Counting Hamiltonian Cycles in the Matroid Basis Graph

Cristina G. Fernandes, César Hernández-Vélez, José C de Pina, Jorge Luis Ramírez Alfonsín

Counting Hamiltonian cycles in the matroid basis graph[☆]

Cristina G. Fernandes[a], César Hernández-Vélez[b,*],
José C. de Pina[a], Jorge Luis Ramírez Alfonsín[c]

[a] *Instituto de Matemática e Estatística, Universidade de São Paulo, Brazil*
[b] *Facultad de Ciencias, Universidad Autónoma de San Luis Potosí, Mexico*
[c] *Institut Montpelliérain Alexander Grothendieck, Université de Montpellier, CNRS, Montpellier, France*

Abstract

We present superfactorial and exponential lower bounds on the number of Hamiltonian cycles passing through any edge of the basis graph of generalized Catalan, uniform, and graphic matroids. All lower bounds were obtained by a common general strategy based on counting appropriated cycles of length four in the corresponding matroid basis graph.

Keywords: matroid basis graph, generalized Catalan matroid, Hamiltonian cycle

1. Introduction

Basis graphs of matroids have been extensively studied. Gel′fand and Serganova [11] proved that basis graphs are 1-skeletons of basis polytopes [6, 7]. Maurer [15] gave a characterization of basis graphs, Liu [12, 13, 14] investigated their connectivity, and Donald, Holzmann, and Tobey [9] gave a characterization of basis graphs of uniform matroids.

A graph is *edge-Hamiltonian* if it has at least three vertices and every edge is in a Hamiltonian cycle. According to Bondy and Ingleton [2], Haff (unpublished) showed that every basis graph is edge-Hamiltonian, unless it is K_1 or K_2, generalizing a result due to Cummins [8] and Shank [17] for graphic matroids. So, basis graphs with at least three vertices are edge-Hamiltonian. In fact, the work of Bondy and Ingleton [2, Theorems 1 and 2] implies the edge-Hamiltonicity proved by Haff. Alspach and Liu [1] proved that basis graphs are Hamilton-connected and edge-pancyclic. In this paper, we investigate further the edge-Hamiltonicity of basis graphs.

A *matroid* $M = (E, \mathcal{B})$ of *rank* $r = r(M)$ is a finite set E together with a nonempty collection $\mathcal{B} = \mathcal{B}(M)$ of r-subsets of E, called the *bases* of M, satisfying the following *basis exchange axiom*:

[☆]Research partially supported by CNPq (Proc. 308523/2012-1 and 456792/2014-7), FAPESP (Proc. 2012/24597-3, Proc. 2013/03447-6, and Proc. 2015/10323-7), PICS (Grant PICS06316), PRODEP (DSA/103.5/16/10419), and Project MaCLinC of NUMEC/USP.

[*]Corresponding author

Email addresses: cris@ime.usp.br (C.G. Fernandes), cesar.velez@uaslp.mx (C. Hernández-Vélez), coelho@ime.usp.br (J.C. de Pina), jramirez@um2.fr (J.L. Ramírez Alfonsín)

(*BEA*) If B_1 and B_2 are members of $\mathcal{B}$ and $e \in B_1 \setminus B_2$,
then there is an element $g \in B_2 \setminus B_1$ such that $(B_1 - e) + g \in \mathcal{B}$.

A *loop* is an element that does not belong to any base and an *isthmus* is an element that belongs to all bases. For general background in matroid theory, we refer the reader to Oxley [16] and Welsh [19].

The *basis graph* $\mathrm{BG}(M)$ *of a matroid* M is the graph having as vertex set the bases of M and two vertices (bases) B_1 and B_2 are adjacent if and only if the symmetric difference of B_1 and B_2 has cardinality two. A graph is a *basis graph* if it can be labeled to become the basis graph of some matroid. We make no distinction between a basis of M and a vertex of $\mathrm{BG}(M)$.

For a given matroid M let $\mathrm{HC}^*(M) = \min\{\mathrm{HC}_e(M) : e \in E(\mathrm{BG}(M))\}$, where $\mathrm{HC}_e(M)$ denotes the number of Hamiltonian cycles in $\mathrm{BG}(M)$ containing edge $e \in E(\mathrm{BG}(M))$. Bondy and Ingleton state that $\mathrm{HC}^*(M) \geq 1$ for every matroid M with at least three bases.

We start, in Section 2, by presenting the general strategy we use. In Section 3, we investigate $\mathrm{HC}^*(M)$ when M is in the class of lattice path matroids. We present a lower bound on $\mathrm{HC}_e(M)$ when M is a generalized Catalan matroid (Theorem 5). In particular, the derived lower bound for the k-Catalan matroid is superfactorial on k. In Section 4, we give lower bounds on $\mathrm{HC}^*(M_G)$ where M_G is the cycle matroid obtained from a k-edge-connected graph G. The lower bound for $k = 2$ is exponential on the number of vertices of G (Theorem 9). The lower bound for $k \geq 3$ is exponential on both k and the number of vertices of G (Theorem 12). Section 5 presents some concluding remarks.

2. General strategy

In order to give a lower bound on $\mathrm{HC}^*(M)$, we follow the strategy described below, which has the same spirit as the one used by Bondy and Ingleton [2].

Let M be a matroid and $\mathrm{BG}(M)$ be its basis graph. Let B_1 and B_2 be adjacent vertices (bases) in $\mathrm{BG}(M)$. By (*BEA*), there exist elements e and g of M, with $e \in B_1 \setminus B_2$ and $g \in B_2 \setminus B_1$, such that $B_2 = B_1 - e + g$. We define an (X, Y)-bipartition (determined by e) of the bases of M, with $X = \{B \in \mathcal{B}(M) \colon e \in B\}$ and $Y = \{B \in \mathcal{B}(M) \colon e \notin B\}$. The bases in X (Y, respectively) correspond exactly to the bases of the matroid $M' = M/e$ obtained by *contracting* e ($M'' = M \setminus e$, obtained by *deleting* e, respectively). Moreover, $\mathrm{BG}(M')$ is $\mathrm{BG}(M)[X]$ ($\mathrm{BG}(M'')$ is $\mathrm{BG}(M)[Y]$, respectively), which is the subgraph of $\mathrm{BG}(M)$ induced by X (Y, respectively). We do not distinguish between $\mathrm{BG}(M')$ and $\mathrm{BG}(M)[X]$ ($\mathrm{BG}(M'')$ and $\mathrm{BG}(M)[Y]$, respectively).

