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# A multilevel algebraic error estimator and the corresponding iterative solver with $p$ -robust behavior\*

Ani Miraçi<sup>†‡</sup>      Jan Papež<sup>†‡</sup>      Martin Vohralík<sup>†‡</sup>

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## Abstract

In this work, we consider conforming finite element discretizations of arbitrary polynomial degree  $p \geq 1$  of the Poisson problem. We propose a multilevel a posteriori estimator of the algebraic error. We prove that this estimator is reliable and efficient (represents a two-sided bound of the error), with a constant independent of the degree  $p$ . We next design a multilevel iterative algebraic solver from our estimator and we show that this solver contracts the algebraic error on each iteration by a factor bounded independently of  $p$ . Actually, we show that these two results are equivalent. The  $p$ -robustness results rely on the work of Schöberl *et al.* [IMA J. Numer. Anal., 28 (2008), pp. 1–24] for one given mesh. We combine this with the design of an algebraic residual lifting constructed over a hierarchy of nested unstructured simplicial meshes. This includes a global coarse-level lowest-order solve together with local contributions from the subsequent mesh levels. These contributions, highest-order on the finest mesh, are given as solutions of mutually independent Dirichlet problems posed over patches of elements around vertices. This residual lifting is the core of our a posteriori estimator. It also determines the descent direction for the next iteration of our multilevel solver, which we consider with optimal step size. Its construction can be seen as one geometric V-cycle multigrid step with zero pre- and one post-smoothing by (damped) additive Schwarz. Numerical tests are presented to illustrate the theoretical findings.

**Key words:** finite element method, stable decomposition, multilevel method, Schwarz method, a posteriori estimate,  $p$ -robustness

## 1 Introduction

The finite element method (FEM) is a widespread approach for discretizing problems given in the form of partial differential equations, and has been used in engineering for more than fifty years. For a thorough overview on the topic, we refer to e.g. Ciarlet [17], Ern and Guermond [19], or Brenner and Scott [14]. Many iterative methods have been suggested to treat the linear systems arising from finite element discretizations, see e.g. Bramble *et al.* [11] and [12], Hackbusch [24], Bank *et al.* [6], Brandt *et al.* [13], Oswald [35], or Zhang [48], and the references therein. A systematic description of iterative solvers is given by Xu in [46]. For convergence results on unstructured and graded meshes, we refer to e.g. Wu and Chen [45], Hiptmair *et al.* [25], Chen *et al.* [16], and Xu *et al.* [47]. The convergence of these methods is typically robust with respect to the size of the mesh ( $h$ -robustness). In fact, this is one of the key advantages of multigrid methods. For the conjugate gradient method on the other hand,  $h$ -robustness is not intrinsic; this problem can be bypassed with the development of appropriate preconditioners.

If we are to consider methods of arbitrary polynomial degree, an additional question arises: how does the polynomial degree  $p$  affect the performance of the method? In this regard, results for  $p$ -version FEM

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include Foresti *et al.* [21] for two-dimensional domains, Mandel [32] for three-dimensional domains, and Babuška *et al.* [5] for two-dimensional domains. For the latter, the condition number of the preconditioned system grows at most by  $1 + \log^2(p)$ , and a generalization of this work is given for  $hp$ -FEM by Ainsworth [1], where the  $p$ -dependence is still present. An early version of a polynomial-degree robust ( $p$ -robust) solver was introduced by Quarteroni and Sacchi Landriani [39] for a specific domain configuration (decomposable into rectangles without internal cross points). Notable development on  $p$ -robustness was later made by Pavarino [38] for quadrilateral/hexahedral meshes, where the author introduced a  $p$ -robust additive Schwarz method. The generalization of this result for triangular/tetrahedral meshes was achieved by Schöberl *et al.* [40], once more by introducing an additive Schwarz preconditioner. More recent works were carried based on these approaches. In Antonietti *et al.* [3] (see also the references therein), the  $p$ -robust approach for rectangular/hexahedral meshes was used for high-order discontinuous Galerkin (DG) methods; moreover the spectral bounds of the preconditioner are also robust with respect to the method's penalization coefficient. We also mention the introduction of multilevel preconditioners yielded by block Gauss–Seidel smoothers in Kanschat [28] for rectangular/hexahedral meshes and DG discretizations. Further multilevel approaches for rectangular/hexahedral meshes using overlapping or non-overlapping Schwarz smoothers can be found in e.g. Janssen and Kanschat [26] and Lucero Lorca and Kanschat [31]. For a study on more general meshes, see e.g. Antonietti and Pennesi [2], where a multigrid approach behaves  $p$ -robustly under the condition that the number of smoothing steps (depending on  $p$ ) is chosen big enough. Another notable contribution is the design of algebraic multigrid methods (AMG) via aggregation techniques, see e.g. Notay and Napov [34], Bastian *et al.* [7], and the references therein. The numerical results of the latter give a satisfactory indication of  $p$ -robustness.

An associated topic is the development of estimates on the algebraic error. In this regard, a posteriori tools have primarily been used to estimate the algebraic error for existing solvers. One particular goal is the development of reliable stopping criteria, allowing to avoid unnecessary iterations. This is achieved with a combination of a posteriori error estimators for the discretization error. Some contributions on this matter (see also references therein) include Becker *et al.* [8] where adaptive error control is achieved for a multigrid solver, Bornemann and Deuffhard [9] where a one-way multigrid method is presented by integrating an adaptive stopping criteria based on a posteriori tools. Further developments were made in Meidner *et al.* [33], where goal-oriented error estimates are used in the framework of the dual weighted residual (DWR) method. In Jiránek *et al.* [27] and later in Papež *et al.* [37], upper and lower bounds for both the algebraic and total errors are computed, which allow to derive guaranteed upper and lower bounds on the discretization error, and consecutively construct safe stopping criteria for iterative algebraic solvers. Arioli *et al.* [4] propose practical stopping criteria which guarantee that the considered inexact adaptive FEM algorithm converges, for inexact solvers of the Krylov subspace type. To the best of the authors' knowledge, though, dedicated proofs of efficiency of a posteriori estimators of the algebraic error have not been presented so far.

In this work, we present an a posteriori algebraic error estimator and a multilevel iterative solver associated to it. The cornerstone of their definitions lies in the multilevel construction of a residual algebraic lifting, motivated partly by the approach of Papež *et al.* [36]. The lifting can be seen as an approximation of the algebraic error by piecewise polynomials of degree  $p$ , obtained by a V-cycle multigrid method with no pre-smoothing step and a single post-smoothing step. The coarse correction is given by a lowest-order (piecewise affine) function. Our smoothing is chosen in the family of damped additive Schwarz (block Jacobi) methods applied to overlapping subdomains composed of patches of elements (two options for defining the patches will be given in due time). Note that additive Schwarz-type smoothing allows for a parallelizable implementation at each level of the mesh hierarchy. Once this lifting is built, the a posteriori estimator is easily derived as a natural guaranteed lower bound on the algebraic error, following [36]. As our first main result, we show that up to a  $p$ -robust constant, the estimator is also an upper bound on the error.

Our solver is then defined as a linear iterative method. Because we have at hand the residual lifting, which approximates the algebraic error, we use it as a descent direction (the asymmetric approach in defining the lifting, because no pre-smoothing is used, will not be a problem for the analysis). The step size is then chosen by a line search in the direction of the lifting. Our choice presents a resemblance with the conjugate gradient method, in that we choose the step size that ensures the best error contraction in the energy norm at the next iteration. As our second main result, we prove that this solver contracts the error at each iteration by a  $p$ -robust constant. Actually, we also show that the  $p$ -robust efficiency of the estimator is equivalent to the  $p$ -robust convergence of the solver. All these results are defined for a general hierarchy

of nested, unstructured, possibly highly refined matching simplicial meshes, and no assumption beyond  $u \in H_0^1(\Omega)$  is imposed on the weak solution.

The work is structured as follows. In Section 2, we introduce the setting in which we will be working as well as the notations employed throughout the paper. Then, we introduce our multilevel residual lifting construction in Section 3 following Papež *et al.* [36]. In Section 4, we present the a posteriori estimator on the algebraic error and the corresponding multilevel solver based on the residual lifting. Our main results are presented in the form of two theorems in Section 5, together with a corollary establishing their equivalence. We provide numerical experiments in Section 6, focusing mainly on showcasing  $p$ -robustness, in agreement with our theoretical results, and on a comparison with several existing approaches as well as a weighted restricted additive Schwarz variant of our solver. The proofs of our main results are given in Section 7. In particular, for the stable decomposition estimate, the  $p$ -robust result introduced by Schöberl *et al.* [40] is crucial. Finally, Section 8 brings forth our conclusions and outlook for future work.

## 2 Setting

We will consider in this work the Poisson problem defined over  $\Omega \subset \mathbb{R}^d$ ,  $1 \leq d \leq 3$ , an open bounded polytope with a Lipschitz-continuous boundary. Let there be given a hierarchy of nested matching simplicial meshes of  $\Omega$ ,  $\{\mathcal{T}_j\}_{0 \leq j \leq J}$ . This means that the intersection of two distinct elements of each mesh  $\mathcal{T}_j$  is either an empty set or a common vertex, edge, or face, and that  $\mathcal{T}_j$  is a refinement of  $\mathcal{T}_{j-1}$ ,  $1 \leq j \leq J$ . Two further assumptions are given below.

### 2.1 Model problem

Let  $f \in L^2(\Omega)$  be the source term. We consider the following problem: find  $u : \Omega \rightarrow \mathbb{R}$  such that

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (2.1)$$

In the weak formulation, we search for  $u \in H_0^1(\Omega)$  such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega), \quad (2.2)$$

where  $(\cdot, \cdot)$  is the  $L^2(\Omega)$  or  $[L^2(\Omega)]^d$  scalar product. The existence and uniqueness of the solution of (2.2) follows from the Riesz representation theorem.

### 2.2 Finite element discretization

Fixing an integer  $p \geq 1$ , we introduce the finite element space of continuous piecewise  $p$ -degree polynomials

$$V_J^p := \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega), \quad (2.3)$$

where  $\mathbb{P}_p(\mathcal{T}_J) := \{v_J \in L^2(\Omega), v_J \in \mathbb{P}_p(K) \forall K \in \mathcal{T}_J\}$ . We set  $N_J := \dim(V_J^p)$ . The discrete problem consists in finding  $u_J \in V_J^p$  such that

$$(\nabla u_J, \nabla v_J) = (f, v_J) \quad \forall v_J \in V_J^p. \quad (2.4)$$

### 2.3 Algebraic system, approximate solution, and algebraic residual

If one introduces  $\psi_J^l$ ,  $1 \leq l \leq N_J$ , a basis of  $V_J^p$ , then problem (2.4) is equivalent to solving a system of linear algebraic equations. Denoting by  $(\mathbb{A}_J)_{lm} := (\nabla \psi_J^m, \nabla \psi_J^l)$  the symmetric, positive definite (stiffness) matrix,  $(\mathbb{F}_J)_l := (f, \psi_J^l)$  the right-hand side (load) vector, one obtains  $u_J = \sum_{m=1}^{N_J} (\mathbb{U}_J)_m \psi_J^m$ , where  $\mathbb{U}_J \in \mathbb{R}^{N_J}$  is the solution of

$$\mathbb{A}_J \mathbb{U}_J = \mathbb{F}_J.$$

For any approximation  $\mathbb{U}_J^i \in \mathbb{R}^{N_J}$  of  $\mathbb{U}_J$  given by an arbitrary algebraic solver at iteration step  $i \geq 0$ , the associated continuous piecewise polynomial of degree  $p$  is  $u_J^i = \sum_{m=1}^{N_J} (\mathbb{U}_J^i)_m \psi_J^m \in V_J^p$ . The associated algebraic residual vector is given by

$$\mathbb{R}_J^i := \mathbb{F}_J - \mathbb{A}_J \mathbb{U}_J^i.$$

Note, however, that  $R_j^i$  depends on the choice of the basis functions  $\psi_j^l$ ,  $1 \leq l \leq N_j$ . To avoid this dependence, we work instead with a residual functional on  $V_j^p$  given by

$$v_J \mapsto (f, v_J) - (\nabla u_J^i, \nabla v_J) \in \mathbb{R}, \quad v_J \in V_J^p. \quad (2.5)$$

We emphasize that the forthcoming results are independent of the choice of the basis.

## 2.4 A hierarchy of meshes and spaces

We introduced in the beginning of this section our mesh hierarchy  $\{\mathcal{T}_j\}_{0 \leq j \leq J}$ . For any element  $K \in \mathcal{T}_j$ , we denote  $h_K := \text{diam}(K)$ . Hereafter, we assume that the hierarchy of meshes is such that the size of each parent element  $K \in \mathcal{T}_j$  is comparable to the size of each of its children, and shape regularity:

**Assumption 2.1** (Strength of refinement). *For any  $j \in \{1, \dots, J\}$  and for all  $K \in \mathcal{T}_{j-1}$  and  $K^* \in \mathcal{T}_j$  such that  $K^* \subset K$ , we have*

$$C_{\text{ref}} h_K \leq h_{K^*} \leq h_K, \quad (2.6)$$

with  $C_{\text{ref}} \leq 1$  a fixed positive real number.

**Assumption 2.2** (Shape regularity). *There exists  $\kappa_{\mathcal{T}} > 0$  such that*

$$\max_{K \in \mathcal{T}_j} \frac{h_K}{\rho_K} \leq \kappa_{\mathcal{T}} \text{ for all } 0 \leq j \leq J, \quad (2.7)$$

where  $\rho_K$  denotes the diameter of the largest ball inscribed in  $K$ .

