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► **To cite this version:**

Martin Genet. A relaxed growth modeling framework for controlling growth-induced residual stresses. Clinical Biomechanics, 2019, 10.1016/j.clinbiomech.2019.08.015 . hal-02069113

**HAL Id: hal-02069113**

**<https://hal.science/hal-02069113v1>**

Submitted on 15 Mar 2019

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# A relaxed growth modeling framework for controlling growth-induced residual stresses

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## Abstract

(Abstract word count: 245.)

*Background.* Constitutive models of the mechanical response of soft tissues have been established and are widely accepted, but models of soft tissues remodeling are more controversial. Specifically for growth, one important question arises pertaining to residual stresses: existing growth models inevitably introduce residual stresses, but it is not entirely clear if this is physiological or merely an artifact of the modeling framework. As a consequence, in simulating growth, some authors have chosen to keep growth-induced residual stresses, and others have chosen to remove them.

*Methods.* In this paper, we introduce a novel “relaxed growth” framework allowing for a fine control of the amount of residual stresses generated during tissue growth. It is a direct extension of the classical framework of the multiplicative decomposition of the transformation gradient, to which an additional sub-transformation is introduced in order to let the original unloaded configuration evolve, hence relieving some residual stresses. We provide multiple illustrations of the framework mechanical response, on time-driven constrained growth as well as the strain-driven growth problem of the artery under internal pressure, including the opening angle experiment.

*Findings.* The novel relaxed growth modeling framework introduced in this paper allows for a better control of growth-induced residual stresses compared to standard growth models based on the multiplicative decomposition of the transformation gradient.

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1 *Interpretation.* Growth-induced residual stresses should be better handled  
2 in soft tissues biomechanical models, especially in patient-specific models  
3 of diseased organs that are aimed at augmented diagnosis and treatment  
4 optimization.

5 *Keywords:*

6 Finite growth, Residual stresses, Relaxation, Finite element method

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7 (Total word count: *ca.* 3000.)

## 8 **1. Introduction**

9 Biomechanical models are widely considered as good candidates to im-  
10 prove patient diagnosis and treatment of various diseases, for instance in  
11 vascular [Taylor and Figueroa, 2009], cardiac [Smith et al., 2011; Krishna-  
12 murthy et al., 2013], or respiratory [Roth et al., 2017] mechanics. To do so,  
13 these models must first be formulated with patient-agnostic constitutive laws  
14 and parameters that are developed alongside with experiments. And once  
15 these generic models are set up, clinical data must be assimilated into to  
16 personalize them and hence produce diagnosis [Xi et al., 2011; Genet et al.,  
17 2015a] and/or perform *in silico* treatment optimization [Sermesant et al.,  
18 2012].

19 Today, most personalized modeling pipelines focus on the current state  
20 of the tissue/organ, and few work focused on predicting the long term evo-  
21 lution of the system of interest, *i.e.*, prognosis [Clatz et al., 2005; Rausch  
22 et al., 2017]. Similarly, constitutive models of the mechanical response of  
23 soft tissues have been established and are widely accepted in the community,  
24 but models of soft tissues remodeling are more controversial [Witzenburg  
25 and Holmes, 2017]. This is explained by the increased complexity and as-  
26 sociated decreased understanding of the physical mechanisms at play during  
27 remodeling. Specifically for growth, a remodeling mechanism associated with  
28 the addition and/or removal of matter with unchanged properties [Taber,  
29 1995; Kuhl, 2014], there are competing approaches for the very description  
30 of growth (multiplicative decomposition of the transformation gradient [Ro-  
31 driguez et al., 1994; Kuhl, 2014] *vs.* constrained mixture theory [Humphrey  
32 and Rajagopal, 2002; Valentín and Holzapfel, 2012; Cyron et al., 2016]), the  
33 growth driving force (chemistry *vs.* mechanics [Maillet et al., 2013; Kuhl,  
34 2014], strain *vs.* stress [Rodriguez et al., 1994; Göktepe et al., 2010; Kerck-  
35 hoffs et al., 2012], static *vs.* oscillatory loading [Maillet et al., 2013; Lee et al.,

1 2015], *etc.*). Nevertheless, the next generation of personalized modeling-  
2 based clinical tools might include remodeling mechanisms such as growth for  
3 prognosis and treatment optimization.

4 One important question associated with the modeling of growth is about  
5 the induced residual stresses, or equivalently, prestrain. On the one hand  
6 they are present in living tissues [Fung, 1993], on the other hand growth nat-  
7 urally induces residual stresses [Rodriguez et al., 1994; Skalak et al., 1996];  
8 however it is still largely unknown how much of the residual stresses in-  
9 duced by the physiological or pathological growth remain, and how much are  
10 relaxed away by some relaxation mechanism [Fung, 1993; Taber, 1995]. Ac-  
11 tually, it was found experimentally, in studies involving hypertension-induced  
12 pathological growth of cardiovascular tissues, that opening angle, and hence  
13 residual stresses, were only very little correlated with growth [Liu and Fung,  
14 1989; Omens et al., 1996]. As a consequence, in growth models of the lit-  
15 erature, some authors have chosen to keep growth-induced residual stresses  
16 [Rodriguez et al., 1994; Lee et al., 2014; Genet et al., 2015b], and others have  
17 chosen to remove them [Kroon et al., 2009; Lee et al., 2015, 2016]. Other  
18 approaches have been proposed to better deal with growth-induced resid-  
19 ual stresses, notably a recent growth modeling framework that include some  
20 “fluid-like” growth [Böl and Bolea Albero, 2014; Bolea Albero et al., 2014].

21 In this paper, we introduce a novel “relaxed growth” model, in the gen-  
22 eral framework of the multiplicative decomposition of the transformation  
23 gradient, which allows one to control the amount of growth-induced resid-  
24 ual stresses. The general idea is to add another sub-transformation in the  
25 decomposition, which relaxes the tissue by transforming its unloaded config-  
26 uration. Illustrations are provided on time-driven constrained growth as well  
27 as the strain-driven growth problem of the artery under internal pressure.