A basis sequence $B_1B_2B_3B_4$ is a *good cycle* for B_1B_2 if it is a cycle in $\mathrm{BG}(M)$, each of B_1 and B_4 contains e, and none of B_2 and B_3 contains e (Figure 1).

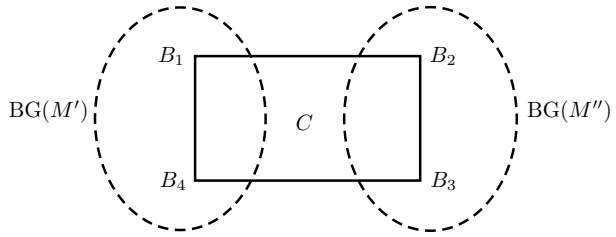


Figure 1: A good cycle $C = B_1B_2B_3B_4$ for B_1B_2.

The symmetric difference between two cycles C_1 and C_2 is the graph induced by the edges in the symmetric difference of $E(C_1)$ and $E(C_2)$.

If $C = B_1B_2B_3B_4$ is good, then the symmetric difference of a Hamiltonian cycle of $\mathrm{BG}(M')$ passing through the edge B_1B_4, the good cycle C, and a Hamiltonian cycle of $\mathrm{BG}(M'')$ passing through the edge B_2B_3 is a Hamiltonian cycle of $\mathrm{BG}(M)$.

So, if $\mathcal{C}(B_1, B_2)$ is the set of good cycles for B_1B_2, then

$$\mathrm{HC}_{B_1B_2}(M) \geq \mathrm{HC}^*(M') \cdot |\mathcal{C}(B_1, B_2)| \cdot \mathrm{HC}^*(M'').$$

This inequality suggests an inductive way to achieve a lower bound on $\mathrm{HC}^*(M)$. A key part in this approach involves proving a lower bound on the number of good cycles for any edge of $\mathrm{BG}(M)$.

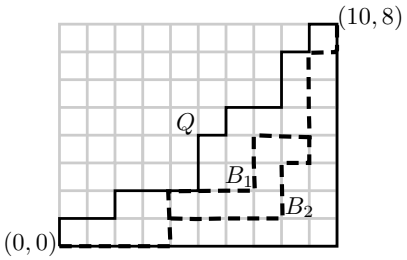
3. Generalized Catalan matroids

In this section we address a special class of transversal matroids introduced by Bonin, de Mier, and Noy [5]. We specialize the description of Bonin and de Mier [4] and Stanley [18].

A *lattice path* L in $\mathbb{Z}^2$ is a sequence $v_0, \ldots, v_k \in \mathbb{Z}^2$ such that each consecutive difference $s_j = v_j - v_{j-1}$ lies in $\{(1,0),(0,1)\}$. We say that L goes from v_0 to v_k and call s_j the jth step of L. All lattice paths we consider go from $(0,0)$ to a certain (m,r). If we write *East* (E) for $(1,0)$ and *North* (N) for $(0,1)$, then L can be represented by a word of length $m+r$ on the alphabet $\{E,N\}$ or by the subset $\{j\colon\ j\text{th step of } L \text{ is } N\}$ of $\{1,\ldots,m+r\} = [m{+}r]$.

Let Q be a lattice path from $(0,0)$ to (m,r) and $\mathcal{P}$ be the set of all lattice paths from $(0,0)$ to (m,r) that do not go above Q. For each i with $1 \leq i \leq r$, let $A_i = \{j\colon\ j\text{th step is the } i\text{th North for some path in } \mathcal{P}\}$. Each A_i is the interval $[a_i, m+i]$ in $[m{+}r]$ where a_i is the position of the ith North step of Q (Figure 2).

Let M_Q be the transversal matroid on the set $[m{+}r]$ with standard presentation $(A_1,\ldots,A_r)$. Note that M_Q has rank r and corank (or nullity) m. The bases of M_Q are the subsets of $[m{+}r]$



$A_8 = \{17, 18\}$ $\quad Q = \{1, 4, 8, 9, 11, 14, 15, 17\}$
$A_7 = \{15, 16, 17\}$ $\quad Q = NEENEEENNENEENNENE$
$A_6 = \{14, 15, 16\}$
$A_5 = \{11, 12, 13, 14, 15\}$ $\quad B_1 = \{5, 6, 10, 11, 14, 15, 16, 18\}$
$A_4 = \{9, 10, 11, 12, 13, 14\}$ $\quad B_1 = EEEENNEEENNEENNNEN$
$A_3 = \{8, 9, 10, 11, 12, 13\}$
$A_2 = \{4, 5, 6, 7, 8, 9, 10, 11, 12\}$ $\quad B_2 = \{5, 10, 11, 13, 14, 15, 16, 18\}$
$A_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ $\quad B_2 = EEEENEEEENNENNNNEN$

Figure 2: Lattice path Q from $(0, 0)$ to $(10, 8)$ and the corresponding sets $A_1, \ldots, A_8$. Representation of Q as a subset of $[10 + 8]$ and as a word of length $10 + 8$ in the alphabet $\{E, N\}$. Lattice paths B_1 and B_2 that do not go above Q and their representations as a subset and as a word.

that represent lattice paths in $\mathcal{P}$. A *generalized Catalan matroid* is a matroid M_Q for some lattice path Q. The class of generalized Catalan matroids is minor-closed [4, Theorem 4.2]. The *k-Catalan matroid* is the generalized Catalan matroid M_Q with $Q = (NE)^k$.

