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Solvability for a drift-diffusion system with Robin boundary conditions

Arnaud Heibig* Adrien Petrov* Christian Reichert*

Abstract

This paper focuses on a drift-diffusion system subjected to boundedly non dissipative Robin boundary conditions. A general existence result with large initial conditions is established by using suitable L^1 , L^2 and trace estimates. Finally, two examples coming from the corrosion and the self-gravitation model are analyzed.

Key words. Drift-diffusion system, Robin boundary conditions, complex interpolation, corrosion model, self-gravitation model.

AMS Subject Classification. 35B45, 35A65, 35D30, 76R50

1 Introduction

The drift-diffusion equations have a vast phenomenology and are currently studied. When coupled with fluid flows equations, the resulting systems are usually quite complex due to the micro-macro effect. A short reference list is [7, 9, 10, 11, 12, 16, 24] for some problems arising in different context, including the theory of dilute or melt polymers. Apart from the theory of stochastic process – mainly the Fokker-Planck equation – a privileged field of application is the theory of semi conductors. This includes systems of Debye type studied for instance in [3, 4, 5, 13, 14, 15, 17]. Let's also mention, in the area of chemotaxis, the Patlak-Keller-Segel system (see [6, 19, 25, 26], and references therein).

In this paper, we focus on the following problem

$$\partial_t u = \nabla \cdot (\nabla u + Du \otimes \nabla \mathcal{V}), \quad (t, x) \in (0, T) \times \Omega, \quad (1.1a)$$

$$u(0) = u_0, \quad (1.1b)$$

with $\Omega \subset \mathbb{R}^d$ a smooth bounded domain, $u = (u^1, \dots, u^n)$ and $Du = (\alpha^1 u^1, \dots, \alpha^n u^n)$ ($\alpha^i \in \mathbb{R}$ for $i = 1, \dots, n$). The potential \mathcal{V} is given by $\mathcal{V}(t) = \mathcal{B}(t, u(t))$ for a.e $t \in (0, T)$, with $\mathcal{B} : \mathbb{R}_+ \times \mathbf{L}^1(\Omega) \rightarrow W^{1,\infty}(\Omega) \cap W^{2,1}(\Omega)$ a suitable smoothing, nonlinear operator. Denoting by ζ the outward normal to $\partial\Omega$, the Robin boundary conditions on the fluxes read as follows

$$\left(\frac{\partial u^i}{\partial \zeta} + \alpha^i u^i \frac{\partial \mathcal{V}}{\partial \zeta} \right)(t, x) = \sigma^i(t, x, u|_{\partial\Omega}(t, x), \mathcal{V}|_{\partial\Omega}(t, x)), \quad (t, x) \in [0, T] \times \partial\Omega. \quad (1.2)$$

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The fluxes σ^i are endowed with boundedly non dissipative conditions, reminiscent of Kruřkov entropy conditions: for all $(t, x, v, \psi) \in [0, T] \times \partial\Omega \times \mathbb{R} \times \mathbb{R}$

$$\sigma^i(t, x, v, \psi) \chi^+(v - k^i) \leq \Lambda_T, \quad (1.3a)$$

$$\sigma^i(t, x, v, \psi) \chi^-(v) \leq 0, \quad (1.3b)$$

where χ^+ is the Heaviside function and $\chi^-(v) = -\chi^+(-v)$ and $k^i > 0$. The goal of the paper is to prove well posedness of such a system in a L^2 frame (see Theorem 4.2). Let's mention the close connexion of the above equations and the theory of the Navier-Stokes equations as developed by [18, 27] (see also [23]). Nevertheless, we will not use this proximity in the present paper, but rather some features of the L^1 theory of Kruřkov for scalar conservation laws. See [20] and compare with assumptions (1.3).

The above problem is a compromise between realistic equations such as the Debye system, and a more abstract setting. Notice that the usual 2×2 semi-conductor model (see [4]) corresponds to the resolvent of the Poisson-Dirichlet problem, i.e $\mathcal{B}(t, \cdot) = \Delta_D^{-1}$. Such a resolvent has relatively bad smoothing properties in a L^∞ frame. But, as a compensation, the system admits opposite sign on the nonlinearities ensuring large data global solutions. Contrarily, this sign condition is not fulfilled for the present system (1.1), (1.2) and we assume the above smoothing assumption on the operator \mathcal{B} . This assumption prevents us to apply our results to the case $\mathcal{B} = \Delta_D^{-1}$ for $d \geq 2$. Nevertheless, due to the special properties of the 1-D Laplace operator, our results apply to the one dimensional Debye type system considered in [8], a problem we had primarily in view (see also [1]). In that case, our existence result improves the former result in [8]. Actually, since we work in a L^2 frame and remove the sign condition of the Debye 2×2 system, we obtain an existence result for a general $n \times n$ system ($d = 1$). We also remove the restrictive conditions on the initial data in [8]. Finally, to conclude this section, note that in the case $\mathcal{B} = \Delta_D^{-1}$, $d \geq 2$, a mollifying process can be used on \mathcal{B} in order to recover some classical results of the theory. We treat the simple case of the self gravitational system at the end of the paper (Section 5).

Compared with former works on the subject (see [4]), the novelty of [8] and of the present paper lies in the Robin boundary condition (1.2). The issue, when dealing with such a non dissipative condition is to derive an $L^\infty(0, T, L^1(\Omega))$ estimate on the function u since no decrease or conservation of $\|u(t)\|_{L^1(\Omega)}$ can be expected. Thus, the main task is to define and evaluate the nonlinear term $\sigma^i(u|_{\partial\Omega}^i)$. When working in the classical setting $u \in L^2(0, t_0; \mathbf{H}^1(\Omega)) \cap C^0([0, t_0]; \mathbf{L}^2(\Omega))$, a simple interpolation procedure shows that the natural trace space for u is $L^q(0, T, L^q(\partial\Omega))$ with $1 \leq q < 2 + \frac{2}{d}$. This corresponds to a restricted class of admissible fluxes, essentially defined as follows. For any $(t, x, v, \psi) \in [0, T] \times \partial\Omega \times \mathbb{R} \times \mathbb{R}$ with $0 \leq t \leq T$ and $|\psi| \leq M$

$$|\sigma^i(t, x, v, \psi) - \sigma^i(t, x, \bar{v}, \bar{\psi})| \leq C_{T,M}((1+|v|^\rho + |\bar{v}|^\rho)|v - \bar{v}| + (1+|v|^{\rho+1} + |\bar{v}|^{\rho+1})|\psi - \bar{\psi}|), \quad (1.4)$$

with $0 \leq \rho < 1 + \frac{2}{d}$. Within such a class of fluxes, the classical existence results of [21] do not apply to the natural linearized versions of the system (1.1a), at least for ρ close to $1 + \frac{2}{d}$. As a matter of fact, such fluxes leads to rather discontinuous right hand sides in the variational formulations, so that getting an existence result require an indirect procedure and the use of all the conditions (1.3) and (1.4) on the flux.

The paper is organized as follows. The equations are described in Subsection 2.1 while a first simplified set of constitutive assumptions is described in Subsection 2.2. Essentially, we replace condition (1.4) by a global Lipschitz condition, in order to get a tractable proof of the existence result given in Section 4. The proof of this existence result relies on the aforementioned

$L^\infty(0, T, L^1(\Omega))$ estimate. Since the extensions we have in view (Section 4) require some uniform estimates, we keep track of the constants (Lemma 2.5). Some trace inequalities are established in Section 3, leading to the definition of an extended set of assumptions. Under these conditions, a general existence theorem with large initial data is established in Section 4 by using some ad-hoc density argument. The final Section 5 is devoted to two realistic examples. The first one deals with a drift-diffusion system with Robin boundary conditions with an application to a corrosion model (cf. [1, 8]), while the second one is the classical equation of self-gravitation system studied for instance in [5].

2 Mathematical formulation

2.1 The model

Let $T > 0$, let Ω be a smooth bounded subset of \mathbb{R}^d , and let $\alpha^i \in \mathbb{R}$ be some real given numbers ($d \in \mathbb{N}^*, i \in \{1, \dots, n\}$). Let $u^i(t, x)$ and $\mathcal{V}(t)$ be scalar valued functions depending on time t . Set $u \stackrel{\text{def}}{=} (u^1, \dots, u^n)$ and denote by ∂_j the partial derivative with respect to the j^{th} spatial variable ($i \in \{1, \dots, n\}$). The mathematical problem is formulated as follows:

$$\forall i \in \{1, \dots, n\} : \partial_t u^i - \sum_{j=1}^d \partial_j (\partial_j u^i + \alpha^i u^i \partial_j \mathcal{V}) = 0, \quad (t, x) \in (0, T) \times \Omega, \quad (2.1a)$$

$$\mathcal{V}(t) = \mathcal{B}(t, u(t)) \quad \text{for a.e } t \in (0, T). \quad (2.1b)$$

The operator \mathcal{B} as well as the fluxes σ^i in equation (2.2) below will be precised in the next subsection.