## 28 **2. Methods**

### 29 *2.1. Review of classical growth modeling*

#### 30 *2.1.1. Kinematics*

31 In this part, we review the standard formulation of finite growth model-  
32 ing based on the multiplicative decomposition of the transformation gradient,  
33 originally proposed by [Rodriguez et al., 1994], and illustrated on Figure 1.  
34 Thus, let us consider a physical body  $\mathcal{B}$  that initially occupies a domain  
35  $\Omega_0$  and that, after deformation and growth, occupies a domain  $\Omega$ . The ge-  
36 ometrical transformation between material neighborhoods of  $\mathcal{B}$  in  $\Omega_0$  and

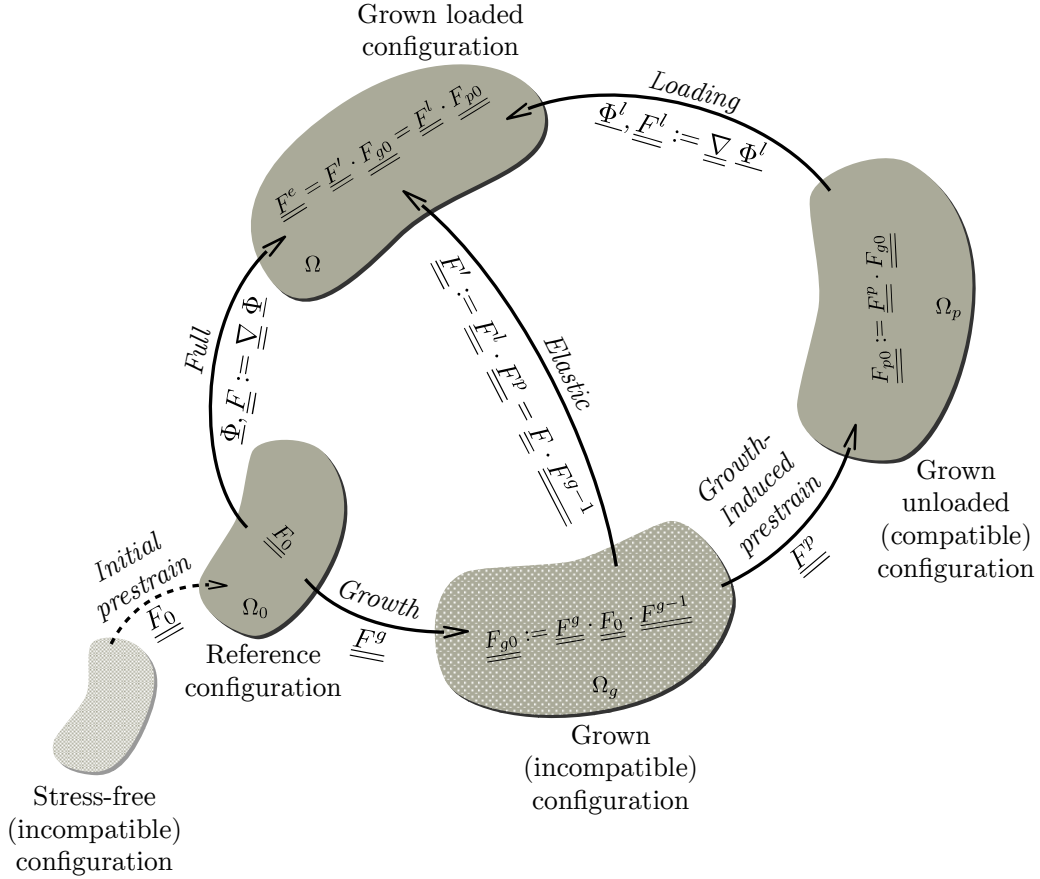


Figure 1: Schematic of the multiplicative decomposition of the transformation gradient into growth, prestrain and loading parts, revealing the elastic part of the transformation [Rodriguez et al., 1994]. Because the growth part represents the transformation of material neighborhoods independently from each others and hence is incompatible (*i.e.*, does not derive from a continuous mapping), the grown configuration is incompatible (*i.e.*, discontinuous everywhere). Consequently, the prestrain and elastic parts of the transformation are also incompatible. Conversely, both the full transformation and the loading part of the transformation are compatible, though if the loading part does represent a continuous mapping between material points, the full transformation only represents a continuous mapping between material neighborhoods, in which mass has potentially been added or removed. This decomposition is generally represented without initial prestrain, *i.e.*, with  $\underline{\underline{F}}_0 = \underline{\underline{1}}$ , in which case  $\underline{\underline{F}}_{g0} = \underline{\underline{1}}$  and  $\underline{\underline{F}}_{p0} = \underline{\underline{F}}_p$ , such that  $\underline{\underline{F}}^e = \underline{\underline{F}}^l$ .

1  $\Omega$  is denoted  $\underline{\Phi}$ . Following the classical scheme of Figure 1, the transfor-  
 2 mation gradient  $\underline{F} := \underline{Grad}(\underline{\Phi})$  is multiplicatively decomposed into growth,  
 3 prestrain, and load-induced parts:

$$\underline{F} = \underline{F}^l \cdot \underline{F}^p \cdot \underline{F}^g, \quad (1)$$

4 where, in general, both  $\underline{F}^g$  (the internal variable describing tissue growth)  
 5 and  $\underline{F}^p$  (the prestrain) are incompatible second order tensor fields, and  
 6  $\underline{F}^l := \underline{Grad}(\underline{\Phi}^l)$  is the gradient of the (compatible) transformation induced  
 7 by mechanical loading. The prestrain and load-induced parts can be multi-  
 8 plicatively combined to form the elastic part of the transformation:

$$\underline{F}' := \underline{F}^l \cdot \underline{F}^p, \quad (2)$$

9 such that the full transformation gradient can be multiplicatively decomposed  
 10 into growth and elastic parts:

$$\underline{F} = \underline{F}' \cdot \underline{F}^g. \quad (3)$$

11 Equivalently, the elastic part of the transformation can be expressed from  
 12 the full transformation gradient and the growth internal variable:

$$\underline{F}' = \underline{F} \cdot \underline{F}^{g-1}. \quad (4)$$

13 The full right Cauchy-Green dilatation tensor is denoted  $\underline{C} := {}^t\underline{F} \cdot \underline{F}$ . As  
 14 for the transformation gradient, the elastic part of the dilatation tensor can  
 15 be expressed from the full dilatation tensor and the growth internal variable:  
 16

$$\underline{C}' := {}^t\underline{F}' \cdot \underline{F}' = {}^t\underline{F}^{g-1} \cdot \underline{C} \cdot \underline{F}^{g-1}. \quad (5)$$