Lemma 1. *Let M_Q be a generalized Catalan matroid of rank r and corank m with neither a loop nor an isthmus, for $m \geq r \geq 2$. Then every edge of $\mathrm{BG}(M_Q)$ is in $r - 1$ good cycles.*

Proof. As M_Q has neither a loop nor an isthmus, the first step of Q is North and the last one is East. Let B_1B_2 be an edge of $\mathrm{BG}(M_Q)$. Say $B_1 = x_1 \cdots x_{m+r}$ and $B_2 = y_1 \cdots y_{m+r}$, with x_i, $y_i \in \{N, E\}$ for $i \in [m+r]$. There exist indices e and g such that $x_e = y_g = N$, $x_g = y_e = E$, and $x_\ell = y_\ell$ for $\ell \neq e, g$. So $B_2 = B_1 - e + g$. We may assume that $e < g$.

If there exists an index $f < e$ such that $x_f = y_f = N$, let f be as small as possible. For every index w such that $x_w = y_w = E$, basis B_4 rises by switching x_w for N and x_f for E in B_1 and basis B_3 rises by switching y_w for N and y_f for E in B_2; that is, $B_4 = B_1 - f + w$ and $B_3 = B_2 - f + w$ (Figure 3). Since the first step of Q is North and the last one is East, the paths corresponding to the words B_3 and B_4 are in $M[Q]$. Thus, for every common E step of B_1 and B_2, we obtain a good cycle. Therefore, there are $m - 1$ good cycles passing through the edge B_1B_2. Similarly, if there exists an index $f > g$ such that $x_f = y_f = E$, by replacing North by East and vice-versa in the previsous argument, one can conclude that there are $r - 1$ good cycles passing through the edge B_1B_2.

Otherwise x_e is the first N in B_1 and x_g is the last E. Let x_h be the penultimate E in B_1. Such x_h exists because $m \geq r \geq 2$. As $y_g = N$, y_h is the last E in B_2. We will obtain a good cycle for each N in B_1, except for x_e, proving that there are $r - 1$ good cycles passing through B_1B_2.

We partition $B_1 - e$ in maximal *blocks* $x_i \cdots x_{w-1}$ such that $x_i = \cdots = x_{w-1} = N$ with $i > e$ and consider three cases.

Case $w < g$: For every $f \in \{i, \ldots, w - 1\}$, basis B_4 rises by switching x_f for E and x_w for N

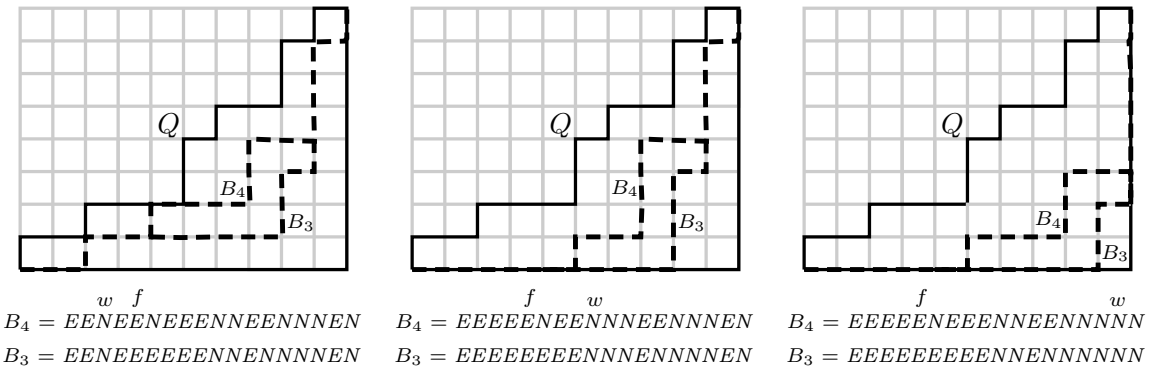


Figure 3: B_3 and B_4 for $w = 3, 9, 17$ and B_1 and B_2 as in Figure 2, with $e = 6$, $g = 13$, and $f = 5$.

in B_1 and basis B_3 rises by switching y_f for E and y_w for N in B_2; that is, $B_4 = B_1 - f + w$ and $B_3 = B_2 - f + w$.

Case $w = g$: Let x_h be the penultimate E in B_1. As $y_w = y_g = N$, y_h is the last E in B_2. For every $f \in \{i, \ldots, g-1\}$, basis B_4 rises by switching x_f for E and x_g for N in B_1, and basis B_3 rises by switching y_f for E and y_h for N in B_2; that is, $B_4 = B_1 - f + g$ and $B_3 = B_2 - f + h$.

Case $i = g + 1$: For every element $f \in \{g+1, \ldots, m+r\}$, basis B_4 rises by switching x_f for E and x_g for N in B_1, and basis B_3 rises by switching y_f for E and y_h for N in B_2; that is, $B_4 = B_1 - f + g$ and $B_3 = B_2 - f + h$. □

Bonin and de Mier [4] observed that the class of all generalized Catalan matroids is closed under duals. In particular, a basis B^* of the dual of M_Q corresponds to the E steps of the basis B in M_Q. Therefore, the following is a consequence of this fact and Lemma 1.

Corollary 2. *For r, $m \geq 2$, let M_Q be a generalized Catalan matroid of rank r and corank m, with neither a loop nor an isthmus. Then every edge of* $\mathrm{BG}(M_Q)$ *is in* $\min\{r-1, m-1\}$ *good cycles.*

Let M_Q be a generalized Catalan matroid and e be an element of M_Q. From an observation of Bonin and de Mier [4], if e is neither a loop nor an isthmus, then $M_Q \setminus e$ is the matroid $M_{Q'}$ where Q' is formed by deleting from Q the first E step that is at or after step e, and M_Q/e is the matroid $M_{Q''}$ where Q'' is formed by deleting from Q the last N step that is at or before step e.

