \right) \geq \frac{\delta}{2}. \tag{3}
\]
In what follows, we condition on the event
\[
\left\{ \inf_{k \geq n_0} k \hat{\varphi}_k(p_c) > c \right\}.
\]
Note that on this event, 0 is connected to infinity by a \( p_c \)-open path. For \( H \) a subgraph of \( \mathbb{Z}^d \), we define
\[
\partial^o H = \left\{ e \in \partial H, \ e \text{ is open} \right\}.
\]
Note that if \( H \subset C_\infty \), then \( \partial C_\infty H = \partial^o H \). Moreover, if \( H \) is equal to \( C(0) \), the open cluster of 0, then \( \partial C(0)H = \partial^o H = \emptyset \). We define next an exploration
process of the cluster of 0. We set \( C_0 = \{0\} \), \( A_0 = \emptyset \). Let us assume that \( C_0, \ldots, C_l \) and \( A_0, \ldots, A_l \) are already constructed. We define
\[
A_{l+1} = \{ x \in \mathbb{Z}^d : \exists y \in C_l \quad \langle x, y \rangle \in \partial^o C_l \}
\]
and
\[
C_{l+1} = C_l \cup A_{l+1}.
\]
We have
\[
\partial^o C_l \subset \{ \langle x, y \rangle \in \mathbb{E}^d : x \in A_{l+1} \}
\]
so that \( |\partial^o C_l| \leq 2d |A_{l+1}| \). Since \( A_{l+1} \) and \( C_l \) are disjoint, we have
\[
|C_{l+1}| = |C_l| + |A_{l+1}| \geq |C_l| + \frac{|\partial^o C_l|}{2d}.
\]
(4)

Let us set \( \alpha = \frac{1}{n_0^d} \) so that \( |C_0| = \alpha n_0^d \). Let \( k \) be the smallest integer greater than \( 2^{d+1}d/c \). We recall that \( c \) and \( n_0 \) were defined in (2) and (3). Let us prove by induction on \( n \) that
\[
\forall n \geq n_0 \quad |C_{(n-n_0)k}| \geq \alpha n^d.
\]
(5)

This is true for \( n = n_0 \). Let us assume that this inequality is true for some integer \( n \geq n_0 \). If \( |C_{(n+1-n_0)k}| \geq n^d \), then we are done. Suppose that \( |C_{(n+1-n_0)k}| < n^d \).

In this case, for any integer \( l \leq k \), we have also \( |C_{(n-n_0)k+l}| < n^d \), and since \( C_{(n-n_0)k+l} \) is a valid subgraph of \( C(0) \) and \( \hat{\varphi}_n(p_c) > c/n \), we conclude that
\[
\frac{|\partial^o C_{(n-n_0)k+l}|}{|C_{(n-n_0)k+l}|} \geq \frac{c}{n}
\]
and so \( |\partial^o C_{(n-n_0)k+l}| \geq \alpha cn^{d-1} \). Thanks to inequality (4) applied \( k \) times, we have
\[
|C_{(n+1-n_0)k}| \geq \alpha \left(n^d + \frac{ck}{2d}n^{d-1}\right).
\]
As \( k \geq 2^{d+1}d/c \), we get
\[
|C_{(n+1-n_0)k}| \geq \alpha (n^d + 2^{d+1}n^{d-1}) \geq \alpha (n+1)^d.
\]
This concludes the induction.