### 17 2.1.2. Free energy and Stresses

18 The elastic response is governed by the strain energy potential,  $W^e$ . Any  
 19 hyperelastic potential can be used. The main modeling assumption is that  
 20 *in fine* the free energy  $\rho_0\Psi$  is not a function of the total transformation but  
 21 only its elastic part  $\underline{F}^e$  [Rodriguez et al., 1994; Göktepe et al., 2010]:

$$\rho_0\Psi(\underline{C}, \underline{F}^g) = W^e(\underline{C}^e := {}^t\underline{F}^e \cdot \underline{F}^e). \quad (6)$$

22 Consequently, the second Piola-Kirchhoff stress tensor can be expressed as

$$\underline{\Sigma} = \frac{\partial \rho_0\Psi}{\partial \underline{E}} = 2 \frac{\partial \rho_0\Psi}{\partial \underline{C}} = 2 \frac{\partial W^e}{\partial \underline{C}^e} : \frac{\partial \underline{C}^e}{\partial \underline{C}}. \quad (7)$$

1 In the canonical case described so far, we have

$$\underline{\underline{F}}^e = \underline{\underline{F}}' = \underline{\underline{F}} \cdot \underline{\underline{F}}^{g^{-1}}, \quad (8)$$

2 such that

$$\underline{\underline{C}}^e = \underline{\underline{C}}' = {}^t \underline{\underline{F}}^{g^{-1}} \cdot \underline{\underline{C}} \cdot \underline{\underline{F}}^{g^{-1}}, \quad (9)$$

3 and

$$\underline{\underline{\Sigma}} = \underline{\underline{F}}^{g^{-1}} \cdot 2 \frac{\partial W^e}{\partial \underline{\underline{C}}^e} \cdot {}^t \underline{\underline{F}}^{g^{-1}}. \quad (10)$$

#### 4 2.1.3. Growth evolution law

5 An evolution law must be formulated for the growth tensor to close the  
6 system. Several types have been proposed, including simple rates [Kuhl,  
7 2014], strain- [Göktepe et al., 2010; Kerckhoffs et al., 2012; Genet et al.,  
8 2016] and stress- [Göktepe et al., 2010] driven laws. See for instance [Kuhl,  
9 2014] for a recent review, and [Witzenburg and Holmes, 2017] for a detailed  
10 comparison of existing laws.

11 In case of tissues subjected to cyclic loading, it is necessary to uncouple  
12 the temporal scales of loading and remodeling, by alternating between load-  
13 ing (*i.e.*, fast time scale) and growth (*i.e.*, slow time scale) steps [Kerckhoffs  
14 et al., 2012; Lee et al., 2016]. Thus, for a given growth step, the driving  
15 force for growth must be computed from the previous loading step. Aver-  
16 age [Lee et al., 2015] and maximum [Kerckhoffs et al., 2012] values of stress  
17 and strain over a cycle have been proposed, in agreement with experimental  
18 analysis [Holmes, 2004].

#### 19 2.1.4. Case of initially prestrained material

20 For the sake of completeness, let us consider the case where the initial  
21 configuration  $\Omega_0$  has some potentially incompatible initial prestrain, denoted  
22  $\underline{\underline{F}}_0$ , or, equivalently, some autobalanced initial prestress  $\underline{\underline{\sigma}}_0 = \frac{1}{J_0} \underline{\underline{F}}_0 \cdot \underline{\underline{\Sigma}}_0 \cdot \underline{\underline{F}}_0^{-1}$   
23 with  $J_0 := \det(\underline{\underline{F}}_0)$  and  $\underline{\underline{\Sigma}}_0 = 2 \frac{\partial \rho_0 W^e}{\partial \underline{\underline{C}}^e}(\underline{\underline{C}}^e = \underline{\underline{C}}_0)$  where  $\underline{\underline{C}}_0 := {}^t \underline{\underline{F}}_0 \cdot \underline{\underline{F}}_0$  (See  
24 Figure 1). Note that here, the prestrain must be seen as a local tensor state  
25 variable more than a gradient of some mapping. And in this case, the full  
26 elastic transformation,  $\underline{\underline{F}}^e$ , contains both  $\underline{\underline{F}}'$  and the prestrain  $\underline{\underline{F}}_0$ :

$$\underline{\underline{F}}^e = \underline{\underline{F}} \cdot \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g^{-1}}, \quad (11)$$

1 such that the free energy is

$$\rho_0 \Psi(\underline{\underline{C}}, \underline{\underline{F}}^g, \underline{\underline{F}}_0) = W^e(\underline{\underline{C}}^e = {}^t \underline{\underline{F}}^{g-1} \cdot {}^t \underline{\underline{F}}_0 \cdot \underline{\underline{C}} \cdot \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1}), \quad (12)$$

2 and the second Piola-Kirchhoff stress tensor is

$$\underline{\underline{\Sigma}} = \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1} \cdot 2 \frac{\partial W^e}{\partial \underline{\underline{C}}^e} \cdot {}^t \underline{\underline{F}}^{g-1} \cdot {}^t \underline{\underline{F}}_0. \quad (13)$$

3 It is also interesting to express the prestrain in the grown configuration  
4  $\Omega_g$ :

$$\underline{\underline{F}}_{g0} := \underline{\underline{F}}^g \cdot \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1}, \quad (14)$$

5 such that the full elastic transformation can also be expressed as

$$\underline{\underline{F}}^e = \underline{\underline{F}}' \cdot \underline{\underline{F}}_{g0}. \quad (15)$$

6 Similarly, the total prestrain in the new grown unloaded configuration  $\Omega_p$  is

$$\underline{\underline{F}}_{p0} := \underline{\underline{F}}^p \cdot \underline{\underline{F}}_{g0} = \underline{\underline{F}}^p \cdot \underline{\underline{F}}^g \cdot \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1}, \quad (16)$$

7 such that the full elastic transformation can also be expressed as

$$\underline{\underline{F}}^e = \underline{\underline{F}}^l \cdot \underline{\underline{F}}_{p0}. \quad (17)$$