We now turn to define the boundary conditions. In the sequel we denote by $\frac{\partial}{\partial \zeta}$ the derivative with respect to the outward normal to $\partial\Omega$. The trace of $u(t)$ on $\partial\Omega$ is denoted by $u|_{\partial\Omega}(t)$, or more often and abusively, by $u(t)$. The Robin boundary conditions for u^i are prescribed by

$$\forall i \in \{1, \dots, n\} : \left(\frac{\partial u^i}{\partial \zeta} + \alpha^i u^i \frac{\partial \mathcal{V}}{\partial \zeta} \right)(t, x) = \sigma^i(t, x, u|_{\partial\Omega}, \mathcal{V}|_{\partial\Omega}), \quad (t, x) \in [0, T] \times \partial\Omega, \quad (2.2)$$

and our problem is completed by the following initial conditions:

$$\forall i \in \{1, \dots, n\} : u^i(0) = u_0^i. \quad (2.3)$$

In the sequel, we will often use notations such as $u \stackrel{\text{def}}{=} (u^1, \dots, u^n)$ or $\sigma(t, x, u(t, x), \mathcal{V}(t, x)) \stackrel{\text{def}}{=} (\sigma^1(t, x, u^1(t, x), \mathcal{V}(t, x)), \dots, \sigma^n(t, x, u^n(t, x), \mathcal{V}(t, x)))$ without any comments. We define $D : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $Du \stackrel{\text{def}}{=} (\alpha^1 u^1, \dots, \alpha^n u^n)$. Let $\nabla \mathcal{V}$ be the gradient of \mathcal{V} . With these last notations, equations (2.1a) can be written in the more compact form

$$\partial_t u = \nabla \cdot (Du \otimes \nabla \mathcal{V} + \nabla u), \quad (t, x) \in (0, T) \times \Omega. \quad (2.4)$$

Thus, we introduce the following notation, used throughout this paper: if X is a space of scalar functions, the bold-face notation \mathbf{X} denotes the space X^n . Define the following sets:

$$\forall t \in (0, T] : \mathcal{Q}_t \stackrel{\text{def}}{=} (0, t) \times \Omega \quad \text{and} \quad \Gamma_t \stackrel{\text{def}}{=} (0, t) \times \partial\Omega.$$

In this paper, equation (2.4) will often be considered in the following variational sense. Let $T > 0$ and let $u_0 \in \mathbf{L}^2(\Omega)$. Then for $t_0 \in]0, T]$, the problem (\mathcal{P}_{t_0}) is

$$(\mathcal{P}_{t_0}) \quad \begin{cases} \text{Find } u \in L^2(0, t_0; \mathbf{H}^1(\Omega)) \cap C^0([0, t_0]; \mathbf{L}^2(\Omega)) \text{ with } \frac{du}{dt} \in L^2(0, t_0; (\mathbf{H}^1(\Omega))') \\ \text{such that } u(0) = u_0 \text{ and for any } w \in L^2(0, t_0; \mathbf{H}^1(\Omega)) : \\ \int_0^{t_0} \left\langle \frac{du}{dt}(\tau), w(\tau) \right\rangle d\tau + \int_{\mathcal{Q}_{t_0}} (\nabla u + Du \otimes \nabla \mathcal{V})(\tau, x) : \nabla w(\tau, x) dx d\tau \\ = \int_{\Gamma_{t_0}} \sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x)) \cdot w(\tau, x) d\mu d\tau, \end{cases}$$

with $\mathcal{V}(t) \stackrel{\text{def}}{=} \mathcal{B}(t, u(t))$ for a.e $t \in (0, t_0)$.

In the above, and throughout this paper, notations such as $(\nabla v + Dv \otimes \nabla \gamma) : \nabla w$ stands for $\sum_{i,j} (\partial_j v^i + \alpha^i v^j \partial_j \gamma) \partial_j w^i$, and the dot usually denotes the canonical scalar product in \mathbb{R}^n . Notation $(\mathbf{H}^1(\Omega))'$ denotes the topological dual of $\mathbf{H}^1(\Omega)$. We always abridge the notation $\langle \cdot, \cdot \rangle_{(\mathbf{H}^1(\Omega))', \mathbf{H}^1(\Omega)}$ in $\langle \cdot, \cdot \rangle$. Last, notation μ or $d\mu$ stands for the usual measure on $\partial\Omega$.

2.2 The simplified case

In this section, we introduce some assumptions on the constitutive functions of the problem and give a few simple consequences of these assumptions. The assumption (A-2) will be relaxed at the end of the paper by using a density argument.

(A-1) The operator $\mathcal{B} : \mathbb{R}_+ \times \mathbf{L}^1(\Omega) \rightarrow W^{1,\infty}(\Omega) \cap W^{2,1}(\Omega)$ is, locally uniformly in t , Lipschitz continuous with respect to the second variable, i.e, for any $(v, w) \in \mathbf{L}^1(\Omega) \times \mathbf{L}^1(\Omega)$, and almost every $t \in [0, T]$, we have

$$\|\mathcal{B}(t, 0)\|_{W^{1,\infty}(\Omega) \cap W^{2,1}(\Omega)} \leq C_T, \quad (2.5a)$$

$$\|\mathcal{B}(t, v) - \mathcal{B}(t, w)\|_{W^{1,\infty}(\Omega) \cap W^{2,1}(\Omega)} \leq C_T \|v - w\|_{L^1(\Omega)}. \quad (2.5b)$$

(A-2) The fluxes $\sigma^i : [0, \infty) \times \partial\Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable, locally bounded functions and satisfy

$$\begin{aligned} \forall M > 0, \quad \exists K_M > 0 : \\ \forall (t, x) \in [0, M] \times \partial\Omega, \quad \forall (v, \psi) \in \mathbb{R} \times [-M, M], \quad \forall (\bar{v}, \bar{\psi}) \in \mathbb{R} \times [-M, M] : \\ |\sigma^i(t, x, v, \psi) - \sigma^i(t, x, \bar{v}, \bar{\psi})| \leq K_M (|v - \bar{v}| + |\psi - \bar{\psi}|). \end{aligned} \quad (2.6)$$

(A-3) The fluxes σ^i are boundedly non dissipative (at height k^i) in the following sense:

$$\begin{aligned} \exists \Lambda_T > 0, \quad \forall i \in \{1, \dots, n\}, \quad \exists k^i > 0 : \\ \forall (t, x, v, \psi) \in [0, T] \times \partial\Omega \times \mathbb{R} \times \mathbb{R} : \sigma^i(t, x, v, \psi) \chi^+(v - k^i) \leq \Lambda_T, \end{aligned} \quad (2.7a)$$

$$\forall (t, x, v, \psi) \in [0, T] \times \partial\Omega \times \mathbb{R} \times \mathbb{R} : \sigma^i(t, x, v, \psi) \chi^-(v) \leq 0, \quad (2.7b)$$

where $\chi^+ : \mathbb{R} \rightarrow \mathbb{R}$ and $\chi^- : \mathbb{R} \rightarrow \mathbb{R}$ are defined by

$$\chi^+(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \quad \text{and} \quad \chi^-(x) \stackrel{\text{def}}{=} -\chi^+(-x).$$

Let us make a few comments about these assumptions. Notice first that we could replace the assumption (A-1) on the operator \mathcal{B} by the following lemma, which is practically all what we need in the sequel. In this lemma, and throughout this paper, $\|\mathcal{B}\|$ denotes the (best) constant C_T in (2.5a) and (2.5b).

Lemma 2.1. *Let $T > 0$. Assume that (A-1) holds. Let u and \bar{u} belongs to $L^2(0, T; \mathbf{H}^1(\Omega))$. Set $\mathcal{V}(t) \stackrel{\text{def}}{=} \mathcal{B}(t, u(t))$ and $\bar{\mathcal{V}}(t) \stackrel{\text{def}}{=} \mathcal{B}(t, \bar{u}(t))$ for almost every $t \in (0, T)$. Let $s \geq 1$. Then, for a.e. $t \in [0, T]$*

$$\|\mathcal{V}(t)\|_{W^{1,\infty}(\Omega)} + \|\mathcal{V}(t)\|_{W^{2,1}(\Omega)} \leq C\|u(t)\|_{L^s(\Omega)} + C, \quad (2.8a)$$

$$\|(\mathcal{V} - \bar{\mathcal{V}})(t)\|_{W^{1,\infty}(\Omega) \cap W^{2,1}(\Omega)} \leq C\|(u - \bar{u})(t)\|_{L^s(\Omega)}, \quad (2.8b)$$

with $C \stackrel{\text{def}}{=} C(T, \|\mathcal{B}\|)$.

Lemma 2.1 will be mostly used with $s = 2$, but the case $s = 1$ will be required when proving a uniform L^∞ bound on a family of potential function $\{\mathcal{V}_p\}_{p \in \mathbb{N}^*}$.

Note also that in assumption (A-2) we solely demand the local Lipschitz continuity with respect to the ψ variable, in contrast with the global Lipschitz continuity with respect to the v variable. This stems from the fact that in the sequel, the functions u^i may not be bounded while we will always have $\mathcal{V}|_{\partial\Omega} \in L^\infty(0, T, L^\infty(\partial\Omega))$, due to the regularizing effect of \mathcal{B} (see (A-1)).

In assumption (A-3), we have written the bounded non-dissipative conditions at the height k^i . This will provide suitable a priori estimates on u^i since for lower values of u^i on $\partial\Omega$, we quite directly derive the upper bound $u^i \leq k^i$ in Ω . This upper bound will be completed by the usual lower bound $u^i \geq 0$.

2.3 Global existence: the simplified case

The aim of this subsection consists in showing a global well-posedness for problem (\mathcal{P}_T) under the simplified set of assumptions (A-1)–(A-3) (see Corollary 2.6).

The following local existence theorem can be proved by using a linear existence theorem (see [22] chap.3, and p.268), trace lemmas and the Picard fixed point theorem. It's proof is omitted, since in the simplified case (i.e under assumption (A-2)), trace terms are easy to handle.

Theorem 2.2. *Assume that (A-1) and (A-2) hold. Assume that $u_0 \in L^2(\Omega)$, and let $T > 0$ be fixed. Then for $t_0 \in]0, T]$ small enough, the problem (\mathcal{P}_{t_0}) admits exactly one solution. Moreover, the time existence is a function of $\|u_0\|_{L^2(\Omega)}$ only.*

We now proceed with the proof of global existence. Let $g \in C^\infty(\mathbb{R})$ be an increasing function such that

$$g(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 1. \end{cases}$$

For any $x \in \mathbb{R}$, let us define $\chi_\varepsilon^+(x) \stackrel{\text{def}}{=} g(x/\varepsilon)$, $\chi_\varepsilon^-(x) \stackrel{\text{def}}{=} -\chi_\varepsilon^+(-x)$ and $(x)_\varepsilon^\pm \stackrel{\text{def}}{=} \int_0^x \chi_\varepsilon^\pm(s) ds$. We denote by $(\cdot)^+$ and $(\cdot)^-$ the positive and negative part functions. The following simple lemma collects some useful properties of these functions.

