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A multilevel algebraic error estimator and the corresponding iterative solver with p -robust behavior*

Ani Miraçi^{†‡} Jan Papež^{†§} Martin Vohralík^{†‡}

March 18, 2019

Abstract

In this work, we consider conforming finite element discretizations of arbitrary polynomial degree $p \geq 1$ of the Poisson problem. We propose a multilevel a posteriori estimator of the algebraic error. We prove that this estimator is reliable and efficient (represents a two-sided bound of the error), with a constant independent of the degree p . We next design a multilevel iterative algebraic solver from our estimator and we show that this solver contracts the algebraic error on each iteration by a factor bounded independently of p . Actually, we show that these two results are equivalent. The p -robustness results rely on the work of Schöberl *et al.* [IMA J. Numer. Anal., 28 (2008), pp. 1–24] for one given mesh. We combine this with the design of an algebraic residual lifting constructed over a hierarchy of nested, unstructured simplicial meshes, in the spirit of Papež *et al.* [HAL Preprint 01662944, 2017]. This includes a global coarse-level lowest-order solve, with local higher-order contributions from the subsequent mesh levels. These higher-order contributions are given as solutions of mutually independent Dirichlet problems posed over patches of elements around vertices. This residual lifting is the core of our a posteriori estimator and determines the descent direction for the next iteration of our multilevel solver. Its construction can be seen as one geometric V-cycle multigrid step with zero pre- and one post-smoothing by damped additive Schwarz. Numerical tests are presented to illustrate the theoretical findings.

Key words: finite element method, stable decomposition, multilevel method, Schwarz method, a posteriori estimate, p -robustness

1 Introduction

The finite element method (FEM) is a widespread approach for discretizing problems given in the form of partial differential equations, and has been used in engineering for more than fifty years. For a thorough overview on the topic, we refer to e.g. Ciarlet [15], Ern and Guermond [17], or Brenner and Scott [13]. Many iterative methods have been suggested to treat the linear systems arising from finite element discretizations, see e.g. Bramble *et al.* [10] and [11], Hackbusch [20], Bank *et al.* [5], Brandt *et al.* [12], Oswald [28], or Zhang [36], and the references therein. A systematic description of iterative solvers is given by Xu in [35]. The convergence of these methods is typically robust with respect to the size of the mesh (h -robustness). In fact, this is one of the key advantages of multigrid methods. For the conjugate gradient method on the other hand, h -robustness is not intrinsic; this problem can be bypassed with the development of appropriate preconditioners.

If we are to consider methods of arbitrary polynomial degree, an additional question arises: how does the polynomial degree p affect the performance of the method? In this regard, a few results for p -version FEM

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include Foresti *et al.* [19] for two-dimensional domains, Mandel [26] for three-dimensional domains, and Babuška *et al.* [4] for two-dimensional domains. For the latter, the condition number of the preconditioned system grows at most by $1 + \log^2(p)$, and a generalization of this work is given for hp -FEM by Ainsworth [1], where the p -dependence is still present. An early version of a polynomial-degree robust (p -robust) solver was introduced by Quarteroni and Sacchi Landriani [32] for a specific domain configuration (decomposable into rectangles without internal cross points). Notable development on p -robustness was later made by Pavarino [31] for quadrilateral/hexahedral meshes, where the author introduced a p -robust additive Schwarz method. The generalization of this result for triangular/tetrahedral meshes was achieved by Schöberl *et al.* [33], once more by introducing an additive Schwarz preconditioner. More recent works were carried based on these approaches. In Antonietti *et al.* [2] (see also references therein), the p -robust approach for rectangular/hexahedral meshes was used for high-order discontinuous Galerkin (DG) methods; moreover the spectral bounds of the preconditioner are also robust with respect to the method's penalization coefficient. We also mention the introduction of multilevel preconditioners yielded by block Gauss–Seidel smoothers in Kanschat [23] for rectangular/hexahedral meshes and DG discretizations. Further multilevel approaches for rectangular/hexahedral meshes using overlapping or non-overlapping Schwarz smoothers can be found in e.g. Janssen and Kanschat [21] and Lucero Lorca and Kanschat [25]. Another notable contribution is the design of algebraic multigrid methods (AMG) via aggregation techniques, see e.g. Bastian *et al.* [6] and the references therein. The numerical results of these works give a satisfactory indication of p -robustness.

An associated topic is the development of error estimates. In this regard, a posteriori tools have primarily been used to estimate the algebraic error for existing solvers. One particular goal is the development of reliable stopping criteria, allowing to avoid unnecessary iterations. This is achieved with a combination of a posteriori error estimators for the discretization error. Some contributions on this matter (see also references therein) include Becker *et al.* [7] where adaptive error control is achieved for a multigrid solver, Bornemann and Deuffhard [9] where a one-way multigrid method is presented by integrating an adaptive stopping criteria based on a posteriori tools. Further developments were made in Meidner *et al.* [27], where goal-oriented error estimates are used in the framework of the dual weighted residual (DWR) method. In Jiránek *et al.* [22] and Papež *et al.* [30], upper and lower bounds for both the algebraic and total errors are computed, which allow to derive guaranteed upper and lower bounds on the discretization error, and consecutively construct efficient stopping criteria for iterative algebraic solvers. Arioli *et al.* [3] propose practical stopping criteria which guarantee that the considered inexact adaptive FEM algorithm converges, for inexact solvers of Krylov subspace type. To the best of the authors' knowledge, though, the proof of efficiency of a posteriori estimators of the algebraic error has not been presented so far.

In this work, we present an a posteriori algebraic error estimator and a multilevel iterative solver. The cornerstone of their definitions lies in the multilevel construction of a residual algebraic lifting, motivated by the approach of Papež *et al.* [29]. The lifting can be seen as an approximation of the algebraic error by piecewise polynomials of degree p , obtained by a V-cycle multigrid method with no pre-smoothing steps and a single post-smoothing step. The coarse correction is given by a lowest-order (piecewise affine) function. The smoothing is chosen in the family of damped additive Schwarz (block Jacobi) methods applied to overlapping subdomains composed of patches of elements (two options for defining the patches will be given in due time). Note that additive Schwarz-type smoothing allows for a parallelizable implementation at each level of the mesh hierarchy. Once this lifting is built, the a posteriori estimator is easily derived as a natural guaranteed lower bound on the algebraic error, following [29]. As our first main result, we show that up to a p -robust constant, the estimator is also an upper bound on the error. The solver is then defined as a linear iterative method. Because we have at hand the residual lifting, which approximates the algebraic error, we use it as a descent direction, (the asymmetric approach in defining the lifting, because no pre-smoothing is used, will not be a problem for the analysis). In order to determine the linear solver, it remains to choose a step size. We do so by choosing a step that depends on the lifting and ensures the best error contraction for the energy norm at the next iteration. As our second main result, we prove that this solver contracts the error at each iteration by a p -robust constant. In particular, the p -robust efficiency of the estimator is equivalent to the p -robust convergence of the solver. All the results are defined for a general hierarchy of nested, unstructured, possibly highly refined matching simplicial meshes.

The work is structured as follows. In Section 2, we introduce the setting in which we will be working as well as the notations employed throughout the paper. Then, we recall the results shown by Papež *et al.* [29] and introduce our multilevel residual lifting construction in Section 3. In Section 4, we present the a posteriori estimator on the algebraic error and the corresponding multilevel solver based on the

residual lifting. Our main results are presented in the form of two theorems in Section 5. We also give here a corollary establishing their equivalence. We provide numerical experiments in Section 6, focusing mainly on showcasing p -robustness, in agreement with our theoretical results. We also design and check numerically a variant of our approach corresponding to weighted restricted additive Schwarz smoothing instead of the damped additive Schwarz one. A numerical comparison between the original and modified solver is presented, suggesting p -robustness is preserved while a convergence speed-up is obtained. The proofs of our main results are given in Section 7. In particular, for the stable decomposition estimate, the p -robust result introduced by Schöberl *et al.* [33] is crucial. Finally, Section 8 brings forth our conclusions and outlook for future work.

2 Setting

We will consider in this work the Poisson problem defined on $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, an open bounded polytope with a Lipschitz-continuous boundary. Let there be given a hierarchy of nested matching simplicial meshes of Ω , $\{\mathcal{T}_j\}_{0 \leq j \leq J}$. This means the intersection of two distinct elements of each mesh \mathcal{T}_j is either an empty set or a common vertex, edge, or face, and that \mathcal{T}_j is a refinement of \mathcal{T}_{j-1} , $1 \leq j \leq J$. Two further assumptions are given below.

2.1 Model problem

Let $f \in L^2(\Omega)$ be the source term. We consider the following problem: find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

In the weak formulation, we search for $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where (\cdot, \cdot) is the $L^2(\Omega)$ or $[L^2(\Omega)]^d$ scalar product. The existence and uniqueness of the solution of (2.2) follows from the Riesz representation theorem.

2.2 Finite element discretization

Fixing an integer $p \geq 1$, we introduce the finite element space of continuous piecewise p -degree polynomials

$$V_J^p := \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega), \tag{2.3}$$

where $\mathbb{P}_p(\mathcal{T}_J) := \{v_J \in L^2(\Omega), v_J \in \mathbb{P}_p(K) \forall K \in \mathcal{T}_J\}$. We set $N_J := \dim(V_J^p)$. The discrete problem consists in finding $u_J \in V_J^p$ such that

$$(\nabla u_J, \nabla v_J) = (f, v_J) \quad \forall v_J \in V_J^p. \tag{2.4}$$

2.3 Algebraic system, approximate solution, and algebraic residual

If one introduces ψ_J^l , $1 \leq l \leq N_J$, a basis of V_J^p , then problem (2.4) is equivalent to solving a system of linear algebraic equations. Denoting by $(\mathbb{A}_J)_{lm} := (\nabla \psi_J^m, \nabla \psi_J^l)$ the symmetric, positive definite matrix, $(\mathbb{F}_J)_l := (f, \psi_J^l)$ the right-hand side vector, one obtains $u_J = \sum_{m=1}^{N_J} (U_J)_m \psi_J^m$, where $U_J \in \mathbb{R}^{N_J}$ is the solution of

$$\mathbb{A}_J U_J = \mathbb{F}_J.$$

For any approximation $U_J^i \in \mathbb{R}^{N_J}$ of U_J given by an arbitrary algebraic solver at iteration step $i \geq 0$, the associated continuous piecewise polynomial of degree p is $u_J^i = \sum_{m=1}^{N_J} (U_J^i)_m \psi_J^m \in V_J^p$. The associated residual vector is given by

$$\mathbb{R}_J^i := \mathbb{F}_J - \mathbb{A}_J U_J^i,$$

which depends on the choice of the basis functions. To avoid this dependence, we prefer to work instead with a residual functional on V_j^p given by

$$v_J \mapsto (f, v_J) - (\nabla u_J^i, \nabla v_J) \in \mathbb{R}. \quad (2.5)$$

We emphasize that the forthcoming results are independent of the choice of the basis.

2.4 A hierarchy of meshes and spaces

We introduced in the beginning of this section our mesh hierarchy $\{\mathcal{T}_j\}_{0 \leq j \leq J}$. For any element $K \in \mathcal{T}_j$, we denote $h_K := \text{diam}(K)$. Hereafter, we assume that the hierarchy of meshes is such that the size of each parent element $K \in \mathcal{T}_j$ is comparable to the size of each of its children's, and shape regularity:

Assumption 2.1 (Strength of refinement). *For any $j \in \{1, \dots, J\}$ and for all $K \in \mathcal{T}_{j-1}$, $K^* \in \mathcal{T}_j$, such that $K^* \subset K$, we have*

$$C_{\text{ref}} h_K \leq h_{K^*} \leq h_K, \quad (2.6)$$

with $C_{\text{ref}} \leq 1$ a fixed positive number.

Assumption 2.2 (Shape regularity). *There exists $\kappa_{\mathcal{T}} > 0$ such that*

$$\max_{K \in \mathcal{T}_j} \frac{h_K}{\rho_K} \leq \kappa_{\mathcal{T}}, \text{ for all } 0 \leq j \leq J, \quad (2.7)$$

where ρ_K denotes the diameter of the largest ball inscribed in K .