**Remark 2.3** (Mesh hierarchy). *Note that no quasi-uniformity assumption is introduced for the meshes: each  $\mathcal{T}_j$  can possibly be highly graded. Moreover, we do not make any specific assumption on the hierarchy of meshes. They merely need to be nested, in particular some mesh elements may not be refined at all.*

In the following, we will need to introduce a hierarchy of finite element spaces associated to the mesh hierarchy. For this purpose, let  $p' \in \{1, \dots, p\}$  be a polynomial degree between 1 and  $p$  that we employ for the intermediate levels. In particular, let:

$$\text{for } 1 \leq j \leq J-1: \quad V_j^{p'} := \mathbb{P}_{p'}(\mathcal{T}_j) \cap H_0^1(\Omega) \quad (p'\text{-th order spaces}), \quad (2.8a)$$

$$\text{for } j = 0: \quad V_0 := \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega) \quad (\text{lowest-order space}), \quad (2.8b)$$

where  $\mathbb{P}_{p'}(\mathcal{T}_j) := \{v_j \in L^2(\Omega), v_j \in \mathbb{P}_{p'}(K) \quad \forall K \in \mathcal{T}_j\}$ . Note that  $V_0 \subset V_1^{p'} \subset \dots \subset V_{J-1}^{p'} \subset V_J^p$ . Let  $\mathcal{V}_j$  be the set of vertices of the mesh  $\mathcal{T}_j$ , which we decompose into the sets of boundary vertices and interior vertices denoted by  $\mathcal{V}_j^{\text{ext}}$  and  $\mathcal{V}_j^{\text{int}}$ , respectively. We denote by  $\psi_j^{\mathbf{a}}$  the standard hat function associated to the vertex  $\mathbf{a} \in \mathcal{V}_j$ ,  $0 \leq j \leq J$ . This is the piecewise affine function with respect to the mesh  $\mathcal{T}_j$ , that takes value 1 in the vertex  $\mathbf{a}$  and 0 in all other  $j$ -th level vertices of  $\mathcal{V}_j$ .

## 2.5 Two types of patches

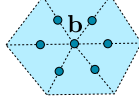
For the following, we need to define two types of patches of elements. In order to facilitate the work with both, we introduce a switching parameter  $s \in \{0, 1\}$ . Given a vertex  $\mathbf{a} \in \mathcal{V}_{j-s}$ ,  $j \in \{1, \dots, J\}$ , we denote the patch related to  $\mathbf{a}$  by  $\mathcal{T}_{j,s}^{\mathbf{a}}$ , the corresponding open patch subdomain by  $\omega_{j,s}^{\mathbf{a}}$ , and the associated local space  $V_{j,s}^{\mathbf{a}}$ . Let  $\mathcal{V}_K$  be the set of vertices of element  $K$ . Then

$$\mathcal{T}_{j,s}^{\mathbf{a}} := \{K \in \mathcal{T}_{j-s}, \mathbf{a} \in \mathcal{V}_K\}, \quad (2.9)$$

$$V_{j,s}^{\mathbf{a}} := \mathbb{P}_{p'}(\mathcal{T}_j) \cap H_0^1(\omega_{j,s}^{\mathbf{a}}), \quad j \in \{1, \dots, J-1\} \text{ and } V_{j,s}^{\mathbf{a}} := \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\omega_{J,s}^{\mathbf{a}}). \quad (2.10)$$

Consider a ‘‘finer’’ ( $s = 0$ ) vertex  $\mathbf{b} \in \mathcal{V}_j$ , and ‘‘coarser’’ ( $s = 1$ ) one  $\mathbf{a} \in \mathcal{V}_{j-1} \subset \mathcal{V}_j$ . For simplicity, we refer to  $\mathcal{T}_{j,0}^{\mathbf{b}}$  as a ‘‘small’’ patch, whereas we refer to  $\mathcal{T}_{j,1}^{\mathbf{a}}$  as a ‘‘big’’ patch. An illustration is given in Figure 1.

patch  $\mathcal{T}_{j,0}^{\mathbf{b}}$  and the  
patch subdomain  $\omega_{j,0}^{\mathbf{b}}$   
for a vertex  $\mathbf{b} \in \mathcal{V}_j$



patch  $\mathcal{T}_{j,1}^{\mathbf{a}}$  and the  
patch subdomain  $\omega_{j,1}^{\mathbf{a}}$   
for a vertex  $\mathbf{a} \in \mathcal{V}_{j-1}$

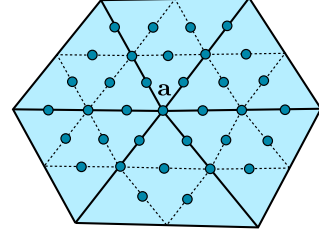


Figure 1: Illustration of degrees of freedom ( $p' = p = 2$ ) for the space  $V_{j,0}^{\mathbf{b}}$  associated to the “small” patch  $\mathcal{T}_{j,0}^{\mathbf{b}}$  (left) and for  $V_{j,1}^{\mathbf{a}}$  associated to the “big” patch  $\mathcal{T}_{j,1}^{\mathbf{a}}$  (right). The mesh  $\mathcal{T}_{j-1}$  and its refinement  $\mathcal{T}_j$  are defined in bold and dotted lines, respectively.

### 3 Multilevel lifting of the algebraic residual

In the spirit of Papež *et al.* [36], we design a multilevel lifting of the algebraic residual given by (2.5). This lifting will lead to the construction of an a posteriori error estimator, it will also serve as a descent direction for the solver we introduce in the next section.

#### 3.1 Coarse solve

The first step of our construction is to solve a global lowest-order problem on the coarsest mesh. Let  $u_J^i \in V_J^p$  be given. Define  $\rho_0^i \in V_0$  by

$$(\nabla \rho_0^i, \nabla v_0) = (f, v_0) - (\nabla u_J^i, \nabla v_0) \quad \forall v_0 \in V_0. \quad (3.1)$$

#### 3.2 Motivation

Let us first motivate our multilevel construction. Consider, for a given  $u_J^i \in V_J^p$ , the following (infeasible in practice but illustrative) hierarchical construction  $\tilde{\rho}_{J,\text{alg}}^i \in V_J^p$

$$\tilde{\rho}_{J,\text{alg}}^i := \rho_0^i + \sum_{j=1}^J \tilde{\rho}_j^i, \quad (3.2)$$

where  $\rho_0^i$  is given by (3.1), and for all  $j = \{1, \dots, J-1\}$ ,  $\tilde{\rho}_j^i \in V_j^{p'}$  as well as  $\tilde{\rho}_J^i \in V_J^p$  are the solutions of

$$(\nabla \tilde{\rho}_j^i, \nabla v_j) = (f, v_j) - (\nabla u_J^i, \nabla v_j) - \sum_{k=0}^{j-1} (\nabla \tilde{\rho}_k^i, \nabla v_j) \quad \forall v_j \in V_j^{p'}, \quad (3.3a)$$

$$(\nabla \tilde{\rho}_J^i, \nabla v_J) = (f, v_J) - (\nabla u_J^i, \nabla v_J) - \sum_{k=0}^{J-1} (\nabla \tilde{\rho}_k^i, \nabla v_J) \quad \forall v_J \in V_J^p. \quad (3.3b)$$

Here,  $\tilde{\rho}_0^i := \rho_0^i$  by convention. This construction (see also [36]) returns the algebraic error, i.e. we actually have  $\tilde{\rho}_{J,\text{alg}}^i = u_J - u_J^i$ . This, in turn, means that  $\tilde{\rho}_{J,\text{alg}}^i$  satisfies  $(\nabla \tilde{\rho}_{J,\text{alg}}^i, \nabla v_J) = (f, v_J) - (\nabla u_J^i, \nabla v_J)$  for all  $v_J \in V_J^p$ . Moreover, there holds  $(\nabla \tilde{\rho}_j^i, \nabla \tilde{\rho}_k^i) = 0$ ,  $0 \leq k, j \leq J$ ;  $j \neq k$ . These observations in particular lead to the orthogonal decomposition

$$\|\nabla(u_J - u_J^i)\|^2 = \|\nabla \tilde{\rho}_{J,\text{alg}}^i\|^2 = \sum_{j=0}^J \|\nabla \tilde{\rho}_j^i\|^2. \quad (3.4)$$

#### 3.3 Multilevel algebraic residual lifting

Let us now introduce our construction that mimics (3.2)–(3.3) in a local way and produces  $\rho_{J,\text{alg}}^i \in V_J^p$  that is hopefully close to  $\tilde{\rho}_{J,\text{alg}}^i$ . The construction relies on the use of coarse solution of (3.1) and on local

contributions arising from all the finer mesh levels. These local contributions are defined on patches of elements. Since we consider two definitions of patches with switching parameter  $s \in \{0, 1\}$  (see Section 2.5), two constructions of  $\rho_{J,\text{alg}}^i$  are implied.

**Definition 3.1** (Construction of the algebraic residual lifting). *Let  $u_j^i \in V_j^p$  be arbitrary. We introduce  $\rho_{J,\text{alg}}^i \in V_J^p$  by*

$$\rho_{J,\text{alg}}^i := \rho_0^i + \sum_{j=1}^J \rho_j^i, \quad (3.5)$$

where  $\rho_0^i \in V_0$  solves (3.1) and  $\rho_j^i \in V_j^{p'}$ , for  $j \in \{1, \dots, J-1\}$ , and  $\rho_j^i \in V_j^p$  are given by

$$\rho_j^i := \frac{1}{J(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \rho_{j,\mathbf{a}}^i, \quad 1 \leq j \leq J, \quad (3.6)$$

with the local contributions  $\rho_{j,\mathbf{a}}^i \in V_{j,s}^{\mathbf{a}}$  given by patch problems, for all  $v_{j,\mathbf{a}} \in V_{j,s}^{\mathbf{a}}$

$$(\nabla \rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} = (f, v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - (\nabla u_j^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}}. \quad (3.7)$$

**Remark 3.2** (Construction of  $\rho_{J,\text{alg}}^i$ ). *The construction (3.5)–(3.7) of  $\rho_{J,\text{alg}}^i$  can be seen as an approximation of  $\tilde{\rho}_{J,\text{alg}}^i$  from (3.2)–(3.3) by one iteration of a V-cycle multigrid, with no pre-smoothing and a single post-smoothing step, corresponding to a “damped” additive Schwarz iteration, with the damping factor  $1/(J(d+1))$ . The subdomains of this Schwarz iteration correspond to the patch domains where the local problems in (3.7) are defined. Two patch options of Figure 1 are considered. In particular, for  $p = 1$  and “small” patches (Figure 1, left), this corresponds to one-step Jacobi (diagonal) smoother, whereas when  $p' = p > 1$ , the smoother is block Jacobi. The construction of Definition 3.1 differs from that of Papež et al. [36, Definition 6.5] by replacing the weightings via hat functions therein by the damping factor  $1/(J(d+1))$ ; we will test numerically the weighted variant in Section 6.*

**Remark 3.3** (Value of the damping parameter). *The value  $1/(J(d+1))$  of the damping in (3.6) is based on the proofs in Section 7 below, where it seems curcial. This is what also seems to be needed in our approach numerically. On the other hand, other options for the weights are theoretically possible, including a second damping factor of the form  $1/w_2$  for the third term on the right-hand side of (3.7), where  $w_2 \rightarrow +\infty$  is allowed.*

## 4 An a posteriori estimator on the the algebraic error and a multilevel solver

We present below how the residual lifting  $\rho_{J,\text{alg}}^i$  of Definition 3.1 can be used to define an a posteriori estimator as well as a multilevel solver.

### 4.1 A posteriori estimate on the algebraic error

We begin by introducing  $\eta_{\text{alg}}^i$ , an a posteriori estimator defined using the residual lifting  $\rho_{J,\text{alg}}^i$ .

**Definition 4.1** (Lower bound algebraic error estimator). *Let  $u_j^i \in V_j^p$  be arbitrary, and let  $\rho_{J,\text{alg}}^i$  be the algebraic residual lifting given by Definition 3.1. If  $\rho_{J,\text{alg}}^i = 0$ , we define the lower bound algebraic error estimator  $\eta_{\text{alg}}^i := 0$ . Otherwise, set*

$$\eta_{\text{alg}}^i := \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_j^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|}. \quad (4.1)$$

Following Papež et al. [36, Theorem 5.3], the estimator  $\eta_{\text{alg}}^i$  is immediately a guaranteed lower bound on the algebraic error.



**Lemma 4.2** (Guaranteed lower bound on the algebraic error). *There holds:*

$$\|\nabla(u_J - u_J^i)\| \geq \eta_{\text{alg}}^i. \quad (4.2)$$

*Proof.* Note that if  $\rho_{J,\text{alg}}^i = 0$ , then  $\|\nabla(u_J - u_J^i)\| \geq 0 = \eta_{\text{alg}}^i$ . Otherwise

$$\|\nabla(u_J - u_J^i)\| = \max_{\substack{v_J \in V_J^p, \\ \|\nabla v_J\| \neq 0}} \frac{(\nabla(u_J - u_J^i), \nabla v_J)}{\|\nabla v_J\|} \geq \frac{(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|} \stackrel{(2.4)}{=} \stackrel{(4.1)}{=} \eta_{\text{alg}}^i.$$

□

## 4.2 Multilevel solver

We will now reuse the construction of  $\rho_{J,\text{alg}}^i$ , given in Definition 3.1 to obtain an approximation of  $u_J$  on a next step, in view of constructing a multilevel solver. Note that for any  $u_J^i \in V_J^p$ , the lifting  $\rho_{J,\text{alg}}^i$  is built to approximate the algebraic error  $\tilde{\rho}_{J,\text{alg}}^i$  given in (3.2), where recall,  $u_J = u_J^i + \tilde{\rho}_{J,\text{alg}}^i$ . Thus, it seems reasonable to consider a linear iterative solver of the form

$$u_J^{i+1} = u_J^i + \lambda \rho_{J,\text{alg}}^i \quad (4.3)$$

where  $\lambda \in \mathbb{R}$  is a real parameter. The optimal choice of  $\lambda$  is given in the following lemma.