Observation 3. *If the k-Catalan matroid is a minor of the generalized Catalan matroid M_Q, then the $(k-1)$-Catalan matroid is a minor of both $M_Q \backslash e$ and M_Q/e for every element e of M_Q.*

For the class of generalized Catalan matroids, we define the function

$$\mathrm{hc}(k) = \min\{\mathrm{HC}^*(M_Q)\colon\ M_Q \text{ has a } k\text{-Catalan matroid as a minor}\}.$$

Proposition 4. *For $k \geq 2$, $\mathrm{hc}(k) \geq (k-1)(\mathrm{hc}(k-1))^2$.*

Proof. Let M_Q be a generalized Catalan matroid such that $\mathrm{HC}^*(M_Q) = \mathrm{hc}(k)$. We may assume that M_Q has neither a loop nor an isthmus. Thus, both the rank and corank of M are at least k. By Corollary 2, there are $\min\{r-1, m-1\} \geq k-1$ good cycles for every edge of $\mathrm{BG}(M_Q)$. Let B_1B_2 be an edge of $\mathrm{BG}(M_Q)$, say $B_1 = B_2 - e + g$, and let $M' = M_Q\backslash e$ and $M'' = M_Q/e$. It follows from Observation 3 that both M' and M'' contain a $(k-1)$-Catalan matroid as a minor. Thus $\mathrm{HC}^*(M') \geq \mathrm{hc}(k-1)$ and $\mathrm{HC}^*(M'') \geq \mathrm{hc}(k-1)$. Therefore we conclude that $\mathrm{hc}(k) \geq (k-1)(\mathrm{hc}(k-1))^2$. □

The *superfactorial* $\mathrm{sf}(x)$ of a nonnegative integer x is the number $x!(x-1)!\cdots 0!$

Theorem 5. *For $k \geq 2$, $\mathrm{hc}(k) \geq \mathrm{sf}(k-1)\,\mathrm{sf}(k-2)$.*

Proof. The proof is by induction on k. Let M_Q be a generalized Catalan matroid such that $\mathrm{HC}^*(M_Q) = \mathrm{hc}(k)$. We may assume that M_Q has neither a loop nor a isthmus. In particular, M_Q has both rank and corank at least k. Let $k = 2$. So $\mathrm{BG}(M_Q)$ has at least three vertices and is edge-Hamiltonian. Thus $\mathrm{hc}(2) \geq 1 = \mathrm{sf}(1)\,\mathrm{sf}(0)$.

Now let $k \geq 3$. Let B_1B_2 be an edge of $\mathrm{BG}(M_Q)$, say $B_2 = B_1 - e + g$. By Corollary 2, the edge B_1B_2 is in $\min\{r-1, m-1\} \geq k-1$ good cycles. Consider $M' = M_Q\backslash e$ and $M'' = M_Q/e$. By Observation 3, the $(k-1)$-Catalan matroid is a minor of both M' and M''. Thus, by the induction hypothesis, $\mathrm{HC}^*(M'), \mathrm{HC}^*(M'') \geq \mathrm{hc}(k-1) \geq \mathrm{sf}(k-2)\,\mathrm{sf}(k-3)$. Hence, every edge of $\mathrm{BG}(M_Q)$ is in $(k-1)\big(\mathrm{sf}(k-2)\,\mathrm{sf}(k-3)\big)^2 \geq \mathrm{sf}(k-1)\,\mathrm{sf}(k-2)$ Hamiltonian cycles. □

The uniform matroid $U_{r,n}$ is a lattice path matroid M_Q, where $Q = E^rN^{n-r}$. By applying the same techniques, we can achieve better lower bounds for them [10].

Theorem 6. *For $n > r \geq 1$, every edge of $\mathrm{BG}(U_{r,n})$ is in $\big((n-r-1)!(r-1)!\big)^{\min\{n-r-1,r-1\}}$ Hamiltonian cycles.* □

4. Graphic matroids

In this section, we consider a graphic matroid M_G where G is a loopless k-edge-connected multigraph of order n; that is, the elements of M_G are the edges of G and a basis of M_G corresponds to a spanning tree of G. We refer to G as a *graph* instead of a *loopless multigraph*.

For readability, we do not distinguish between a basis of M_G and a spanning tree of G. If B is a basis of M_G and g is an edge of G not in B, then $B+g$ induces a unique cycle (circuit) $C(g,B)$ in G (in M_G, respectively) called the *fundamental cycle* (*circuit*, respectively) with respect to g and B [16].

Let X and Y be disjoint subsets of the vertex set $V(G)$. We denote by $E[X,Y]$ $(=E[Y,X])$ the set of edges of G with one end in X and the other end in Y, and by $e(X,Y)$ the number of edges in $E[X,Y]$.

4.1. General structure of good cycles

Here we fix the structure that we will use in the rest of Section 4.

Let G be a connected graph and B_1 and B_2 be bases of M_G such that $B_2 = B_1 - e + g$. Let f be an edge of $B_1 - e$. Let X be the vertex set of the component of $B_1 - e$ that contains no end of f. Let Z be the vertex set of the component of $B_1 - f$ that contains no end of e. Let $Y = V(G) \setminus (X \cup Z)$.

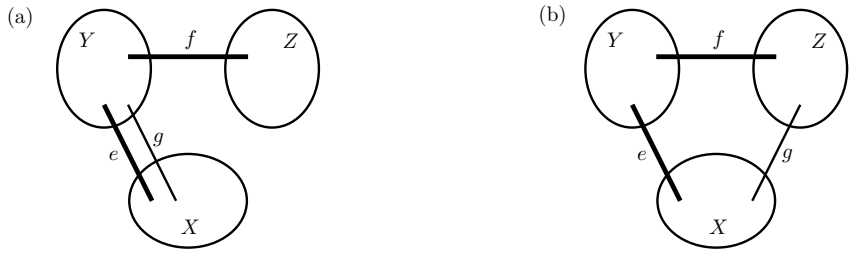


Figure 4: Components X, Y, Z provided by the edges e and f in B_1, and possible positions for the edge g.