Remark 2.3 (Mesh hierarchy). *Note that no quasi-uniformity assumption is introduced on the meshes: each \mathcal{T}_j can possibly be highly refined. Moreover, we do not make any specific assumption on the hierarchy of meshes. They merely need to be nested, in particular some mesh elements may not be refined at all. We solely need (2.6), meaning that any parent element is only refined a fixed number of times.*

For the following, we will need to introduce a hierarchy of finite element spaces associated to the mesh hierarchy:

$$\begin{aligned} \text{for } 1 \leq j \leq J: \quad & V_j^p := \mathbb{P}_p(\mathcal{T}_j) \cap H_0^1(\Omega), \quad (p\text{-order spaces}) \\ \text{for } j = 0: \quad & V_0 := \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega), \quad (\text{lowest-order spaces}) \end{aligned}$$

where $\mathbb{P}_p(\mathcal{T}_j) := \{v_j \in L^2(\Omega), v_j \in \mathbb{P}_p(K) \forall K \in \mathcal{T}_j\}$, for $0 \leq j \leq J$. Note that V_j^p is the same as defined in (2.3). Let \mathcal{V}_j be the set of vertices of the mesh \mathcal{T}_j , which can be decomposed into the sets of boundary vertices and interior vertices denoted by $\mathcal{V}_j^{\text{ext}}$ and $\mathcal{V}_j^{\text{int}}$, respectively. We denote by $\psi_j^{\mathbf{a}}$ the standard hat function associated to the vertex $\mathbf{a} \in \mathcal{V}_j$, $0 \leq j \leq J$, i.e. the piecewise affine function that takes value 1 in \mathbf{a} and 0 in all other j -th level vertices of \mathcal{V}_j .

2.5 Two types of patches

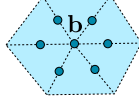
For the following, we need to define two types of patches of elements. In order to facilitate the work with both, we introduce a switching parameter $s \in \{0, 1\}$ to distinguish them. Given a vertex $\mathbf{a} \in \mathcal{V}_{j-s}$, $j \in \{1, \dots, J\}$, we denote the patch related to \mathbf{a} by $\mathcal{T}_{j,s}^{\mathbf{a}}$, the corresponding patch subdomain by $\omega_{j,s}^{\mathbf{a}}$, and the associated local space $V_{j,s}^{\mathbf{a}}$. Let \mathcal{V}_K be the set of vertices of element K . Then

$$\mathcal{T}_{j,s}^{\mathbf{a}} := \{K \in \mathcal{T}_{j-s}, \mathbf{a} \in \mathcal{V}_K\}, \quad (2.8)$$

$$V_{j,s}^{\mathbf{a}} := H_0^1(\omega_{j,s}^{\mathbf{a}}) \cap \mathbb{P}^p(\mathcal{T}_j). \quad (2.9)$$

Consider a ‘‘finer’’ ($s = 0$) vertex $\mathbf{b} \in \mathcal{V}_j$, and ‘‘coarser’’ ($s = 1$) one $\mathbf{a} \in \mathcal{V}_{j-1} \subset \mathcal{V}_j$. For simplicity, we refer to $\mathcal{T}_{j,0}^{\mathbf{b}}$ as a ‘‘small’’ patch, whereas we refer to $\mathcal{T}_{j,1}^{\mathbf{a}}$ as a ‘‘big’’ patch. An illustration is given in Figure 1.

patch $\mathcal{T}_{j,0}^{\mathbf{b}}$ and the patch subdomain $\omega_{j,0}^{\mathbf{b}}$ for a vertex $\mathbf{b} \in \mathcal{V}_j$



patch $\mathcal{T}_{j,1}^{\mathbf{a}}$ and the patch subdomain $\omega_{j,1}^{\mathbf{a}}$ for a vertex $\mathbf{a} \in \mathcal{V}_{j-1}$

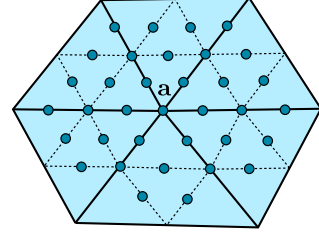


Figure 1: Illustration of degrees of freedom ($p = 2$) for the space $V_{j,0}^{\mathbf{b}}$ associated to the “small” patch $\mathcal{T}_{j,0}^{\mathbf{b}}$ (left) and for $V_{j,1}^{\mathbf{a}}$ associated to the “big” patch $\mathcal{T}_{j,1}^{\mathbf{a}}$ (right). The mesh \mathcal{T}_{j-1} and its refinement \mathcal{T}_j are defined in bold and dotted lines, respectively.

3 Multilevel lifting of the algebraic residual

Similarly to Papež *et al.* [29], we design a multilevel lifting of the algebraic residual given by (2.5). This lifting will lead to the construction of an a posteriori error estimator, as well as will serve as a descent direction for the solver we introduce in the next section.

3.1 Coarse solve

The first step of our construction is to solve a global lowest-order problem on the coarsest mesh. Let $u_J^i \in V_J^p$ be given. Define $\rho_0^i \in V_0$ by

$$(\nabla \rho_0^i, \nabla v_0) = (f, v_0) - (\nabla u_J^i, \nabla v_0) \quad \forall v_0 \in V_0. \quad (3.1)$$

3.2 Motivation

Let us first motivate our multilevel construction. Following Papež *et al.* [29], consider for a given $u_J^i \in V_J^p$ the following (infeasible in practice but illustrative) hierarchical construction $\tilde{\rho}_{J,\text{alg}}^i \in V_J^p$

$$\tilde{\rho}_{J,\text{alg}}^i := \rho_0^i + \sum_{j=1}^J \tilde{\rho}_j^i, \quad (3.2)$$

where ρ_0^i is given by (3.1) and for all $j = \{1, \dots, J\}$, $\tilde{\rho}_j^i \in V_j^p$ is the solution of

$$(\nabla \tilde{\rho}_j^i, \nabla v_j) = (f, v_j) - (\nabla u_J^i, \nabla v_j) - \sum_{k=0}^{j-1} (\nabla \tilde{\rho}_k^i, \nabla v_j), \quad \forall v_j \in V_j^p. \quad (3.3)$$

Here, $\tilde{\rho}_0^i := \rho_0^i$ by convention. This construction (see also [29]) gives us the algebraic error, i.e. we have $\tilde{\rho}_{J,\text{alg}}^i = u_J - u_J^i$. This, in turn, means that $\tilde{\rho}_{J,\text{alg}}^i$ satisfies $(\nabla \tilde{\rho}_{J,\text{alg}}^i, \nabla v_J) = (f, v_J) - (\nabla u_J^i, \nabla v_J)$, for all $v_J \in V_J^p$. Moreover, there holds $(\nabla \tilde{\rho}_j^i, \nabla \tilde{\rho}_k^i) = 0$, $0 \leq k, j \leq J$; $j \neq k$. These observations lead to the orthogonal decomposition

$$\|\nabla(u_J - u_J^i)\|^2 = \|\nabla \tilde{\rho}_{J,\text{alg}}^i\|^2 = \sum_{j=0}^J \|\nabla \tilde{\rho}_j^i\|^2. \quad (3.4)$$

This construction will be an important tool for the theoretical study of Section 7.

3.3 Multilevel algebraic residual lifting

Let us now introduce our construction that mimics (3.2)–(3.3) in a local way and produces $\rho_{J,\text{alg}}^i \in V_J^p$ that is hopefully close to $\tilde{\rho}_{J,\text{alg}}^i$. The construction relies on the use of coarse solution of (3.1) and on local contributions arising from all the finer mesh levels. These local contributions are defined on patches of elements. Since we consider two definitions of patches with switching parameter $s \in \{0, 1\}$ (see Section 2.5), two constructions of $\rho_{J,\text{alg}}^i$ are implied.

Definition 3.1 (Construction of the algebraic residual lifting). *Let $u_j^i \in V_J^p$ be arbitrary. We introduce $\rho_{J,\text{alg}}^i \in V_J^p$ by*

$$\rho_{J,\text{alg}}^i := \rho_0^i + \sum_{j=1}^J \rho_j^i, \quad (3.5)$$

where $\rho_0^i \in V_0$ solves (3.1) and $\rho_j^i \in V_j^p$, $j \in \{1, \dots, J\}$, are given by

$$\rho_j^i := \frac{1}{J(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \rho_{j,\mathbf{a}}^i, \quad 1 \leq j \leq J, \quad (3.6)$$

and local contributions $\rho_{j,\mathbf{a}}^i \in V_{j,s}^{\mathbf{a}}$ given as solutions of patch problems, for all $v_{j,\mathbf{a}} \in V_{j,s}^{\mathbf{a}}$

$$(\nabla \rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} = (f, v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - (\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}}. \quad (3.7)$$

Remark 3.2 (Construction of $\rho_{J,\text{alg}}^i$). *The construction (3.5)–(3.7) of $\rho_{J,\text{alg}}^i$ can be seen as an approximation of $\tilde{\rho}_{J,\text{alg}}^i$ from (3.2)–(3.3) by one iteration of a V-cycle multigrid, with no pre-smoothing and a single post-smoothing step, corresponding to a “damped” additive Schwarz iteration, with the damping factor $1/(J(d+1))$. The subdomains of this Schwarz iteration correspond to the patch domains where the local problems in (3.7) are defined. Two patch options of Figure 1 are considered. In particular, for $p = 1$ and “small” patches (Figure 1, left), this corresponds to one-step Jacobi (diagonal) smoother, whereas the other situations are associated to a block Jacobi smoother. The construction of Definition 3.1 differs from that of Papež et al. [29, Definition 6.5] by replacing the damping factor by weightings via hat functions.*

Remark 3.3 (Value of the damping parameter). *The value $1/(J(d+1))$ of the damping in (3.6) proves to be crucial for our theoretical analysis. It is based on the proofs in Section 7 below. This is what seems to be needed in our approach, also numerically, though other options are theoretically possible (including a second damping factor of the form $1/w_2$ for the third term on the right-hand side of (3.7), where $w_2 \rightarrow +\infty$ is allowed).*

4 An a posteriori estimator on the the algebraic error and a multilevel solver

We present below how the residual lifting $\rho_{J,\text{alg}}^i$ of Definition 3.1 can be used to define an a posteriori estimator as well as a multilevel solver.

4.1 A posteriori estimate on the algebraic error

We begin by introducing η_{alg}^i , an a posteriori estimator defined using the residual lifting $\rho_{J,\text{alg}}^i$.

Definition 4.1 (Lower bound on the algebraic error). *Let $u_j^i \in V_J^p$ be arbitrary, and let $\rho_{J,\text{alg}}^i$ be the algebraic residual lifting given by Definition 3.1. If $\rho_{J,\text{alg}}^i = 0$, we define the lower bound algebraic error estimator $\eta_{\text{alg}}^i := 0$. Otherwise define*

$$\eta_{\text{alg}}^i := \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|}. \quad (4.1)$$

Following Papež et al. [29, Theorem 5.3], the introduced estimator η_{alg}^i can be showed to be a guaranteed lower bound on the algebraic error.

Lemma 4.2 (Guaranteed lower bound on the algebraic error). *There holds:*

$$\|\nabla(u_J - u_J^i)\| \geq \eta_{\text{alg}}^i. \quad (4.2)$$

Proof. Note that if $\rho_{J,\text{alg}}^i = 0$, then $\|\nabla(u_J - u_J^i)\| \geq 0 = \eta_{\text{alg}}^i$. Otherwise

$$\|\nabla(u_J - u_J^i)\| = \max_{\substack{v_J \in V_J^p, \\ \|\nabla v_J\| \neq 0}} \frac{(\nabla(u_J - u_J^i), \nabla v_J)}{\|\nabla v_J\|} \geq \frac{(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|} \stackrel{(2.4)}{=} \eta_{\text{alg}}^i.$$

□

4.2 Multilevel solver

We now intend to reuse the construction of $\rho_{J,\text{alg}}^i$, given in Definition 3.1 to obtain an approximation of u_J on a next step, in view of constructing a multilevel solver. Note that for any $u_J^i \in V_J^p$, the lifting $\rho_{J,\text{alg}}^i$ is built to approximate the algebraic error $\tilde{\rho}_{J,\text{alg}}^i$ given in (3.2), where $u_J = u_J^i + \tilde{\rho}_{J,\text{alg}}^i$. Thus, it seems reasonable to consider a linear iterative solver of the form

$$u_J^{i+1} = u_J^i + \lambda \rho_{J,\text{alg}}^i \tag{4.3}$$

where $\lambda \in \mathbb{R}$ is a real parameter. The optimal choice of λ is given in the lemma below.

Lemma 4.3 (Optimal step size). *Consider a solver of the general form (4.3) and suppose $\rho_{J,\text{alg}}^i \neq 0$. Then the choice $\lambda := [(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)] / \|\nabla \rho_{J,\text{alg}}^i\|^2$ leads to minimal algebraic error with respect to the energy norm.*

Proof. We write the algebraic error of the next iteration as a function of λ

$$\|\nabla(u_J - u_J^{i+1})\|^2 = \|\nabla(u_J - u_J^i)\|^2 - 2\lambda(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i) + \lambda^2 \|\nabla \rho_{J,\text{alg}}^i\|^2,$$

and realize that this function has a minimum at

$$\lambda_{\min} = \frac{(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|^2} \stackrel{(2.4)}{=} \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|^2}.$$

□

We are now ready to define our multilevel solver.