Lemma 2.3. (i) *Let $U \subset \mathbb{R}^m$ ($m \in \mathbb{N}^*$). For any $\varepsilon > 0$*

$$0 \leq \chi_\varepsilon^\pm \chi_\varepsilon^\pm \leq 1 \quad \text{and} \quad \chi_\varepsilon^\pm \chi_\varepsilon^{\pm 2} = \chi_\varepsilon^\pm, \quad (2.9a)$$

$$\forall (f, w) \in L^1(U) \times L^1(U) : \chi_\varepsilon^\pm(w) f \xrightarrow{\varepsilon \rightarrow 0} \chi^\pm(w) f \quad \text{in } L^1(U). \quad (2.9b)$$

(ii) Let $T > 0$, $z \in \mathbb{R}$, $j \in \{1, \dots, n\}$, $\varepsilon > 0$, $\phi \in L^2(0, T; H^1(\Omega))$, $h \in L^\infty((0, T) \times \Omega)$. Then

$$\int_{Q_T} (\phi - z) h \partial_j (\chi_\varepsilon^\pm (\phi - z)) dx d\tau \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (2.10)$$

Proof. We only prove (ii) for χ_ε^+ . Let us introduce the following notation: $\mathcal{I}_\varepsilon \stackrel{\text{def}}{=} |\int_{Q_T} (\phi - z) h \partial_j (\chi_\varepsilon^+ (\phi - z)) dx d\tau|$. Since the support of $(\chi_\varepsilon^+)'$ is included in $[0, \varepsilon]$, and since $|(\chi_\varepsilon^+)'| \leq \frac{C}{\varepsilon}$ we readily obtain

$$\mathcal{I}_\varepsilon \leq \int_{[0, T] \times \Omega} \mathbf{1}_{0 < \phi - z \leq \varepsilon} |\varepsilon (C/\varepsilon) h \partial_j (\phi - z)| dx d\tau$$

where $\mathbf{1}_{0 < \phi - z \leq \varepsilon}$ denotes the indicator function of the set $0 < \phi - z \leq \varepsilon$. By dominated convergence, this last integral tends to zero with ε . In fact, $|h \partial_j (\phi - z) \mathbf{1}_{0 < \phi - z \leq \varepsilon}| \leq |h \partial_j (\phi - z)| \in L^1([0, T] \times \Omega)$ and $h \partial_j (\phi - z) \mathbf{1}_{0 < \phi - z \leq \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0$ a.e. \square

We now prove some global in time L^1 and L^2 estimates for the solutions of the problem (P_T) . In the following statement, our main assumptions are conditions (A-1) and (A-3). Since the composition operators have to be well defined, we also assume that the assumption (A-2) also holds true. Nevertheless, notice that the estimates of Lemma 2.4 do not depend on the constants K_M of continuity of the functions σ^i , a fact that will be used in the next section. In the sequel, for $f = (f^1, \dots, f^N) \in \mathbf{L}^1(U)$, we write $\|f\|_{\mathbf{L}^1(U)} = \sum_{k=1}^N \|f^k\|_{L^1(U)}$.

Lemma 2.4. *Assume that (A-1)–(A-3) hold, and assume that $u_0 \in \mathbf{L}^2(\Omega)$. Let $T > 0$ be given, and let u be any solution to problem (P_T) .*

(i) *Then, for any $t \in [0, T]$, we have*

$$\|(u^i)^-(t)\|_{L^1(\Omega)} \leq \|(u_0^i)^-(t)\|_{L^1(\Omega)} \quad (2.11a)$$

$$\begin{aligned} \|(u^i - k^i)^+(t)\|_{L^1(\Omega)} &\leq \|(u_0^i - k^i)^+\|_{L^1(\Omega)} - \int_{\Gamma_t} k^i \alpha^i \chi^+(u^i - k^i) \nabla \mathcal{V} \cdot \zeta d\mu d\tau \\ &+ \int_{Q_t} k^i \alpha^i \chi^+(u^i - k^i) \Delta \mathcal{V} dx d\tau + \mu(\partial\Omega) \Lambda_T t. \end{aligned} \quad (2.11b)$$

(ii) *Assume moreover that $u^i \geq 0$, $i \in \{1, \dots, n\}$. Then, we have*

$$\begin{aligned} \frac{1}{2} \|u^i(t)\|_{L^2(\Omega)}^2 + \|\nabla u^i\|_{L^2(Q_t)}^2 &\leq \frac{1}{2} \|u_0^i\|_{L^2(\Omega)}^2 - \alpha^i \int_{Q_t} u^i \nabla \mathcal{V} \cdot \nabla u^i dx d\tau \\ &+ (\Lambda_T + \sup_{A_i(T, \|\mathcal{V}\|_{L^\infty(0, T, L^\infty(\partial\Omega))})} |\sigma^i|) \int_{\Gamma_t} u^i d\mu d\tau, \end{aligned} \quad (2.12)$$

with, for any $Z \in \mathbb{R}_+$, $A_i(T, Z) \stackrel{\text{def}}{=} [0, T] \times \partial\Omega \times [0, k^i] \times [-Z, Z]$.

Proof. We prove (2.11a) and (2.11b) at the same time. In the sequel, $(\chi_\varepsilon^\pm, (\cdot)_\varepsilon^\pm, \chi^\pm, (\cdot)^\pm, z^i)$ denotes either $(\chi_\varepsilon^+, (\cdot)_\varepsilon^+, \chi^+, (\cdot)^+, k^i)$ or $(\chi_\varepsilon^-, (\cdot)_\varepsilon^-, \chi^-, (\cdot)^-, 0)$. Since $\chi_\varepsilon^\pm(u^i - z^i) \in L^2(0, T; H^1(\Omega))$, (P_T) provides for any $t \in [0, T]$

$$\begin{aligned} &\int_0^t \left\langle \frac{du^i}{dt}(\tau), \chi_\varepsilon^\pm(u^i - z^i)(\tau) \right\rangle d\tau \\ &= - \int_{Q_t} (\nabla(u^i - z^i) \cdot \nabla(\chi_\varepsilon^\pm(u^i - z^i)) + \alpha^i u^i \nabla \mathcal{V} \cdot \nabla(\chi_\varepsilon^\pm(u^i - z^i))) dx d\tau \\ &+ \int_{\Gamma_t} \sigma^i(\tau, x, u^i(\tau, x), \mathcal{V}(\tau, x)) \chi_\varepsilon^\pm(u^i(\tau, x) - z^i) d\mu d\tau. \end{aligned} \quad (2.13)$$

We estimate the various terms appearing in the equality (2.13). Note first that

$$- \int_{\mathcal{Q}_t} \nabla(u^i - z^i) \cdot \nabla(\chi_\varepsilon^\pm(u^i - z^i)) dx d\tau \leq 0, \quad (2.14)$$

due to $(\chi_\varepsilon^\pm)' \geq 0$. Next, since $u^i \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, we have

$$\begin{aligned} & - \int_{\mathcal{Q}_t} u^i \nabla \mathcal{V} \cdot \nabla(\chi_\varepsilon^\pm(u^i - z^i)) dx d\tau = - \int_{\mathcal{Q}_t} (u^i - z^i) \nabla \mathcal{V} \cdot \nabla(\chi_\varepsilon^\pm(u^i - z^i)) dx d\tau \\ & + z^i \int_{\mathcal{Q}_t} \Delta \mathcal{V} \chi_\varepsilon^\pm(u^i - z^i) dx d\tau - z^i \int_{\Gamma_t} \chi_\varepsilon^\pm(u^i - z^i) \nabla \mathcal{V} \cdot \zeta d\mu d\tau \\ & \xrightarrow{\varepsilon \rightarrow 0} z^i \int_{\mathcal{Q}_t} \Delta \mathcal{V} \chi^\pm(u^i - z^i) dx d\tau - z^i \int_{\Gamma_t} \chi^\pm(u^i - z^i) \nabla \mathcal{V} \cdot \zeta d\mu d\tau, \end{aligned} \quad (2.15)$$

due to Lemma 2.3, (2.9b) and (2.10), Lemma 2.1 and a trace lemma. For the boundary term, using Lemma 2.3, (2.9a) and assumption (A-3), we get

$$\begin{aligned} & \int_{\Gamma_t} \sigma^i(\tau, x, u^i(\tau, x), \mathcal{V}(\tau, x)) \chi_\varepsilon^\pm(u^i(\tau, x) - z^i) d\mu d\tau \\ & = \int_{\Gamma_t} \sigma^i(\tau, x, u^i(\tau, x), \mathcal{V}(\tau, x)) \chi^\pm(u^i(\tau, x) - z^i) (\chi^\pm \chi_\varepsilon^\pm)(u^i(\tau, x) - z^i) d\mu d\tau \\ & \leq \int_{\Gamma_t} A_T d\mu d\tau = \mu(\partial\Omega) t A_T, \end{aligned} \quad (2.16)$$

with

$$A_T \stackrel{\text{def}}{=} 0 \quad \text{for } z^i = 0 \quad \text{and} \quad A_T \stackrel{\text{def}}{=} \Lambda_T \quad \text{for } z^i = k^i. \quad (2.17)$$

Last, we observe that for any $w \in C^\infty([0, T]; H^1(\Omega))$, we have

$$\begin{aligned} & \int_0^t \left\langle \frac{d(w - z^i)}{dt}(\tau), \chi_\varepsilon^\pm(w(\tau) - z^i) \right\rangle d\tau = \int_{\mathcal{Q}_t} \frac{d}{dt} (w - z^i)_\varepsilon^\pm dx d\tau \\ & = \int_\Omega ((w(t) - z^i)_\varepsilon^\pm - (w(0, \cdot) - z^i)_\varepsilon^\pm) dx. \end{aligned} \quad (2.18)$$

Let $E(u^i) \in L^2(\mathbb{R}; H^1(\Omega)) \cap H^1(\mathbb{R}; H^1(\Omega)')$ be an extension of u^i . Denote by $\theta \in \mathcal{D}(\mathbb{R})$ a probability density, and for any $\eta > 0, t \in \mathbb{R}$, write $\theta_\eta(t) \stackrel{\text{def}}{=} \eta^{-1} \theta(\eta^{-1} t)$. Set in equality (2.18) $w \stackrel{\text{def}}{=} (E(u^i)) \star \theta_\eta$, where \star denotes the convolution with respect to the time variable. Letting η tends to 0, we see that (2.18) holds true with $w = u^i$. Furthermore, Lemma 2.3 implies that

$$\int_0^t \left\langle \frac{d(u^i - z^i)}{dt}(\tau), \chi_\varepsilon^\pm(u^i(\tau) - z^i) \right\rangle d\tau \xrightarrow{\varepsilon \rightarrow 0} \int_\Omega ((u^i(t) - z^i)^\pm - (u_0^i - z^i)^\pm) dx, \quad (2.19)$$

for all $i \in \{1, \dots, n\}$. It follows from (2.13)–(2.16), and (2.19) that

$$\begin{aligned} & \int_\Omega (u^i - z^i)^\pm dx - \int_\Omega (u_0^i - z^i)^\pm dx \leq z^i \alpha^i \int_{\mathcal{Q}_T} \Delta \mathcal{V} \chi^\pm(u^i - z^i) dx d\tau \\ & - z^i \alpha^i \int_{\mathcal{Q}_T} \chi^\pm(u^i - z^i) \nabla \mathcal{V} \cdot \zeta dx d\tau + \mu(\partial\Omega) t A_T. \end{aligned} \quad (2.20)$$