Let \( \eta > 0 \) be a constant that we will choose later. In [1], Barsky, Grimmett and Newman proved that there is no percolation in a half-space at criticality. An important consequence of the result of Grimmett and Marstrand [4] is that the critical value for bond percolation in a half-space equals to the critical parameter \( p_c(d) \) of bond percolation in the whole space, i.e., we have
\[
\mathbb{P}(0 \text{ is connected to infinity by a } p_c\text{-open path in } \mathbb{N} \times \mathbb{Z}^{d-1}) = 0,
\]
so that for \( n \) large enough,
\[
\mathbb{P}(\exists \gamma \text{ a } p_c\text{-open path starting from } 0 \text{ in } \mathbb{N} \times \mathbb{Z}^{d-1} \text{ such that } |\gamma| \geq n) \leq \eta.
\]
In what follows, we will consider an integer \( n \) such that the above inequality holds. By construction the set \( C_n \) is inside the box \([-n,n]^d\). Starting from this
cluster, we are going to resume our exploration but with the constraint that we
do not explore anything outside the box \([-n, n]^d\). We set \(C_0 = C_n\) and \(A_0 = \emptyset\).
Let us assume \(C_0', \ldots, C_l'\) and \(A_0', \ldots, A_l'\) are already constructed. We define
\[
A_{l+1}' = \{ x \in [-n, n]^d : \exists y \in C_l' \quad \langle x, y \rangle \in \partial C_l' \}
\]
and
\[
C_{l+1}' = C_l' \cup A_{l+1}'.
\]
We stop the process when \(A_{l+1}' = \emptyset\). As the number of vertices in the box
\([-n, n]^d\) is finite, this process of exploration will eventually stop for some inte-
ger \(l\). We have that \(|C_l'| \leq n^d\) and \(n \phi_k(p_c) > c\) so that
\[
|\partial^c C_l'| \geq \frac{c}{n} |C_l'| \geq \frac{c}{n} |C_n|.
\]
Moreover, for \(n \geq kn_0\), we have, thanks to inequality (5),
\[
|C_n| \geq |C_{\lfloor \frac{n}{k} \rfloor}| \geq |C_{\lfloor \frac{n}{k} \rfloor-n_0}| \geq \alpha \left( \frac{n}{k} \right)^d.
\]
We suppose that \(n\) is large enough so that \(n \geq kn_0\) and \(\left\lfloor \frac{n}{k} \right\rfloor \geq n/2k\). Combining
the two previous display inequalities, we conclude that
\[
|\partial^c C_l'| \geq \frac{c \alpha}{2d k^d} n^{d-1}.
\]
Therefore, for \(n\) large enough, there exists one face of \([-n, n]^d\) such that there
are at least \(c n^{d-1}/(2d k^d n^{d-1})\) vertices that are connected to 0 by a \(p_c\)-open path
that remains inside the box \([-n, n]^d\) and so
\[
P \left( \text{there exists one face of } [-n, n]^d \text{ with at least } \frac{c n^{d-1}}{2d k^d n^{d-1}} \text{ vertices that are connected to 0 by a } p_c-\text{open path that remains inside the box } [-n, n]^d \right) \geq \frac{\delta}{2}. \tag{6}
\]
Let us denote by \(X_n\) the number of vertices in the face \([-n] \times [-n, n]^{d-1}\) that
are connected to 0 by a \(p_c\)-open path inside the box \([-n, n]^d\). We have
\[
E(X_n) \leq \left| \{-n\} \times [-n, n]^{d-1} \cap \mathbb{Z}^d \right| P \left( \exists \gamma \text{ a } p_c-\text{open path starting} \right.
\]
\[
\left. \text{from 0 in } \mathbb{N} \times \mathbb{Z}^{d-1} \text{ such that } |\gamma| \geq n \right) \leq (2n+1)^{d-1} \eta. \tag{7}
\]
Moreover, we have
\[
E(X_n) \geq \frac{c \alpha}{2d^2 k^d n^{d-1}} P \left( X_n > \frac{c \alpha}{2d^2 k^d n^{d-1}} \right). \tag{8}
\]
Finally, combining inequalities (7) and (8), we get
\[
P \left( X_n > \frac{c \alpha}{2d^2 k^d n^{d-1}} \right) \leq \frac{2d n^{3d-1} 2d k^d}{c \alpha}.
\]
Therefore, we can choose \(\eta\) small enough such that
\[
P \left( X_n > \frac{c \alpha}{2d^2 k^d n^{d-1}} \right) \leq \frac{\delta}{10d}
\]
and so using the symmetry of the lattice

\[
P\left(\text{there exists one face of } [-n, n]^d \text{ such there are at least } \frac{c_\alpha d^{-1}}{2^dkd2d} \text{ vertices that are connected to } 0 \text{ by a } p_c \text{-open path that remains inside the box } [-n, n]^d\right)
\leq 2dP\left(X_n > \frac{c_\alpha}{2d2^dkd}n^{d-1}\right) \leq \frac{\delta}{5}.
\]

This contradicts inequality (6) and yields the result. \(\Box\)

References