Definition 4.4 (Multilevel solver). *1. Initialize $u_J^0 \in V_0$ as the solution of $(\nabla u_J^0, \nabla v_0) = (f, v_0)$, $\forall v_0 \in V_0$.
2. Let $i \geq 0$ be the iteration number, and let $\rho_{J,\text{alg}}^i$ be constructed from u_J^i as in Definition 3.1. If $\rho_{J,\text{alg}}^i = 0$ we define $u_J^{i+1} := u_J^i$. Otherwise, let*

$$u_J^{i+1} := u_J^i + \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|^2} \rho_{J,\text{alg}}^i. \tag{4.4}$$

Remark 4.5 (Multilevel solver). *Note that the solver of Definition 4.4 is not initialized randomly but via a coarse solve. The descent direction is the residual lifting $\rho_{J,\text{alg}}^i$, constructed via no pre-smoothing, one post-smoothing V-cycle step. This minimalist and asymmetrical procedure will not be an issue for the forthcoming analysis.*

5 Main results

In this section we present the main results concerning our a posteriori estimator η_{alg}^i of Definition 4.1 and our multilevel solver of Definition 4.4. We shall also see how these two main results are related. For the estimator it holds

Theorem 5.1 (*p*-robust reliable and efficient bound on the algebraic error). *Let $u_J \in V_J^p$ be the (unknown) solution of (2.4) and let $u_J^i \in V_J^p$ be arbitrary, $i \geq 0$. Let η_{alg}^i be given by Definition 4.1. Then, in addition to $\|\nabla(u_J - u_J^i)\| \geq \eta_{\text{alg}}^i$ of (4.2), there holds*

$$\eta_{\text{alg}}^i \geq \beta \|\nabla(u_J - u_J^i)\|, \quad (5.1)$$

where $0 < \beta < 1$ only depends on the space dimension d , the mesh shape regularity parameter $\kappa_{\mathcal{T}}$, and the number of mesh levels J .

The theorem allows to write η_{alg}^i as a two-sided bound of the algebraic error (up to a generic constant for the upper bound), meaning that the estimator is robustly efficient with respect to the polynomial degree p . For the solver, in turn, we have

Theorem 5.2 (*p*-robust error contraction of the multilevel solver). *Let $u_J \in V_J^p$ be the (unknown) solution of (2.4) and let $u_J^i \in V_J^p$ be arbitrary, $i \geq 0$. Take u_J^{i+1} to be constructed from u_J^i using one step of the multilevel solver of Definition 4.4, by (4.4). Then there holds*

$$\|\nabla(u_J - u_J^{i+1})\| \leq \alpha \|\nabla(u_J - u_J^i)\|, \quad (5.2)$$

where $0 < \alpha < 1$ only depends on the space dimension d , the mesh shape regularity parameter $\kappa_{\mathcal{T}}$, and the number of mesh levels J .

In the above theorem, α represents an estimation of the algebraic error contraction factor at each step i . As α only depends on d , $\kappa_{\mathcal{T}}$, and J , this means that the solver of Definition 4.4 contracts the algebraic error at each iteration step in a robust way both with respect to the number of mesh elements in \mathcal{T}_J (to mesh size h) and with respect to the polynomial degree p . Theorems 5.1 and 5.2 are connected as follows

Corollary 5.3 (Equivalence of the *p*-robust estimator efficiency and *p*-robust solver contraction). *Let the assumptions of Theorems 5.1 and 5.2 be satisfied. Then (5.1) holds if and only if (5.2) holds. Moreover, when $\rho_{J,\text{alg}}^i$ of Definition 3.1 is not zero, then $\beta = \sqrt{1 - \alpha^2}$.*

Proof. Let $u_J \in V_J^p$ be the solution of (2.4), let $u_J^i \in V_J^p$ be arbitrary, and let $u_J^{i+1} \in V_J^p$ constructed from u_J^i by our multilevel solver of Definition 4.4.

Case $\rho_{J,\text{alg}}^i = 0$. By Definitions 4.4, 4.1, we have $u_J^{i+1} = u_J^i$ and $\eta_{\text{alg}}^i = 0$. In particular, this means that $\|\nabla(u_J - u_J^{i+1})\| = \|\nabla(u_J - u_J^i)\|$. These observations allow us to write, starting from (5.2) with $0 < \alpha < 1$,

$$\begin{aligned} \|\nabla(u_J - u_J^{i+1})\| &\leq \alpha \|\nabla(u_J - u_J^i)\| \Leftrightarrow \|\nabla(u_J - u_J^i)\| \leq \alpha \|\nabla(u_J - u_J^i)\| \\ &\Leftrightarrow \|\nabla(u_J - u_J^i)\|(1 - \alpha) \leq 0 \Leftrightarrow \|\nabla(u_J - u_J^i)\|(1 - \alpha) \leq \eta_{\text{alg}}^i \end{aligned}$$

Case $\rho_{J,\text{alg}}^i \neq 0$. First, note that by the construction of the lifting given in Definition 3.1 and by (2.4), we have $[u_J = u_J^i \Rightarrow \rho_{J,\text{alg}}^i = 0]$. Thus $[\rho_{J,\text{alg}}^i \neq 0 \Rightarrow u_J \neq u_J^i]$. This ensures that neither $\|\nabla \rho_{J,\text{alg}}^i\|$ nor $\|\nabla(u_J - u_J^i)\|$ is zero, and we can therefore use them in the denominators below. We write the relation between the algebraic errors associated to u_J^{i+1} and u_J^i

$$\begin{aligned} \|\nabla(u_J - u_J^{i+1})\|^2 &\stackrel{(4.4)}{=} \left\| \nabla(u_J - u_J^i) - \frac{(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|^2} \nabla \rho_{J,\text{alg}}^i \right\|^2 \\ &\stackrel{(2.4)}{=} \|\nabla(u_J - u_J^i)\|^2 - 2 \frac{(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i)^2}{\|\nabla \rho_{J,\text{alg}}^i\|^2} + \frac{(\nabla(u_J - u_J^i), \nabla \rho_{J,\text{alg}}^i)^2}{\|\nabla \rho_{J,\text{alg}}^i\|^4} \|\nabla \rho_{J,\text{alg}}^i\|^2 \\ &\stackrel{(4.1)}{=} \|\nabla(u_J - u_J^i)\|^2 \left(1 - \frac{(\eta_{\text{alg}}^i)^2}{\|\nabla(u_J - u_J^i)\|^2} \right). \end{aligned}$$

This means that

$$\|\nabla(u_J - u_J^{i+1})\| \leq \alpha \|\nabla(u_J - u_J^i)\| \Leftrightarrow \left(1 - \frac{(\eta_{\text{alg}}^i)^2}{\|\nabla(u_J - u_J^i)\|^2} \right)^{\frac{1}{2}} \leq \alpha \Leftrightarrow \eta_{\text{alg}}^i \geq \sqrt{1 - \alpha^2} \|\nabla(u_J - u_J^i)\|.$$

□

In view of Corollary 5.3, we will prove in Section 7 only Theorem 5.1.

6 Numerical experiments

In this section we report some numerical illustrations of the theoretical results of Section 5. In particular, we focus on the p -robustness. In the following tests, we consider the model problem (2.1) with three different choices of the domain $\Omega \subset \mathbb{R}^2$ and exact solution u :

$$\text{Sine:} \quad u(x, y) := \sin(2\pi x) \sin(2\pi y); \quad \Omega := (-1, 1)^2, \quad (6.1)$$

$$\text{Peak:} \quad u(x, y) := x(x-1)y(y-1)e^{-100((x-0.5)^2 - (y-0.117)^2)}; \quad \Omega := (0, 1)^2, \quad (6.2)$$

$$\text{L-shape:} \quad u(r, \theta) := r^{2/3} \sin(2\theta/3); \quad \Omega := (-1, 1)^2 \setminus ([0, 1] \times [-1, 0]). \quad (6.3)$$

For the L-shape problem (6.3), we impose an inhomogeneous Dirichlet boundary condition corresponding to the exact solution, which is expressed here in polar coordinates. For each of the test cases, we start with an initial Delaunay triangulation of Ω . Then, we consider J uniform refinements where all triangles are decomposed into four congruent subtriangles. Implementation-wise, we opt for Lagrange basis functions with non-uniformly distributed nodes because of their better behavior with respect to high-order methods. Recall that this choice has no influence on the theoretical results of Section 5 as well as presented numerical results (in exact arithmetics).

6.1 An alternative construction of the residual lifting $\rho_{J,\text{alg}}^i$

A crucial component in the definition of our a posteriori estimator and multilevel solver is the construction of the residual lifting $\rho_{J,\text{alg}}^i$ of Definition 3.1. Recall that this corresponds to a multigrid V-cycle with no pre-smoothing and one post-smoothing Schwarz iteration step. Different Schwarz methods have been developed over the years. In the literature, we can find interesting works that compare the performances of these methods. Notable works include Cai and Sarkis [14], where the restricted Schwarz method performs better as a preconditioner compared to the classical additive Schwarz approach. Later work by Efstathiou and Gander [16] confirmed this result for piecewise affine approximations, while indicating that additive Schwarz used as a solver does not converge in general. An overview of Schwarz methods for spectral element methods (cf. Bernardi and Maday [8]) was given by Loisel *et al.* [24].

Based on the above works (see also the references therein), we believe it is worthwhile to test numerically the typically more efficient weighted restricted Schwarz method, in addition to the “damped” additive Schwarz construction. Recall that our construction (3.6) of $\rho_{J,\text{alg}}^i$ using a “damped” additive Schwarz iteration is

$$\text{dAS} \quad \rho_j^i := \frac{1}{J(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \rho_{j,\mathbf{a}}^i, \quad 1 \leq j \leq J, \quad (6.4)$$

with $1/(J(d+1))$ the damping parameter. The weighted restricted additive Schwarz as proposed in Papež *et al.* [29], consists in

$$\text{wRAS} \quad \rho_j^i := \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \mathcal{I}_j^p(\psi_{j-s}^{\mathbf{a}} \rho_{j,\mathbf{a}}^i), \quad 1 \leq j \leq J, \quad (6.5)$$

where \mathcal{I}_j^p is the \mathbb{P}^p Lagrange interpolation operator on the mesh level j . Note that the local contributions $\rho_{j,\mathbf{a}}^i \in V_{j,s}^{\mathbf{a}}$ remain unchanged, defined by (3.7), and we once more build the residual lifting as $\rho_{J,\text{alg}}^i := \sum_{j=0}^J \rho_j^i$ by (3.5). For simplicity, we use the shorthand notation dAS and wRAS to refer to these two constructions of $\rho_{J,\text{alg}}^i$ using (6.4) and (6.5), respectively.

6.2 Testing for p -robustness

As stated in Corollary 5.3, the study of the contraction factor on each step i of the solver, given by $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$, also allows to study the constant of efficiency of the a posteriori estimator η_{alg}^i of Definition 4.1, given by $\eta_{\text{alg}}^i/\|\nabla(u_J - u_J^i)\|$. Keeping this in mind, the tests we perform below only focus on the contraction factor.

Our goal is to present numerical confirmation of p -robustness of our multilevel solver of Definition 4.4. In this case, a common choice for the stopping criterion is

$$\frac{\|\mathbb{F}_J - \mathbb{A}_J \mathbf{U}_J^{i_{\text{stop}}}\|}{\|\mathbb{F}_J\|} \leq 10^{-5} \frac{\|\mathbb{F}_J - \mathbb{A}_J \mathbf{U}_J^0\|}{\|\mathbb{F}_J\|}, \quad (6.6)$$

where i_{stop} is the number of iterations needed to satisfy this criterion. We also introduce the average error contraction factor, given by

$$\bar{\alpha} := \frac{1}{i_{\text{stop}}} \sum_{i=0}^{i_{\text{stop}}-1} \frac{\|\nabla(u_J - u_J^{i+1})\|}{\|\nabla(u_J - u_J^i)\|}. \quad (6.7)$$

We expect a p -robust solver to converge in a similar number of iterations, and have similar error contraction factors at all iterations, for different polynomial degrees p , whenever tested for a given number of mesh levels J .

In the forthcoming subsections, we summarize the results obtained for each of the problems (6.1)–(6.3), in the Tables 1–3, respectively. Therein, we present the number of iterations i_{stop} needed to satisfy the stopping criterion (6.6) and the average contraction factors $\bar{\alpha}$, for the polynomial degrees p up to 9. The tests cover different number of mesh levels J , both constructions dAS (6.4) and wRAS (6.5) of the lifting, using the “small” as well as the “big” patches of Figure 1. For each of the cases, we also give the conditioning of the matrix \mathbb{A}_J approximated using the Matlab function `condest`. Next, we illustrate in Figures 2–7 the algebraic error in the energy norm and the error contraction factors $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$ on each iteration step i of our solver.