**Lemma 4.3** (Optimal step size). *Consider a solver of the general form (4.3) and suppose  $\rho_{J,\text{alg}}^i \neq 0$ . Then the choice  $\lambda := [(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)] / \|\nabla \rho_{J,\text{alg}}^i\|^2$  leads to minimal algebraic error with respect to the energy norm.*

*Proof.* We write the algebraic error of the next iteration as a function of  $\lambda$

$$\|\nabla(u_J - u_J^{i+1})\|^2 = \|\nabla(u_J - u_J^i)\|^2 - 2\lambda(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i) + \lambda^2 \|\nabla \rho_{J,\text{alg}}^i\|^2,$$

and realize that this function has a minimum at

$$\lambda_{\min} = \frac{(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|^2} \stackrel{(2.4)}{=} \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|^2}.$$

□

We are now ready to define our multilevel solver.

**Definition 4.4** (Multilevel solver).

1. Initialize  $u_J^0 \in V_0$  as the solution of  $(\nabla u_J^0, \nabla v_0) = (f, v_0) \quad \forall v_0 \in V_0$ .
2. Let  $i \geq 0$  be the iteration number, and let  $\rho_{J,\text{alg}}^i$  be constructed from  $u_J^i$  following Definition 3.1. If  $\rho_{J,\text{alg}}^i = 0$ , we define  $u_J^{i+1} := u_J^i$ . Otherwise, let

$$u_J^{i+1} := u_J^i + \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|^2} \rho_{J,\text{alg}}^i. \quad (4.4)$$

**Remark 4.5** (Multilevel solver). *Note that the solver of Definition 4.4 is not initialized randomly but via a coarse solve. The descent direction is the residual lifting  $\rho_{J,\text{alg}}^i$ , constructed via no pre-smoothing, one post-smoothing V-cycle step. This minimalist and asymmetrical procedure will not be an issue for the forthcoming analysis.*



## 5 Main results

In this section, we present the main results concerning our a posteriori estimator  $\eta_{\text{alg}}^i$  of Definition 4.1 and our multilevel solver of Definition 4.4. We shall also see how these two main results are related. For the estimator it holds

**Theorem 5.1** (*p*-robust reliable and efficient bound on the algebraic error). *Let  $u_J \in V_J^p$  be the (unknown) solution of (2.4) and let  $u_J^i \in V_J^p$  be arbitrary,  $i \geq 0$ . Let  $\eta_{\text{alg}}^i$  be given by Definition 4.1. Then, in addition to  $\|\nabla(u_J - u_J^i)\| \geq \eta_{\text{alg}}^i$  of (4.2), there holds*

$$\eta_{\text{alg}}^i \geq \beta \|\nabla(u_J - u_J^i)\|, \quad (5.1)$$

where  $0 < \beta < 1$  only depends on the space dimension  $d$ , the mesh shape regularity parameter  $\kappa_{\mathcal{T}}$ , and the number of mesh levels  $J$ .

The theorem allows to write  $\eta_{\text{alg}}^i$  as a two-sided bound of the algebraic error (up to the generic constant  $\beta$  for the upper bound), meaning that the estimator is robustly efficient with respect to the polynomial degree  $p$ . For the solver, in turn, we have:

**Theorem 5.2** (*p*-robust error contraction of the multilevel solver). *Let  $u_J \in V_J^p$  be the (unknown) solution of (2.4) and let  $u_J^i \in V_J^p$  be arbitrary,  $i \geq 0$ . Take  $u_J^{i+1}$  to be constructed from  $u_J^i$  using one step of the multilevel solver of Definition 4.4, by (4.4). Then there holds*

$$\|\nabla(u_J - u_J^{i+1})\| \leq \alpha \|\nabla(u_J - u_J^i)\|, \quad (5.2)$$

where  $0 < \alpha < 1$  only depends on the space dimension  $d$ , the mesh shape regularity parameter  $\kappa_{\mathcal{T}}$ , and the number of mesh levels  $J$ .

In the above theorem,  $\alpha$  represents an estimation of the algebraic error contraction factor at each step  $i$ . As  $\alpha$  only depends on  $d$ ,  $\kappa_{\mathcal{T}}$ , and  $J$ , this means that the solver of Definition 4.4 contracts the algebraic error at each iteration step in a robust way both with respect to the number of mesh elements in  $\mathcal{T}_J$  (to the mesh size  $h$ ) and with respect to the polynomial degree  $p$ . Theorems 5.1 and 5.2 are connected as follows

**Corollary 5.3** (Equivalence of the *p*-robust estimator efficiency and *p*-robust solver contraction). *Let the assumptions of Theorems 5.1 and 5.2 be satisfied. Then (5.1) holds if and only if (5.2) holds, and  $\beta = \sqrt{1 - \alpha^2}$ .*

*Proof.* Let  $u_J \in V_J^p$  be the solution of (2.4), let  $u_J^i \in V_J^p$  be arbitrary, and let  $u_J^{i+1} \in V_J^p$  be constructed from  $u_J^i$  by our multilevel solver of Definition 4.4.

*Case  $\rho_{J,\text{alg}}^i = 0$ .* By Definitions 4.4 and 4.1, we have  $u_J^{i+1} = u_J^i$  and  $\eta_{\text{alg}}^i = 0$ . In particular, this means that  $\|\nabla(u_J - u_J^{i+1})\| = \|\nabla(u_J - u_J^i)\|$ . These observations allow us to write, starting from (5.2) with  $0 < \alpha < 1$ ,

$$\begin{aligned} \|\nabla(u_J - u_J^{i+1})\|^2 &\leq \alpha^2 \|\nabla(u_J - u_J^i)\|^2 \Leftrightarrow \|\nabla(u_J - u_J^i)\|^2 \leq \alpha^2 \|\nabla(u_J - u_J^i)\|^2 \\ &\Leftrightarrow \|\nabla(u_J - u_J^i)\|^2 (1 - \alpha^2) \leq 0 \Leftrightarrow \|\nabla(u_J - u_J^i)\|^2 (1 - \alpha^2) \leq (\eta_{\text{alg}}^i)^2 \end{aligned}$$

*Case  $\rho_{J,\text{alg}}^i \neq 0$ .* First, note that by the construction of the lifting given in Definition 3.1 and by (2.4), we have  $[u_J = u_J^i \Rightarrow \rho_{J,\text{alg}}^i = 0]$ . Thus  $[\rho_{J,\text{alg}}^i \neq 0 \Rightarrow u_J \neq u_J^i]$ . This ensures that neither  $\|\nabla \rho_{J,\text{alg}}^i\|$  nor  $\|\nabla(u_J - u_J^i)\|$  is zero, and we can therefore use them in the denominators below. We write the relation between the algebraic errors associated to  $u_J^{i+1}$  and  $u_J^i$

$$\begin{aligned} \|\nabla(u_J - u_J^{i+1})\|^2 &\stackrel{(4.4)}{=} \left\| \nabla(u_J - u_J^i) - \frac{(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|^2} \nabla \rho_{J,\text{alg}}^i \right\|^2 \\ &\stackrel{(2.4)}{=} \|\nabla(u_J - u_J^i)\|^2 - 2 \frac{(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i)^2}{\|\nabla \rho_{J,\text{alg}}^i\|^2} + \frac{(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i)^2}{\|\nabla \rho_{J,\text{alg}}^i\|^4} \|\nabla \rho_{J,\text{alg}}^i\|^2 \\ &\stackrel{(4.1)}{=} \|\nabla(u_J - u_J^i)\|^2 \left( 1 - \frac{(\eta_{\text{alg}}^i)^2}{\|\nabla(u_J - u_J^i)\|^2} \right). \end{aligned}$$

This means that

$$\begin{aligned} \|\nabla(u_J - u_J^{i+1})\| &\leq \alpha \|\nabla(u_J - u_J^i)\| \Leftrightarrow \left(1 - \frac{(\eta_{\text{alg}}^i)^2}{\|\nabla(u_J - u_J^i)\|^2}\right)^{\frac{1}{2}} \leq \alpha \\ &\Leftrightarrow \eta_{\text{alg}}^i \geq \sqrt{1 - \alpha^2} \|\nabla(u_J - u_J^i)\|. \end{aligned}$$

□

In view of Corollary 5.3, we will prove in Section 7 only Theorem 5.1.

## 6 Numerical experiments

In this section we report some numerical illustrations of the theoretical results of Section 5. In particular, we focus on the  $p$ -robustness. In the following tests, we consider the model problem (2.1) with three different choices of the domain  $\Omega \subset \mathbb{R}^2$  and of the exact solution  $u$ :

$$\text{Sine:} \quad u(x, y) := \sin(2\pi x) \sin(2\pi y), \quad \Omega := (-1, 1)^2, \quad (6.1)$$

$$\text{Peak:} \quad u(x, y) := x(x-1)y(y-1)e^{-100((x-0.5)^2 - (y-0.117)^2)}, \quad \Omega := (0, 1)^2, \quad (6.2)$$

$$\text{L-shape:} \quad u(r, \theta) := r^{2/3} \sin(2\theta/3), \quad \Omega := (-1, 1)^2 \setminus ([0, 1] \times [-1, 0]). \quad (6.3)$$

For the L-shape problem (6.3), we impose an inhomogeneous Dirichlet boundary condition corresponding to the exact solution, which is expressed here in polar coordinates. For each of the test cases, we start with an initial Delaunay triangulation of  $\Omega$ . Then, we consider  $J$  uniform refinements where all triangles are decomposed into four congruent subtriangles. Implementation-wise, we opt for Lagrange basis functions with non-uniformly distributed nodes because of their better behavior with respect to high-order methods, see Warburton [44]. Recall that this choice has no influence on the theoretical results of Section 5 as well as presented numerical results (in exact arithmetics). Though it is not the focus of this work, we also remark that the solver can be implemented in a matrix-free way and can also be parallelized.

### 6.1 A weighted restrictive additive Schwarz construction of the residual lifting $\rho_{J,\text{alg}}^i$

A crucial component in the definition of our a posteriori estimator and multilevel solver is the construction of the residual lifting  $\rho_{J,\text{alg}}^i$  of Definition 3.1, where we have used damped additive Schwarz to cope with overlapping. In practice, hat functions weighting via a restrictive additive Schwarz often performs better, cf. Cai and Sarkis [15], Efstathiou and Gander [18], or Loisel *et al.* [30]. Thus, in addition to the damped additive Schwarz construction (3.6) of  $\rho_{J,\text{alg}}^i$ , i.e.,

$$\text{dAS} \quad \rho_j^i := \frac{1}{J(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \rho_{j,\mathbf{a}}^i, \quad 1 \leq j \leq J, \quad (6.4)$$

we also consider the weighted restricted additive Schwarz

$$\text{wRAS} \quad \rho_j^i := \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \mathcal{I}_j^p(\psi_{j-s}^{\mathbf{a}} \rho_{j,\mathbf{a}}^i), \quad 1 \leq j \leq J, \quad (6.5)$$

where  $\mathcal{I}_j^p$  is the  $\mathbb{P}_p$  Lagrange interpolation operator on the mesh level  $j$ . Here the local contributions  $\rho_{j,\mathbf{a}}^i \in V_{j,s}^{\mathbf{a}}$  remain unchanged, defined by (3.7), and we once more build the residual lifting as  $\rho_{J,\text{alg}}^i := \sum_{j=0}^J \rho_j^i$  by (3.5).

## 6.2 Testing for $p$ -robustness

As stated in Corollary 5.3, the contraction factor of the solver on each step  $i$   $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$  reveals the efficiency of the a posteriori estimator  $\eta_{\text{alg}}^i$  of Definition 4.1. Keeping this in mind, we only focus on the contraction factor.

We will follow a common choice for the stopping criterion

$$\frac{\|F_J - \mathbb{A}_J U_J^{i_{\text{stop}}}\|}{\|F_J\|} \leq 10^{-5} \frac{\|F_J - \mathbb{A}_J U_J^0\|}{\|F_J\|}. \quad (6.6)$$

We also introduce the average error contraction factor

$$\bar{\alpha} := \frac{1}{i_{\text{stop}}} \sum_{i=0}^{i_{\text{stop}}-1} \frac{\|\nabla(u_J - u_J^{i+1})\|}{\|\nabla(u_J - u_J^i)\|}. \quad (6.7)$$

We expect a  $p$ -robust solver to converge in a similar number of iterations and have similar error contraction factors at all iterations for different polynomial degrees  $p$ .

We summarize the results obtained for each of the problems (6.1)–(6.3) in Figures 2–4 and in Tables 1–3. The tests cover different number of mesh levels  $J = 3, 4, 5$ , polynomial degrees  $p = 1, 3, 6, 9$ , both constructions dAS (6.4) and wRAS (6.5) of the lifting, and the use of the “small” as well as of the “big” patches of Figure 1. We focus on the choice  $p' = p$  in (2.8a); the choice  $p' = 1$  is tested in Section 6.3 below. For each of the cases, we also give the condition number of the matrix  $\mathbb{A}_J$  approximated using the Matlab function `condest`. Finally, in the last two columns of the tables, we present a comparison with two standard smoothers for multigrid, namely the Jacobi and the Gauss–Seidel ones. Here, we employ no pre-smoothing step, one post-smoothing step, and a coarse solve with polynomials of order 1 as in (3.1) to compare with our approach.