Now, (2.11a) and (2.11b) follow from (2.17) and (2.20). Finally, (2.12) follows from (\mathcal{P}_T) with $w = (0, \dots, 0, u^i, 0, \dots, 0)$ and the estimate

$$\int_{\Gamma_t} \sigma^i(\tau, x, u^i(\tau, x), \mathcal{V}(\tau, x)) u^i d\mu d\tau \leq (\Lambda_T + \sup_{A_i(T, \|\mathcal{V}\|_{L^\infty(0, T, L^\infty(\partial\Omega))})} |\sigma^i|) \int_{\Gamma_t} u^i d\mu d\tau$$

is a consequence of condition (A-3), $u^i \geq 0$ and the definition of $A_i(T, \mathcal{V})$. \square

Let $t \in [0, T]$. We now prove our main L^2 estimates, which hold in the functional spaces $E_t \stackrel{\text{def}}{=} L^2(0, t; \mathbf{H}^1(\Omega)) \cap L^\infty([0, t]; L^2(\Omega))$. For any $v \stackrel{\text{def}}{=} (v^1, \dots, v^n) \in E_t$, set

$$\|v\|_{E_t}^2 \stackrel{\text{def}}{=} \sum_{i=1}^n (\|v^i\|_{L^\infty(0,t;L^2(\Omega))}^2 + \|\nabla v^i\|_{L^2(Q_t)}^2). \quad (2.21)$$

Observe that until the end of the paper, $|\cdot|_1$ denotes the ℓ^1 norm in \mathbb{R}^n .

Before proceeding, notice the following inequalities, valid for $x \geq 0$ and $z \in \mathbb{R}$: $(x-z)^+ - |z| \leq x \leq (x-z)^+ + z$. As a consequence, for any $p \in [1, \infty)$, $z \in \mathbb{R}$, $v \in L^p(U)$ with $v \geq 0$ (and U a bounded domain)

$$\|(v-z)^+\|_{L^p(U)} - |z|U^{1/p} \leq \|v\|_{L^p(U)} \leq \|(v-z)^+\|_{L^p(U)} + |z|U^{1/p}. \quad (2.22)$$

Together with Lemma 2.4, this provide lemma 2.5 below. In the statement of this lemma, we keep track of the dependences with respect to the constitutive constants appearing in conditions (A-1)–(A-3), since this will turn out to be useful in the next section. Nevertheless, we drop in our writings the extraneous dependences such those with respect to Ω .

Lemma 2.5. *Under the assumptions of Lemma 2.4, and assuming that $u_0^i \geq 0$ a.e for any $i \in \{1, \dots, n\}$, we have*

$$u^i(t) \geq 0 \text{ for any } t \in [0, T], \text{ } x \text{ a.e,} \quad (2.23a)$$

$$\|u(t)\|_{L^1(\Omega)} \leq C_1 e^{C_2 t}, \quad (2.23b)$$

$$\|u\|_{E_t} \leq C_3 e^{C_4 t} \text{ for all } t \in [0, T], \quad (2.23c)$$

with the notations $C_1 \stackrel{\text{def}}{=} C_1(\|u_0\|_{L^1}, \Lambda_T, k^1, \dots, k^n, T, \|\mathcal{B}\|)$, $C_2 \stackrel{\text{def}}{=} C_2(\Lambda_T, k^1, \dots, k^n, T, \|\mathcal{B}\|)$, $C_3 \stackrel{\text{def}}{=} C_3(\|u_0\|_{L^1}, \|u_0\|_{L^2}, \Lambda_T^*, k^1, \dots, k^n, T, \|\mathcal{B}\|)$, $C_4 \stackrel{\text{def}}{=} C_4(\|u_0\|_{L^1}, \Lambda_T^*, k^1, \dots, k^n, T, \|\mathcal{B}\|)$, $\Lambda_T^* \stackrel{\text{def}}{=} \Lambda_T + \sum_{i=1}^n \sup_{A_i(T, C_*)} |\sigma^i|$ and $C_* \stackrel{\text{def}}{=} C_*(\|u_0\|_{L^1}, \Lambda_T, k^1, \dots, k^n, T, \|\mathcal{B}\|)$. Moreover, the constants C_i and C_* are non-decreasing functions of their arguments.

Proof. The Lemma 2.4 and assumption $u_0^i \geq 0$ a.e imply that (2.23a) holds true.

We now prove inequality (2.23b). Before proceeding, remark that (2.8) and trace lemmas entail that, for any $t \in [0, T)$, we have

$$\|\mathcal{V}(t)\|_{W^{2,1}(\Omega) \cap W^{1,\infty}(\Omega)} + \|\nabla \mathcal{V}(t)\|_{L^1(\partial\Omega)} \leq C(T, \|\mathcal{B}\|, \Omega) (\|u(t)\|_{L^1(\Omega)} + 1) \quad (2.24)$$

We derive from inequalities (2.11b) and (2.24) that, for any $t \in [0, T)$, $i \in \{1, \dots, n\}$

$$\begin{aligned} \|(u^i - k^i)^+(t)\|_{L^1(\Omega)} &\leq \|(u_0^i - k^i)^+\|_{L^1(\Omega)} \\ &+ |k^i \alpha^i| C(T, \|\mathcal{B}\|, \Omega) (\|u\|_{L^1(0,t;L^1(\Omega))} + t) + \mu(\partial\Omega) \Lambda_T t. \end{aligned} \quad (2.25)$$

Taking the sum over $i \in \{1, \dots, n\}$ and using (2.22), we get

$$\begin{aligned} \|u(t)\|_{L^1(\Omega)} &\leq \|u_0\|_{L^1(\Omega)} + |k|_1 |\alpha|_1 C(T, \|\mathcal{B}\|, \Omega) (\|u\|_{L^1(0,t;L^1(\Omega))} + t) \\ &+ n\mu(\partial\Omega) \Lambda_T t + 2n|\Omega| |k|_1. \end{aligned} \quad (2.26)$$

Appealing to Grönwall lemma, we obtain (2.23b).

We finally prove inequality (2.23c). Inequality (2.24) together with inequality (2.23b) and a trace lemma, give

$$\|\mathcal{V}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \|\mathcal{V}\|_{L^\infty(\Gamma_T)} \leq C_* \quad (2.27)$$

with $C_* \stackrel{\text{def}}{=} C_*(\|u_0\|_{L^1(\Omega)}, \Lambda_T, k^1, \dots, k^n, T, \|\mathcal{B}\|, \Omega)$. Recall that for any $w \in H^1(\Omega)$, we have

$$\|w\|_{L^2(\partial\Omega)} \leq C\|w\|_{H^{3/2}(\Omega)} \leq \varepsilon\|w\|_{H^1(\Omega)} + C_\varepsilon\|w\|_{L^2(\Omega)} \quad (2.28)$$

for any $\varepsilon > 0$, so that, integrating (2.28) and using Young inequality, we get

$$\|u^i\|_{L^1(0,t,L^2(\partial\Omega))} \leq t + C_\eta\|u^i\|_{L^2(\mathcal{Q}_t)}^2 + \eta\|\nabla u^i\|_{L^2(0,t,L^2(\Omega))}^2 \quad (2.29)$$

for any $\eta > 0$. Hence, identity (2.29), (2.27), (2.12) and Young inequalities imply that

$$\begin{aligned} & \frac{1}{2}\|u^i(t)\|_{L^2(\Omega)}^2 + \|\nabla u^i\|_{L^2(\mathcal{Q}_t)}^2 \\ & \leq \frac{1}{2}\|u_0^i\|_{L^2(\Omega)}^2 + C_*|\alpha^i|(C_\eta\|u^i\|_{L^2(\mathcal{Q}_t)}^2 + \eta\|\nabla u^i\|_{L^2(0,t,L^2(\Omega))}^2) \\ & \quad + (\Lambda_T + \sup_{A_i(T,C_*)}|\sigma^i|)(t + C_\eta\|u^i\|_{L^2(\mathcal{Q}_t)}^2 + \eta\|\nabla u^i\|_{L^2(0,t,L^2(\Omega))}^2). \end{aligned} \quad (2.30)$$

Now, inequality (2.23c) follows by choosing $\eta > 0$ small enough in (2.30) and Grönwall lemma. \square

Corollary 2.6. *Let $T > 0$ be fixed. Assume that (A–1)–(A–3) hold true. Assume that $u_0 \in L^2(\Omega)$ and $u_0^i \geq 0$ for any $i \in \{1, \dots, n\}$. Then, the problem (\mathcal{P}_T) admits exactly one solution. Moreover, $u(t) \geq 0$ for any $t \in [0, T]$.*

Proof. As quoted in Theorem 2.2, the time existence t_0 is a function of $\|u_0\|_{L^2(\Omega)}$ only. Due to Lemma 2.5, inequality (2.23c), global well-posedness follows. \square

3 Trace integrals inequalities.

Our goal is to prove that Corollary 2.6 holds true under a relaxed assumption (A–2). This shall be done in Section 4 below. Since we argue by density, we first have to determine the relevant estimates for the trace terms. Our trace integral estimates (cf. Lemma 3.2) are consequences of a simple continuity lemma (see Lemma 3.1 below). Since in the sequel we loose an arbitrary small order of derivation by the use of the Aubin-Lions lemma, we introduce a somewhat larger space than E_T . For $t \in (0, T)$ and $\alpha \geq 0$, define

$$E_t^\alpha \stackrel{\text{def}}{=} L^\infty(0, t, \mathbf{H}^{-\alpha}(\Omega)) \cap L^2(0, t, \mathbf{H}^{1-\alpha}(\Omega)) \quad \text{and} \quad \dot{E}_t \stackrel{\text{def}}{=} \bigcap_{0 < \alpha \leq 1} E_t^\alpha. \quad (3.1)$$

The space \dot{E}_t is endowed with its natural Fréchet structure. In particular, $f_n \xrightarrow[n \rightarrow \infty]{} f$ in \dot{E}_t iff $f_n \xrightarrow[n \rightarrow \infty]{} f$ in all the E_t^α , $\alpha \in (0, 1)$. By interpolation, for any $s \in [2, \infty)$ and $r \in (0, 1)$, we have

$$L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) = E_T^0 \hookrightarrow \dot{E}_T \hookrightarrow L^s(0, T; L^2(\Omega)) \cap L^2(0, T; H^r(\Omega)). \quad (3.2)$$

Until this end of the paper, we always abridge the notation E_T^0 in E_T (see (2.21)) and still denote by $\mathcal{C}_{T,R}$ the closed ball with radius $R > 0$ in E_T .