Let $\mathcal{C} = \mathcal{C}(B_1B_2)$ be the set of good cycles for B_1B_2. An arbitrary element of $\mathcal{C}$ is represented as $B_1B_2B_3B_4$. For $f \in B_1 - e$, let $\mathcal{C}(f)$ be the set of good cycles $B_1B_2B_3B_4$ in $\mathcal{C}$ for which $f \notin B_4$. For every $f' \in B_1 - e$ with $f' \neq f$, if $B_1B_2B_3B_4 \in \mathcal{C}(f)$, then f' belongs to B_4, and so $B_1B_2B_3B_4 \notin \mathcal{C}(f')$. Thus $\mathcal{C}(f) \cap \mathcal{C}(f') = \emptyset$. For every $w \notin B_1 + g = B_2 + e$, we denote by $\mathcal{C}(f,w)$ the set of cycles in $\mathcal{C}(f)$ such that $w \in B_3$. Similarly, $\mathcal{C}(f,w) \cap \mathcal{C}(f,w') = \emptyset$ for every $w' \notin B_1 + g$ with $w' \neq w$. Therefore

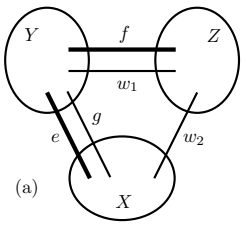
$$\mathcal{C} \;=\; \mathcal{C}(B_1B_2) \;=\; \dot{\bigcup}\{\mathcal{C}(f) : f \in B_1 - e\} \;=\; \dot{\bigcup}\{\mathcal{C}(f,w) : f \in B_1 - e,\ w \notin B_1 + g\}. \qquad (1)$$

4.2. 2-edge-connected graphs

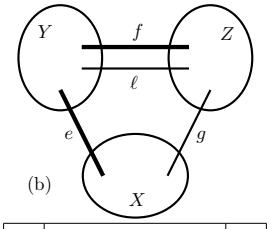
We give a lower bound on $\mathrm{HC}^*(M_G)$ where M_G is the cycle matroid obtained from a 2-edge-connected graph G. This bound will be used in the next subsection to achieve a lower bound for k-edge-connected graphs with $k \geq 3$.

Lemma 7. *If G is a 2-edge-connected graph of order $n \geq 4$ and size at least $n+2$, then every edge of $\mathrm{BG}(M_G)$ is in two good cycles.*

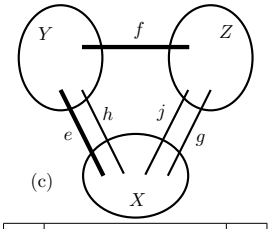
Proof. Let B_1 and $B_2 = B_1 - e + g$ be adjacent bases of $\mathrm{BG}(M_G)$. Suppose e and g are parallel edges. As $n \geq 4$, there are edges f and f' in $B_1 - e$ not in $C(g, B_1)$ with $f \neq f'$. Consider the edge f and let X, Y, and Z be as in Figure 4(a). Since G is 2-edge-connected, there is at least one edge $w \in E[X \cup Y, Z] \setminus \{f\}$, and a good cycle in $\mathcal{C}(f, w)$ (Figure 5(a)). Similarly, there is a good cycle in $\mathcal{C}(f', w')$ for some w'. By (1), these two good cycles are distinct and we are done. So we may assume that e and g are not parallel edges and $C(g, B_1)$ has at least three edges. For each edge w not in $B_1 + g$, if w has its two ends in $C(g, B_1)$, then there is an $f \in C(g, B_1)$ with $f \notin \{e, g\}$. Let X, Y, and Z be as in Figure 4(b). Edge w is as edge ℓ in Figure 5(b) or as edges h or j in Figure 5(c). In each of these cases, there is a good cycle in $\mathcal{C}(f, w)$. On the other hand, if w has at most one end in $C(g, B_1)$, there is an edge $f \in B_1$ not in $C(g, B_1)$ such that $f \in C(w, B_1)$, and there is a good cycle in $\mathcal{C}(f, w)$ as in Figure 5(a). Since G has size at least $n+2$, there are at least two edges not in $B_1 + g$ and, by (1), the corresponding good cycles are distinct and we are done. □



B_1	$\{e, f, \ldots\}$	$\{g, f, \ldots\}$	B_2
B_4	$\{e, w, \ldots\}$	$\{g, w, \ldots\}$	B_3



B_1	$\{e, f, \ldots\}$	$\{g, f, \ldots\}$	B_2
B_4	$\{e, \ell, \ldots\}$	$\{g, \ell, \ldots\}$	B_3
B_1	$\{e, f, \ldots\}$	$\{g, f, \ldots\}$	B_2
B_4	$\{e, g, \ldots\}$	$\{g, \ell, \ldots\}$	B_3



B_1	$\{e, f, \ldots\}$	$\{g, f, \ldots\}$	B_2
B_4	$\{e, g, \ldots\}$	$\{g, h, \ldots\}$	B_3
B_1	$\{e, f, \ldots\}$	$\{g, f, \ldots\}$	B_2
B_4	$\{e, j, \ldots\}$	$\{j, f, \ldots\}$	B_3

Figure 5: The bold edges are in B_1. (a) For $f \notin C(g, B_1)$ and $w = w_1$ or $w = w_2$ in $E[X \cup Y, Z]$, the table shows a good cycle in $\mathcal{C}(f, w)$. (b) For $f \in C(g, B_1)$ and ℓ in $E[Y, Z]$, the table shows two good cycles in $\mathcal{C}(f, \ell)$. (c) For $f \in C(g, B_1)$ and h in $E[X, Y]$ or j in $E[X, Z]$, the table shows a good cycle in $\mathcal{C}(f, h)$ and one in $\mathcal{C}(f, j)$.

The 1-*sum* $C \oplus_1 C'$ of two cycles C and C' is the graph obtained from identifying a vertex of C with a vertex of C'.