## 8 *2.2. Relaxed growth modeling*

### 9 *2.2.1. Kinematics*

10 We now describe a new formulation for finite growth, involving an ad-  
11 ditional internal sub-transformation for local relaxation. The associated  
12 schematic is represented on the Figure 2. The total transformation gradient  
13 is now decomposed into growth, relaxation, prestrain and loading parts:

$$\underline{\underline{F}} = \underline{\underline{F}}^l \cdot \underline{\underline{F}}^p \cdot \underline{\underline{F}}^r \cdot \underline{\underline{F}}^g, \quad (18)$$

14 where, again, both  $\underline{\underline{F}}^g$  (the internal variable describing tissue growth),  $\underline{\underline{F}}^r$   
15 (the internal variable describing tissue relaxation) and  $\underline{\underline{F}}^p$  (the prestrain) are  
16 incompatible second order tensor fields, and  $\underline{\underline{F}}^l := \underline{\underline{Grad}}(\Phi^l)$  is the gradi-  
17 ent of the (compatible) transformation induced by mechanical loading. The  
18 elastic part of the transformation is still

$$\underline{\underline{F}}' := \underline{\underline{F}}^l \cdot \underline{\underline{F}}^p, \quad (19)$$



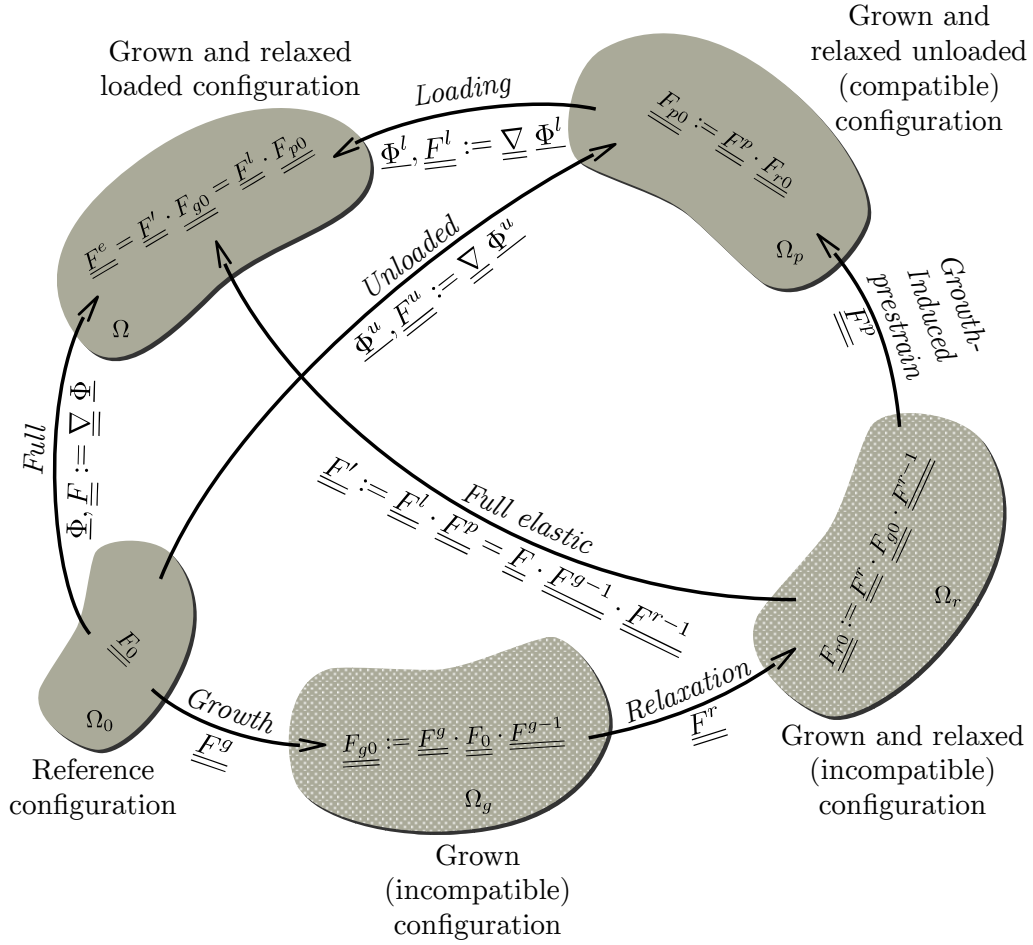


Figure 2: Schematic of the proposed multiplicative decomposition of the transformation gradient into growth, relaxation, prestrain and loading parts.

1 such that

$$\underline{\underline{F}} = \underline{\underline{F'}} \cdot \underline{\underline{F}}^r \cdot \underline{\underline{F}}^g. \quad (20)$$

2 Thus, here the elastic part of the transformation can be expressed from the  
3 full transformation gradient, and both the growth and relaxation internal  
4 variables:

$$\underline{\underline{F'}} := \underline{\underline{F}} \cdot \underline{\underline{F}}^{g-1} \cdot \underline{\underline{F}}^{r-1}. \quad (21)$$

5 Let us point out that the order of the decomposition is arbitrary.

6 The role of the relaxation is to update the unloaded configuration of  
7 the grown body in order to control the amount of growth-induced residual  
8 stresses. Thus, the updated unloaded configuration is now  $\Omega_p$ , with prestrain  
9

$$\underline{\underline{F}}_{p0} := \underline{\underline{F}}^p \cdot \underline{\underline{F}}_{r0}, \quad (22)$$

10 where

$$\underline{\underline{F}}_{r0} := \underline{\underline{F}}^r \cdot \underline{\underline{F}}_{g0} \cdot \underline{\underline{F}}^{r-1}, \quad (23)$$

11 and

$$\underline{\underline{F}}_{g0} := \underline{\underline{F}}^g \cdot \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1}. \quad (24)$$

### 12 2.2.2. Free energy and Stresses

13 As for simple growth, we assume that the free energy  $\rho_0\Psi$  is only a func-  
14 tion of the elastic part of the transformation  $\underline{\underline{F}}^e$ , see Equation (6). However,  
15 in the case of relaxed growth, we have

$$\underline{\underline{F}}^e = \underline{\underline{F'}} \cdot \underline{\underline{F}}_{r0} = \underline{\underline{F}} \cdot \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1} \cdot \underline{\underline{F}}^{r-1}, \quad (25)$$