Lemma 3.1. *Let $T > 0$. We assume that $1 \leq p < 2 + \frac{2}{d}$ and $m \in [1, \infty[$. Then, there exists $\alpha = \alpha_{p,m} \in (0, 1)$ and $C = C_{p,m} > 0$ such that for any $(v, \bar{v}) \in \dot{E}_T \times \dot{E}_T$, we have*

$$\|v\|_{L^p(\Gamma_T)} \leq C\|v\|_{E_T^\alpha}, \quad (3.3a)$$

$$\|\mathcal{V} - \bar{\mathcal{V}}\|_{L^m(\Gamma_T)} \leq C\|v - \bar{v}\|_{E_T^\alpha}, \quad (3.3b)$$

with $\mathcal{V}(t) \stackrel{\text{def}}{=} \mathcal{B}(t, v(t))$ and $\bar{\mathcal{V}}(t) \stackrel{\text{def}}{=} \mathcal{B}(t, \bar{v}(t))$ for almost every $t \in [0, T]$. If moreover, we assume that $v \in \mathcal{C}_{T,R}$ ($R > 0$) then

$$\|\mathcal{V}\|_{L^m(\Gamma_T)} \leq C_{R,T}. \quad (3.4)$$

Proof. According to Hölder's inequality, it's enough to prove (3.3a) for $p \in]2, 2 + \frac{2}{d}[$. We first assume that $p \in]2, \infty)$. Let $s \in [1, \infty)$, $\theta \in]0, 1[$, $r \in]0, 1[$ and $q \in]0, 1[$ such that

$$[L^s(0, T; L^2(\Omega)), L^2(0, T; H^r(\Omega))]_\theta = L^p(0, T; H^q(\Omega))$$

which is

$$\frac{1}{p} = \frac{1-\theta}{s} + \frac{\theta}{2}, \quad (3.5a)$$

$$q = \theta r, \quad (3.5b)$$

where $[\cdot, \cdot]_\theta$ denotes the holomorphic interpolation functor (see [2, p. 107]). Note that the existence of such s, θ, r, q is granted by the condition $p \in]2, \infty)$. Now, for any $v \in \dot{E}_T$, the interpolation and the Young inequalities give

$$\begin{aligned} \|v\|_{L^p(0,T;H^q(\Omega))} &\leq C \|v\|_{L^s(0,T;L^2(\Omega))}^{1-\theta} \|v\|_{L^2(0,T;H^r(\Omega))}^\theta \\ &\leq C (\|v\|_{L^s(0,T;L^2(\Omega))} + \|v\|_{L^2(0,T;H^r(\Omega))}) \\ &\leq C \|v\|_{E_T^\alpha} \end{aligned} \quad (3.6)$$

for some $\alpha = \alpha_{s,r} \in (0, 1)$ (see (3.2)). We now turn to determine the best exponents p and q in (3.6). Notice that a limiting value for θ in (3.5a) is $\theta = \frac{2}{p}$. Hence $q = \frac{2}{p}$ is the corresponding limiting value for q in (3.5b). Therefore, we can take $q = \frac{p}{2} - \eta$, with $\eta > 0$ arbitrary small. Finally, we have

$$v \in L^p(0, T, H^{2/p-\eta}(\Omega)). \quad (3.7)$$

We now restrict to $1 \leq p < 2 + \frac{2}{d}$ and write $p^{-1} = \frac{d}{2(d+1)} + \delta$ with $\delta > 0$. Choose $\eta = \frac{(1+d)\delta}{2}$. With these notations, we easily compute

$$\frac{2}{p} - \eta - d\left(\frac{1}{2} - \frac{1}{p}\right) = \frac{1}{p} + \eta. \quad (3.8)$$

It follows from (3.8), Sobolev injections and trace lemmas that

$$L^p(0, T, H^{2/p-\eta}(\Omega)) \hookrightarrow L^p(0, T, W^{1/p+\eta,p}(\Omega)) \rightarrow L^p(0, T, L^p(\partial\Omega)),$$

where the last arrow is also continuous. Together with (3.7) and (3.6), it proves (3.3a).

We now prove (3.3b). According to (2.8b) and (3.2), we see that

$$\|\mathcal{V} - \bar{\mathcal{V}}\|_{L^m(0,T;W^{1,\infty}(\Omega))} \leq C_T \|v - \bar{v}\|_{L^m(0,T;L^2(\Omega))} \leq C \|v - \bar{v}\|_{E_T^\alpha} \quad (3.9)$$

for some $\alpha = \alpha_m$. Now, inequality (3.3b) follows from (3.9) and a trace lemma. The proof of inequality (3.4) is omitted. \square

In the sequel, we denote $\mathcal{H} \stackrel{\text{def}}{=} 2 + \frac{2}{d}$. The following technical lemma will be used in the proof of the general existence theorem. It has essentially the same meaning as Lemma 3.1, i.e. boundary integrals of $|u^i|^p$ can be bounded by (functions of) $\|u\|_{E_T}$ for $0 \leq p < 2 + \frac{2}{d}$.

Lemma 3.2. *Assume that condition (A–1) is satisfied. Assume that $a \geq 0$, $b \geq 0$, $1 \leq a+b < \mathcal{H}$ and $\theta \geq 0$. Let $R > 0$. There exists two constants $C = C_{T,R,\|\mathcal{B}\|,a,b,\theta} > 0$ and $\alpha = \alpha_{\theta,b} \in (0,1)$ such that, for any $u = (u^1, \dots, u^n) \in \mathcal{C}_{T,R}$ and $\bar{u} = (\bar{u}^1, \dots, \bar{u}^n) \in \mathcal{C}_{T,R}$, we have*

$$\int_{\Gamma_t} (1 + |u^i|^a) |u^j - \bar{u}^j|^b |\mathcal{V} - \bar{\mathcal{V}}|^\theta d\mu d\tau \leq C \|u - \bar{u}\|_{E_T^\alpha}^{b+\theta}, \quad (3.10a)$$

$$\int_{\Gamma_t} (1 + |u^i|^a) |u^j|^b |\mathcal{V}|^\theta d\mu d\tau \leq C, \quad (3.10b)$$

for any $(i, j) \in \{1, \dots, n\}^2$ and $t \in (0, T)$. As usual, we have written $\mathcal{V}(t) = \mathcal{B}(t, u(t))$ and $\bar{\mathcal{V}}(t) = \mathcal{B}(t, \bar{u}(t))$ for a.e. $t \in]0, T]$.

Proof. In the following, we denote with a prime a conjugate exponent. It is enough to prove inequality (3.10) for $t = T$ and to estimate the integral

$$\mathcal{I}(a, b, \theta) = \int_{\Gamma_T} |u^i|^a |u^j - \bar{u}^j|^b |\mathcal{V} - \bar{\mathcal{V}}|^\theta d\mu d\tau.$$

We restrict to the case $a > 0$ and $b > 0$, since the cases $a = 0$ or $b = 0$ are easier. Let $\varepsilon > 0$ such that $1 \leq (1 + \varepsilon)(a + b) < \mathcal{H}$. The Hölder inequality leads to

$$\mathcal{I}(a, b, \theta) \leq \|\mathcal{V} - \bar{\mathcal{V}}\|_{L^{(1+\varepsilon)'\theta}(\Gamma_T)}^\theta \mathcal{I}((1 + \varepsilon)a, (1 + \varepsilon)b, 0)^{\frac{1}{1+\varepsilon}}.$$

Hence, by using (3.3b), we find

$$\mathcal{I}(a, b, \theta) \leq C \|u - \bar{u}\|_{E_T^\alpha}^\theta \mathcal{I}((1 + \varepsilon)a, (1 + \varepsilon)b, 0)^{\frac{1}{1+\varepsilon}}. \quad (3.11)$$

Therefore, setting $a_1 \stackrel{\text{def}}{=} (1 + \varepsilon)a > 0$ and $b_1 \stackrel{\text{def}}{=} (1 + \varepsilon)b > 0$, and recalling that $1 \leq a_1 + b_1 < \mathcal{H}$, we just have to prove (3.10a) for $\mathcal{I}(a_1, b_1, 0)$, or simply $\mathcal{I}(a, b, 0)$ which we now estimate. Since $a > 0$, $b > 0$ and $1 \leq a + b < \mathcal{H}$, we have $\frac{a\mathcal{H}}{\mathcal{H}-b} < \mathcal{H}$. Let $\gamma \in]\max(1, \frac{a\mathcal{H}}{\mathcal{H}-b}), \mathcal{H}[$. For $u \in \mathcal{C}_{T,R}$, inequality (3.3a) and (3.2) ensure that $u \in L^\gamma((0, T) \times \partial\Omega)$ with

$$\|u^j\|_{L^\gamma(\Gamma_T)} \leq C_{R,T}. \quad (3.12)$$

Set $q = \frac{\gamma}{a} > \frac{\mathcal{H}}{\mathcal{H}-b} > 1$. We have

$$q' < \left(\frac{\mathcal{H}}{\mathcal{H}-b}\right)' = \frac{\mathcal{H}}{b}. \quad (3.13)$$

By Hölder inequality

$$\mathcal{I}(a, b, 0) \leq \|u^i\|_{L^\gamma(\Gamma_T)}^a \|u^j - \bar{u}^j\|_{L^{bq'}(\Gamma_T)}^b, \quad (3.14)$$

with a slight abuse of notation in the case $0 < bq' < 1$. Appealing to (3.13) and (3.3a) (and Hölder inequality in the case $0 < bq' < 1$), we find that

$$\|u^j - \bar{u}^j\|_{L^{bq'}(\Gamma_T)}^b \leq C \|u - \bar{u}\|_{E_T^\alpha}^b. \quad (3.15)$$