The results confirm the expected independence of the polynomial degree  $p$  for our multilevel solver which uses the construction dAS (6.4) of the lifting. We observe an inferior quality of the contraction factors for the case of  $p = 1$  and the use of “small” patches. This behavior, though, improves considerably for the “big” patches, whose price is still very reasonable for  $p = 1$ . This is in line with some precedents in literature, where numerically  $p$ -robust solvers also perform worse for order 1 approximations; we mention, for example, Griebel *et al.* [23, Table 1] and Kronbichler and Wall [29, Table 1]. Recall that we consider no pre-smoothing and only one post-smoothing step; an important drop of the number of iterations appears if more smoothing steps are employed (not presented since we want to promote the extremely cheap and parameter-free (0,1) case). Another observation is that the number of iterations depends on the number of mesh levels  $J$ , in accordance with the theoretical result of Section 7, though in a rather mild way. Moreover, the results for the modified solver, defined using the wRAS (6.5) construction of the lifting, indicate an improvement in the error contraction factors and, moreover, complete independence of  $J$ . In contrast to these results, we see that the multigrid with standard smoothers degrades violently with respect to  $p$ . Note also that for  $p = 1$ , the only difference between wRAS of (6.5) with small patches and standard Jacobi lies in the optimally chosen step size of Lemma 4.3. This allows for a spectacular gain in the number of iterations, and as we see in Tables 1–3, and often makes the method convergent when the standard Jacobi fails.

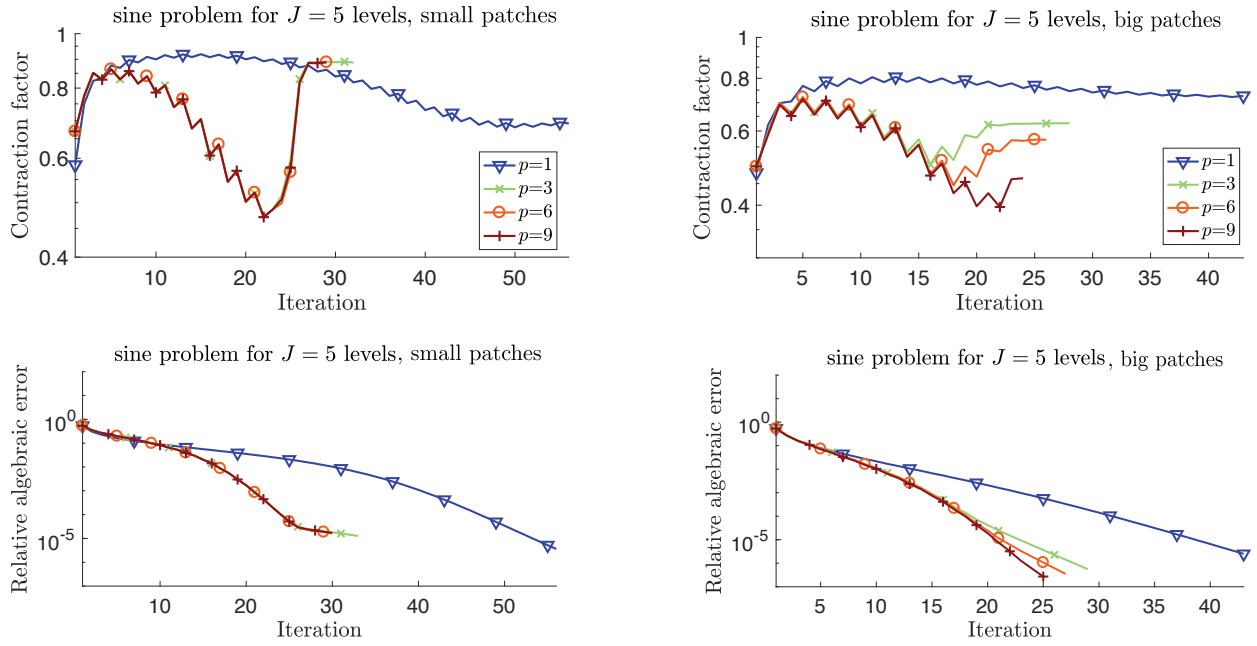


Figure 2: Sine problem (6.1): results of the multilevel solver (4.4) for  $p' = p$  in (2.8a), “small” (left) and “big” (right) patches, and stopping criterion (6.6); top: error contraction factors  $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$ ; bottom: relative algebraic error  $\|\nabla(u_J - u_J^i)\|/\|\nabla u_J\|$ .

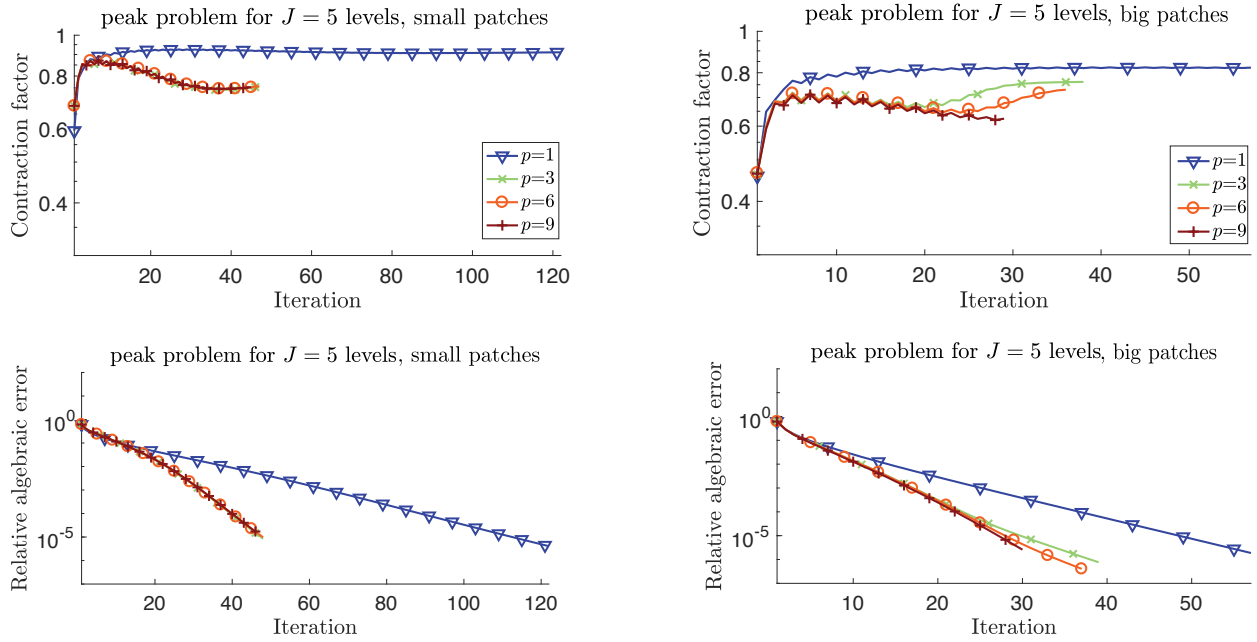


Figure 3: Peak problem (6.2): results of the multilevel solver (4.4) for  $p' = p$  in (2.8a), “small” (left) and “big” (right) patches, and stopping criterion (6.6); top: error contraction factors  $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$ ; bottom: relative algebraic error  $\|\nabla(u_J - u_J^i)\|/\|\nabla u_J\|$ .

$J$	$p$	DoF	$\text{cond}(\mathbb{A}_J)$	“small” patches				“big” patches				MG(0,1)	
				dAS (6.4)		wRAS (6.5)		dAS (6.4)		wRAS (6.5)		Jacobi	GS
				$i_{\text{stop}}$	$\bar{\alpha}$	$i_{\text{stop}}$	$\bar{\alpha}$	$i_{\text{stop}}$	$\bar{\alpha}$	$i_{\text{stop}}$	$\bar{\alpha}$	$i_{\text{stop}}$	$i_{\text{stop}}$
3	1	4625	$3.35 \times 10^3$	48	0.79	21	0.57	34	0.70	9	0.27	-	10
	3	42 289	$6.28 \times 10^4$	23	0.63	15	0.44	24	0.59	6	0.13	-	81
	6	169 825	$7.05 \times 10^5$	23	0.63	13	0.40	22	0.55	6	0.13	-	470
	9	382 609	$6.72 \times 10^6$	23	0.63	13	0.40	19	0.50	6	0.12	-	+600
4	1	18 721	$1.34 \times 10^4$	52	0.80	23	0.60	40	0.74	9	0.28	-	11
	3	169 825	$2.51 \times 10^5$	27	0.68	15	0.43	26	0.60	6	0.13	-	81
	6	680 641	$2.82 \times 10^6$	26	0.66	13	0.39	24	0.57	6	0.13	-	468
	9	1 532 449	$2.69 \times 10^7$	26	0.67	13	0.39	21	0.53	5	0.13	-	+600
5	1	75 329	$5.35 \times 10^4$	56	0.81	22	0.59	43	0.75	9	0.27	-	11
	3	680 641	$1.00 \times 10^6$	32	0.73	15	0.43	28	0.61	6	0.13	-	81
	6	2 725 249	$1.13 \times 10^7$	29	0.71	13	0.39	26	0.58	6	0.12	-	470
	9	6 133 825	$1.07 \times 10^8$	29	0.71	13	0.39	24	0.56	5	0.13	-	+600

Table 1: Sine problem (6.1): comparison of the solver of Definition 4.4 with the constructions dAS (6.4) and wRAS (6.5) for  $p' = p$  in (2.8a), “small” and “big” patches.  $i_{\text{stop}}$ : the number of iterations needed to reach the stopping criterion (6.6);  $\bar{\alpha}$ : average error contraction factor given by (6.7). Last two columns: comparison with standard multigrid method with piecewise affine (lowest-order) coarse grid correction (3.1), initialized by the coarse grid solution (3.1), no pre-smoothing, one post-smoothing step.

$J$	$p$	DoF	$\text{cond}(\mathbb{A}_J)$	“small” patches				“big” patches				MG(0,1)	
				dAS (6.4)		wRAS (6.5)		dAS (6.4)		wRAS (6.5)		Jacobi	GS
				$i_{\text{stop}}$	$\bar{\alpha}$	$i_{\text{stop}}$	$\bar{\alpha}$	$i_{\text{stop}}$	$\bar{\alpha}$	$i_{\text{stop}}$	$\bar{\alpha}$	$i_{\text{stop}}$	$i_{\text{stop}}$
3	1	4625	$3.01 \times 10^3$	87	0.88	19	0.52	36	0.72	9	0.25	68	8
	3	42 289	$5.80 \times 10^4$	39	0.75	15	0.43	28	0.63	6	0.12	-	70
	6	169 825	$6.52 \times 10^5$	39	0.75	14	0.41	26	0.60	6	0.11	-	462
	9	382 609	$6.22 \times 10^6$	39	0.75	14	0.40	20	0.55	5	0.12	-	+600
4	1	18 721	$1.20 \times 10^4$	109	0.90	20	0.54	48	0.77	9	0.25	-	10
	3	169 825	$2.32 \times 10^5$	43	0.77	15	0.42	34	0.68	5	0.12	-	79
	6	680 641	$2.61 \times 10^6$	43	0.78	14	0.40	31	0.64	5	0.12	-	460
	9	1 532 449	$2.49 \times 10^7$	43	0.78	14	0.40	25	0.61	5	0.12	-	+600
5	1	75 329	$4.81 \times 10^4$	122	0.91	20	0.53	57	0.80	9	0.24	-	11
	3	680 641	$9.28 \times 10^5$	47	0.79	15	0.42	38	0.70	5	0.12	-	80
	6	2 725 249	$1.04 \times 10^7$	47	0.79	14	0.39	36	0.68	5	0.12	-	461
	9	6 133 825	$9.95 \times 10^7$	45	0.79	13	0.39	29	0.65	5	0.12	-	+600

Table 2: Peak problem (6.2): comparison of the solver of Definition 4.4 with the constructions dAS (6.4) and wRAS (6.5) for  $p' = p$  in (2.8a), “small” and “big” patches.  $i_{\text{stop}}$ : the number of iterations needed to reach the stopping criterion (6.6);  $\bar{\alpha}$ : average error contraction factor given by (6.7). Last two columns: comparison with standard multigrid method with piecewise affine (lowest-order) coarse grid correction (3.1), initialized by the coarse grid solution (3.1), no pre-smoothing, one post-smoothing step.

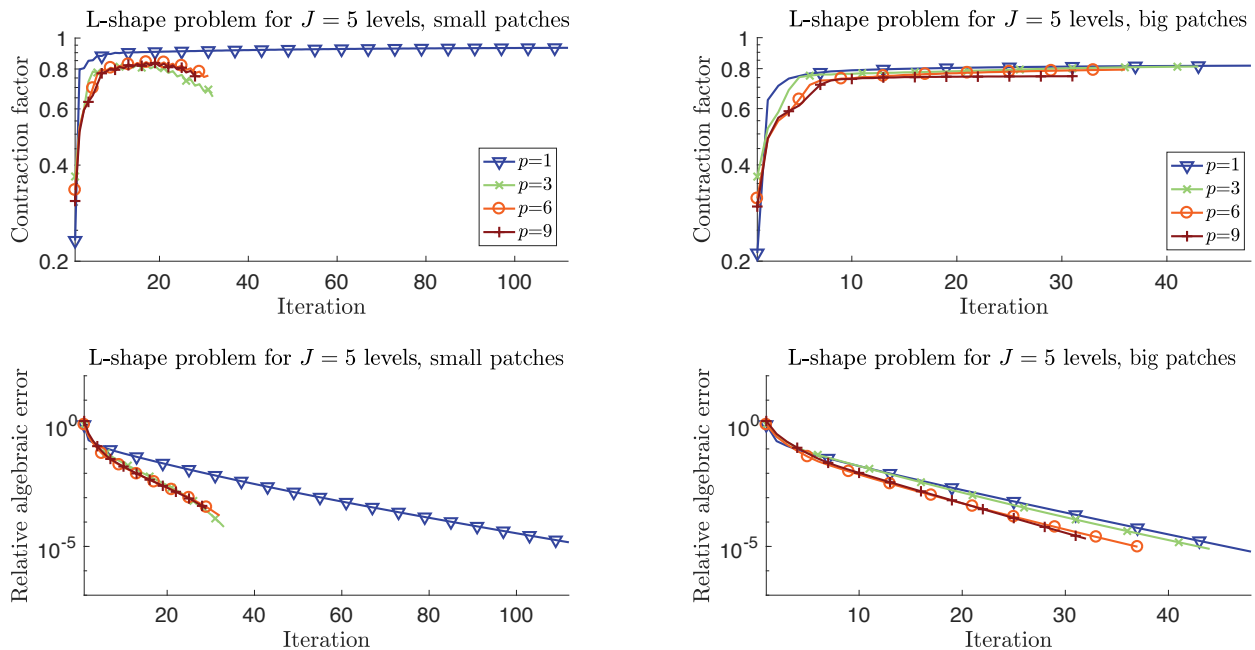


Figure 4: L-shape problem (6.3): results of the multilevel solver (4.4) for  $p' = p$  in (2.8a), “small” (left) and “big” (right) patches, and stopping criterion (6.6); top: error contraction factors  $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$ ; bottom: relative algebraic error  $\|\nabla(u_J - u_J^i)\|/\|\nabla u_J\|$ .