16 such that

$$\underline{\underline{C}}^e = {}^t\underline{\underline{F}}^{r-1} \cdot {}^t\underline{\underline{F}}^{g-1} \cdot {}^t\underline{\underline{F}}_0 \cdot \underline{\underline{C}} \cdot \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1} \cdot \underline{\underline{F}}^{r-1}, \quad (26)$$

17 and

$$\underline{\underline{\Sigma}} = \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1} \cdot \underline{\underline{F}}^{r-1} \cdot 2 \frac{\partial W^e}{\partial \underline{\underline{C}}^e} \cdot {}^t\underline{\underline{F}}^{r-1} \cdot {}^t\underline{\underline{F}}^{g-1} \cdot {}^t\underline{\underline{F}}_0. \quad (27)$$

18 Similarly, for the unloaded configuration the total elastic deformation is sim-  
19 ply

$$\underline{\underline{F}}_{p0} = \underline{\underline{F}}^u \cdot \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1} \cdot \underline{\underline{F}}^{r-1}, \quad (28)$$

20 such that

$$\underline{\underline{C}}_{p0} := {}^t\underline{\underline{F}}_{p0} \cdot \underline{\underline{C}}_{p0} = {}^t\underline{\underline{F}}^{r-1} \cdot {}^t\underline{\underline{F}}^{g-1} \cdot {}^t\underline{\underline{F}}_0 \cdot \underline{\underline{C}}^u \cdot \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1} \cdot \underline{\underline{F}}^{r-1}, \quad (29)$$

1 and

$$\underline{\underline{\Sigma}}^u = \underline{\underline{F}}_0 \cdot \underline{\underline{F}}^{g-1} \cdot \underline{\underline{F}}^{r-1} \cdot 2 \frac{\partial W^e}{\partial \underline{\underline{C}}_{p0}} \cdot {}^t \underline{\underline{F}}^{r-1} \cdot {}^t \underline{\underline{F}}^{g-1} \cdot {}^t \underline{\underline{F}}_0. \quad (30)$$

2 *2.2.3. Relaxation evolution law*

3 As for the growth evolution, many choices can be made for the evolution  
 4 of the relaxation internal variable. Since the role of the relaxation is to  
 5 regulate the prestrain, in this article we will consider the simplest possible  
 6 evolution law, directly based on the current prestrain level:

$$\dot{\underline{\underline{F}}}^r = \frac{1}{\tau^r} \cdot \underline{\underline{E}}_{p0}, \quad (31)$$

7 where  $\tau^r$  is a characteristic time for relaxation, and  $\underline{\underline{E}}_{p0} := \frac{1}{2} \left( \underline{\underline{C}}_{p0} - \underline{\underline{1}} \right)$  is  
 8 the Green-Lagrange prestrain. The underlying hypothesis is that the tissue is  
 9 somehow able to sense the current level of prestrain. This is analogous to the  
 10 hypothesis made in constrained mixture theory-based growth models, where  
 11 newly deposited matter has its own reference state that can be the one of  
 12 the surrounding matter, the current state of deformation, or a combination  
 13 of both [Valentín et al., 2013; Cyron et al., 2016].

14 *2.3. Numerical resolution*

15 The relaxed growth problem can be solved like any nonlinear problem  
 16 with internal variables in mechanics. The only subtlety here is induced by  
 17 the choice of the relaxation evolution law, which requires the computation  
 18 of the new unloaded configuration together with the loaded configuration.  
 19 We propose a mixed formulation of the problem, the unknowns being  $\underline{\underline{U}}$  (the  
 20 total displacement),  $\underline{\underline{U}}^u$  (the displacement of the unloaded configuration),  
 21  $\underline{\underline{F}}^g$  (the growth tensor) and  $\underline{\underline{F}}^r$  (the relaxation tensor). After quasi-static  
 22 assumption, and implicit, mid-point rule temporal discretization, the mixed

1 variational formulation of the problem to solve at each time step is:

$$\left\{ \int_{\Omega_0} \underline{\underline{\Sigma}} : d_{\underline{U}; \underline{U}^*} \underline{\underline{E}} d\Omega_0 = \mathcal{W}_e(\underline{U}; \underline{U}^*) \quad \forall \underline{U}^* \right. \quad (32a)$$

$$\left. \int_{\Omega_0} \underline{\underline{\Sigma}}^u : d_{\underline{U}^u; \underline{U}^{u*}} \underline{\underline{E}}^u d\Omega_0 = 0 \quad \forall \underline{U}^{u*} \right. \quad (32b)$$

$$\left. \int_{\Omega_0} \left( \underline{\underline{F}}^g - \left( \underline{\underline{F}}^{g, \text{old}} + \underline{\underline{\dot{F}}}^{g, \text{mid}} \Delta t \right) \right) : \underline{\underline{F}}^{g*} d\Omega_0 = 0 \quad \forall \underline{\underline{F}}^{g*} \right. \quad (32c)$$

$$\left. \int_{\Omega_0} \left( \underline{\underline{F}}^r - \left( \underline{\underline{F}}^{r, \text{old}} + \underline{\underline{\dot{F}}}^{r, \text{mid}} \Delta t \right) \right) : \underline{\underline{F}}^{r*} d\Omega_0 = 0 \quad \forall \underline{\underline{F}}^{r*} \right. \quad (32d)$$

2 where  $d_{\underline{U}; \underline{U}^*} \underline{\underline{E}}$  and  $d_{\underline{U}^u; \underline{U}^{u*}} \underline{\underline{E}}^u$  denote the first variations of the total and un-  
3 loaded Green-Lagrange strain tensors with respect to the total and unloaded  
4 displacements, and  $\mathcal{W}_e$  denotes the virtual work (semi-linear form) associ-  
5 ated to external loads.  $\underline{\underline{\dot{F}}}^{g, \text{mid}}$  will be specified for each growth evolution law,  
6 and relaxation evolution law (31) leads to:

$$\underline{\underline{\dot{F}}}^{r, \text{mid}} = \frac{1}{\tau^r} \underline{\underline{E}}_{p0}^{\text{mid}} \quad (33)$$

7 with  $\underline{\underline{E}}_{p0}^{\text{mid}} = \frac{1}{2} \left( \underline{\underline{E}}_{p0}^{\text{old}} + \underline{\underline{E}}_{p0} \right)$ .