The estimate on $\mathcal{I}(a, b, 0)$ follows from (3.12), (3.14) and (3.15). The proof of (3.10b) is similar. \square

Our new assumptions are motivated by Lemma 3.2. Assumptions (A–1) and (A–3) are not modified, while assumption (A–2) becomes

(A-4) The fluxes $\sigma^i : [0, \infty) \times \partial\Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable, locally bounded functions. Moreover, there exists $\rho \in [0, 1 + \frac{2}{d})$ such that

$$\begin{aligned} & \forall M > 0, \quad \exists K_M > 0 : \\ & \forall (t, x) \in [0, M] \times \partial\Omega, \quad \forall (v, \psi) \in \mathbb{R} \times [-M, M], \quad \forall (\bar{v}, \bar{\psi}) \in \mathbb{R} \times [-M, M] : \\ & |\sigma^i(t, x, v, \psi) - \sigma^i(t, x, \bar{v}, \bar{\psi})| \\ & \leq K_M((1 + |v|^\rho + |\bar{v}|^\rho)|v - \bar{v}| + (1 + |v|^{\rho+1} + |\bar{v}|^{\rho+1})|\psi - \bar{\psi}|). \end{aligned} \quad (3.16)$$

As an immediate consequence of Lemma 3.2, we have

Corollary 3.3. *Let $T > 0$ and $R > 0$. Assume that the conditions (A-1) and (A-4) are satisfied. Then*

- (i) *For any $1 \leq s < \frac{\mathcal{H}}{\rho+1}$, there exist $\alpha = \alpha_{s(\rho+1)} \in (0, 1)$ such that the application $\mathcal{G} : \mathcal{C}_{T,R} \rightarrow \mathbf{L}^s(\Gamma_T)$ with $(\mathcal{G}(v))(t, x) \stackrel{\text{def}}{=} \sigma(t, x, v(t, x), \mathcal{B}(t, v(t))(x))$ is well defined and Lipschitz continuous for $\mathcal{C}_{T,R}$ endowed with the E_T^α norm.*
- (ii) *Assume that $\rho \in [0, \frac{2}{d})$. For any $u \in \mathcal{C}_{T,R}$, $\bar{u} \in \mathcal{C}_{T,R}$, $\mathcal{V}(t) \stackrel{\text{def}}{=} \mathcal{B}(t, u(t))$ and $\bar{\mathcal{V}}(t) \stackrel{\text{def}}{=} \mathcal{B}(t, \bar{u}(t))$ for a.e $t \in (0, T)$, we have*

$$\begin{aligned} & \int_{\Gamma_t} |\sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x)) - \sigma(\tau, x, \bar{u}(\tau, x), \bar{\mathcal{V}}(\tau, x))|_1 |u - \bar{u}(\tau, x)|_1 d\mu d\tau \\ & \leq C \|u - \bar{u}\|_{L^2(0,t;\mathbf{L}^2(\Omega))}^2 + \eta \|u - \bar{u}\|_{E_t}^2. \end{aligned}$$

Proof. Property (i) is a direct consequence of the property (A-4) and Lemma 3.2. Similarly, (ii) follows from the property (A-4), the inequality (3.10a) and, for $\alpha \in (0, 1)$, the inequality

$$\|\cdot\|_{E_t^\alpha}^2 \leq C_{\alpha,\eta} \|\cdot\|_{L^2(0,t;\mathbf{L}^2(\Omega))}^2 + \eta \|\cdot\|_{E_t}^2.$$

□

4 Global existence: the general case

Assume that conditions (A-1), (A-3) and (A-4) are satisfied for the fluxes σ^i . We still denote by Λ_T and k^i , $i = 1, \dots, n$, the constants appearing in the condition (A-3). In order to apply Corollary 2.6, we define a family of functions $\sigma_p^i : [0, \infty) \times \partial\Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $p \in \mathbb{N}^*$, $i \in \{1, \dots, n\}$ endowed with conditions (A-2) and (A-3). Let $h \in \mathcal{D}(\mathbb{R})$ with $h(x) = 1$ for $|x| \leq 1$, $h(x) = 0$ for $|x| \geq 2$ and $0 \leq h \leq 1$. For any $(t, x, v, \psi) \in [0, \infty) \times \partial\Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ set

$$\sigma_p^i(t, x, v, \psi) = \sigma^i(t, x, v, \psi) h\left(\frac{v}{p}\right).$$

As easily checked, the function σ_p^i satisfies the two conditions (A-2) and (A-3), with constants $\Lambda_{T,p} = \Lambda_T$ and $k_p^i = k^i$ ($i = 1, \dots, n$) independent of p . Moreover, for any $p \in \mathbb{N}^*$, we have $|\sigma_p^i| \leq |\sigma^i|$.

Let now $T > 0$, and let $u_0 \in \mathbf{L}^2(\Omega)$ with $u_0^i \geq 0$ for $i \in \{1, \dots, n\}$. For any $p \in \mathbb{N}^*$, Corollary 2.6 asserts the existence of a unique solution $u_p \in E_T$ to problem (\mathcal{P}_T) . Now, it follows from the Lemma 2.5 that the sequence $\{\|u_p\|_{E_T}\}_{p \in \mathbb{N}^*}$ is bounded. In the sequel, we denote by $R \stackrel{\text{def}}{=} \sup_{p \in \mathbb{N}^*} \|u_p\|_{E_T} < \infty$. Thus, for any $p \in \mathbb{N}^*$, we have

$$u_p \in \mathcal{C}_{T,R}, \quad (4.1)$$

and by Lemma 2.1 and a trace lemma, we get

$$\|\mathcal{V}_p\|_{L^\infty(0,T,\mathbf{W}^{1,\infty}(\Omega)\cap\mathbf{W}^{2,1}(\Omega))} + \|\mathcal{V}_p\|_{L^\infty(\Gamma_T)} \leq C_{R,T}. \quad (4.2)$$

This allows us to use all the previous results of the paper. The rest of this section is devoted to the proof of the convergence of $\{u_p\}_{p \in \mathbf{N}^*}$ towards an exact solution.

Lemma 4.1. *Under the assumptions (A-1), (A-2) and (A-4), and with the previous notations, there exists $u \in \mathcal{C}_{T,R}$ such that, extracting if necessary a subsequence*

$$u_p \xrightarrow[p \rightarrow \infty]{} u \text{ strongly in } \dot{E}_T \text{ and } C^0(0,T,\mathbf{H}^{-1/4}(\Omega)), \quad (4.3a)$$

$$u_p \xrightarrow[p \rightarrow \infty]{} u \text{ weakly in } L^2(0,T,\mathbf{H}^1(\Omega)) \text{ and weakly-} \star \text{ in } L^\infty(0,T,\mathbf{L}^2(\Omega)). \quad (4.3b)$$

Proof. Since u_p satisfies equation (2.4) in the sense of distributions, we deduce from (4.1) and (4.2) that

$$\left\{ \frac{du_p}{dt} \right\}_{p \in \mathbf{N}^*} \text{ is bounded in } L^2(0,T,\mathbf{H}^{-1}(\Omega)). \quad (4.4)$$

With (4.1), and using the Aubin-Lions lemma and a diagonal process, we extract from $\{u_p\}_{p \in \mathbf{N}^*}$ a converging (and not relabeled) subsequence in \dot{E}_T . Still by the Aubin-Lions lemma, we can also assume that $\{u_p\}_{p \in \mathbf{N}^*}$ converges strongly in $C^0(0,T,\mathbf{H}^{-1/4}(\Omega))$. Properties (4.3b) and $u \in \mathcal{C}_{T,R}$ follow from (4.1). \square

Until the end of the paper, for the sake of clarity, we sometimes go back to the notation $v|_{\partial\Omega}(t)$ for the trace of $v(t)$ on $\partial\Omega$. We now motivate and introduce a space of test functions compatible with the boundary conditions. Since in the sequel we mainly have to estimate integrals such as $\int_{\Gamma_T} |u|^{\rho+1} w d\mu d\tau$ with $|u|^{(\rho+1)} \in L^{p/(\rho+1)}(\Gamma_T)$, $0 \leq p < \mathcal{H}$, the limiting conjugate exponent for the function w is $(\mathcal{H}/(\rho+1))' = \mathcal{H}/[\mathcal{H} - (\rho+1)]$. Hence, for $T > 0$ and $0 \leq \rho < \mathcal{H} - 1$, we set

$$\mathbf{b}(T, \rho) \stackrel{\text{def}}{=} \left\{ v \in L^2(0,T,\mathbf{H}^1(\Omega)) \text{ such that there exists } r > \frac{\mathcal{H}}{\mathcal{H} - (\rho+1)} \right. \\ \left. \text{depending of } v, \text{ with } v|_{\partial\Omega} \in L^r(\Gamma_T) \right\}. \quad (4.5)$$

Notice that for $0 \leq \rho < \mathcal{H} - 2$, we have $\frac{\mathcal{H}}{\mathcal{H} - (\rho+1)} < \mathcal{H}$. Hence, by Lemma 3.1, we have $E_T \subset \mathbf{b}(T, \rho)$ for $0 \leq \rho < 2/d$. It follows that

$$\mathbf{H}^1([0,T] \times \Omega) \subset \mathbf{b}(T, \rho) \quad \text{for } 0 \leq \rho < 2/d. \quad (4.6)$$

Notice also that, for $0 \leq \rho < \frac{\mathcal{H}-2}{2}$, we have $\frac{\mathcal{H}}{\mathcal{H} - (\rho+1)} < 2$. Hence, $L^2(0,T,\mathbf{H}^1(\Omega)) \subset \mathbf{b}(T, \rho)$, and since the opposite inclusion is also true, we have

$$\mathbf{b}(T, \rho) = L^2(0,T,\mathbf{H}^1(\Omega)) \quad \text{for } 0 \leq \rho < 1/d. \quad (4.7)$$

Last, the inclusion

$$C^1(\bar{Q}_T) \subset \mathbf{b}(T, \rho) \quad \text{for } 0 \leq \rho < 1 + 2/d \quad (4.8)$$

holds true. We are ready to prove our existence theorem.