These results confirm the expected independence of the polynomial degree p for our multilevel solver which uses the construction dAS (6.4) of the lifting. As pointed out in Remark 3.2, we expect a much inferior quality of the contraction factors for the case of $p = 1$ and the use of “small” patches. The behavior, though, improves considerably for the “big” patches whose price is still very reasonable for $p = 1$. Recall that our solver can be seen as a multigrid no pre-smoothing and only one post-smoothing step. We expect an important drop on the number of iterations if more smoothing steps were employed. Another observation is that the number of iterations depends on the number of mesh levels J , in accordance with the theoretical result of Section 7, but rather in a mild way. Moreover, the results for the modified solver, defined using the wRAS (6.5) construction of the lifting, indicate an improvement in the error contraction factors and complete independence of J .

6.3 Sine problem

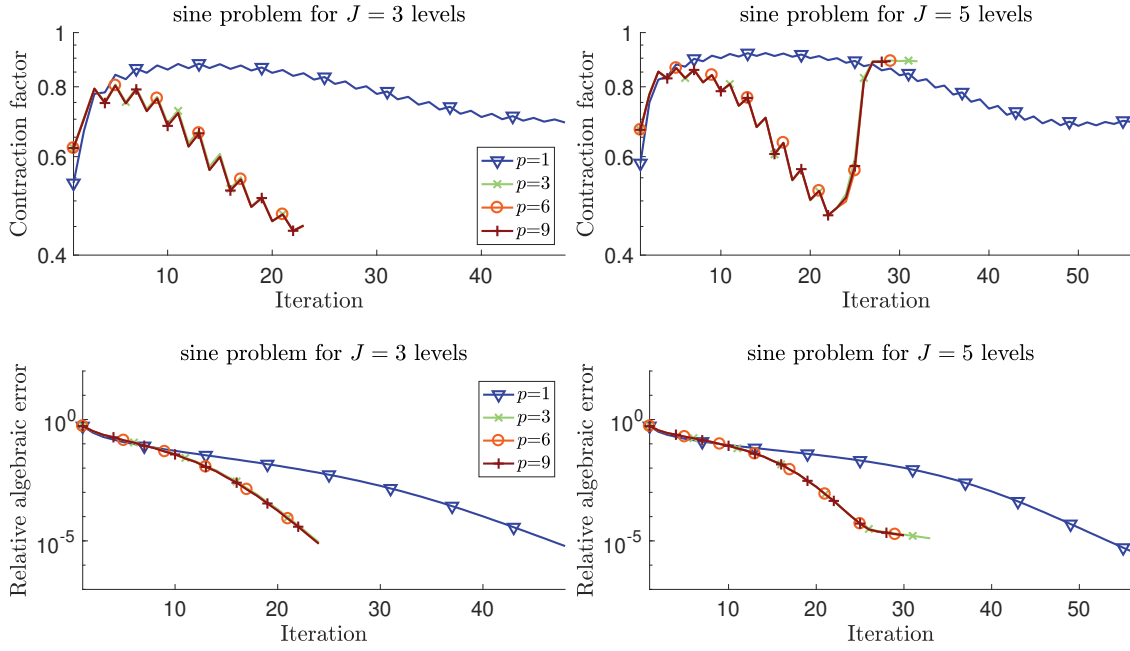


Figure 2: Sine problem (6.1): Results of the multilevel solver (4.4) for “small” patches with stopping criterion (6.6) for mesh levels $J = 3$ and $J = 5$; top: error contraction factor $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$; bottom: relative algebraic error in energy norm $\|\nabla(u_J - u_J^i)\|/\|\nabla u_J\|$.

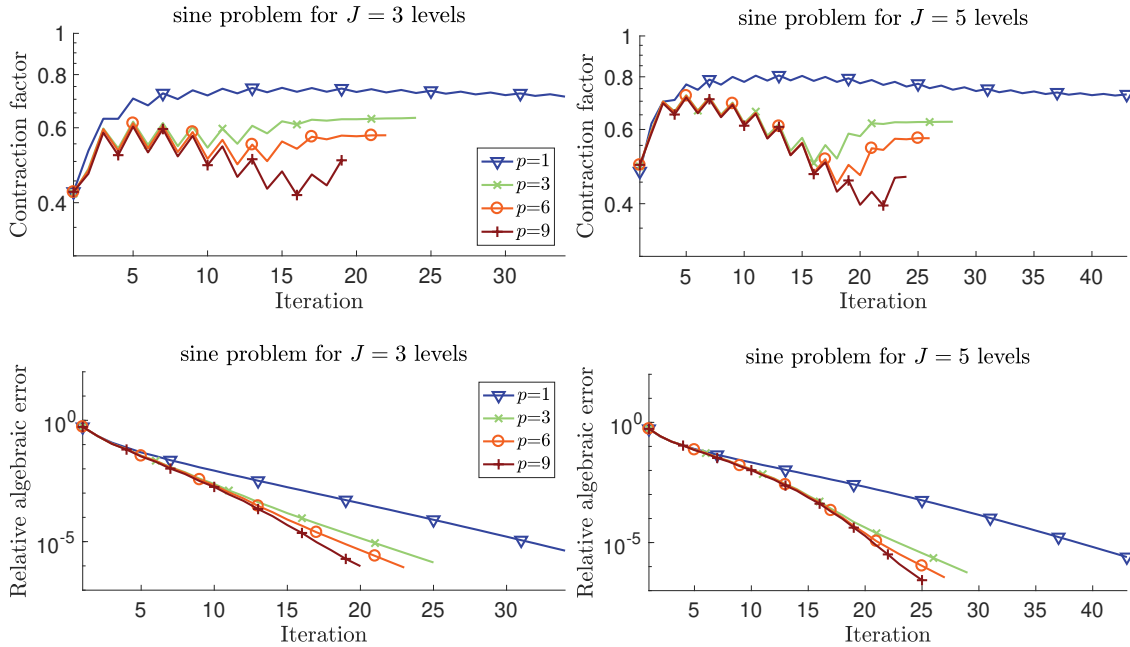


Figure 3: Sine problem (6.1): Results of the multilevel solver (4.4) for “big” patches with stopping criterion (6.6) for mesh levels $J = 3$ and $J = 5$; top: error contraction factor $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$; bottom: relative algebraic error in energy norm $\|\nabla(u_J - u_J^i)\|/\|\nabla u_J\|$.

J	p	DoF	$\text{cond}(A_J)$	“small” patches				“big” patches			
				dAS (6.4)		wRAS (6.5)		dAS (6.4)		wRAS (6.5)	
				i_{stop}	$\bar{\alpha}$	i_{stop}	$\bar{\alpha}$	i_{stop}	$\bar{\alpha}$	i_{stop}	$\bar{\alpha}$
3	1	4625	3.35×10^3	48	0.79	21	0.57	34	0.70	9	0.27
	3	42 289	6.28×10^4	23	0.63	15	0.44	24	0.59	6	0.13
	6	169 825	7.05×10^5	23	0.63	13	0.40	22	0.55	6	0.13
	9	382 609	6.72×10^6	23	0.63	13	0.40	19	0.50	6	0.12
4	1	18 721	1.34×10^4	52	0.80	23	0.60	40	0.74	9	0.28
	3	169 825	2.51×10^5	27	0.68	15	0.43	26	0.60	6	0.13
	6	680 641	2.82×10^6	26	0.66	13	0.39	24	0.57	6	0.13
	9	1 532 449	2.69×10^7	26	0.67	13	0.39	21	0.53	5	0.13
5	1	75 329	5.35×10^4	56	0.81	22	0.59	43	0.75	9	0.27
	3	680 641	1.00×10^6	32	0.73	15	0.43	28	0.61	6	0.13
	6	2 725 249	1.13×10^7	29	0.71	13	0.39	26	0.58	6	0.12
	9	6 133 825	1.07×10^8	29	0.71	13	0.39	24	0.56	5	0.13

Table 1: Sine problem (6.1): Comparison of the multilevel solver of Definition 4.4 corresponding to the constructions with smoothing dAS (6.4) and wRAS (6.5) for “small” and “big” patches. i_{stop} : the number of iterations needed to reach the stopping criterion (6.6); $\bar{\alpha}$: average error contraction factor given by (6.7).

6.4 Peak problem

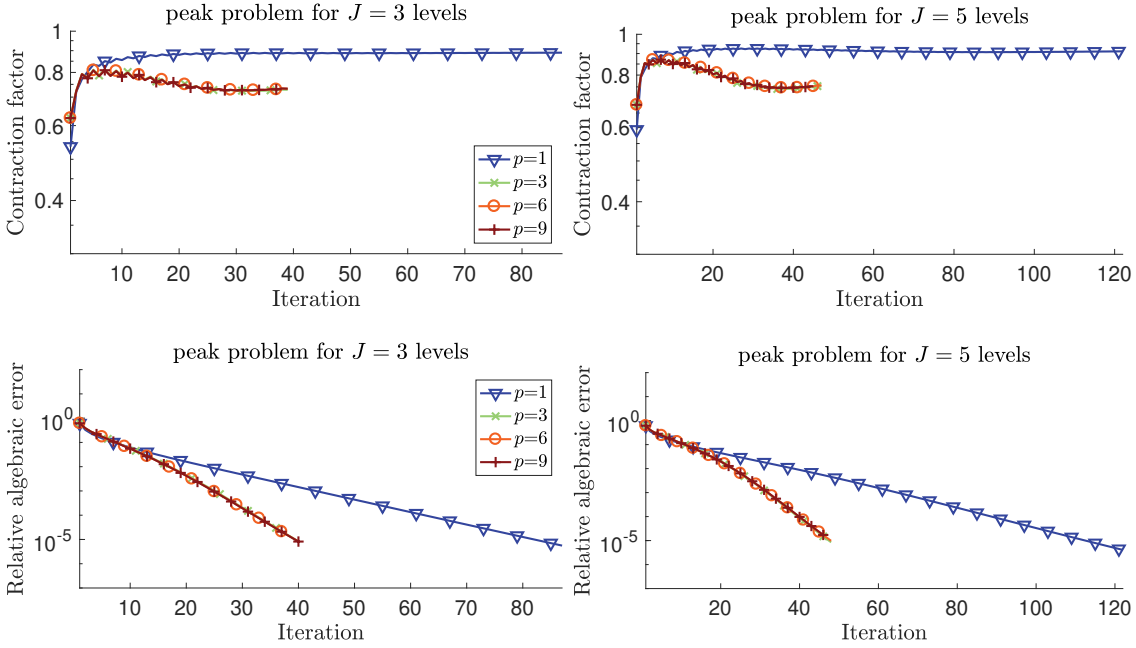


Figure 4: Peak problem (6.2): Results of the multilevel solver (4.4) for “small” patches with stopping criterion (6.6) for mesh levels $J = 3$ and $J = 5$; top: error contraction factor $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$; bottom: relative algebraic error in energy norm $\|\nabla(u_J - u_J^i)\|/\|\nabla u_J\|$.

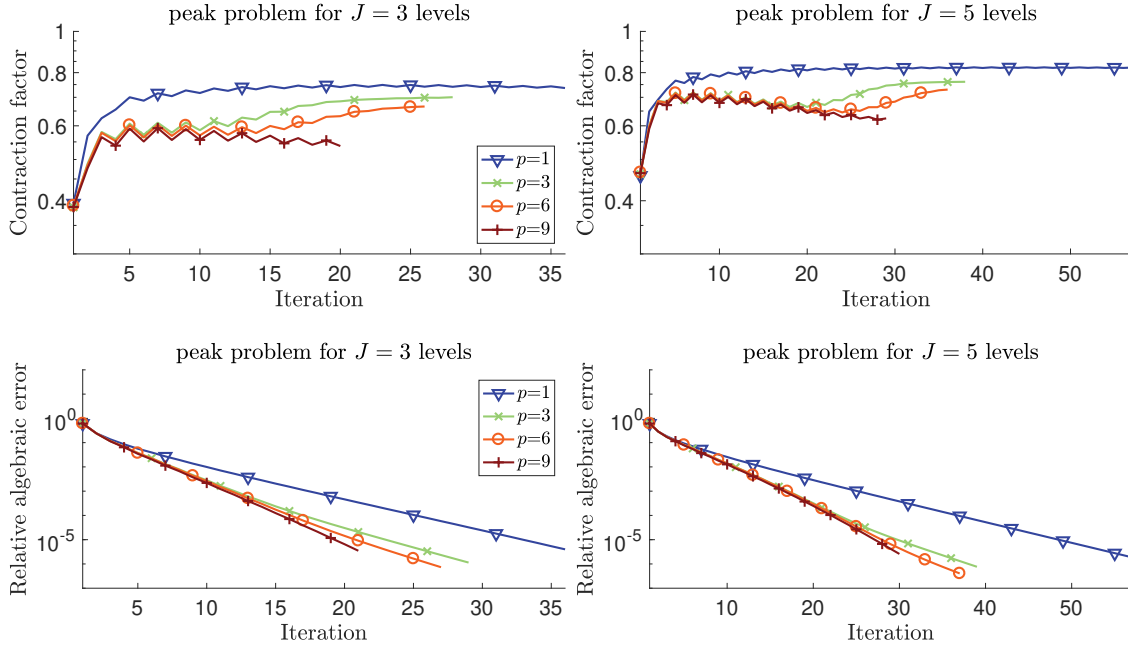


Figure 5: Peak problem (6.2): Results of the multilevel solver (4.4) for “big” patches with stopping criterion (6.6) for mesh levels $J = 3$ and $J = 5$; top: error contraction factor $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$; bottom: relative algebraic error in energy norm $\|\nabla(u_J - u_J^i)\|/\|\nabla u_J\|$.