$J$	$p$	DoF	$\text{cond}(A_J)$	“small” patches				“big” patches				MG(0,1)			
				dAS (6.4)	$i_{\text{stop}}$	$\bar{\alpha}$	wRAS (6.5)	$i_{\text{stop}}$	$\bar{\alpha}$	dAS (6.4)	$i_{\text{stop}}$	$\bar{\alpha}$	wRAS (6.5)	$i_{\text{stop}}$	$\bar{\alpha}$
3	1	5057	$2.26 \times 10^3$	76	0.87	17	0.50	32	0.72	8	0.22	44	9	-	-
	3	46273	$4.40 \times 10^4$	26	0.68	12	0.41	27	0.63	5	0.09	-	49	-	-
	6	185857	$4.95 \times 10^5$	23	0.69	10	0.36	22	0.60	5	0.08	-	228	-	-
	9	418753	$4.72 \times 10^6$	21	0.68	10	0.35	18	0.55	5	0.07	-	586	-	-
4	1	20481	$9.04 \times 10^3$	95	0.89	18	0.52	40	0.77	8	0.22	-	9	-	-
	3	185857	$1.76 \times 10^5$	29	0.71	12	0.41	35	0.68	5	0.09	-	42	-	-
	6	744961	$1.98 \times 10^6$	27	0.73	10	0.36	29	0.64	5	0.08	-	186	-	-
	9	1677313	$1.89 \times 10^7$	25	0.72	9	0.33	24	0.61	5	0.07	-	454	-	-
5	1	82433	$3.62 \times 10^4$	112	0.91	17	0.51	48	0.80	8	0.22	-	8	-	-
	3	744961	$7.05 \times 10^5$	32	0.75	12	0.40	43	0.70	5	0.09	-	35	-	-
	6	2982913	$7.92 \times 10^6$	31	0.77	9	0.34	36	0.68	5	0.08	-	147	-	-
	9	6713857	$7.55 \times 10^7$	28	0.76	8	0.31	31	0.71	4	0.07	-	333	-	-

Table 3: L-shape problem (6.3): comparison of the solver of Definition 4.4 with the constructions dAS (6.4) and wRAS (6.5) for  $p' = p$  in (2.8a), “small” and “big” patches.  $i_{\text{stop}}$ : the number of iterations needed to reach the stopping criterion (6.6);  $\bar{\alpha}$ : average error contraction factor given by (6.7). Last two columns: comparison with standard multigrid method with piecewise affine (lowest-order) coarse grid correction (3.1), initialized by the coarse grid solution (3.1), no pre-smoothing, one post-smoothing step.

### 6.3 Comparison with other multilevel solvers

Some recent comparisons of state-of-the-art solvers for Poisson problems with multigrid methods in the high-order setting include Gholami *et al.* [22], Sundar *et al.* [42], and Kronbichler and Wall [29]. In Sundar *et al.* [42], it was in particular reported that none of the methods considered behaves fully independently of the polynomial degree. In this subsection, we compare our developments with 4 well-established options. We

focus on the number of iterations, but we also indicate CPU times of our vectorized Matlab implementation<sup>1</sup>, trusting the reader to understand the trickiness inherent to such implementation- and machine-dependent measurement. The timings below involve the solution time only; i.e. they do not include the assembly time of the matrices. The methods we consider for the comparison are:

**wRAS ( $\sim$ MG(0,1)-bJ):** cf. Definition 4.4 with small patches,  $p'=p$  in (2.8a) (to illustrate the associated space hierarchy, we use the notation “ $1, p \rightarrow p$ ”), and wRAS construction (6.5).

**wRAS<sub>1</sub> ( $\sim$ MG(0,1)-bJ):** cf. Definition 4.4 with small patches,  $p'=1$  in (2.8a) (to illustrate the associated space hierarchy, we use the notation “ $1 \rightarrow 1, p$ ”), and wRAS construction (6.5).

**PCG(MG(3,3)-bJ):** Preconditioned CG solver; the preconditioner is multigrid V-cycle(3,3) with weighted block Jacobi smoother associated to small patches; the space hierarchy relies on order  $p$  discretization, including the coarsest space (we use the notation “ $p \rightarrow p$ ”); the iterations start with the zero vector. This choice of the solver is motivated by Antonietti and Pennesi [2], adapted to the conforming finite elements setting.

**MG(1,1)-PCG(iChol):** Multigrid solver V-cycle(1,1); the smoother is PCG with incomplete zero level fill-in Cholesky preconditioner; the space hierarchy is of increasing order: from order 1 for coarsest level to order  $p$  for the finest level (“ $1 \nearrow p$ ”); the iterations start with the zero vector. This choice of the solver is motivated by Botti *et al.* [10], adapted for a symmetric setting.

**MG(0,1)-bGS:** Multigrid solver V-cycle(0,1); the smoother is block Gauss–Seidel associated to small patches; the space hierarchy consists of order 1 for all levels except the finest level, which is of order  $p$  (“ $1 \rightarrow 1, p$ ”), i.e., as in (2.8a) with  $p'=1$ ; the iterations start with the zero vector. This choice of the solver is motivated by NGSolve [41], however, the multigrid is used here as a solver instead of a preconditioner.

**MG(3,3)-GS:** Multigrid solver V-cycle(3,3); the smoother is a standard Gauss–Seidel; the space hierarchy is of increasing order: from order 1 for coarse level to order  $p$  for the finest level (“ $1 \nearrow p$ ”); the iterations start with the zero vector.

Table 4: Comparison of various multilevel solvers (described in Section 6.3) for the L-shape case (6.3),  $i_{\text{stop}}$ : the number of iterations to reach the stopping criterion (6.6).

$J$	$p$	DoF	wRAS $1, p \rightarrow p$		wRAS <sub>1</sub> $1 \rightarrow 1, p$		PCG(MG (3,3)-bJ) $p \rightarrow p$		MG(1,1)- PCG(iChol) $1 \nearrow p$		MG(0,1)- bGS $1 \rightarrow 1, p$		MG(3,3)- GS $1 \nearrow p$	
			$i_{\text{stop}}$	time	$i_{\text{stop}}$	time	$i_{\text{stop}}$	time	$i_{\text{stop}}$	time	$i_{\text{stop}}$	time	$i_{\text{stop}}$	time
3	1	5057	17	0.0 s	17	0.0 s	7	0.0 s	4	0.1 s	9	0.0 s	3	0.0 s
	3	46 273	12	0.2 s	18	0.2 s	3	0.2 s	14	0.5 s	8	1.0 s	4	0.1 s
	6	185 857	10	1.5 s	15	1.7 s	2	2.0 s	21	7.6 s	7	2.4 s	9	1.6 s
	9	418 753	10	7.2 s	14	7.7 s	2	10.5 s	63	1.2m	6	7.4 s	9	4.3 s
4	1	20 481	18	0.0 s	18	0.0 s	8	0.1 s	7	0.1 s	9	0.0 s	3	0.0 s
	3	185 857	12	1.0 s	18	1.0 s	3	0.8 s	29	4.1 s	8	4.3 s	4	0.3 s
	6	744 961	10	8.4 s	15	7.5 s	3	11.4 s	49	58.9 s	7	11.9 s	5	2.9 s
	9	1 677 313	9	29.7 s	13	36.1 s	2	30.3 s	167	12.5m	6	29.2 s	8	16.0 s
5	1	82 433	17	0.2 s	17	0.2 s	8	0.3 s	19	0.8 s	8	0.1 s	3	0.1 s
	3	744 961	12	3.4 s	17	3.6 s	3	3.6 s	77	57.7 s	8	16.1 s	4	1.5 s
	6	2 982 913	9	24.3 s	14	26.8 s	3	38.9 s	129	11.6m	7	44.5 s	4	10.0 s
	9	6 713 857	8	2.2m	12	2.2m	2	3.5m	+200	+1.0 h	6	2.1m	8	1.2m

The presented methods split into two groups: numerically  $p$ -robust (wRAS, wRAS<sub>1</sub>, PCG(MG(3,3)-bJ), MG(0,1)-bGS) and not (MG(1,1)-PCG(iChol), MG(3,3)-GS). Note that the present choice of

<sup>1</sup>The codes were prepared to benefit as much as possible from Matlab’s fast operations on matrices and vectors. The experiments were run on one Dell C6220 dual-Xeon E5-2650 node of Inria Sophia Antipolis - Méditerranée “NEF” computation cluster, however, in a sequential Matlab script.



3 pre- and 3 post-smoothing steps makes every iteration of **PCG(MG(3,3)-bJ)** and **MG(3,3)-GS** considerably more expensive than those of the methods **wRAS**, **wRAS<sub>1</sub>**, **MG(0,1)-bGS**, where the minimalist (0,1) choice is sufficient. In **PCG(MG(3,3)-bJ)**, in addition, the coarse grid correction is more expensive as it uses order  $p$  approximations. The inversion of the Jacobi blocks in **PCG(MG(3,3)-bJ)** on the finest level  $J$ , corresponds to solving the patch problems of order  $p$  as in (3.7), so that its cost is the same as for the local problems of **wRAS<sub>1</sub>**. As for **MG(1,1)-PCG(iChol)**, we find the method to be quite satisfactory for lower-order approximations, but as soon as  $p$  and  $J$  increase, the number of iterations degrades considerably. **MG(0,1)-bGS**, like **wRAS<sub>1</sub>**, only employs one post-smoothing step per iteration and it uses the same polynomial degree distribution over the levels. In contrast to **wRAS<sub>1</sub>** however, **MG(0,1)-bGS** is a multiplicative Schwarz method, and is thus less suitable for parallelization. The classical **MG(3,3)-GS** is a combination of  $h$ - and  $p$ -multigrid and gives the best timings in our experiments. The numbers of pre- and post-smoothing steps, however, remain parameters, and their tuning might not be straightforward in order to get an efficient and numerically robust multigrid solver in general (cf. the poor results of the very similar, up to a different number of pre- and post-smoothing steps and a stronger hierarchy, **MG(0,1)-GS** version in Tables 1–3). The Gauss–Seidel smoother used therein again makes the method harder to parallelize.

## 7 Proof of Theorem 5.1

As shown in Corollary 5.3, the results of Theorem 5.1 and Theorem 5.2 are equivalent. Therefore it suffices to prove the first one. Our approach to proving Theorem 5.1 consists in studying the uncomputable exact residual lifting  $\tilde{\rho}_{J,\text{alg}}^i$  given by (3.2) and its approximation  $\rho_{J,\text{alg}}^i$  given by Definition 3.1. In particular, we will estimate  $p$ -robustly the quantities  $\|\nabla\tilde{\rho}_{J,\text{alg}}^i\|$ ,  $\|\nabla\rho_{J,\text{alg}}^i\|$ , and  $(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla\rho_{J,\text{alg}}^i)$  by local contributions  $\rho_{j,\mathbf{a}}^i$  of (3.7) used to construct  $\rho_{J,\text{alg}}^i$ . This will allow us to prove the claim of the theorem:  $\eta_{\text{alg}}^i \geq \beta\|\nabla(u_J - u_J^i)\|$ , with  $\eta_{\text{alg}}^i$  our a posteriori estimator of Definition 4.1 and  $\beta$  a  $p$ -independent constant. Actually, due to (3.4) and Definition 4.1, this is equivalent to showing

$$\begin{aligned} \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla\rho_{J,\text{alg}}^i)}{\|\nabla\rho_{J,\text{alg}}^i\|} &\geq \beta\|\nabla\tilde{\rho}_{J,\text{alg}}^i\| && \text{when } \rho_{J,\text{alg}}^i \neq 0, \\ \|\nabla\tilde{\rho}_{J,\text{alg}}^i\| &= 0 && \text{when } \rho_{J,\text{alg}}^i = 0. \end{aligned}$$

Hereafter, we will use the notation  $x_1 \lesssim x_2$  when there exists  $c$ , a positive real constant only depending on the mesh shape regularity parameter  $\kappa_{\mathcal{T}}$  and the space dimension  $d$  such that  $x_1 \leq cx_2$ . Similarly,  $x_1 \gtrsim x_2$  means  $x_2 \lesssim x_1$  and  $x_1 \approx x_2$  means that  $x_1 \lesssim x_2$  and  $x_2 \lesssim x_1$  simultaneously. If these constants additionally depend on the number of mesh levels  $J$ , we use the notations  $\lesssim_J$ ,  $\gtrsim_J$ , and  $\approx_J$ , respectively.