Theorem 4.2. *Let $T > 0$ be fixed. Let $0 \leq \rho < 1 + \frac{2}{d}$, and assume that (A-1), (A-3) and (A-4) hold true. Let $u_0 \in \mathbf{L}^2(\Omega)$ with $u_0^i \geq 0$, $i \in \{1, \dots, n\}$.*

(i) *The problem*

$$(\mathcal{R}_T) \quad \begin{cases} \text{Find } u \in L^2(0, T; \mathbf{H}^1(\Omega)) \cap L^\infty([0, T]; \mathbf{L}^2(\Omega)) \text{ such that for any} \\ w \in C^1(\bar{\mathcal{Q}}_T), \text{ and a.e } t \in (0, T) \\ - \int_{\mathcal{Q}_t} u \cdot \partial_t w \, dx \, d\tau + \int_{\mathcal{Q}_t} (\nabla u + Du \otimes \nabla \mathcal{V})(\tau, x) : \nabla w(\tau, x) \, dx \, d\tau \\ = \int_{\Gamma_t} \sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x)) \cdot w(\tau, x) \, d\mu \, d\tau + \int_{\Omega} u_0 \cdot w(0) \, dx - \int_{\Omega} u(t) \cdot w(t) \, dx, \end{cases}$$

(with $\mathcal{V}(\tau) = \mathcal{B}(\tau, u(\tau))$ for a.e $\tau \in (0, T)$) admits at least one solution. In the case $0 \leq \rho < \frac{2}{d}$, one can choose any $w \in H^1((0, T) \times \Omega)$ as a test function.

(ii) Assume that $0 \leq \rho < \frac{1}{d}$. Then, the problem (\mathcal{P}_T) admits exactly one solution.

Proof. We still denote by u and u_p the functions of Lemma 4.1. As usual, we write $\mathcal{V}(t) = \mathcal{B}(t, u(t))$ and $\mathcal{V}_p(t) = \mathcal{B}(t, u_p(t))$ for a.e $t \in (0, T)$. We must prove that u is a solution of problem \mathcal{R}_T or \mathcal{P}_T .

(i) **Existence in the case $0 \leq \rho < 1 + \frac{2}{d}$.**

Let $w \in \mathbf{b}(T, \rho)$. Since the sequence $\{u_p\}_{p \in \mathbf{N}^*}$ converges in \dot{E}_T (see Lemma 4.1), we deduce from (3.2) its convergence in $L^4(0, T, L^2(\Omega))$. Therefore, by Lemma 2.1, we also have $\nabla \mathcal{V}_p \xrightarrow[p \rightarrow \infty]{} \nabla \mathcal{V}$ in $L^4(0, T, L^\infty(\Omega))$. Last, recall that $\{u_p\}_{p \in \mathbf{N}^*}$ converges weakly in $L^2(0, T, \mathbf{H}^1(\Omega))$. As a consequence of these convergences

$$\int_{\mathcal{Q}_T} (\nabla u_p + Du_p \otimes \nabla \mathcal{V}_p) : \nabla w \, dx \, d\tau \xrightarrow[p \rightarrow \infty]{} \int_{\mathcal{Q}_T} (\nabla u + Du \otimes \nabla \mathcal{V}) : \nabla w \, dx \, d\tau. \quad (4.9)$$

We now prove that

$$D_p \stackrel{\text{def}}{=} \int_{\Gamma_T} (\sigma_p(\tau, x, u_p(\tau, x), \mathcal{V}_p(\tau, x)) - \sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x))) \cdot w(x) \, d\mu \, d\tau \xrightarrow[p \rightarrow \infty]{} 0. \quad (4.10)$$

Since $w \in \mathbf{b}(T, \rho)$, we have $w|_{\partial\Omega} \in L^r(\Gamma_T)$ for some $r > \frac{\mathcal{H}}{\mathcal{H} - (\rho + 1)}$. Hence, it is enough to show that

$$J_p \stackrel{\text{def}}{=} \int_{\Gamma_T} |\sigma_p(\tau, x, u_p(\tau, x), \mathcal{V}_p(\tau, x)) - \sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x))|_1^s \, d\mu \, d\tau \xrightarrow[p \rightarrow \infty]{} 0, \quad (4.11)$$

for $s = r' < \left(\frac{\mathcal{H}}{\mathcal{H} - (\rho + 1)}\right)' = \frac{\mathcal{H}}{\rho + 1}$. Using $|h| \leq 1$, we see that

$$J_p \leq J_{p,1} + J_{p,2}, \quad (4.12)$$

where

$$J_{p,1} \stackrel{\text{def}}{=} C_s \int_{\Gamma_T} |\sigma(\tau, x, u_p(\tau, x), \mathcal{V}_p(\tau, x)) - \sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x))|_1^s \, d\mu \, d\tau, \quad (4.13a)$$

$$J_{p,2} \stackrel{\text{def}}{=} C_s \int_{\Gamma_T} |1 - h(u_p(\tau, x)/p)|^s |\sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x))|_1^s \, d\mu \, d\tau. \quad (4.13b)$$

Notice that

$$J_{p,1} \xrightarrow[p \rightarrow \infty]{} 0, \quad (4.14)$$

due to $1 \leq s < \frac{\mathcal{H}}{\rho+1}$, Corollary 3.3(i), and the convergence of $\{u_p\}_{p \in \mathbb{N}^*}$ in \dot{E}_T . Next, since $1 \leq s < \mathcal{H}$, invoking again the convergence of $\{u_p\}_{p \in \mathbb{N}^*}$ in \dot{E}_T and (3.3a), we obtain the following convergence

$$u_{p|_{\partial\Omega}} \xrightarrow{p \rightarrow \infty} u|_{\partial\Omega} \quad \text{in } L^s(\Gamma_T). \quad (4.15)$$

Extracting if necessary a subsequence, we get $u_{p|_{\partial\Omega}} \xrightarrow{p \rightarrow \infty} u|_{\partial\Omega}$ a.e. Since $h(0) = 0$, we finally obtain

$$|1 - h(u_p(\tau, x)/p)|^s |\sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x))|_1^s \xrightarrow{p \rightarrow \infty} 0, \quad (4.16)$$

for a.e. $(\tau, x) \in \Gamma_T$. Moreover,

$$|1 - h(u_p(\tau, x)/p)|^s |\sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x))|_1^s \leq |\sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x))|_1^s, \quad (4.17)$$

with $\int_{\Gamma_T} |\sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x))|_1^s d\mu d\tau < \infty$ (see Corollary 3.3). From (4.16), (4.17) and Lebesgue theorem, we derive that $J_{p,2}$ tends to 0. With (4.14), (4.12) and (4.11), this proves (4.10).

Now, since the function u_p is a solution of the problem (\mathcal{P}_T) with σ_p in place of σ , using (4.9) and (4.10) we see that

$$\begin{aligned} \int_0^T \left\langle \frac{du_p(\tau)}{d\tau}, w(\tau) \right\rangle d\tau &\xrightarrow{p \rightarrow \infty} \int_{\mathcal{Q}_T} (\nabla u + Du \otimes \nabla \mathcal{V}) : \nabla w dx d\tau \\ &+ \int_{\Gamma_T} \sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x)) \cdot w(x) d\mu d\tau. \end{aligned} \quad (4.18)$$

Starting with $t \in [0, T]$ in place of T , we conclude that (4.18) holds true for any $t \in [0, T]$. Restricting to $w \in C^1(\bar{\mathcal{Q}}_T)$ (see (4.8)), we have, for any $t \in (0, T)$

$$\int_0^t \left\langle \frac{du_p}{d\tau}(\tau), w(\tau) \right\rangle d\tau = - \int_{\mathcal{Q}_t} u_p \cdot \partial_t w dx d\tau - \int_{\Omega} u_0 \cdot w(0) dx + \int_{\Omega} u_p(t) \cdot w(t) dx. \quad (4.19)$$

In order to pass to the limit in (4.19) notice the convergence $\int_{\Omega} u_p \cdot w dx \xrightarrow{p \rightarrow \infty} \int_{\Omega} u \cdot w dx$ in $L^1(0, T)$, consequence of $u_p \xrightarrow{p \rightarrow \infty} u$ in $L^2(\mathcal{Q}_T)$ (see Lemma 4.1). Hence, extracting if necessary a subsequence, we get

$$\int_{\Omega} u_p(t) \cdot w(t) dx \xrightarrow{p \rightarrow \infty} \int_{\Omega} u(t) \cdot w(t) dx \quad \text{for a.e } t \in (0, T). \quad (4.20)$$

Appealing to (4.19), (4.20) and (4.3b) with t in place of T , we obtain

$$\int_0^t \left\langle \frac{du_p}{d\tau}(\tau), w(\tau) \right\rangle d\tau \xrightarrow{p \rightarrow \infty} - \int_{\mathcal{Q}_t} u \cdot \partial_t w dx d\tau - \int_{\Omega} u_0 \cdot w(0) dx + \int_{\Omega} u(t) \cdot w(t) dx \quad (4.21)$$

for a.e $t \in (0, T)$. Appealing to (4.18), it implies that (\mathcal{R}_T) holds true. The proof in the case $0 \leq \rho < \frac{2}{d}$ is similar (see (4.6)).

(ii) **Existence and uniqueness in the case $0 \leq \rho < \frac{1}{d}$.** We begin with the existence part. From one hand, since $u_p \xrightarrow{p \rightarrow \infty} u$ in $L^2(\mathcal{Q}_T)$, we obtain the convergence in $L^2(0, T, H^1(\Omega)')$. Hence we find $\frac{du_p}{d\tau} \xrightarrow{p \rightarrow \infty} \frac{du}{d\tau}$ in $H^{-1}(0, T, H^1(\Omega)')$. On the other hand, since $0 \leq \rho < \frac{1}{d}$ we have $\mathbf{b}(T, \rho) = L^2(0, T, H^1(\Omega))$, and by (4.18), $\{\frac{du_p}{d\tau}\}_{p \in \mathbb{N}^*}$ converge towards some f in $L^2(0, T, H^1(\Omega)')$

weak- \star . By identification, $f = \frac{du}{dt}$, and the variational existence part follows from (4.18). Next, $u \in C^0(0, T, \mathbf{L}^2(\Omega))$ follows classically from $u \in L^2(0, T, \mathbf{H}^1(\Omega))$ and $\frac{du}{dt} \in L^2(0, T, \mathbf{H}^1(\Omega)')$. Last, writing $u_p(0) = u_0$ and using (4.3a), we get $u(0) = u_0$.