8 In case of incompressible elastic deformation, two new unknowns are  
9 introduced:  $p$  (the Lagrange multiplier associated to the incompressibility  
10 constraint of the elastic strain, equal to the hydrostatic pressure within the  
11 tissue) and  $p^u$  (the Lagrange multiplier/hydrostatic pressure in the unloaded  
12 configuration), and the full mixed variational formulation becomes:

$$\left\{ \int_{\Omega_0} \left( \underline{\underline{\Sigma}} - p J^e \underline{\underline{C}}^{e-1} \right) : d_{\underline{U}; \underline{U}^*} \underline{\underline{E}} d\Omega_0 = \mathcal{W}_e(\underline{U}; \underline{U}^*) \quad \forall \underline{U}^* \right. \quad (34a)$$

$$\left. \int_{\Omega_0} (J^e - 1) p^* d\Omega_0 = 0 \quad \forall p^* \right. \quad (34b)$$

$$\left\{ \int_{\Omega_0} \left( \underline{\underline{\Sigma}}^u - p^u J_{p0} \underline{\underline{C}}_{p0}^{-1} \right) : d_{\underline{U}^u; \underline{U}^{u*}} \underline{\underline{E}}^u d\Omega_0 = 0 \quad \forall \underline{U}^{u*} \right. \quad (34c)$$

$$\left. \int_{\Omega_0} (J_{p0} - 1) p^{u*} d\Omega_0 = 0 \quad \forall p^{u*} \right. \quad (34d)$$

$$\left. \int_{\Omega_0} \left( \underline{\underline{F}}^g - \left( \underline{\underline{F}}^{g, \text{old}} + \underline{\underline{\dot{F}}}^{g, \text{mid}} \Delta t \right) \right) : \underline{\underline{F}}^{g*} d\Omega_0 = 0 \quad \forall \underline{\underline{F}}^{g*} \right. \quad (34e)$$

$$\left. \int_{\Omega_0} \left( \underline{\underline{F}}^r - \left( \underline{\underline{F}}^{r, \text{old}} + \underline{\underline{\dot{F}}}^{r, \text{mid}} \Delta t \right) \right) : \underline{\underline{F}}^{r*} d\Omega_0 = 0 \quad \forall \underline{\underline{F}}^{r*} \right. \quad (34f)$$

1 Problems (32) and (34) are spatially discretized using the standard finite  
 2 element method. Second order elements are used for displacement unknowns  
 3 (both  $\underline{U}$  &  $\underline{U}^u$ ), and first order elements for pressure unknowns (both  $p$   
 4 &  $p^u$ ), preventing numerical locking in the incompressible limit [Hughes,  
 5 2000; Chapelle and Bathe, 2010]. First order elements are used for internal  
 6 variables (both  $\underline{F}^g$  &  $\underline{F}^r$ ) as well.

7 The full scheme has been implemented in python, based on the FEniCS  
 8 library [Logg et al., 2012; Alnæs et al., 2015], and is freely available<sup>1</sup>.

### 9 3. Results

10 We will now present multiple illustrations of the relaxed growth response  
 11 in 2D.

#### 12 3.1. Model response: constrained growth

13 Let us first illustrate the constitutive behavior described by the relaxed  
 14 growth model. To do so, we consider a single material point for which the  
 15 total deformation is blocked to zero, and subjected to time-driven isotropic  
 16 growth:

$$\begin{cases} \underline{F}^g = (1 + \theta^g) \underline{\underline{1}} \\ \theta^g(t=0) = 0 \\ \dot{\theta}^g = \frac{1}{\tau^g} \end{cases}, \quad (35)$$

17 where  $\tau^g$  is a characteristic time for growth. This growth evolution law leads,  
 18 after temporal discretization, to the following expression:

$$\underline{\underline{\dot{F}^{g,\text{mid}}}} = \frac{1}{\tau^g} \underline{\underline{1}} \quad (36)$$

19 For the sake of simplicity, and to focus on the relaxed growth framework  
 20 introduced in this paper, we consider a simple compressible neo-hookean  
 21 strain energy potential under the plane strain assumption [Ciarlet and Gey-  
 22 monat, 1982]:

$$W^e(\underline{\underline{C}}^e) = \frac{\lambda}{4} (J^{e2} - 1 - 2 \ln J^e) + \frac{\mu}{2} (\text{tr}(\underline{\underline{C}}^e) - 2 - 2 \ln J^e), \quad (37)$$

---

<sup>1</sup>[https://gitlab.inria.fr/mgenet/dolfin\\_cm](https://gitlab.inria.fr/mgenet/dolfin_cm)

1 where  $\lambda$  &  $\mu$  are the bulk and shear modulus, taken as unity. Compressible  
 2 mixed variational formulation (32) is used, with evolution laws (33) & (36).

3 This is the purest example of constrained growth. In case of simple  
 4 growth, this would lead to the development of compressive residual stresses,  
 5 which arise from constraining the grown tissue to its original volume. This  
 6 is also what happens in the relaxed growth framework; however, here the  
 7 amount of growth-induced stresses is controlled by the ratio of the charac-  
 8 teristic time of relaxation to the characteristic time of growth. If relaxation  
 9 is much slower than growth, the response is similar to simple growth; con-  
 10 versely, if relaxation is much faster than growth, then constrained growth  
 11 happens with almost no induced residual stress. This is well illustrated on  
 12 the Figure 3, which shows the normalized (with respect to shear modulus)  
 13 hydrostatic pressure within the material point as a function of normalized  
 14 (with respect to growth time constant) time, for various ratios of relaxation  
 15 over growth characteristic times. The relaxed growth framework allows to  
 16 control the amount of growth-induced residual stresses in constrained growth.