### 7.1 Upper bound on $\|\nabla\rho_{J,\text{alg}}^i\|$ by patchwise contributions from all mesh levels

We present here properties of the constructed residual lifting  $\rho_{J,\text{alg}}^i$  and its level-wise components  $\rho_j^i$ , where  $1 \leq j \leq J$ .

**Lemma 7.1** (Estimating  $\|\nabla\rho_{J,\text{alg}}^i\|$  and  $\|\nabla\rho_j^i\|$  by local contributions). *Let  $\rho_{J,\text{alg}}^i$  and  $\rho_j^i$  be given by Definition 3.1, for  $j \in \{1, \dots, J\}$ . There holds*

$$\|\nabla\rho_j^i\|^2 \leq \frac{1}{J^2(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla\rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2, \quad (7.2)$$

$$\|\nabla\rho_{J,\text{alg}}^i\|^2 \leq 2 \left( \|\nabla\rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla\rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \right). \quad (7.3)$$

*Proof.* Definition 3.1 and the inequality  $|\sum_{k=1}^{d+1} a_k|^2 \leq (d+1) \sum_{k=1}^{d+1} |a_k|^2$  lead to

$$\begin{aligned} \|\nabla \rho_j^i\|^2 &= \sum_{K \in \mathcal{T}_{j-s}} \|\nabla \rho_j^i\|_K^2 = \sum_{K \in \mathcal{T}_{j-s}} \left\| \frac{1}{J(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_K} \nabla \rho_{j,\mathbf{a}}^i \right\|_K^2 \\ &\leq \frac{d+1}{J^2(d+1)^2} \sum_{K \in \mathcal{T}_{j-s}} \sum_{\mathbf{a} \in \mathcal{V}_K} \|\nabla \rho_{j,\mathbf{a}}^i\|_K^2 = \frac{1}{J^2(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2. \end{aligned}$$

Note that this allows us to write

$$\left\| \sum_{j=1}^J \nabla \rho_j^i \right\|^2 \leq J \sum_{j=1}^J \|\nabla \rho_j^i\|^2 \leq \frac{J}{J^2(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2. \quad (7.4)$$

This property facilitates the writing of the sought estimate for  $\rho_{J,\text{alg}}^i$ ,

$$\begin{aligned} \|\nabla \rho_{J,\text{alg}}^i\|^2 &= \left\| \nabla \rho_0^i + \sum_{j=1}^J \nabla \rho_j^i \right\|^2 \leq 2\|\nabla \rho_0^i\|^2 + 2 \left\| \sum_{j=1}^J \nabla \rho_j^i \right\|^2 \\ &\stackrel{(7.4)}{\leq} 2\|\nabla \rho_0^i\|^2 + \frac{2}{J(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \\ &\leq 2 \left( \|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \right). \end{aligned}$$

□

## 7.2 Lower bound on $(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)$ by patchwise contributions from all mesh levels

While studying  $(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)$ , the interaction of different level contributions of the lifting  $\rho_{J,\text{alg}}^i$  arises naturally. In order to estimate these terms, the damping  $J(d+1)$  used in the construction (3.6) of our lifting proves to be of the essence.

**Lemma 7.2** (*p*-robust estimate on  $(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)$  from below by patchwise contributions). *Let  $\rho_{J,\text{alg}}^i$  be given by Definition 3.1. Then*

$$(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i) \gtrsim_J \|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2. \quad (7.5)$$

*Proof.* We begin by using the construction of  $\rho_{J,\text{alg}}^i$  given in Definition 3.1 to write

$$\begin{aligned} (f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i) &= (f, \rho_0^i) - (\nabla u_J^i, \nabla \rho_0^i) + \sum_{j=1}^J \left( (f, \rho_j^i) - (\nabla u_J^i, \nabla \rho_j^i) \right) \\ &\stackrel{(3.1)}{=} \|\nabla \rho_0^i\|^2 + \frac{1}{J(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \left( (f, \rho_{j,\mathbf{a}}^i)_{\omega_{j,s}^{\mathbf{a}}} - (\nabla u_J^i, \nabla \rho_{j,\mathbf{a}}^i)_{\omega_{j,s}^{\mathbf{a}}} \right) \\ &\stackrel{(3.7)}{=} \|\nabla \rho_0^i\|^2 + \frac{1}{J(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \left( \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 + \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla \rho_{j,\mathbf{a}}^i)_{\omega_{j,s}^{\mathbf{a}}} \right) \\ &= \|\nabla \rho_0^i\|^2 + \frac{1}{J(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 + \sum_{j=1}^J \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla \rho_j^i). \end{aligned}$$

The first two terms above are of the right form to prove the result, one needs to be a bit more careful with the third one, however. We estimate it using Young's inequality, where the parameter is chosen in function of the damping factor  $1/(J(d+1))$ , and the sum interchange  $\sum_{j=2}^J \sum_{k=1}^{j-1} = \sum_{k=1}^{J-1} \sum_{j=k+1}^J$

$$\begin{aligned}
\sum_{j=1}^J \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla \rho_j^i) &= \sum_{j=2}^J \sum_{k=1}^{j-1} (\nabla \rho_k^i, \nabla \rho_j^i) + \sum_{j=1}^J (\nabla \rho_0^i, \nabla \rho_j^i) \\
&\geq \sum_{j=2}^J \sum_{k=1}^{j-1} \left( -\frac{1}{2} \|\nabla \rho_k^i\|^2 - \frac{1}{2} \|\nabla \rho_j^i\|^2 \right) + \sum_{j=1}^J \left( -\frac{1}{2 \frac{2J}{3}} \|\nabla \rho_0^i\|^2 - \frac{\frac{2J}{3}}{2} \|\nabla \rho_j^i\|^2 \right) \\
&= -\frac{1}{2} \sum_{k=1}^{J-1} (J-k) \|\nabla \rho_k^i\|^2 - \frac{1}{2} \sum_{j=2}^J (j-1) \|\nabla \rho_j^i\|^2 - \frac{3}{4J} J \|\nabla \rho_0^i\|^2 - \frac{J}{3} \sum_{j=1}^J \|\nabla \rho_j^i\|^2 \\
&= -\frac{1}{2} \sum_{j=1}^J (J-j) \|\nabla \rho_j^i\|^2 - \frac{1}{2} \sum_{j=1}^J (j-1) \|\nabla \rho_j^i\|^2 - \frac{3}{4} \|\nabla \rho_0^i\|^2 - \frac{J}{3} \sum_{j=1}^J \|\nabla \rho_j^i\|^2,
\end{aligned}$$

where we added the terms in the sum corresponding to  $k = J$  and  $j = 1$  since they are zero, and then renamed the summation index when there is no confusion. A few more manipulations on the right-hand side and the use of (7.2), give us

$$\sum_{j=1}^J \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla \rho_j^i) \geq -\frac{3}{4} \|\nabla \rho_0^i\|^2 - \frac{5J}{6} \frac{1}{J^2(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2.$$

Returning to the main estimate, we obtain the desired result

$$(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i) \geq \frac{1}{6J(d+1)} \left( \|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \right). \quad (7.6)$$

□

### 7.3 Upper bound on $\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|$ by patchwise contributions from all mesh levels

Recall that  $\tilde{\rho}_{J,\text{alg}}^i$ , introduced in (3.2), is the unknown exact algebraic error. We estimate here  $\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|$  from above. First, we summarize for our setting the remarkable result of Schöberl et al. [40], stating the existence of a  $p$ -robust stable decomposition for a fixed mesh. Then, we adapt this result to the multilevel case. This, together with inter-level and local properties of  $\tilde{\rho}_{J,\text{alg}}^i$  introduced hereafter, allows to obtain a  $p$ -robust estimate on the algebraic error.

#### 7.3.1 Polynomial-degree-robust stable decomposition on a fixed mesh level $j$

We begin by presenting in the form of a lemma the  $p$ -robust stable decomposition result of Schöberl et al. [40, Proof of Theorem 2.1]. This decomposition of any function on a given level into a continuous piecewise affine part and a sum of local continuous piecewise polynomials of degree  $p$  will be particularly important in the following. We also use the local spaces  $V_{j,s}^{\mathbf{a}}$  introduced in (2.10).

**Lemma 7.3** (One-level  $p$ -robust stable decomposition). *Let  $1 \leq j \leq J$  and  $v_j \in V_j^{p'}$  when  $1 \leq j \leq J-1$ , or  $v_J \in V_J^p$  be arbitrary. Suppose  $v_j = v_j^\# + v_j^{\mathbf{b}}$ , with  $v_j^\# \in V_j^1$  such that*

$$\|\nabla v_j^\#\|^2 + \|\nabla(v_j - v_j^\#)\|^2 + \sum_{K \in \mathcal{T}_j} h_K^{-1} \|(v_j - v_j^\#)\|_K^2 \leq C_{\text{LO}} \|\nabla v_j\|^2, \quad (7.7)$$

for a positive constant  $C_{\text{LO}}$  only depending on mesh shape regularity parameter  $\kappa_{\mathcal{T}}$  and space dimension  $d$ . Then, there are  $v^{\mathbf{b}} \in V_{j,0}^{\mathbf{b}}$ ,  $\mathbf{b} \in \mathcal{V}_j$ , such that  $v_j = v_j^\# + \sum_{\mathbf{b} \in \mathcal{V}_j} v_j^{\mathbf{b}}$ , and this decomposition is stable in the sense

$$\exists C_{\text{SD}} > 0 \quad \text{such that} \quad \|\nabla v_j^\#\|^2 + \sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla v_j^{\mathbf{b}}\|_{\omega_{j,0}^{\mathbf{b}}}^2 \leq C_{\text{SD}} \|\nabla v_j\|^2, \quad (7.8)$$

where  $C_{\text{SD}}$  only depends on the constant  $C_{\text{LO}}$ , mesh shape regularity parameter  $\kappa_{\mathcal{T}}$ , and space dimension  $d$ . Additionally, we can assume that  $C_{\text{SD}} > 1$  (otherwise, we work with  $\tilde{C}_{\text{SD}} := \max(1, C_{\text{SD}})$  which still satisfies (7.8)).

**Remark 7.4.** Note that this decomposition is straightforward for  $v_j \in V_j^1$  and  $1 \leq j \leq J-1$ , i.e., for the choice  $p' = 1$ .

### 7.3.2 Polynomial-degree-robust stable decomposition on a hierarchy of meshes

Because we intend to obtain a decomposition similar to Lemma 7.3 in a multilevel setting, we will work with  $v^\# \in V_0$ . For this purpose, we first define a coarse space interpolator.

**Lemma 7.5** (Coarse space interpolation). *For  $v \in H_0^1(\Omega)$ , define the vertex values*

$$\begin{aligned} (\mathcal{C}_0 v)(\mathbf{a}) &:= \frac{1}{|\mathcal{T}_{0,0}^{\mathbf{a}}|} \sum_{K \in \mathcal{T}_{0,0}^{\mathbf{a}}} \bar{v}_K \quad \text{for } \mathbf{a} \in \mathcal{V}_0^{\text{int}}, \quad \text{where } \bar{v}_K := \frac{(v, 1)_K}{|K|}, \\ (\mathcal{C}_0 v)(\mathbf{a}) &:= 0 \quad \text{for } \mathbf{a} \in \mathcal{V}_0^{\text{ext}}. \end{aligned}$$

where  $\mathcal{T}_{0,0}^{\mathbf{a}}$  is the patch of elements sharing the vertex  $\mathbf{a}$ , cf. (2.9). The operator  $\mathcal{C}_0 : H_0^1(\Omega) \rightarrow V_0$ , given by  $\mathcal{C}_0(v) := \sum_{\mathbf{a} \in \mathcal{V}_0} \mathcal{C}_0 v(\mathbf{a}) \psi_0^{\mathbf{a}}$ , satisfies for all  $v \in H_0^1(\Omega)$

$$\|\nabla \mathcal{C}_0(v)\|^2 + \|\nabla(v - \mathcal{C}_0(v))\|^2 + \sum_{K \in \mathcal{T}_0} h_K^{-1} \|(v - \mathcal{C}_0(v))\|_K^2 \leq \tilde{C}_{\text{LO}} \|\nabla v\|^2. \quad (7.9)$$

*Proof.* We start by estimating the first term in (7.9). Consider  $v \in H_0^1(\Omega)$ ,  $K \in \mathcal{T}_0$ . Note that the hat functions form a partition of unity:  $\sum_{\mathbf{a} \in \mathcal{V}_0} \psi_0^{\mathbf{a}} = 1$ . Then

$$\begin{aligned} \|\nabla \mathcal{C}_0(v)\|_K &= \|\nabla \mathcal{C}_0(v) - \bar{v}_K \nabla 1\|_K = \left\| \nabla \left( \sum_{\mathbf{a} \in \mathcal{V}_K} (\mathcal{C}_0 v)(\mathbf{a}) \psi_0^{\mathbf{a}} \right) - \bar{v}_K \nabla \left( \sum_{\mathbf{a} \in \mathcal{V}_K} \psi_0^{\mathbf{a}} \right) \right\|_K \\ &= \left\| \sum_{\mathbf{a} \in \mathcal{V}_K} \{((\mathcal{C}_0 v)(\mathbf{a}) - \bar{v}_K) \nabla \psi_0^{\mathbf{a}}\} \right\|_K \leq \sum_{\mathbf{a} \in \mathcal{V}_K} \{ |(\mathcal{C}_0 v)(\mathbf{a}) - \bar{v}_K| \|\nabla \psi_0^{\mathbf{a}}\|_K \} \\ &\lesssim h_K^{-1} |K|^{\frac{1}{2}} \left( \sum_{\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{int}}} |\mathcal{T}_{0,0}^{\mathbf{a}}|^{-1} \sum_{K^* \in \mathcal{T}_{0,0}^{\mathbf{a}}} |\bar{v}_{K^*} - \bar{v}_K| + \sum_{\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{ext}}} |\mathcal{T}_{0,0}^{\mathbf{a}}|^{-1} \sum_{K^* \in \mathcal{T}_{0,0}^{\mathbf{a}}} |\bar{v}_K| \right), \end{aligned}$$

where  $|\mathcal{T}_{0,0}^{\mathbf{a}}|$  is uniformly bounded by the mesh shape regularity parameter  $\kappa_{\mathcal{T}}$  and space dimension  $d$ . We distinguish two cases.