J	p	DoF	$\text{cond}(A_J)$	“small” patches				“big” patches			
				dAS (6.4)		wRAS (6.5)		dAS (6.4)		wRAS (6.5)	
				i_{stop}	$\bar{\alpha}$	i_{stop}	$\bar{\alpha}$	i_{stop}	$\bar{\alpha}$	i_{stop}	$\bar{\alpha}$
3	1	4625	3.01×10^3	87	0.88	19	0.52	36	0.72	9	0.25
	3	42 289	5.80×10^4	39	0.75	15	0.43	28	0.63	6	0.12
	6	169 825	6.52×10^5	39	0.75	14	0.41	26	0.60	6	0.11
	9	382 609	6.22×10^6	39	0.75	14	0.40	20	0.55	5	0.12
4	1	18 721	1.20×10^4	109	0.90	20	0.54	48	0.77	9	0.25
	3	169 825	2.32×10^5	43	0.77	15	0.42	34	0.68	5	0.12
	6	680 641	2.61×10^6	43	0.78	14	0.40	31	0.64	5	0.12
	9	1 532 449	2.49×10^7	43	0.78	14	0.40	25	0.61	5	0.12
5	1	75 329	4.81×10^4	122	0.91	20	0.53	57	0.80	9	0.24
	3	680 641	9.28×10^5	47	0.79	15	0.42	38	0.70	5	0.12
	6	2 725 249	1.04×10^7	47	0.79	14	0.39	36	0.68	5	0.12
	9	6 133 825	9.95×10^7	45	0.79	13	0.39	29	0.65	5	0.12

Table 2: Peak problem (6.2). Comparison of the multilevel solver of Definition 4.4 corresponding to the constructions with smoothing dAS (6.4) and wRAS (6.5) for “small” and “big” patches. i_{stop} : the number of iterations needed to reach the stopping criterion (6.6); $\bar{\alpha}$: average error contraction factor given by (6.7).

6.5 L-shape problem

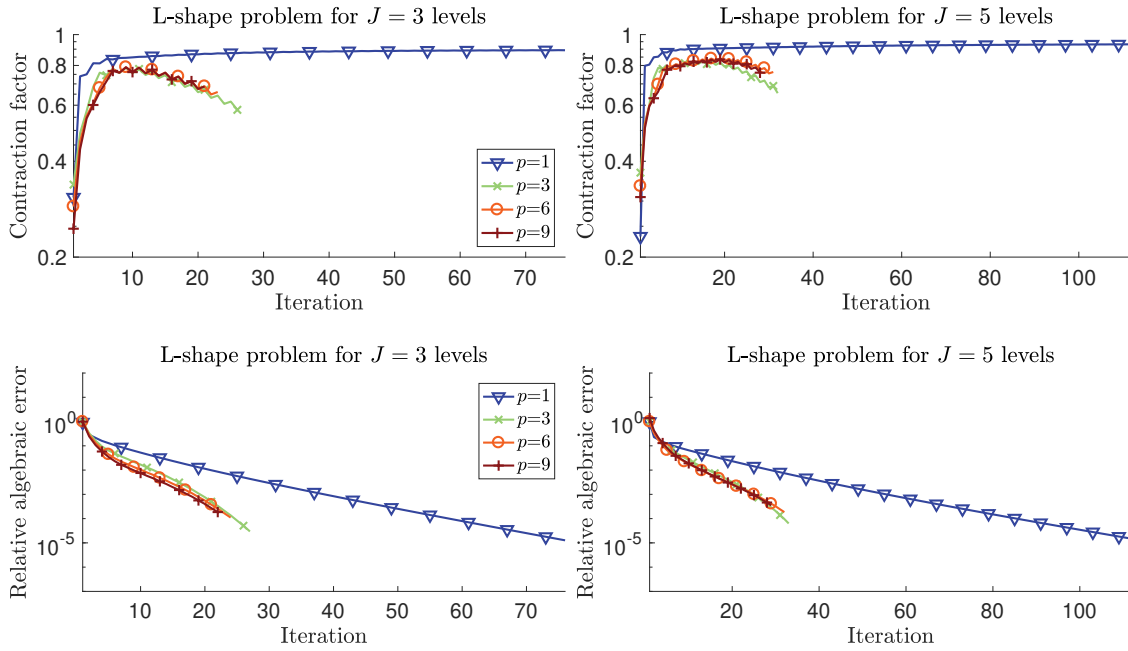


Figure 6: L-shape problem (6.3): Results of the multilevel solver (4.4) for “small” patches with stopping criterion (6.6) for mesh levels $J = 3$ and $J = 5$; top: error contraction factor $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$; bottom: relative algebraic error in energy norm $\|\nabla(u_J - u_J^i)\|/\|\nabla u_J\|$.

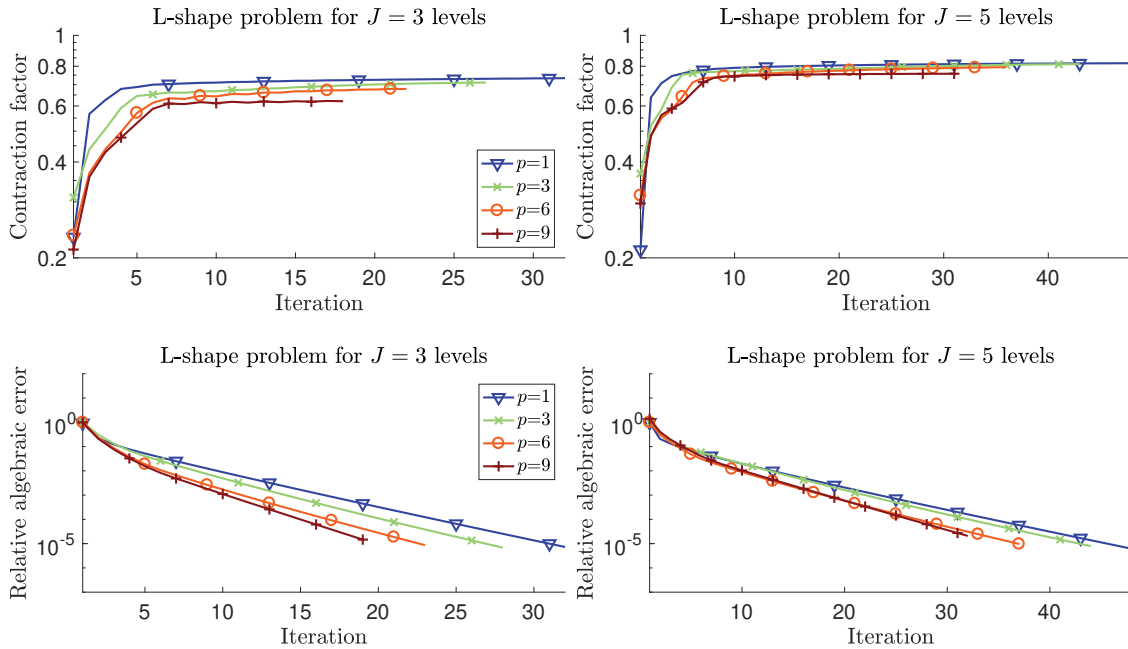


Figure 7: L-shape problem (6.3): Results of the multilevel solver (4.4) for “big” patches with stopping criterion (6.6) for mesh levels $J = 3$ and $J = 5$; top: error contraction factor $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$; bottom: relative algebraic error in energy norm $\|\nabla(u_J - u_J^i)\|/\|\nabla u_J\|$.

J	p	DoF	$\text{cond}(\mathbb{A}_J)$	“small” patches				“big” patches			
				dAS (6.4)		wRAS (6.5)		dAS (6.4)		wRAS (6.5)	
				i_{stop}	$\bar{\alpha}$	i_{stop}	$\bar{\alpha}$	i_{stop}	$\bar{\alpha}$	i_{stop}	$\bar{\alpha}$
3	1	5057	2.26×10^3	76	0.87	17	0.50	32	0.72	8	0.22
	3	46 273	4.40×10^4	26	0.68	12	0.41	27	0.63	5	0.09
	6	185 857	4.95×10^5	23	0.69	10	0.36	22	0.60	5	0.08
	9	418 753	4.72×10^6	21	0.68	10	0.35	18	0.55	5	0.07
4	1	20 481	9.04×10^3	95	0.89	18	0.52	40	0.77	8	0.22
	3	185 857	1.76×10^5	29	0.71	12	0.41	35	0.68	5	0.09
	6	744 961	1.98×10^6	27	0.73	10	0.36	29	0.64	5	0.08
	9	1 677 313	1.89×10^7	25	0.72	9	0.33	24	0.61	5	0.07
5	1	82 433	3.62×10^4	112	0.91	17	0.51	48	0.80	8	0.22
	3	744 961	7.05×10^5	32	0.75	12	0.40	43	0.70	5	0.09
	6	2 982 913	7.92×10^6	31	0.77	9	0.34	36	0.68	5	0.08
	9	6 713 857	7.55×10^7	28	0.76	8	0.31	31	0.71	4	0.07

Table 3: L-shape problem (6.3). Comparison of the multilevel solver of Definition 4.4 corresponding to the constructions with smoothing dAS (6.4) and wRAS (6.5) for “small” and “big” patches. i_{stop} : the number of iterations needed to reach the stopping criterion (6.6); $\bar{\alpha}$: average error contraction factor given by (6.7).

7 Proof of Theorem 5.1

As shown in Corollary 5.3, the results of Theorem 5.1 and Theorem 5.2 are equivalent. Therefore it suffices to prove the first one. Our approach to proving Theorem 5.1 consists in studying the uncomputable exact residual lifting $\tilde{\rho}_{J,\text{alg}}^i$ given by (3.2) and its approximation $\rho_{J,\text{alg}}^i$ given by Definition 3.1. In particular, we will estimate p -robustly the quantities $\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|$, $\|\nabla \rho_{J,\text{alg}}^i\|$, and $(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)$ by local contributions $\rho_{j,\mathbf{a}}^i$ of (3.7) used to construct $\rho_{J,\text{alg}}^i$. This will allow us to prove the claim of the theorem: $\eta_{\text{alg}}^i \geq \beta \|\nabla(u_J - u_J^i)\|$, with η_{alg}^i our a posteriori estimator of Definition 4.1 and β a p -independent constant. Actually, due to (3.4) and Definition 4.1, this is equivalent to showing

$$\begin{aligned} \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|} &\geq \beta \|\nabla \tilde{\rho}_{J,\text{alg}}^i\| && \text{when } \rho_{J,\text{alg}}^i \neq 0, \\ \|\nabla \tilde{\rho}_{J,\text{alg}}^i\| &= 0 && \text{when } \rho_{J,\text{alg}}^i = 0. \end{aligned}$$

Hereafter, we will use the notation $x_1 \lesssim x_2$ when there exists c , a positive real constant only depending on the mesh shape regularity parameter $\kappa_{\mathcal{T}}$ and the space dimension d such that $x_1 \leq cx_2$. Similarly, $x_1 \gtrsim x_2$ means $x_2 \lesssim x_1$ and $x_1 \approx x_2$ means that $x_1 \lesssim x_2$ and $x_2 \lesssim x_1$ simultaneously. If these constants additionally depend on the number of mesh levels J , we use the notations \lesssim_J , \gtrsim_J , and \approx_J , respectively.

7.1 Upper bound on $\|\nabla \rho_{J,\text{alg}}^i\|$ by patchwise contributions from all mesh levels

We present here properties of the constructed residual lifting $\rho_{J,\text{alg}}^i$ and its level-wise components ρ_j^i , where $1 \leq j \leq J$.