17 *3.2. A time-driven relaxed growth example: constrained vs. unconstrained*  
 18 *growth*

19 Let us now illustrate the relaxed growth response in a first structural  
 20 case, inspired from [Kuhl, 2014]. Here we consider a simple square geometry,  
 21 with three fixed boundaries (left, bottom and right) and one free boundary  
 22 (top), subjected to the same time-driven growth (35). This case contains  
 23 both constrained (toward the bottom, where the total strain is restricted,  
 24 thus developing compressive residual stresses), and unconstrained (toward  
 25 the top, where the free edge limits the stress level, and large strains will  
 26 develop) growth [Kuhl, 2014]. In terms of elasticity, we now consider a simple  
 27 incompressible neo-hookean strain energy potential under the plane strain  
 28 assumption:

$$W^e(\underline{\underline{C}}^e) = \frac{\mu}{2} (\text{tr}(\underline{\underline{C}}^e) - 2), \quad (38)$$

29 where  $\mu$  is the shear modulus, taken as unity, and use the incompressible  
 30 mixed variational formulation (34) is used with evolution laws (33) & (36).

31 Figure 4 shows the response of the tissue modeled by relaxed growth, as a  
 32 function of normalized (with respect to growth time constant) time, and for  
 33 various ratios of relaxation over growth characteristic times. Since growth is  
 34 purely time-driven, the growth pattern is very similar in all cases. However,

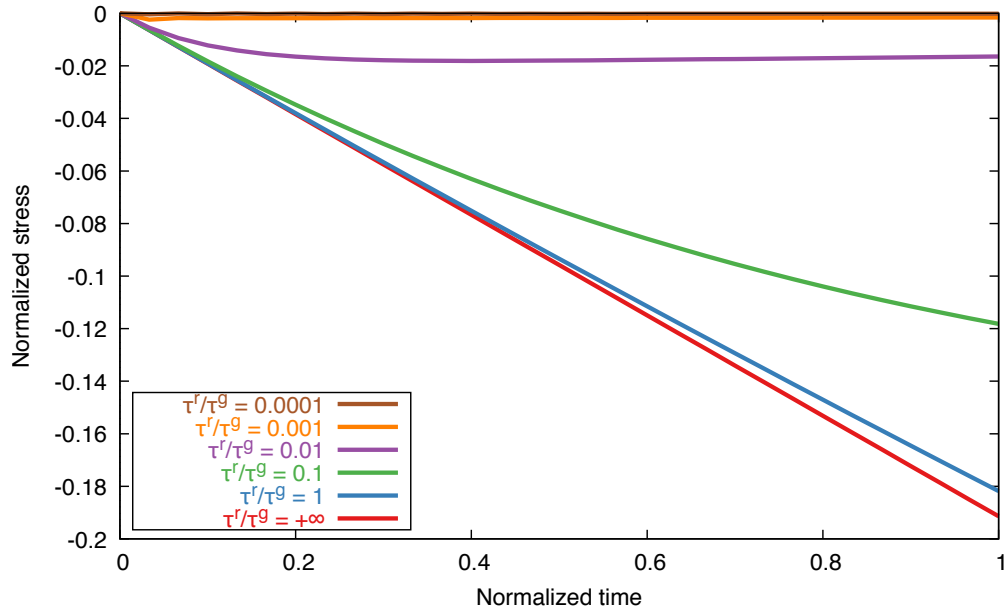


Figure 3: Relaxed growth material response under constrained growth: normalized (with respect to shear modulus) hydrostatic pressure within a material point as a function of normalized (with respect to growth time constant  $\tau^g$ ) time, for various ratios of relaxation over growth characteristic times. For slow relaxation (*i.e.*, for large relaxation time constant  $\tau^r$ ), the model behaves as standard growth, with the development of residual stresses; conversely, for fast relaxation (*i.e.*, for small relaxation time constant  $\tau^r$ ), constrained growth happens with almost no induced residual stresses.

1 the level of growth-induced residual stresses varies drastically with the re-  
 2 laxation characteristic time: for very fast relaxation (compared to growth),  
 3 the muffin grows without developing large residual stresses, even in the con-  
 4 strained region.

### 5 3.3. A strain-driven relaxed growth example: the artery

6 The final example is the canonical case of residual stresses in pressurized  
 7 arteries. We start from a simple disc representing an unloaded, stress-free  
 8 artery. The artery is loaded with some internal pressure, and allowed to grow  
 9 and relax for some time while the internal pressure is maintained. Then, the  
 10 loading is removed, and an opening angle experiment is simulated by making  
 11 a radial cut in the model, and letting the artery spring open by releasing  
 12 some residual stresses. Here we consider the simplest form of strain-driven  
 13 growth evolution law:

$$\begin{cases} \underline{\underline{F}}^g = (1 + \theta^g) \underline{\underline{1}} \\ \theta^g(t=0) = 0 \\ \dot{\theta}^g = \frac{\|\underline{\underline{E}}^e\|}{\tau^g} \end{cases}, \quad (39)$$

14 where  $\tau^g$  still represents the growth time constant, and  $\underline{\underline{E}}^e := \frac{1}{2}(\underline{\underline{C}}^e - \underline{\underline{1}})$   
 15 is the Green-Lagrange elastic strain. This growth evolution law leads, after  
 16 temporal discretization, to the following expression:

$$\underline{\underline{\dot{F}}}^{g,\text{mid}} = \frac{\|\underline{\underline{E}}^{e,\text{mid}}\|}{\tau^g} \underline{\underline{1}} \quad (40)$$

17 We use the incompressible neo-hookean strain energy potential (38), and the  
 18 incompressible mixed variational formulation (34) with evolution laws (33)  
 19 & (40).

20 Figure 5 shows the response of the artery over time, for various levels of  
 21 relaxation characteristic times: it first inflates due to the applied pressure,  
 22 grows, deflates as the pressure is removed, and springs open. Note that since  
 23 residual stresses are here induced by heterogeneous, strain-driven growth de-  
 24 scribed by growth evolution law (39), only a small amount of growth (with  
 25 respect to the pressure-induced deformation) is required to generate realistic  
 26 residual stresses, leading to physiological opening angles. This last example  
 27 illustrates the fact that the relaxed growth model allows to control the open-  
 28 ing angle (*i.e.*, the residual stresses) induced by a given amount of growth.