Lemma 8. *Let G be a 2-edge-connected graph of order $n \geq 4$. There exists an edge in $\mathrm{BG}(M_G)$ not in two good cycles if and only if G is either C_n or $C_2 \oplus_1 C_{n-1}$.*

Proof. Let m denote the number of edges of G. Since G is 2-edge-connected, every edge is in a cycle, so $m \geq n$. If $m = n$, then G is the n-cycle C_n and no edge of $\mathrm{BG}(M_G)$ is in a good cycle. For $m \geq n + 2$, Lemma 7 implies that every edge of $\mathrm{BG}(M_G)$ is in two good cycles. So, we may assume that $m = n + 1$. Because every 2-edge-connected graph has a closed ear-decomposition [3] and G has exactly $n + 1$ edges, the closed ear-decomposition of G consists of exactly two ears. Thus, G is either the 1-sum of two cycles, or the union of three internally

disjoint paths that have the same two end vertices.

First, suppose that G is the 1-sum of two cycles. Since we only consider graphs with no loops, the length of both of these cycles is at least two. If G is $C_2 \oplus_1 C_{n-1}$, then it can be verified that there are adjacent bases in $\mathrm{BG}(M_G)$ for which there is only one good cycle (Figure 6). So assume the length of both cycles is at least three, and let B_1 and $B_2 = B_1 - e + g$ be adjacent bases of $\mathrm{BG}(M_G)$. For each edge $f \in B_1$ not in $C(g, B_1)$, let X, Y, and Z be as in Figure 4(a). Since G is 2-edge-connected, there is at least a $w \in E[X \cup Y, Z] \setminus \{f\}$ and such w corresponds to a good cycle in $\mathcal{C}(f, w)$ as shown in Figure 5(a). As both cycles in G have length at least three, there are at least two such edges f and, by (1), the corresponding good cycles are distinct.

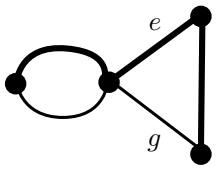


Figure 6: A 2-edge-connected graph G whose basis graph $\mathrm{BG}(M_G)$ has an edge B_1B_2 with no two good cycles. The basis B_1 is the spanning tree in thick edges and $B_2 = B_1 - e + g$.

Now, suppose that G is the union of three internally disjoint paths P_1, P_2, P_3 with the same two end vertices. In this case, every edge of $\mathrm{BG}(M_G)$ is in two good cycles. Indeed, let B_1 and $B_2 = B_1 - e + g$ be adjacent bases of $\mathrm{BG}(M_G)$. First, suppose that e and g are in the same path, say P_1. We may assume that all edges of P_2 are in B_1 and that there exists an edge w in P_3 not in B_1. Let f be an edge of P_2 (and thus of B_1). For X, Y, and Z as in Figure 4(b), edge w is in $E[Y, Z]$ as ℓ in Figure 5(b), so there are two good cycles in $\mathcal{C}(f, w)$. Finally, suppose that e and g are in different paths; say e belongs to P_1 and g belongs to P_2. Hence all edges of P_1 are in B_1, and there exists an edge w in P_3 not in B_1. If there is an $f \in B_1 - e$ in P_1, then w is in $E[X, Z]$ as j in Figure 5(c). If there is an $f \in B_1$ in P_2, then w is in $E[X, Y]$ as h in Figure 5(c). If there is an $f \in B_1$ in P_3, then w is in $E[X \cup Y, Z]$ as w_1 or w_2 in Figure 5(a). In any case we get a good cycle in $\mathcal{C}(f, w)$. Since $n \geq 4$, there are two edges $f, f' \in B_1$ other than e. Therefore, by (1), there are two good cycles, one in $\mathcal{C}(f, w)$ and the other in $\mathcal{C}(f', w)$. □

Theorem 9. *If G is a 2-edge-connected graph of order $n \geq 3$, then every edge of $\mathrm{BG}(M_G)$ is in 2^{n-3} Hamiltonian cycles.*

Proof. The proof is by induction on n. If $n = 3$, then $2^{n-3} = 1$ and the theorem follows from the edge-Hamiltonicity of $\mathrm{BG}(M_G)$. So we may assume that $n \geq 4$.

By Lemma 8, if there exists an edge in $\mathrm{BG}(M_G)$ not in two good cycles, then G is either C_n or $C_2 \oplus_1 C_{n-1}$. If $G = C_n$, then $\mathrm{BG}(M_G) = K_n$ and every edge of $\mathrm{BG}(M_G)$ is in $(n-2)! \geq 2^{n-3}$ Hamiltonian cycles. If $G = C_2 \oplus_1 C_{n-1}$, then $\mathrm{BG}(M_G)$ is the cartesian product of K_2 and K_{n-1},

that is, the graph whose vertex set is $V(K_2) \times V(K_{n-1})$ and whose edge set consists of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(K_2)$ and $v_1 = v_2$, or $v_1v_2 \in E(K_{n-1})$ and $u_1 = u_2$. In this case, every edge of $\mathrm{BG}(M_G)$ is in $(n-2)!(n-3)! \geq 2^{n-3}$ Hamiltonian cycles. Therefore, we may assume that G has at least $n+1$ edges and every edge of $\mathrm{BG}(M_G)$ is in two good cycles.