Lemma 7.1 (Estimating $\|\nabla \rho_{J,\text{alg}}^i\|$ and $\|\nabla \rho_j^i\|$ by local contributions). *Let $\rho_{J,\text{alg}}^i$ and ρ_j^i be given by Definition 3.1, for $j \in \{1, \dots, J\}$. There holds*

$$\|\nabla \rho_j^i\|^2 \leq \frac{1}{J^2(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2, \quad (7.2)$$

$$\|\nabla \rho_{J,\text{alg}}^i\|^2 \leq 2 \left(\|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \right). \quad (7.3)$$

Proof. Definition 3.1 and the inequality $|\sum_{k=1}^{d+1} a_k|^2 \leq (d+1) \sum_{k=1}^{d+1} |a_k|^2$ lead to

$$\begin{aligned} \|\nabla \rho_j^i\|^2 &= \sum_{K \in \mathcal{T}_{j-s}} \|\nabla \rho_j^i\|_K^2 = \sum_{K \in \mathcal{T}_{j-s}} \left\| \frac{1}{J(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_K} \nabla \rho_{j,\mathbf{a}}^i \right\|_K^2 \\ &\leq \frac{d+1}{J^2(d+1)^2} \sum_{K \in \mathcal{T}_{j-s}} \sum_{\mathbf{a} \in \mathcal{V}_K} \|\nabla \rho_{j,\mathbf{a}}^i\|_K^2 = \frac{1}{J^2(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2. \end{aligned}$$

Note that this allows us to write

$$\left\| \sum_{j=1}^J \nabla \rho_j^i \right\|^2 \leq J \sum_{j=1}^J \|\nabla \rho_j^i\|^2 \leq \frac{J}{J^2(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2. \quad (7.4)$$

This property facilitates the writing of the sought estimate for $\rho_{J,\text{alg}}^i$,

$$\begin{aligned} \|\nabla \rho_{J,\text{alg}}^i\|^2 &= \left\| \nabla \rho_0^i + \sum_{j=1}^J \nabla \rho_j^i \right\|^2 \leq 2\|\nabla \rho_0^i\|^2 + 2 \left\| \sum_{j=1}^J \nabla \rho_j^i \right\|^2 \\ &\stackrel{(7.4)}{\leq} 2\|\nabla \rho_0^i\|^2 + \frac{2}{J(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \\ &\leq 2 \left(\|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \right). \end{aligned}$$

□

7.2 Lower bound on $(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)$ by patchwise contributions from all mesh levels

While studying $(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)$, the interaction of different level contributions of the lifting $\rho_{J,\text{alg}}^i$ arises naturally. In order to estimate these terms, the damping $J(d+1)$ used in the construction (3.6) of our lifting proves to be of the essence.

Lemma 7.2 (*p*-robust estimate on $(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)$ from below by patchwise contributions). *Let $\rho_{J,\text{alg}}^i$ be given by Definition 3.1. Then*

$$(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i) \gtrsim_J \|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2. \quad (7.5)$$

Proof. We begin by using the construction of $\rho_{J,\text{alg}}^i$ given in Definition 3.1 to write

$$\begin{aligned} (f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i) &= (f, \rho_0^i) - (\nabla u_J^i, \nabla \rho_0^i) + \sum_{j=1}^J \left((f, \rho_j^i) - (\nabla u_J^i, \nabla \rho_j^i) \right) \\ &\stackrel{(3.1)}{=} \|\nabla \rho_0^i\|^2 + \frac{1}{J(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \left((f, \rho_{j,\mathbf{a}}^i)_{\omega_{j,s}^{\mathbf{a}}} - (\nabla u_J^i, \nabla \rho_{j,\mathbf{a}}^i)_{\omega_{j,s}^{\mathbf{a}}} \right) \\ &\stackrel{(3.7)}{=} \|\nabla \rho_0^i\|^2 + \frac{1}{J(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \left(\|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 + \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla \rho_{j,\mathbf{a}}^i)_{\omega_{j,s}^{\mathbf{a}}} \right) \\ &= \|\nabla \rho_0^i\|^2 + \frac{1}{J(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 + \sum_{j=1}^J \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla \rho_j^i). \end{aligned}$$

The first two terms above are of the right form to prove the result, one needs to be a bit more careful with the third one, however. We estimate it using Young's inequality, where the parameter is chosen in function of the damping factor $J(d+1)$, and the sum interchange $\sum_{j=2}^J \sum_{k=1}^{j-1} = \sum_{k=1}^{J-1} \sum_{j=k+1}^J$

$$\begin{aligned}
\sum_{j=1}^J \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla \rho_j^i) &= \sum_{j=2}^J \sum_{k=1}^{j-1} (\nabla \rho_k^i, \nabla \rho_j^i) + \sum_{j=1}^J (\nabla \rho_0^i, \nabla \rho_j^i) \\
&\geq \sum_{j=2}^J \sum_{k=1}^{j-1} \left(-\frac{1}{2} \|\nabla \rho_k^i\|^2 - \frac{1}{2} \|\nabla \rho_j^i\|^2 \right) + \sum_{j=1}^J \left(-\frac{1}{2 \frac{2J}{3}} \|\nabla \rho_0^i\|^2 - \frac{2J}{2} \|\nabla \rho_j^i\|^2 \right) \\
&= -\frac{1}{2} \sum_{k=1}^{J-1} (J-k) \|\nabla \rho_k^i\|^2 - \frac{1}{2} \sum_{j=2}^J (j-1) \|\nabla \rho_j^i\|^2 - \frac{3}{4J} J \|\nabla \rho_0^i\|^2 - \frac{J}{3} \sum_{j=1}^J \|\nabla \rho_j^i\|^2 \\
&= -\frac{1}{2} \sum_{j=1}^J (J-j) \|\nabla \rho_j^i\|^2 - \frac{1}{2} \sum_{j=1}^J (j-1) \|\nabla \rho_j^i\|^2 - \frac{3}{4} \|\nabla \rho_0^i\|^2 - \frac{J}{3} \sum_{j=1}^J \|\nabla \rho_j^i\|^2,
\end{aligned}$$

where we added the terms in the sum corresponding to $k = J$ and $j = 1$ since they are zero, and then renamed the summation index when there is no confusion. A few more manipulations on the right-hand side and the use of (7.2), give us

$$\sum_{j=1}^J \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla \rho_j^i) \geq -\frac{3}{4} \|\nabla \rho_0^i\|^2 - \frac{5J}{6} \frac{1}{J^2(d+1)} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2.$$

Returning to the main estimate, we obtain the desired result

$$(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i) \geq \frac{1}{6J(d+1)} \left(\|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \right). \quad (7.6)$$

□

7.3 Upper bound on $\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|$ by patchwise contributions from all mesh levels

Recall that $\tilde{\rho}_{J,\text{alg}}^i$, introduced in (3.2), is the unknown exact algebraic error. We estimate here $\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|$ from above. First, we summarize for our setting the remarkable result of Schöberl et al. [33], stating the existence of a p -robust stable decomposition for a fixed mesh. Then, we adapt this result to the multilevel case. This, together with inter-level and local properties of $\tilde{\rho}_{J,\text{alg}}^i$ introduced hereafter, allows to obtain a p -robust estimate on the algebraic error.

7.3.1 Polynomial-degree-robust stable decomposition on a fixed mesh level j

We begin by presenting in the form of a lemma the p -robust stable decomposition result of Schöberl et al. [33, Proof of Theorem 2.1]. This decomposition of any function on a given level into a continuous piecewise affine part and a sum of local continuous piecewise polynomials of degree p will be particularly important in the following. We also use the local spaces $V_{j,s}^{\mathbf{a}}$ introduced in (2.9).

Lemma 7.3 (One-level p -robust stable decomposition). *Let $1 \leq j \leq J$ and let $v_j \in V_j^p$ be arbitrary. Suppose $v_j = v_j^\# + v_j^{\flat}$, with $v_j^\# \in V_j^1$ such that*

$$\|\nabla v_j^\#\|^2 + \|\nabla(v_j - v_j^\#)\|^2 + \sum_{K \in \mathcal{T}_j} h_K^{-1} \|(v_j - v_j^\#)\|_K^2 \leq C_{\text{LO}} \|\nabla v_j\|^2, \quad (7.7)$$

for a positive constant C_{LO} only depending on mesh shape regularity parameter $\kappa_{\mathcal{T}}$ and space dimension d . Then, there are $v^{\mathbf{b}} \in V_{j,0}^{\mathbf{b}}$, $\mathbf{b} \in \mathcal{V}_j$, such that $v_j = v_j^\# + \sum_{\mathbf{b} \in \mathcal{V}_j} v_j^{\mathbf{b}}$, and this decomposition is stable in the sense

$$\exists C_{\text{SD}} > 0 \quad \text{such that} \quad \|\nabla v_j^\#\|^2 + \sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla v_j^{\mathbf{b}}\|_{\omega_{j,0}^{\mathbf{b}}}^2 \leq C_{\text{SD}} \|\nabla v_j\|^2, \quad (7.8)$$

where C_{SD} only depends on the constant C_{LO} , mesh shape regularity parameter $\kappa_{\mathcal{T}}$, and space dimension d . Additionally, we can assume that $C_{\text{SD}} > 1$ (otherwise, we work with $\tilde{C}_{\text{SD}} := \max(1, C_{\text{SD}})$ which still satisfies (7.8)).

7.3.2 Polynomial-degree-robust stable decomposition on a hierarchy of meshes

Because we intend to obtain a decomposition similar to Lemma 7.3 in a multilevel setting, we will work with $v^\# \in V_0$. For this purpose, we first define a coarse space interpolator.

Lemma 7.4 (Coarse space interpolation). *For $v \in H_0^1(\Omega)$, define the vertex values*

$$\begin{aligned} (\mathcal{C}_0 v)(\mathbf{a}) &:= \frac{1}{|\mathcal{T}_{0,0}^{\mathbf{a}}|} \sum_{K \in \mathcal{T}_{0,0}^{\mathbf{a}}} \bar{v}_K \quad \text{for } \mathbf{a} \in \mathcal{V}_0^{\text{int}}, \quad \text{where } \bar{v}_K := \frac{(v, 1)_K}{|K|}, \\ (\mathcal{C}_0 v)(\mathbf{a}) &:= 0 \quad \text{for } \mathbf{a} \in \mathcal{V}_0^{\text{ext}}. \end{aligned}$$

where $\mathcal{T}_{0,0}^{\mathbf{a}}$ is the patch of elements sharing the vertex \mathbf{a} , cf. (2.8). The operator $\mathcal{C}_0 : H_0^1(\Omega) \rightarrow V_0$, given by $\mathcal{C}_0(v) := \sum_{\mathbf{a} \in \mathcal{V}_0} \mathcal{C}_0 v(\mathbf{a}) \psi_0^{\mathbf{a}}$, satisfies for all $v \in H_0^1(\Omega)$

$$\|\nabla \mathcal{C}_0(v)\|^2 + \|\nabla(v - \mathcal{C}_0(v))\|^2 + \sum_{K \in \mathcal{T}_0} h_K^{-1} \|(v - \mathcal{C}_0(v))\|_K^2 \leq \tilde{C}_{\text{LO}} \|\nabla v\|^2. \quad (7.9)$$

Proof. We start by estimating the first term in (7.9). Consider $v \in H_0^1(\Omega)$, $K \in \mathcal{T}_0$. Note that the hat functions form a partition of unity: $\sum_{\mathbf{a} \in \mathcal{V}_0} \psi_0^{\mathbf{a}} = 1$. Then

$$\begin{aligned} \|\nabla \mathcal{C}_0(v)\|_K &= \|\nabla \mathcal{C}_0(v) - \bar{v}_K \nabla 1\|_K = \left\| \nabla \left(\sum_{\mathbf{a} \in \mathcal{V}_K} (\mathcal{C}_0 v)(\mathbf{a}) \psi_0^{\mathbf{a}} \right) - \bar{v}_K \nabla \left(\sum_{\mathbf{a} \in \mathcal{V}_K} \psi_0^{\mathbf{a}} \right) \right\|_K \\ &= \left\| \sum_{\mathbf{a} \in \mathcal{V}_K} \{((\mathcal{C}_0 v)(\mathbf{a}) - \bar{v}_K) \nabla \psi_0^{\mathbf{a}}\} \right\|_K \leq \sum_{\mathbf{a} \in \mathcal{V}_K} \{ |(\mathcal{C}_0 v)(\mathbf{a}) - \bar{v}_K| \|\nabla \psi_0^{\mathbf{a}}\|_K \} \\ &\lesssim h_K^{-1} |K|^{\frac{1}{2}} \left(\sum_{\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{int}}} |\mathcal{T}_{0,0}^{\mathbf{a}}|^{-1} \sum_{K^* \in \mathcal{T}_{0,0}^{\mathbf{a}}} |\bar{v}_{K^*} - \bar{v}_K| + \sum_{\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{ext}}} |\mathcal{T}_{0,0}^{\mathbf{a}}|^{-1} \sum_{K^* \in \mathcal{T}_{0,0}^{\mathbf{a}}} |\bar{v}_K| \right), \end{aligned}$$

where $|\mathcal{T}_{0,0}^{\mathbf{a}}|$ is uniformly bounded by the mesh shape regularity parameter $\kappa_{\mathcal{T}}$ and space dimension d . We distinguish two cases.