*Case  $\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{int}}$ .* There are two possibilities: either  $K^*$  and  $K$  share an interface  $F_1$ , or there is a path of elements in the neighborhood of  $K$  connecting  $K$  and  $K^*$  such that  $|\bar{v}_{K^*} - \bar{v}_K| \leq \sum_{l=0}^{L_1-1} |\bar{v}_{K_{l+1}} - \bar{v}_{K_l}|$  by the triangle inequality, where  $K_{L_1} = K^*$  and  $K_0 = K$ , and  $L_1$  only depends on the mesh shape regularity parameter  $\kappa_{\mathcal{T}}$  and space dimension  $d$ . Thus, we need only treat the case where  $K^*$  and  $K$  share an interface  $F_1$ . Note that, since  $v \in H_0^1(\Omega)$ , its trace is well defined, and we can introduce the notation  $\bar{v}_{F_1} := (v, 1)_{F_1} / |F_1| \in \mathbb{R}$ . Using interpolation estimates for simplices shown in, e.g., Eymard *et al.* [20, Lemma 2], Vohralík [43, Lemma 4.1], we write

$$\begin{aligned} |\bar{v}_{K^*} - \bar{v}_K| &= |\bar{v}_{K^*} - \bar{v}_{F_1} + \bar{v}_{F_1} - \bar{v}_K| \leq |\bar{v}_{K^*} - \bar{v}_{F_1}| + |\bar{v}_{F_1} - \bar{v}_K| \\ &\lesssim \max_{K^\# \in \{K, K^*\}} (h_{K^\#} |K^\#|^{-\frac{1}{2}}) (\|\nabla v\|_{K^*} + \|\nabla v\|_K) \\ &\leq \max_{K^\# \in \{K, K^*\}} (h_{K^\#} |K^\#|^{-\frac{1}{2}}) \|\nabla v\|_{\omega_K}, \end{aligned} \quad (7.10)$$

where  $\omega_K := \cup_{\mathbf{a} \in \mathcal{V}_K} \omega_{0,0}^{\mathbf{a}}$ .

*Case  $\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{ext}}$ .* There are again two possibilities: either the intersection of  $K$  with  $\partial\Omega$  is an interface  $F_2$ , or it is the vertex  $\mathbf{a}$ . For the latter, there is again a path connecting  $K$  with  $\tilde{K}$ , such that the intersection of  $\tilde{K}$  and  $\partial\Omega$  is a face. Similarly to the first case, we can write  $|\bar{v}_K| \leq |\bar{v}_{\tilde{K}}| + \sum_{l=0}^{L_2-1} |\bar{v}_{K_l} - \bar{v}_{K_{l+1}}|$ , where

$K_{L_2} = \tilde{K}$  and  $K_0 = K$ , and  $L_2$  only depends on the mesh shape regularity parameter  $\kappa_{\mathcal{T}}$  and space dimension  $d$ . Note that the terms in the sum can be treated as in (7.10). Thus, it is sufficient to consider  $|\bar{v}_K|$  when  $K \cap \partial\Omega = F_2$ , a face of  $K$ . Since  $v \in H_0^1(\Omega)$ , we have  $\bar{v}_{F_2} := (v, 1)_{F_2}/|F_2| = 0$ ,

$$|\bar{v}_K| = |\bar{v}_K - \bar{v}_{F_2}| \lesssim h_K |K|^{-\frac{1}{2}} \|\nabla v\|_K.$$

In view of the mesh regularity Assumption 2.7, this leads to the desired estimate for the first term in (7.9)

$$\|\nabla \mathcal{C}_0(v)\| \lesssim \|\nabla v\|. \quad (7.11)$$

As for the second term, we use the triangle inequality and (7.11).

It remains to estimate the third term. We have

$$\begin{aligned} \|v - \mathcal{C}_0(v)\|_K &\leq \|v - \bar{v}_K 1\|_K + \|\bar{v}_K 1 - \mathcal{C}_0(v)\|_K \\ &\stackrel{\text{Poincaré}}{\lesssim} h_K \|\nabla v\|_K + \left\| \sum_{\mathbf{a} \in \mathcal{V}_K} (\bar{v}_K - (\mathcal{C}_0 v)(\mathbf{a})) \psi_0^{\mathbf{a}} \right\|_K \\ &\leq h_K \|\nabla v\|_K + \sum_{\mathbf{a} \in \mathcal{V}_K} |\bar{v}_K - (\mathcal{C}_0 v)(\mathbf{a})| \|\psi_0^{\mathbf{a}}\|_K \\ &\leq h_K \|\nabla v\|_K + |K|^{\frac{1}{2}} \left( \sum_{\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{int}}} |\mathcal{T}_{0,0}^{\mathbf{a}}|^{-1} \sum_{K^* \in \mathcal{T}_{0,0}^{\mathbf{a}}} |\bar{v}_K - \bar{v}_{K^*}| + \sum_{\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{ext}}} |\bar{v}_K| \right). \end{aligned}$$

The last two terms in this estimate can be treated similarly to (7.10), allowing us to obtain the inequality  $\|v - \mathcal{C}_0(v)\|_K \lesssim h_K \|\nabla v\|_{\omega_K}$ . Finally, putting the three estimations together, we obtain the desired result. We define  $\tilde{C}_{\text{LO}}$  the constant obtained after these estimations; it only depends on the mesh shape regularity parameter  $\kappa_{\mathcal{T}}$  and space dimension  $d$ .  $\square$

**Lemma 7.6** (Multilevel  $p$ -robust stable decomposition). *For any  $1 \leq j \leq J$  and  $v_j \in V_j^{p'}$  when  $1 \leq j \leq J-1$ , or  $v_j \in V_j^p$ , there exists a constant  $C_{\text{SD},J}$  only depending on the mesh shape regularity parameter  $\kappa_{\mathcal{T}}$ , space dimension  $d$ , and the number of mesh levels  $J$ , such that*

$$v_j = \mathcal{C}_0(v_j) + \sum_{\mathbf{b} \in \mathcal{V}_j} v_j^{\mathbf{b}}, \quad v_j^{\mathbf{b}} \in V_{j,0}^{\mathbf{b}}; \quad \|\nabla \mathcal{C}_0(v_j)\|^2 + \sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla v_j^{\mathbf{b}}\|_{\omega_{j,0}^{\mathbf{b}}}^2 \leq C_{\text{SD},J} \|\nabla v_j\|^2. \quad (7.12)$$

*Proof.* By Lemma 7.5, and since  $h_K \approx_J h_{K^*}$  for all  $K \in \mathcal{T}_0$  and all  $K^* \in \mathcal{T}_j$ ,  $K^* \subset K$  by Assumption 2.6, there holds

$$\|\nabla \mathcal{C}_0(v_j)\|^2 + \|\nabla(v_j - \mathcal{C}_0(v_j))\|^2 + \sum_{K \in \mathcal{T}_j} h_K^{-1} \|(v_j - \mathcal{C}_0(v_j))\|_K^2 \lesssim_J \|\nabla v_j\|^2,$$

where the constant in the estimate above depends on  $\tilde{C}_{\text{LO}}$  of (7.9) and the number of mesh levels  $J$ . Thus, we have  $\mathcal{C}_0(v_j) \in V_0 \subset V_j^1$  which satisfies (7.7). Using the result of Schöberl et al. [40] described in Lemma 7.3, we obtain  $v_j^{\mathbf{b}} \in V_{j,0}^{\mathbf{b}}$ , for  $\mathbf{b} \in \mathcal{V}_j$  such that (7.12) holds with a constant  $C_{\text{SD},J}$  only depending on the mesh shape regularity parameter  $\kappa_{\mathcal{T}}$ , space dimension  $d$ , and number of mesh levels  $J$ .  $\square$

### 7.3.3 Orthogonality of $\tilde{\rho}_j^i$ , local links between $\tilde{\rho}_j^i$ and $\rho_{j,\mathbf{a}}^i$

Two other important components will serve in estimating  $\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|$ . First, the relations of orthogonality of a given mesh error contribution  $\tilde{\rho}_j^i$ ,  $j \in \{1, \dots, J\}$ , with respect to previous mesh level functions. And secondly, the local properties of  $\tilde{\rho}_j^i$  with respect to local functions of the same mesh. In particular, the local properties will allow the transition from the uncomputable  $\tilde{\rho}_j^i$  to the available local contributions of  $\rho_j^i$ .

**Lemma 7.7** (Inter-level properties of  $\tilde{\rho}_j^i$ ). *Consider the hierarchical construction of the error  $\tilde{\rho}_{J,\text{alg}}^i$  given in (3.2). For  $j \in \{1, \dots, J\}$  and  $k \in \{0, \dots, j-1\}$ , there holds*

$$(\nabla \tilde{\rho}_j^i, \nabla v_k) = 0 \quad \forall v_k \in V_k^p. \quad (7.13)$$

*Proof.* Take  $v_k \in V_k^{p'}$ . Note that since  $k \leq j-1$ , we have  $v_k \in V_{j-1}^{p'}$ . If  $j \in \{1, \dots, J-1\}$ ,  $V_{j-1}^{p'} \subset V_j^{p'}$ , and if  $j = J$ ,  $V_{J-1}^{p'} \subset V_J^p$ . The definition given in (3.3) applied to  $\tilde{\rho}_j^i$  and  $\tilde{\rho}_{j-1}^i$  allows us to write

$$\begin{aligned} (\nabla \tilde{\rho}_j^i, \nabla v_k) &= (f, v_k) - (\nabla u_J^i, \nabla v_k) - \sum_{l=0}^{j-2} (\nabla \tilde{\rho}_l^i, \nabla v_k) - (\nabla \tilde{\rho}_{j-1}^i, \nabla v_k) \\ &= (\nabla \tilde{\rho}_{j-1}^i, \nabla v_k) - (\nabla \tilde{\rho}_{j-1}^i, \nabla v_k) = 0. \end{aligned}$$

□

Below, we present the relation between  $\tilde{\rho}_j^i$  and  $\rho_j^i$  locally on patches, more precisely when tested against functions of the local spaces  $V_{j,s}^{\mathbf{a}}$  given by (2.10).

**Lemma 7.8** (Local relation between  $\tilde{\rho}_j^i$  and  $\rho_{j,\mathbf{a}}^i$ ). *Let  $j \in \{1, \dots, J\}$ . Let  $\tilde{\rho}_j^i, \rho_j^i$  be given by (3.3), (3.6), respectively. For all  $\mathbf{a} \in \mathcal{V}_{j-s}$  and all  $v_{j,\mathbf{a}} \in V_{j,s}^{\mathbf{a}}$ , we have*

$$(\nabla \tilde{\rho}_j^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} = (\nabla \rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - \sum_{k=1}^{j-1} (\nabla (\tilde{\rho}_k^i - \rho_k^i), \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}}, \quad (7.14)$$

where  $\rho_{j,\mathbf{a}}^i \in V_{j,s}^{\mathbf{a}}$  is defined as solution of a local problem by (3.7). We use the convention that the sum in the relation above is zero when  $j = 1$ .

*Proof.* We take  $v_{j,\mathbf{a}} \in V_{j,s}^{\mathbf{a}}$ . This implies that  $v_{j,\mathbf{a}}$  is zero on the boundary of the patch domain  $\omega_{j,s}^{\mathbf{a}}$ . Since  $v_{j,\mathbf{a}} \in V_j^{p'}$  when  $j \in \{1, \dots, J-1\}$  and  $v_{J,\mathbf{a}} \in V_J^p$  otherwise, we can use it as a test function in the definition of  $\tilde{\rho}_j^i$  in (3.3) as well as in the definition of  $\rho_{j,\mathbf{a}}^i$  in (3.7). We conclude by subtracting the two following identities once we take into account  $\tilde{\rho}_0^i = \rho_0^i$

$$\begin{aligned} (\nabla \tilde{\rho}_j^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} &= (f, v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - (\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - \sum_{k=0}^{j-1} (\nabla \tilde{\rho}_k^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}}, \\ (\nabla \rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} &= (f, v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - (\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}}. \end{aligned}$$

□

### 7.3.4 Estimating the error on a hierarchy of meshes

The previous results allowed to establish a useful  $p$ -robust stable decomposition for a hierarchy of meshes, and summarize the inter-level and local properties of the error  $\tilde{\rho}_{J,\text{alg}}^i$ . These properties will be useful in the forthcoming lemma in order to give an upper bound to algebraic error  $\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|$  by the local constructed contributions  $\rho_{j,\mathbf{a}}^i$  of the hierarchy of meshes.