Let B_1 and $B_2 = B_1 - e + g$ be adjacent bases of $\mathrm{BG}(M_G)$. Let $G' = G/e$ and $G'' = G \setminus e$. As G' is 2-edge-connected of order $n-1 \geq 3$, by the induction hypothesis, every edge of $\mathrm{BG}(M_{G'})$ is in 2^{n-4} Hamiltonian cycles in $\mathrm{BG}(M_{G'})$. As G'' has $n \geq 4$ vertices and at least n edges, G'' has at least two spanning trees, and therefore $\mathrm{BG}(M_{G''})$ is either K_2 or edge-Hamiltonian. Let $C = B_1B_2B_3B_4$ be a good cycle. If $\mathrm{BG}(M_{G''})$ is K_2, then the symmetric difference of C and a Hamiltonian cycle of $\mathrm{BG}(M_{G'})$ containing B_1B_4 is a Hamiltonian cycle of $\mathrm{BG}(M_G)$. On the other hand, if $\mathrm{BG}(M_{G''})$ is edge-Hamiltonian, then the symmetric difference of C, a Hamiltonian cycle of $\mathrm{BG}(M_{G'})$ containing B_1B_4, and a Hamiltonian cycle of $\mathrm{BG}(M_{G''})$ containing B_2B_3 is a Hamiltonian cycle of $\mathrm{BG}(M_G)$. As every edge of $\mathrm{BG}(M_G)$ is in two good cycles, in either case we conclude that every edge of $\mathrm{BG}(M_G)$ is in $2^{n-4} \cdot 2 \cdot 1 = 2^{n-3}$ Hamiltonian cycles. □

4.3. k-edge-connected graphs

Now, we turn our attention to k-edge-connected graphs for $k \geq 3$.

Lemma 10. *If G is a k-edge-connected graph of order $n \geq 3$ for $k \geq 3$, then every edge of $\mathrm{BG}(M_G)$ is in $(n-2)(k-1)$ good cycles.*

Proof. Let B_1 and $B_2 = B_1 - e + g$ be adjacent bases of $\mathrm{BG}(M_G)$ and $f \in B_1 - e$. We shall prove that there are $k-1$ good cycles in $\mathcal{C}(f)$. By (1) and as there are $n-2$ choices for f, there are $(n-2)(k-1)$ good cycles for every edge of $\mathrm{BG}(M_G)$ and we are done.

Let X, Y, and Z be as in Figure 4. If $f \notin C(g, B_1)$, as G is k-edge-connected, there are distinct edges $w_1, \ldots, w_{k-1}$ in $E[X \cup Y, Z]$ and a distinct good cycle in each $\mathcal{C}(f, w_i)$ for $i = 1, \ldots, k-1$ (Figure 5(a)). If $f \in C(g, B_1)$, there is a good cycle in $\mathcal{C}(f, w)$ for each $w \in (E[X, Y] \cup E[X, Z] \cup E[Y, Z]) \setminus \{e, f, g\}$ (Figures 5(b) and 5(c)). As G is k-edge-connected and $k \geq 3$, there are at least

$$\left\lceil \frac{e(X,Y) + e(X,Z) + e(Y,Z)}{2} \right\rceil - 3 \geq \left\lceil \frac{3k}{2} \right\rceil - 3 \geq k - 1$$

choices for w. □

In order to give a bound on $\mathrm{HC}^*(M_G)$, we define the function

$$\mathrm{hc}(n, k) = \min\{\mathrm{HC}^*(M_G) \colon G \text{ is a } k\text{-edge-connected graph of order } n\}.$$

Proposition 11. *For k, $n \geq 3$, $\mathrm{hc}(n, k) \geq (n-2)(k-1)\,\mathrm{hc}(n-1, k)\,\mathrm{hc}(n, k-1)$.*

Proof. Let G be a k-edge-connected graph of order n such that $\mathrm{HC}^*(M_G) = \mathrm{hc}(n,k)$. By Lemma 10, there are $(n-2)(k-1)$ good cycles for every edge of $\mathrm{BG}(M_G)$. Let B_1 and $B_2 = B_1 - e + g$ be adjacent bases of $\mathrm{BG}(M_G)$ and let $G' = G/e$ and $G'' = G \setminus e$. The symmetric difference of a good cycle $C = B_1B_2B_3B_4$, a Hamiltonian cycle of $\mathrm{BG}(M_{G'})$ containing B_1B_4, and a Hamiltonian cycle of $\mathrm{BG}(M_{G''})$ containing B_2B_3 is a Hamiltonian cycle of $\mathrm{BG}(M_G)$ containing B_1B_2. Hence, B_1B_2 is in $(n-2)(k-1)\,\mathrm{HC}^*(M_{G'})\,\mathrm{HC}^*(M_{G''})$ Hamiltonian cycles of $\mathrm{BG}(M_G)$. Now, as G' is k-edge-connected of order $n-1$, we have that $\mathrm{HC}^*(M_{G'}) \geq \mathrm{hc}(n-1,k)$ and, as G'' is $(k-1)$-edge-connected of order n, we have that $\mathrm{HC}^*(M_{G''}) \geq \mathrm{hc}(n,k-1)$. Therefore we conclude that $\mathrm{hc}(n,k) = \mathrm{HC}^*(M_G) \geq (n-2)(k-1)\,\mathrm{hc}(n-1,k)\,\mathrm{hc}(n,k-1)$. □

Theorem 12. *For n, $k \geq 2$ with $n+k \geq 5$, $\mathrm{hc}(n,k) \geq (2^{n-2}(n-2)!)^{k-2}$.*

Proof. The proof is by induction on $n+k$. If $k=2$ or $n=2$, then the theorem follows from the edge-Hamiltonicity of $\mathrm{BG}(M_G)$. Now assume that $n \geq 3$ and $k \geq 3$. By applying Proposition 11 and the induction hypothesis on both $\mathrm{hc}(n-1,k)$ and $\mathrm{hc}(n,k-1)$, we have that

$$\begin{aligned}
\mathrm{hc}(n,k) &\geq (n-2)(k-1)\,\mathrm{hc}(n-1,k)\,\mathrm{hc}(n,k-1)\\
&\geq (n-2)(k-1)(2^{n-3}(n-3)!)^{k-2}(2^{n-2}(n-2)!)^{k-3}\\
&= (n-2)(k-1)2^{(n-3)(k-2)+(n-2)(k-3)}((n-3)!)^{k-2}((n-2)!)^{k-3}\\
&\geq 2^{(n-3)(k-2)+(n-2)(k-3)+1}(n-2)(n-3)!((n-2)!)^{k-3}\\
&\geq (2^{n-2}(n-2)!)^{k-2},
\end{aligned}$$

where the last inequality holds because $(n-3)(k-2)+(n-2)(k-3)+1 \geq (n-2)(k-2)$ for $k \geq 3$ and $n \geq 3$. This completes the proof of the theorem. □

5. Concluding remarks

The general strategy (Section 2) used in order to give a lower bound on $\mathrm{HC}^*(M_G)$ takes into account only Hamiltonian cycles that cross some particular cuts exactly twice. Using the same strategy and doing some tedious computation, the lower bound presented for 2-edge-connected graphs can be improved to a superfactorial one for k-edge-connected graphs, by proving specific lower bounds on both $\mathrm{hc}(3,k)$ and $\mathrm{hc}(n,3)$ [10]. Namely, one can prove the following theorems.