Case $\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{int}}$. There are two possibilities: either K^* and K share an interface F_1 , or there is a path of elements in the neighborhood of K connecting K and K^* such that $|\bar{v}_{K^*} - \bar{v}_K| \leq \sum_{l=0}^{L_1-1} |\bar{v}_{K_{l+1}} - \bar{v}_{K_l}|$ by the triangle inequality, where $K_{L_1} = K^*$ and $K_0 = K$, and L_1 only depends on the mesh shape regularity parameter $\kappa_{\mathcal{T}}$ and space dimension d . Thus, we need only treat the case where K^* and K share an interface F_1 . Note that, since $v \in H_0^1(\Omega)$, its trace is well defined, and we can introduce the notation $\bar{v}_{F_1} := (v, 1)_{F_1} / |F_1| \in \mathbb{R}$. Using interpolation estimates for simplices shown in e.g., Eymard *et al.* [18, Lemma 2], Vohralík [34, Lemma 4.1], we write

$$\begin{aligned} |\bar{v}_{K^*} - \bar{v}_K| &= |\bar{v}_{K^*} - \bar{v}_{F_1} + \bar{v}_{F_1} - \bar{v}_K| \leq |\bar{v}_{K^*} - \bar{v}_{F_1}| + |\bar{v}_{F_1} - \bar{v}_K| \\ &\lesssim \max_{K^\# \in \{K, K^*\}} (h_{K^\#} |K^\#|^{-\frac{1}{2}}) (\|\nabla v\|_{K^*} + \|\nabla v\|_K) \\ &\leq \max_{K^\# \in \{K, K^*\}} (h_{K^\#} |K^\#|^{-\frac{1}{2}}) \|\nabla v\|_{\omega_K}, \end{aligned} \quad (7.10)$$

where $\omega_K := \cup_{\mathbf{a} \in \mathcal{V}_K} \omega_{0,0}^{\mathbf{a}}$.

Case $\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{ext}}$. There are again two possibilities: either the intersection of K with $\partial\Omega$ is an interface F_2 , or it is the vertex \mathbf{a} . For the latter, there is again a path connecting K with \tilde{K} , such that the intersection of \tilde{K} and $\partial\Omega$ is a face. Similarly to the first case, we can write $|\bar{v}_K| \leq |\bar{v}_{\tilde{K}}| + \sum_{l=0}^{L_2-1} |\bar{v}_{K_l} - \bar{v}_{K_{l+1}}|$, where $K_{L_2} = \tilde{K}$ and $K_0 = K$, and L_2 only depends on the mesh shape regularity parameter $\kappa_{\mathcal{T}}$ and space

dimension d . Note that the terms in the sum can be treated as in (7.10). Thus, it is sufficient to consider $|\bar{v}_K|$ when $K \cap \partial\Omega = F_2$, a face of K . Since $v \in H_0^1(\Omega)$, we have $\bar{v}_{F_2} := (v, 1)_{F_2}/|F_2| = 0$,

$$|\bar{v}_K| = |\bar{v}_K - \bar{v}_{F_2}| \lesssim h_K |K|^{-\frac{1}{2}} \|\nabla v\|_K.$$

In view of the mesh regularity Assumption 2.7, this leads to the desired estimate for the first term in (7.9)

$$\|\nabla \mathcal{C}_0(v)\| \lesssim \|\nabla v\|. \quad (7.11)$$

As for the second term, we use the triangle inequality and (7.11).

It remains to estimate the third term. We have

$$\begin{aligned} \|v - \mathcal{C}_0(v)\|_K &\leq \|v - \bar{v}_K 1\|_K + \|\bar{v}_K 1 - \mathcal{C}_0(v)\|_K \\ &\stackrel{\text{Poincaré}}{\lesssim} h_K \|\nabla v\|_K + \left\| \sum_{\mathbf{a} \in \mathcal{V}_K} (\bar{v}_K - (\mathcal{C}_0 v)(\mathbf{a})) \psi_0^{\mathbf{a}} \right\|_K \\ &\leq h_K \|\nabla v\|_K + \sum_{\mathbf{a} \in \mathcal{V}_K} |\bar{v}_K - (\mathcal{C}_0 v)(\mathbf{a})| \|\psi_0^{\mathbf{a}}\|_K \\ &\leq h_K \|\nabla v\|_K + |K|^{\frac{1}{2}} \left(\sum_{\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{int}}} |\mathcal{T}_{0,0}^{\mathbf{a}}|^{-1} \sum_{K^* \in \mathcal{T}_{0,0}^{\mathbf{a}}} |\bar{v}_K - \bar{v}_{K^*}| + \sum_{\mathbf{a} \in \mathcal{V}_K \cap \mathcal{V}_0^{\text{ext}}} |\bar{v}_K| \right). \end{aligned}$$

The last two terms in this estimate can be treated similarly to (7.10), allowing us to obtain the inequality $\|v - \mathcal{C}_0(v)\|_K \lesssim h_K \|\nabla v\|_{\omega_K}$. Finally, putting the three estimations together, we obtain the desired result. We define \tilde{C}_{LO} the constant obtained after these estimations; it only depends on the mesh shape regularity parameter $\kappa_{\mathcal{T}}$ and space dimension d . \square

Lemma 7.5 (Multilevel p -robust stable decomposition). *For any $v_j \in V_j^p$, there exists a constant $C_{\text{SD},J}$ only depending on the mesh shape regularity parameter $\kappa_{\mathcal{T}}$, space dimension d , and the number of mesh levels J , such that*

$$\begin{aligned} v_j &= \mathcal{C}_0(v_j) + \sum_{\mathbf{b} \in \mathcal{V}_j} v_j^{\mathbf{b}}, \quad v_j^{\mathbf{b}} \in V_{j,0}^{\mathbf{b}}, \\ \|\nabla \mathcal{C}_0(v_j)\|^2 + \sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla v_j^{\mathbf{b}}\|_{\omega_{j,0}^{\mathbf{b}}}^2 &\leq C_{\text{SD},J} \|\nabla v_j\|^2. \end{aligned} \quad (7.12)$$

Proof. By Lemma 7.4, and since $h_K \approx_J h_{K^*}$ for all $K \in \mathcal{T}_0$ and all $K^* \in \mathcal{T}_j$, $K^* \subset K$ by Assumption 2.6, there holds

$$\|\nabla \mathcal{C}_0(v_j)\|^2 + \|\nabla(v_j - \mathcal{C}_0(v_j))\|^2 + \sum_{K \in \mathcal{T}_j} h_K^{-1} \|(v_j - \mathcal{C}_0(v_j))\|_K^2 \lesssim_J \|\nabla v_j\|^2,$$

where the constant in the estimate above depends on \tilde{C}_{LO} of (7.9) and the number of mesh levels J . Thus, we have $\mathcal{C}_0(v_j) \in V_0 \subset V_j^1$ which satisfies (7.7). Using the result of Schöberl et al. [33] described in Lemma 7.3, we obtain $v_j^{\mathbf{b}} \in V_{j,0}^{\mathbf{b}}$, for $\mathbf{b} \in \mathcal{V}_j$ such that (7.12) holds with a constant $C_{\text{SD},J}$ only depending on the mesh shape regularity parameter $\kappa_{\mathcal{T}}$, space dimension d , and number of mesh levels J . \square

7.3.3 Orthogonality of $\tilde{\rho}_j^i$, local links between $\tilde{\rho}_j^i$ and $\rho_{j,\mathbf{a}}^i$

Two other important components will serve in estimating $\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|$. First, the relations of orthogonality of a given mesh error contribution $\tilde{\rho}_j^i$, $j \in \{1, \dots, J\}$, with respect to previous mesh level functions. And secondly, the local properties of $\tilde{\rho}_j^i$ with respect to local functions of the same mesh. In particular, the local properties will allow the transition from the uncomputable $\tilde{\rho}_j^i$ to the available local contributions of ρ_j^i .

Lemma 7.6 (Inter-level properties of $\tilde{\rho}_j^i$). *Consider the hierarchical construction of the error $\tilde{\rho}_{J,\text{alg}}^i$ given in (3.2). For $j \in \{1, \dots, J\}$ and $k \in \{0, \dots, j-1\}$, there holds*

$$(\nabla \tilde{\rho}_j^i, \nabla v_k) = 0 \quad \forall v_k \in V_k^p. \quad (7.13)$$

Proof. Take $v_k \in V_k^p$. Note that since $k \leq j-1$, we have $v_k \in V_{j-1}^p \subset V_j^p$. The definition given in (3.3) applied to $\tilde{\rho}_j^i$ and $\tilde{\rho}_{j-1}^i$ allows us to write

$$\begin{aligned} (\nabla \tilde{\rho}_j^i, \nabla v_k) &= (f, v_k) - (\nabla u_J^i, \nabla v_k) - \sum_{l=0}^{j-2} (\nabla \tilde{\rho}_l^i, \nabla v_k) - (\nabla \tilde{\rho}_{j-1}^i, \nabla v_k) \\ &= (\nabla \tilde{\rho}_{j-1}^i, \nabla v_k) - (\nabla \tilde{\rho}_{j-1}^i, \nabla v_k) = 0. \end{aligned}$$

□

Below, we present the relation between $\tilde{\rho}_j^i$ and ρ_j^i locally on patches, more precisely when tested against functions of the local spaces $V_{j,s}^{\mathbf{a}}$ given by (2.9).

Lemma 7.7 (Local relation between $\tilde{\rho}_j^i$ and ρ_j^i). *Let $j \in \{1, \dots, J\}$. Let $\tilde{\rho}_j^i, \rho_j^i$ be given by (3.3), (3.6), respectively. For all $\mathbf{a} \in \mathcal{V}_{j-s}$ and all $v_{j,\mathbf{a}} \in V_{j,s}^{\mathbf{a}}$, we have*

$$(\nabla \tilde{\rho}_j^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} = (\nabla \rho_j^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - \sum_{k=1}^{j-1} (\nabla (\tilde{\rho}_k^i - \rho_k^i), \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}}, \quad (7.14)$$

where $\rho_{j,\mathbf{a}}^i \in V_{j,s}^{\mathbf{a}}$ is defined as solution of a local problem by (3.7). We use the convention that the sum in the relation above is zero when $j = 1$.

Proof. We take $v_{j,\mathbf{a}} \in V_{j,s}^{\mathbf{a}}$. This implies that $v_{j,\mathbf{a}}$ is zero on the boundary of the patch domain $\omega_{j,s}^{\mathbf{a}}$. Since $v_{j,\mathbf{a}} \in V_j^p$, we can use it as a test function in the definition of $\tilde{\rho}_j^i$ in (3.3) as well as in the definition of $\rho_{j,\mathbf{a}}^i$ in (3.7). We conclude by subtracting the two following identities once we take into account $\tilde{\rho}_0^i = \rho_0^i$

$$\begin{aligned} (\nabla \tilde{\rho}_j^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} &= (f, v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - (\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - \sum_{k=0}^{j-1} (\nabla \tilde{\rho}_k^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}}, \\ (\nabla \rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} &= (f, v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - (\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}} - \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_{j,s}^{\mathbf{a}}}. \end{aligned}$$

□

7.3.4 Estimating the error on a hierarchy of meshes

The previous results allowed to establish a useful p -robust stable decomposition for a hierarchy of meshes, and summarize the inter-level and local properties of the error $\tilde{\rho}_{J,\text{alg}}^i$. These properties will be useful in the forthcoming lemma in order to give an upper bound to algebraic error $\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|$ by the local constructed contributions $\rho_{j,\mathbf{a}}^i$ of the hierarchy of meshes.