**Lemma 7.9** ( $p$ -robust error estimation). *Let  $\tilde{\rho}_{j,\text{alg}}^i$  and  $\rho_{j,\text{alg}}^i$  be defined by (3.2) and Definition 3.1, respectively. There holds*

$$\|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \gtrsim_J \|\nabla \tilde{\rho}_{J,\text{alg}}^i\|^2. \quad (7.15)$$

*Proof.* We begin by estimating  $\|\nabla \tilde{\rho}_j^i\|$ , where  $\tilde{\rho}_j^i \in V_j^p$  solves (3.3) for  $1 \leq j \leq J$ . From Lemma 7.6, using the stable decomposition result applied to  $\tilde{\rho}_j^i$ ,

$$\tilde{\rho}_j^i = \mathcal{C}_0(\tilde{\rho}_j^i) + \sum_{\mathbf{b} \in \mathcal{V}_j} \tilde{\rho}_{j,\mathbf{b}}^i, \quad \tilde{\rho}_{j,\mathbf{b}}^i \in V_{j,0}^{\mathbf{b}}, \quad (7.16a)$$

$$\|\nabla \mathcal{C}_0(\tilde{\rho}_j^i)\|^2 + \sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla \tilde{\rho}_{j,\mathbf{b}}^i\|_{\omega_{j,0}^{\mathbf{b}}}^2 \leq C_{\text{SD},J} \|\nabla \tilde{\rho}_j^i\|^2. \quad (7.16b)$$

In the case  $s = 1$ , i.e. of “big” patches of Figure 1 (right), note that for  $\mathbf{b} \in \mathcal{V}_j$ , we can pick a vertex  $\mathbf{a}_\mathbf{b} \in \mathcal{V}_{j-1}$  such that  $\omega_{j,0}^\mathbf{b} \subset \omega_{j,s}^{\mathbf{a}_\mathbf{b}}$ , and we can extend  $\tilde{\rho}_{j,\mathbf{b}}^i$  by zero to  $\omega_{j,s}^{\mathbf{a}_\mathbf{b}}$ , so that it can be used as test function in the local problems (7.14). There is no need for extension in the case  $s = 0$  of “small” patches of Figure 1 (left), where we can take  $\mathbf{a}_\mathbf{b} = \mathbf{b} \in \mathcal{V}_{j-1}$  and  $\omega_{j,0}^{\mathbf{a}_\mathbf{b}} = \omega_{j,0}^\mathbf{b}$ . We introduce  $\mathbf{q}_{j,s} := \max_{\mathbf{a} \in \mathcal{V}_{j-s}} \#\{\mathbf{b} \in \mathcal{V}_j \mid \omega_{j,0}^\mathbf{b} \subset \omega_{j,s}^\mathbf{a}\}$ . Note that  $\mathbf{q}_{j,s} \geq 1$  when  $s = 1$  and  $\mathbf{q}_{j,s} = 1$  for  $s = 0$ . We have

$$\begin{aligned}
\|\nabla \tilde{\rho}_j^i\|^2 &\stackrel{(7.16a)}{=} (\nabla \tilde{\rho}_j^i, \nabla \mathcal{C}_0(\tilde{\rho}_j^i)) + (\nabla \tilde{\rho}_j^i, \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i) \stackrel{(7.13)}{=} 0 + \sum_{\mathbf{b} \in \mathcal{V}_j} (\nabla \tilde{\rho}_j^i, \nabla \tilde{\rho}_{j,\mathbf{b}}^i)_{\omega_{j,0}^\mathbf{b}} \\
&\stackrel{(7.14)}{=} \sum_{\mathbf{b} \in \mathcal{V}_j} (\nabla \rho_{j,\mathbf{a}_\mathbf{b}}^i, \nabla \tilde{\rho}_{j,\mathbf{b}}^i)_{\omega_{j,s}^{\mathbf{a}_\mathbf{b}}} - \sum_{\mathbf{b} \in \mathcal{V}_j} \sum_{k=0}^{j-1} (\nabla(\tilde{\rho}_k^i - \rho_k^i), \nabla \tilde{\rho}_{j,\mathbf{b}}^i)_{\omega_{j,s}^{\mathbf{a}_\mathbf{b}}} \\
&= \sum_{\mathbf{b} \in \mathcal{V}_j} \left( \sqrt{2C_{\text{SD},J}} \nabla \rho_{j,\mathbf{a}_\mathbf{b}}^i, \frac{\nabla \tilde{\rho}_{j,\mathbf{b}}^i}{\sqrt{2C_{\text{SD},J}}} \right)_{\omega_{j,s}^{\mathbf{a}_\mathbf{b}}} - \sum_{k=1}^{j-1} \left( \nabla(\tilde{\rho}_k^i - \rho_k^i), \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \\
&\leq C_{\text{SD},J} \sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}_\mathbf{b}}^i\|_{\omega_{j,s}^{\mathbf{a}_\mathbf{b}}}^2 + \frac{\sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla \tilde{\rho}_{j,\mathbf{b}}^i\|_{\omega_{j,s}^{\mathbf{a}_\mathbf{b}}}^2}{4C_{\text{SD},J}} - \sum_{k=1}^{j-1} \left( \nabla(\tilde{\rho}_k^i - \rho_k^i), \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \\
&\stackrel{(7.16b)}{\leq} C_{\text{SD},J} \sum_{\substack{\mathbf{a} \in \mathcal{V}_{j-s} \\ \omega_{j,0}^\mathbf{b} \subset \omega_{j,s}^\mathbf{a}}} \sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^\mathbf{a}}^2 + \frac{\|\nabla \tilde{\rho}_j^i\|^2}{4} - \sum_{k=1}^{j-1} \left( \nabla(\tilde{\rho}_k^i - \rho_k^i), \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \\
&\leq \mathbf{q}_{j,s} C_{\text{SD},J} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^\mathbf{a}}^2 + \frac{1}{4} \|\nabla \tilde{\rho}_j^i\|^2 - \sum_{k=1}^{j-1} \left( \nabla(\tilde{\rho}_k^i - \rho_k^i), \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right). \tag{7.17}
\end{aligned}$$

For the special case of  $j = 1$ , the third term is not present since  $\tilde{\rho}_0^i = \rho_0^i$ . This leads to

$$\|\nabla \tilde{\rho}_1^i\|^2 \leq \frac{4}{3} \mathbf{q}_{1,s} C_{\text{SD},1} \sum_{\mathbf{a} \in \mathcal{V}_{1-s}} \|\nabla \rho_{1,\mathbf{a}}^i\|_{\omega_{1,s}^\mathbf{a}}^2 \leq 2 \mathbf{q}_{1,s} C_{\text{SD},1} \sum_{\mathbf{a} \in \mathcal{V}_{1-s}} \|\nabla \rho_{1,\mathbf{a}}^i\|_{\omega_{1,s}^\mathbf{a}}^2. \tag{7.18}$$

This would be enough to conclude the proof if  $J = 1$ , since

$$\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|^2 \stackrel{(3.4)}{=} \|\nabla \rho_0^i\|^2 + \|\nabla \tilde{\rho}_1^i\|^2 \stackrel{(7.18)}{\leq} 2 \mathbf{q}_{1,s} C_{\text{SD},1} \left( \|\nabla \rho_0^i\|^2 + \sum_{\mathbf{a} \in \mathcal{V}_{1-s}} \|\nabla \rho_{1,\mathbf{a}}^i\|_{\omega_{1,s}^\mathbf{a}}^2 \right).$$

When  $J > 1$ , we continue below the estimation for the third term in (7.17) for  $j \in \{2, \dots, J\}$ . The inter-level properties of  $\tilde{\rho}_{j,\text{alg}}^i$  presented in (7.13) are crucial here:

$$\begin{aligned}
& - \sum_{k=1}^{j-1} \left( \nabla(\tilde{\rho}_k^i - \rho_k^i), \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \stackrel{(7.16a)}{=} - \sum_{k=1}^{j-1} \left( \left( \nabla \tilde{\rho}_k^i, \nabla(\tilde{\rho}_j^i - \mathcal{C}_0(\tilde{\rho}_j^i)) \right) - \left( \nabla \rho_k^i, \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \right) \\
& \stackrel{(7.13)}{=} 0 + 0 + \sum_{k=1}^{j-1} \left( \sqrt{2C_{\text{SD},J} J(d+1)} \nabla \rho_k^i, \frac{1}{\sqrt{2C_{\text{SD},J} J(d+1)}} \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \\
& \leq C_{\text{SD},J} J(d+1) \sum_{k=1}^{j-1} \|\nabla \rho_k^i\|^2 + \frac{1}{4C_{\text{SD},J} J(d+1)} \sum_{k=1}^{j-1} \left\| \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right\|^2 \\
& \stackrel{(7.2)}{\leq} \frac{C_{\text{SD},J}}{J} \sum_{k=1}^{j-1} \sum_{\mathbf{a} \in \mathcal{V}_{k-s}} \|\nabla \rho_{k,\mathbf{a}}^i\|_{\omega_{k,s}^\mathbf{a}}^2 + \frac{1}{4} \|\nabla \tilde{\rho}_j^i\|^2.
\end{aligned}$$



Returning to (7.17), we obtain for all  $j \in \{2, \dots, J\}$

$$\|\nabla \tilde{\rho}_j^i\|^2 \leq 2\mathfrak{q}_{j,s} C_{\text{SD},J} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 + \frac{2C_{\text{SD},J}}{J} \sum_{k=1}^{j-1} \sum_{\mathbf{a} \in \mathcal{V}_{k-s}} \|\nabla \rho_{k,\mathbf{a}}^i\|_{\omega_{k,s}^{\mathbf{a}}}^2. \quad (7.19)$$

We achieve the result by summing these estimates on different levels. We denote by  $\mathfrak{q} := \max_{j \in \{1, \dots, J\}} \mathfrak{q}_{j,s}$ . Then

$$\begin{aligned} \|\nabla \tilde{\rho}_{J,\text{alg}}^i\|^2 &\stackrel{(3.4)}{=} \|\nabla \rho_0^i\|^2 + \|\nabla \tilde{\rho}_1^i\|^2 + \sum_{j=2}^J \|\nabla \tilde{\rho}_j^i\|^2 \\ &\stackrel{(7.18)}{\leq} \|\nabla \rho_0^i\|^2 + 2\mathfrak{q} C_{\text{SD},J} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 + \frac{2C_{\text{SD},J}}{J} \sum_{j=2}^J \sum_{k=1}^{j-1} \sum_{\mathbf{a} \in \mathcal{V}_{k-s}} \|\nabla \rho_{k,\mathbf{a}}^i\|_{\omega_{k,s}^{\mathbf{a}}}^2 \\ &\stackrel{(7.19)}{\leq} \|\nabla \rho_0^i\|^2 + 2\mathfrak{q} C_{\text{SD},J} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 + 2C_{\text{SD},J} \sum_{k=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{k-s}} \|\nabla \rho_{k,\mathbf{a}}^i\|_{\omega_{k,s}^{\mathbf{a}}}^2 \\ &\leq 4\mathfrak{q} C_{\text{SD},J} \left( \|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \right). \end{aligned} \quad (7.20)$$

□

## 7.4 Proof of Theorem 5.1

The results of the previous subsections allow us to give a concise proof of Theorem 5.1.

*Proof of Theorem 5.1. Case  $\rho_{J,\text{alg}}^i = 0$ .* By Definition 4.1 this means  $\eta_{\text{alg}}^i = 0$ , so that it suffices to show that  $u_J = u_J^i$ . We do this by using Lemma 7.2 and 7.9 which lead to

$$\begin{aligned} \|\nabla(u_J - u_J^i)\|^2 &\stackrel{(3.4)}{=} \|\nabla \tilde{\rho}_{J,\text{alg}}^i\|^2 \stackrel{(7.15)}{\lesssim_J} \|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \\ &\stackrel{(7.5)}{\lesssim_J} (f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i) = 0. \end{aligned}$$

*Case  $\rho_{J,\text{alg}}^i \neq 0$ .* In this case, we combine the results of Lemmas 7.1, 7.2, and 7.9

$$\begin{aligned} \eta_{\text{alg}}^i &= \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|} \stackrel{(7.3)}{\geq} \frac{1}{6J(d+1)\sqrt{2}} \left( \|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \right)^{\frac{1}{2}} \\ &\stackrel{(7.20)}{\geq} \frac{1}{12J(d+1)\sqrt{2\mathfrak{q}C_{\text{SD},J}}} \|\nabla \tilde{\rho}_{J,\text{alg}}^i\| \stackrel{(3.4)}{=} \beta \|\nabla(u_J - u_J^i)\|, \end{aligned}$$

for  $\beta := \frac{1}{12J(d+1)\sqrt{2\mathfrak{q}C_{\text{SD},J}}} > 0$ , depending only on the mesh shape regularity parameter  $\kappa_{\mathcal{T}}$ , the space dimension  $d$ , and the number of levels  $J$ . □

## 8 Conclusions and outlook

In this work, we presented a hierarchical construction of the algebraic residual lifting in the spirit of Papež *et al.* [36]. This lifting approximates the algebraic error by one iteration of a V-cycle multigrid with no

pre-smoothing steps, a single damped additive Schwarz post-smoothing step, and a coarse solve of lowest polynomial degree. The lifting leads us to an a posteriori estimator on the algebraic error and to a linear iterative solver. We showed that two following results are equivalent: the (reliable) a posteriori estimator is  $p$ -robustly efficient, and the solver contracts  $p$ -robustly the error at each iteration. The provided numerical tests agree with these theoretical findings. Moreover, we also presented numerical results for a modified solver corresponding to a weighted restricted additive Schwarz smoothing. In accordance with the literature, this modified solver is a further speed-up compared to the damped Schwarz smoothing. Although we currently cannot show that our  $p$ -robust theoretical result also applies to this construction, the use of high degree polynomials does not seem to cause a degradation of the solver. Thus far, our theory involves estimates depending on the number of mesh levels  $J$ , which we do not observe in the numerical results for the weighted restricted variant. Further work would explore this dependence and seek a theoretical improvement. In forthcoming works, we plan to develop adaptivity based on the property  $\|\nabla(u_J - u_J^i)\|^2 \approx \|\nabla\rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla\rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2$  shown in Section 7, i.e., a computable splitting equivalent to the error and localized not only level-wise but also patch-wise. Applications to more involved problems are also on our work list.

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