Theorem 13 (Theorem 13 [10])**.** *For $k \geq 3$, $\mathrm{hc}(3,k) \geq \mathrm{sf}(k-1)$.*

Theorem 14 (Theorem 14 [10])**.** *For $n \geq 3$, $\mathrm{hc}(n,3) \geq (n-2)!\,2^{\binom{n-1}{2}}$.*

Theorem 15 (Theorem 15 [10])**.** *For n, $k \geq 4$,*

$$\mathrm{hc}(n,k) \geq \frac{2^{\binom{n+k-4}{n-3}} \cdot 3^{\binom{n+k-7}{k-3}}}{(n-1)k} \prod_{r=4}^{k} (r\,\mathrm{sf}(r-1))^{\binom{n+k-4-r}{n-4}} \cdot \prod_{s=4}^{n} (s-1)!^{\binom{n+k-4-s}{k-4}}.$$

The bound given by Theorem 15 is best possible using the general strategy. For the sake of comparison, the bound on $\mathrm{hc}(10,4)$ provided by Theorem 12 is a number with 15 digits, while the bound provided by Theorem 15 is a number with 61 digits.

The following corollary follows from mathematical manipulations on the right side of the inequality given by Theorem 15 and it gives a more explicit and concise expression.

Corollary 16 (Corollary 16 [10])**.** *For $n > k \geq 5$,*

$$\mathrm{hc}(n,k) > \prod_{r=3}^{n} (\mathrm{sf}(r-1))^{\binom{n+k-5-r}{n-6}+\binom{n+k-4-r}{n-4}+\binom{n+k-5-r}{k-5}}.$$

As our last remark, we point out that the function $\mathrm{hc}(n,k)$ is monotonically increasing.

Lemma 17 (Lemma 11 [10])**.** *Let G be a 3-edge-connected graph of order $n \geq 3$ and let e be an edge of G. Then $\mathrm{HC}^*(M_G) \geq \mathrm{HC}^*(M_{G/e})$ and $\mathrm{HC}^*(M_G) \geq \mathrm{HC}^*(M_{G\setminus e})$.*

References

[1] B. Alspach, G. Liu, Paths and cycles in matroid base graphs, Graphs and Combinatorics 5 (1989) 207–211.

[2] J. A. Bondy, A. W. Ingleton, Pancyclic graphs. II, J. Combinatorial Theory Ser. B 20 (1) (1976) 41–46.

[3] J. A. Bondy, U. S. R. Murty, Graph Theory, vol. 244 of *Graduate Texts in Mathematics*, Springer, New York, 2008.

[4] J. E. Bonin, A. de Mier, Lattice path matroids: structural properties, European J. Combin. 27 (5) (2006) 701–738.

[5] J. E. Bonin, A. de Mier, M. Noy, Lattice path matroids: enumerative aspects and Tutte polynomials, J. Combin. Theory Ser. A 104 (1) (2003) 63–94.

[6] V. Chatelain, J. L. Ramírez Alfonsín, Matroid base polytope decomposition, Adv. in Appl. Math. 47 (1) (2011) 158–172.

[7] V. Chatelain, J. L. Ramírez Alfonsín, Matroid base polytope decomposition II: Sequences of hyperplane splits, Adv. in Appl. Math. 54 (2014) 121–136.

[8] R. L. Cummins, Hamilton circuits in tree graphs, IEEE Trans. Circuit Theory CT-13 (1966) 82–90.

[9] J. D. Donald, C. A. Holzmann, M. D. Tobey, A characterization of complete matroid base graphs, J. Combinatorial Theory Ser. B 22 (2) (1977) 139–158.

[10] C. G. Fernandes, C. Hernández-Vélez, J. C. de Pina, J. L. Ramírez Alfonsín, Matroid basis graph: Counting Hamiltonian cycles, ArXiv, https://arxiv.org/abs/1608.02635, 2016.

[11] I. M. Gel′fand, V. V. Serganova, Combinatorial geometries and the strata of a torus on homogeneous compact manifolds, Uspekhi Mat. Nauk 42 (2(254)) (1987) 107–134, 287.

[12] G. Z. Liu, The connectivity of matroid base graphs, Chinese J. Oper. Res. 3 (1) (1984) 59–60.

[13] G. Z. Liu, Erratum: "A lower bound on connectivities of matroid base graphs", Discrete Math. 71 (2) (1988) 187.

[14] G. Z. Liu, A lower bound on connectivities of matroid base graphs, Discrete Math. 69 (1) (1988) 55–60.

[15] S. B. Maurer, Matroid basis graphs. I, J. Combinatorial Theory Ser. B 14 (1973) 216–240.

[16] J. Oxley, Matroid theory, vol. 21 of *Oxford Graduate Texts in Mathematics*, Oxford University Press, Oxford, second edn., 2011.

[17] H. S. Shank, A note on Hamilton circuits in tree graphs, IEEE Trans. Circuit Theory CT-15 (1968) 86.

[18] R. P. Stanley, Enumerative combinatorics. Volume 1, Cambridge Studies in Advanced Mathematics, Cambridge University Press, second edn., 2012.

[19] D. J. A. Welsh, Matroid theory, Dover Publications Inc., 2009.