Lemma 7.8 (p -robust error estimation). *Let $\tilde{\rho}_{J,\text{alg}}^i$ and $\rho_{J,\text{alg}}^i$ be defined by (3.2) and Definition 3.1, respectively. There holds*

$$\|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \gtrsim_J \|\nabla \tilde{\rho}_{J,\text{alg}}^i\|^2. \quad (7.15)$$

Proof. We begin by estimating $\|\nabla \tilde{\rho}_j^i\|$, where $\tilde{\rho}_j^i \in V_j^p$ solves (3.3) for $1 \leq j \leq J$. From Lemma 7.5, using the stable decomposition result applied to $\tilde{\rho}_j^i$,

$$\tilde{\rho}_j^i = \mathcal{C}_0(\tilde{\rho}_j^i) + \sum_{\mathbf{b} \in \mathcal{V}_j} \tilde{\rho}_{j,\mathbf{b}}^i, \quad \tilde{\rho}_{j,\mathbf{b}}^i \in V_{j,0}^{\mathbf{b}}, \quad (7.16a)$$

$$\|\nabla \mathcal{C}_0(\tilde{\rho}_j^i)\|^2 + \sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla \tilde{\rho}_{j,\mathbf{b}}^i\|_{\omega_{j,0}^{\mathbf{b}}}^2 \leq C_{\text{SD},J} \|\nabla \tilde{\rho}_j^i\|^2. \quad (7.16b)$$

In the case $s = 1$, i.e. of “big” patches of Figure 1 (right), note that for $\mathbf{b} \in \mathcal{V}_j$, we can pick a vertex $\mathbf{a}_\mathbf{b} \in \mathcal{V}_{j-s}$ such that $\omega_{j,0}^\mathbf{b} \subset \omega_{j,s}^{\mathbf{a}_\mathbf{b}}$, and we can extend $\tilde{\rho}_{j,\mathbf{b}}^i$ by zero to $\omega_{j,s}^{\mathbf{a}_\mathbf{b}}$, so that it can be used as test function in the local problems (7.14). There is no need for extension in the case $s = 0$ of “small” patches of Figure 1 (left), where we can take $\mathbf{a}_\mathbf{b} = \mathbf{b} \in \mathcal{V}_{j-s}$ and $\omega_{j,0}^{\mathbf{a}_\mathbf{b}} = \omega_{j,0}^\mathbf{b}$. We introduce $\mathbf{q}_{j,s} := \max_{\mathbf{a} \in \mathcal{V}_{j-s}} \#\{\mathbf{b} \in \mathcal{V}_j \mid \omega_{j,0}^\mathbf{b} \subset \omega_{j,s}^\mathbf{a}\}$. Note that $\mathbf{q}_{j,s} \geq 1$ when $s = 1$ and $\mathbf{q}_{j,s} = 1$ for $s = 0$. We have

$$\begin{aligned}
\|\nabla \tilde{\rho}_j^i\|^2 &\stackrel{(7.16a)}{=} (\nabla \tilde{\rho}_j^i, \nabla \mathcal{C}_0(\tilde{\rho}_j^i)) + (\nabla \tilde{\rho}_j^i, \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i) \stackrel{(7.13)}{=} 0 + \sum_{\mathbf{b} \in \mathcal{V}_j} (\nabla \tilde{\rho}_j^i, \nabla \tilde{\rho}_{j,\mathbf{b}}^i)_{\omega_{j,0}^\mathbf{b}} \\
&\stackrel{(7.14)}{=} \sum_{\mathbf{b} \in \mathcal{V}_j} (\nabla \rho_{j,\mathbf{a}_\mathbf{b}}^i, \nabla \tilde{\rho}_{j,\mathbf{b}}^i)_{\omega_{j,s}^{\mathbf{a}_\mathbf{b}}} - \sum_{\mathbf{b} \in \mathcal{V}_j} \sum_{k=0}^{j-1} (\nabla(\tilde{\rho}_k^i - \rho_k^i), \nabla \tilde{\rho}_{j,\mathbf{b}}^i)_{\omega_{j,s}^{\mathbf{a}_\mathbf{b}}} \\
&= \sum_{\mathbf{b} \in \mathcal{V}_j} \left(\sqrt{2C_{\text{SD},J}} \nabla \rho_{j,\mathbf{a}_\mathbf{b}}^i, \frac{\nabla \tilde{\rho}_{j,\mathbf{b}}^i}{\sqrt{2C_{\text{SD},J}}} \right)_{\omega_{j,s}^{\mathbf{a}_\mathbf{b}}} - \sum_{k=1}^{j-1} \left(\nabla(\tilde{\rho}_k^i - \rho_k^i), \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \\
&\leq C_{\text{SD},J} \sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}_\mathbf{b}}^i\|_{\omega_{j,s}^{\mathbf{a}_\mathbf{b}}}^2 + \frac{\sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla \tilde{\rho}_{j,\mathbf{b}}^i\|_{\omega_{j,s}^{\mathbf{a}_\mathbf{b}}}^2}{4C_{\text{SD},J}} - \sum_{k=1}^{j-1} \left(\nabla(\tilde{\rho}_k^i - \rho_k^i), \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \\
&\stackrel{(7.16b)}{\leq} C_{\text{SD},J} \sum_{\substack{\mathbf{a} \in \mathcal{V}_{j-s} \\ \omega_{j,0}^\mathbf{b} \subset \omega_{j,s}^\mathbf{a}}} \sum_{\mathbf{b} \in \mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^\mathbf{a}}^2 + \frac{\|\nabla \tilde{\rho}_j^i\|^2}{4} - \sum_{k=1}^{j-1} \left(\nabla(\tilde{\rho}_k^i - \rho_k^i), \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \\
&\leq \mathbf{q}_{j,s} C_{\text{SD},J} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^\mathbf{a}}^2 + \frac{1}{4} \|\nabla \tilde{\rho}_j^i\|^2 - \sum_{k=1}^{j-1} \left(\nabla(\tilde{\rho}_k^i - \rho_k^i), \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right). \tag{7.17}
\end{aligned}$$

For the special case of $j = 1$, the third term is not present since $\tilde{\rho}_0^i = \rho_0^i$. This leads to

$$\|\nabla \tilde{\rho}_1^i\|^2 \leq \frac{4}{3} \mathbf{q}_{1,s} C_{\text{SD},1} \sum_{\mathbf{a} \in \mathcal{V}_{1-s}} \|\nabla \rho_{1,\mathbf{a}}^i\|_{\omega_{1,s}^\mathbf{a}}^2 \leq 2 \mathbf{q}_{1,s} C_{\text{SD},1} \sum_{\mathbf{a} \in \mathcal{V}_{1-s}} \|\nabla \rho_{1,\mathbf{a}}^i\|_{\omega_{1,s}^\mathbf{a}}^2. \tag{7.18}$$

This would be enough to conclude the proof if $J = 1$, since

$$\|\nabla \tilde{\rho}_{J,\text{alg}}^i\|^2 \stackrel{(3.4)}{=} \|\nabla \rho_0^i\|^2 + \|\nabla \tilde{\rho}_1^i\|^2 \stackrel{(7.18)}{\leq} 2 \mathbf{q}_{1,s} C_{\text{SD},1} \left(\|\nabla \rho_0^i\|^2 + \sum_{\mathbf{a} \in \mathcal{V}_{1-s}} \|\nabla \rho_{1,\mathbf{a}}^i\|_{\omega_{1,s}^\mathbf{a}}^2 \right).$$

When $J > 1$, we continue below the estimation for the third term in (7.17) for $j \in \{2, \dots, J\}$. The inter-level properties of $\tilde{\rho}_{j,\text{alg}}^i$ presented in (7.13) are crucial here:

$$\begin{aligned}
& - \sum_{k=1}^{j-1} \left(\nabla(\tilde{\rho}_k^i - \rho_k^i), \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \stackrel{(7.16a)}{=} - \sum_{k=1}^{j-1} \left(\left(\nabla \tilde{\rho}_k^i, \nabla(\tilde{\rho}_j^i - \mathcal{C}_0(\tilde{\rho}_j^i)) \right) - \left(\nabla \rho_k^i, \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \right) \\
& \stackrel{(7.13)}{=} 0 + 0 + \sum_{k=1}^{j-1} \left(\sqrt{2C_{\text{SD},J}J(d+1)} \nabla \rho_k^i, \frac{1}{\sqrt{2C_{\text{SD},J}J(d+1)}} \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right) \\
& \leq C_{\text{SD},J}J(d+1) \sum_{k=1}^{j-1} \|\nabla \rho_k^i\|^2 + \frac{1}{4C_{\text{SD},J}J(d+1)} \sum_{k=1}^{j-1} \left\| \sum_{\mathbf{b} \in \mathcal{V}_j} \nabla \tilde{\rho}_{j,\mathbf{b}}^i \right\|^2 \\
& \stackrel{(7.2)}{\leq} \frac{C_{\text{SD},J}}{J} \sum_{k=1}^{j-1} \sum_{\mathbf{a} \in \mathcal{V}_{k-s}} \|\nabla \rho_{k,\mathbf{a}}^i\|_{\omega_{k,s}^\mathbf{a}}^2 + \frac{1}{4} \|\nabla \tilde{\rho}_j^i\|^2.
\end{aligned}$$

Returning to (7.17), we obtain for all $j \in \{2, \dots, J\}$

$$\|\nabla \tilde{\rho}_j^i\|^2 \leq 2\mathfrak{q}_{j,s} C_{\text{SD},J} \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 + \frac{2C_{\text{SD},J}}{J} \sum_{k=1}^{j-1} \sum_{\mathbf{a} \in \mathcal{V}_{k-s}} \|\nabla \rho_{k,\mathbf{a}}^i\|_{\omega_{k,s}^{\mathbf{a}}}^2. \quad (7.19)$$

We achieve the result by summing these estimates on different levels. We denote by $\mathfrak{q} := \max_{j \in \{1, \dots, J\}} \mathfrak{q}_{j,s}$. Then

$$\begin{aligned} \|\nabla \tilde{\rho}_{J,\text{alg}}^i\|^2 &\stackrel{(3.4)}{=} \|\nabla \rho_0^i\|^2 + \|\nabla \tilde{\rho}_1^i\|^2 + \sum_{j=2}^J \|\nabla \tilde{\rho}_j^i\|^2 \\ &\stackrel{(7.18)}{\leq} \|\nabla \rho_0^i\|^2 + 2\mathfrak{q} C_{\text{SD},J} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 + \frac{2C_{\text{SD},J}}{J} \sum_{j=2}^J \sum_{k=1}^{j-1} \sum_{\mathbf{a} \in \mathcal{V}_{k-s}} \|\nabla \rho_{k,\mathbf{a}}^i\|_{\omega_{k,s}^{\mathbf{a}}}^2 \\ &\stackrel{(7.19)}{\leq} \|\nabla \rho_0^i\|^2 + 2\mathfrak{q} C_{\text{SD},J} \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 + 2C_{\text{SD},J} \sum_{k=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{k-s}} \|\nabla \rho_{k,\mathbf{a}}^i\|_{\omega_{k,s}^{\mathbf{a}}}^2 \\ &\leq 4\mathfrak{q} C_{\text{SD},J} \left(\|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \right). \end{aligned} \quad (7.20)$$

□

7.4 Proof of Theorem 5.1

The results of the previous subsections allow us to give a concise proof of Theorem 5.1.

Proof of Theorem 5.1. Case $\rho_{J,\text{alg}}^i = 0$. By Definition 4.1 this means $\eta_{\text{alg}}^i = 0$, so that it suffices to show that $u_J = u_J^i$. We do this by using Lemma 7.2 and 7.8 which lead to

$$\begin{aligned} \|\nabla(u_J - u_J^i)\|^2 &\stackrel{(3.4)}{=} \|\nabla \tilde{\rho}_{J,\text{alg}}^i\|^2 \stackrel{(7.15)}{\lesssim_J} \|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \\ &\stackrel{(7.5)}{\lesssim_J} (f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i) = 0. \end{aligned}$$

Case $\rho_{J,\text{alg}}^i \neq 0$. In this case, we combine the results of Lemmas 7.1, 7.2, and 7.8

$$\begin{aligned} \eta_{\text{alg}}^i &= \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|} \stackrel{(7.3)}{\geq} \frac{1}{6J(d+1)\sqrt{2}} \left(\|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_{j-s}} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_{j,s}^{\mathbf{a}}}^2 \right)^{\frac{1}{2}} \\ &\stackrel{(7.20)}{\geq} \frac{1}{12J(d+1)\sqrt{2\mathfrak{q}C_{\text{SD},J}}} \|\nabla \tilde{\rho}_{J,\text{alg}}^i\| \stackrel{(3.4)}{=} \beta \|\nabla(u_J - u_J^i)\|, \end{aligned}$$

for $\beta := \frac{1}{12J(d+1)\sqrt{2\mathfrak{q}C_{\text{SD},J}}} > 0$, depending only on the mesh shape regularity parameter $\kappa_{\mathcal{T}}$, the space dimension d , and the number of levels J . □

8 Conclusions and outlook

In this work, we presented a hierarchical construction of the algebraic residual lifting in the spirit of Papež *et al.* [29]. This lifting approximates the algebraic error by one iteration of a V-cycle multigrid with no

pre-smoothing steps, a single damped additive Schwarz post-smoothing step, and a coarse solve of lowest polynomial degree. The lifting leads us to an a posteriori estimator on the algebraic error and to a linear iterative solver. We showed that two following results are equivalent: the (reliable) a posteriori estimator is p -robustly efficient, and the solver contracts p -robustly the error at each iteration. The provided numerical tests agree with these theoretical findings. Moreover, we also presented numerical results for a modified solver corresponding to a weighted restricted additive Schwarz smoothing. In accordance with the literature, this modified solver is a further speed-up compared to the damped Schwarz smoothing. Although we currently cannot show that our p -robust theoretical result also applies to this construction, the use of high degree polynomials does not seem to cause a degradation of the solver. Thus far, our theory involves estimates depending on the number of mesh levels J , which is not present in the numerical results for the weighted restricted variant. Further work would explore this dependence and seek a theoretical improvement. Another perspective we aim to develop is the use of adaptivity based on the derived efficient algebraic error estimator, and applications to more involved problems.

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