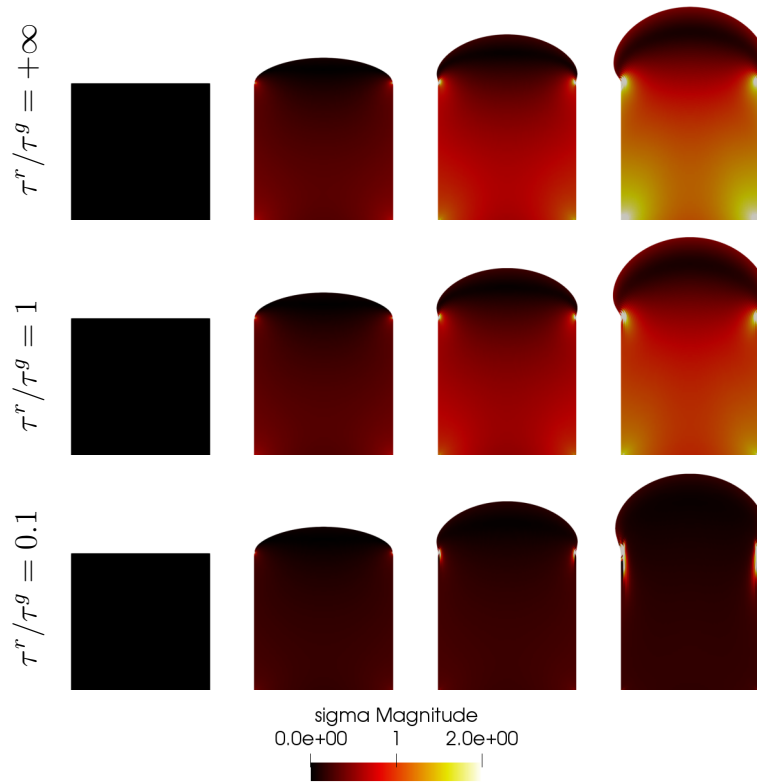


Figure 4: Relaxed growth structural response under constrained and unconstrained growth: norm of Cauchy stress tensor superimposed onto the deformed domain as a function of normalized (with respect to growth time constant  $\tau^r$ ) time, for various ratios of relaxation over growth characteristic times. The faster the relaxation, the less residual stresses develop for the same amount of growth.

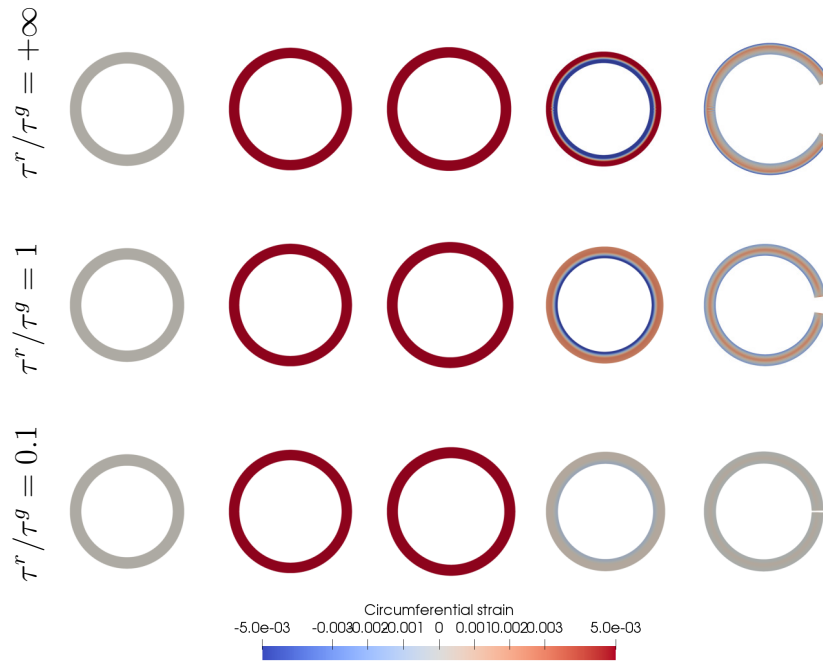


Figure 5: Relaxed growth response of a growing artery under pressure: circumferential strain superimposed onto the deformed artery at each step of the simulation (initial, after applying the internal pressure, after growth and relaxation, after removing the internal pressure, and after making a radial cut and letting the artery spring open), for various ratios of relaxation over growth characteristic times. The faster the relaxation, the less residual stresses, and hence a smaller opening angle develops for the same amount of growth.

## 1 4. Discussion

2 In this paper, we introduced a novel “relaxed growth” framework allowing  
3 for a fine control of the amount of residual stresses generated during tissue  
4 heterogeneous growth. It is a direct extension of the classical multiplicative  
5 decomposition of the transformation gradient framework [Rodriguez et al.,  
6 1994], to which an additional sub-transformation is introduced in order to let  
7 the original unloaded configuration to evolve, hence relaxing away some of  
8 the residual stresses. To solve it numerically, we proposed here a monolithic  
9 mixed formulation, but a standard displacement formulation with internal  
10 variables could be used equivalently.

11 This work could be extended in multiple directions. Most and foremost,  
12 experimental characterization of the amount of residual stresses induced di-  
13 rectly by physiological and/or pathological growth is required to provide  
14 elements of validation to relaxed growth models. Moreover, other relaxation  
15 evolution laws could be formulated and tested, especially in case of cyclic  
16 loading. 3D simulations could be run. More generally, since in the current  
17 numerical procedure both the deformed and unloaded configurations are com-  
18 puted at once, it is straightforward to stop the simulation, for instance when  
19 the total deformation becomes too important and the mesh too distorted,  
20 transfer all the fields to the new unloaded configuration, and restart from  
21 here. That way problems with extreme growth, for instance in organogene-  
22 sis, could be tackled, with a clear control in both growth and growth-induced  
23 residual stresses. Finally, relaxed growth could be added personalized model-  
24 ing pipelines, at both modeling and estimation steps, in order to make more  
25 objective and quantitative the handling of longitudinal clinical data, and  
26 design the next generation of diagnosis and treatment optimization tools.

## 27 Acknowledgements

28 MG would like to thank Prof. Lik Chuan Lee, from Michigan State Uni-  
29 versity, USA, for helpful discussions throughout the development of this work.

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