For the uniqueness part, let $u \in E_T$, $\bar{u} \in E_T$ be two solutions of (\mathcal{P}_T) associated with the same initial data. Denote by \mathcal{V} and $\bar{\mathcal{V}}$ the associated potentials. Notice that $u - \bar{u} \in E_T \subset \mathbf{b}(T, \rho)$ by (4.6). Using (\mathcal{P}_T) , we derive the following energy estimate

$$\begin{aligned} \|u - \bar{u}\|_{E_t}^2 &\leq C_{R,T} \|u - \bar{u}\|_{L^2(\mathcal{Q}_t)}^2 \\ &+ C_{R,T} \int_{\Gamma_t} |\sigma(\tau, x, u(\tau, x), \mathcal{V}(\tau, x)) - \sigma(\tau, x, \bar{u}(\tau, x), \bar{\mathcal{V}}(\tau, x))|_1 |u - \bar{u}|_1 d\mu d\tau. \end{aligned}$$

Appealing to Corollary 3.3 (ii), we get, for $\eta > 0$ small enough

$$\|u - \bar{u}\|_{E_t}^2 \leq C_{R,T} \|u - \bar{u}\|_{L^2(\mathcal{Q}_t)}^2,$$

and uniqueness follows. \square

Remark 4.3. *Theorem 4.2 holds true with a general diffusion term $(\eta^1 \Delta u^1, \dots, \eta^n \Delta u^n)$ in place of Δu ($\eta^i > 0$). This will implicitly be used in Section 5*

5 Examples

In this concluding section, we illustrate our setting by some realistic equations. We focus below on two examples coming from corrosion and self-gravitation. In the first example, well-posedness follows from a direct application of Theorem 4.2. In the second example, Theorem 4.2 is used as a mollifying frame in an existence proof. This method could be used for more complicated systems.

5.1 The drift-diffusion system coming from a corrosion model

We first consider a drift-diffusion system endowed with quite general Robin boundary conditions. Then we illustrate this general setting with a more specific model, namely the corrosion in a nuclear waste repository (cf. [1, 8]).

5.1.1 The drift-diffusion system

Assume that for any $i \in \{1, \dots, n\}$, $\alpha^i \in \mathbb{R}$, $\beta^i \in \mathbb{R}$, θ_i a Borel measure on $[a, b]$ ($(a, b) \in \mathbb{R}^2$) and $u_0^i \in L^2(0, 1)$ with $u_0^i \geq 0$, $(\mathcal{A}_0, \mathcal{A}_1, \mathcal{V}_0, \mathcal{V}_1) \in \mathbb{R}^4$ and $\xi > 0$. Let us define $f^i \in C^1(\partial\Omega \times \mathbb{R} \times [a, b])$ and $g^i \in C^1(\mathbb{R} \times [a, b])$ satisfying the following assumptions:

- (D-1) $\forall (x, \phi, s) \in [0, 1] \times \mathbb{R} \times [a, b] : f^i(x, \phi, s) \leq 0$,
- (D-2) $\exists R \in [0, \infty), \forall (v, s) \in [R, \infty) \times [a, b] : g^i(v, s) \geq 0$,
- (D-3) $\forall (v, s) \in (-\infty, 0] \times [a, b] : g^i(v, s) \leq 0$,
- (D-4) $\forall (v, \bar{v}, s) \in \mathbb{R} \times \mathbb{R} \times [a, b] : |g^i(v, s) - g^i(\bar{v}, s)| \leq K(1 + |v|^\rho + |\bar{v}|^\rho)|v - \bar{v}|$ for some $\rho \in [0, 3)$.

Set

$$\forall i \in \{1, \dots, n\} : \sigma^i(t, x, v, \phi) \stackrel{\text{def}}{=} \int_{[a, b]} f^i(x, \phi, s) g^i(v, s) d\theta_i(s). \quad (5.1)$$

We consider the following drift-diffusion system:

$$\forall i \in \{1, \dots, n\} : \partial_t u^i = \partial_x(\partial_x u^i + \alpha^i(u^i \partial_x \mathcal{V})), \quad (t, x) \in (0, T) \times (0, 1), \quad (5.2a)$$

$$\partial_{xx} \mathcal{V} = \sum_{i=1}^n \beta^i u^i + \xi, \quad (t, x) \in (0, T) \times (0, 1), \quad (5.2b)$$

together with boundary conditions

$$\forall i \in \{1, \dots, n\} : -(\partial_x u^i + \alpha^i u^i \partial_x \mathcal{V})(t, 0) = \sigma^i(t, 0, u^i(t, 0), \mathcal{V}(t, 0)), \quad t \in [0, T], \quad (5.3a)$$

$$\forall i \in \{1, \dots, n\} : (\partial_x u^i + \alpha^i u^i \partial_x \mathcal{V})(t, 1) = \sigma^i(t, 1, u^i(t, 1), \mathcal{V}(t, 1)), \quad t \in [0, T], \quad (5.3b)$$

$$\mathcal{V}(t, 0) + \mathcal{A}_0 \partial_x \mathcal{V}(t, 0) = \mathcal{V}_0, \quad t \in [0, T], \quad (5.3c)$$

$$\mathcal{V}(t, 1) + \mathcal{A}_1 \partial_x \mathcal{V}(t, 1) = \mathcal{V}_1, \quad t \in [0, T], \quad (5.3d)$$

and initial conditions

$$\forall i \in \{1, \dots, n\} : u^i(0, x) = u_0^i, \quad x \in (0, 1). \quad (5.4)$$

We suppose now that $1 + \mathcal{A}_1 - \mathcal{A}_0 \neq 0$ and $\varphi \in L^1(0, 1)$. Then the following problem:

$$\begin{cases} \text{Find } \mathcal{V} \in W^{2,1}(0, 1) \text{ such that} \\ \partial_{xx} \mathcal{V} = \varphi, \\ (\mathcal{V} + \mathcal{A}_0 \partial_x \mathcal{V})(0) = \mathcal{V}_0 \quad \text{and} \quad (\mathcal{V} + \mathcal{A}_1 \partial_x \mathcal{V})(1) = \mathcal{V}_1, \end{cases} \quad (5.5)$$

admits exactly one solution given by

$$\mathcal{V}(x) = \int_0^1 G(x, y) \varphi(y) dy + \frac{x - \mathcal{A}_0}{1 + \mathcal{A}_1 - \mathcal{A}_0} (\mathcal{V}_1 - \mathcal{V}_0) + \mathcal{V}_0, \quad (5.6)$$

where $G \in L^\infty(0, 1; W^{1,\infty}(0, 1)) \cap C^0([0, 1] \times [0, 1])$ is the Green kernel associated with problem (5.5). We may observe that the function G is defined as follows:

$$G(x, y) \stackrel{\text{def}}{=} \frac{(1 + \mathcal{A}_1 - x)(\mathcal{A}_0 - y)}{1 + \mathcal{A}_1 - \mathcal{A}_0}$$

for $0 \leq y \leq x \leq 1$ and $G(x, y) = G(y, x)$ for $0 \leq x \leq y \leq 1$. Notice that (A-2) and (A-4) follow from (5.6) while (A-1) comes from the assumptions (D-1)–(D-3). Since it is quite a routine to verify that (A-1), (A-2) and (A-4) holds true, the verification is let to the reader. Consequently, we may deduce from Theorem 4.2 that (5.2)–(5.4) admits at least one solution $u \stackrel{\text{def}}{=} (u^1, \dots, u^n)$ (in the sense of (\mathcal{R}_T)) belonging to $L^2(0, T; \mathbf{H}^1(0, 1)) \cap L^\infty([0, T]; \mathbf{L}^2(0, 1))$.

5.1.2 A corrosion model

Let u^1 , u^2 , u^3 and \mathcal{V} be the electrons and cations densities, oxygen vacancies and electrical potential, respectively. Following [8], we assume that the boundary conditions on u^1 , u^2 and u^3 have exactly the same form. Let ξ be the density charge in the host lattice and λ and ε be two nonnegative constants such that $\varepsilon \ll 1$. Set

$$\forall i = 1, 2, 3 : \sigma^i(t, x, v, \phi) \stackrel{\text{def}}{=} -(m_x^i e^{-\gamma^i b_x^i \phi} + k_x^i e^{\gamma^i a_x^i \phi})v + m_x^i u_{\max}^i e^{-\gamma^i b_x^i \phi}, \quad (5.7)$$

where $m_x^i > 0$, $k_x^i > 0$, $a_x^i \in [0, 1]$, $b_x^i \in [0, 1]$ and $u_{\max}^i > 0$ with $i = 1, \dots, 3$ and $\gamma^1 = -1$, $\gamma^2 = 3$ and $\gamma^3 = 1$. The mathematical problem is formulated as follows:

$$\forall i = 1, 2, 3 : \varepsilon^{2-i} \partial_t u^i = \partial_x(\partial_x u^i + \gamma^i u^i \partial_x \mathcal{V}), \quad (t, x) \in \mathbb{R}_*^+ \times (0, 1), \quad (5.8a)$$

$$-\lambda \partial_{xx} \mathcal{V} = \gamma^1 u^1 + \gamma^2 u^2 + \gamma^3 u^3 + \xi, \quad (t, x) \in \mathbb{R}_*^+ \times (0, 1). \quad (5.8b)$$

The system (5.8) is endowed with the following boundary conditions

$$\forall i = 1, 2, 3 : -(\partial_x u^i + \gamma^i u^i \partial_x \mathcal{V})(t, 0) = \sigma^i(t, 0, u^i(t, 0), \mathcal{V}(t, 0)), \quad t \in \mathbb{R}_*^+, \quad (5.9a)$$

$$\forall i = 1, 2, 3 : (\partial_x u^i + \gamma^i u^i \partial_x \mathcal{V})(t, 1) = \sigma^i(t, 1, u^i(t, 1), \Psi - \mathcal{V}(t, 1)), \quad t \in \mathbb{R}_*^+, \quad (5.9b)$$

$$(\mathcal{V} - \mathcal{A}_0 \partial_x \mathcal{V})(t, 0) = \Delta \mathcal{V}_0, \quad t \in \mathbb{R}_*^+, \quad (5.9c)$$

$$(\mathcal{V} - \mathcal{A}_1 \partial_x \mathcal{V})(t, 1) = \Psi - \Delta \mathcal{V}_1, \quad t \in \mathbb{R}_*^+, \quad (5.9d)$$

and the following initial conditions

$$\forall i = 1, 2, 3 : u^i(0, x) = u_0^i, \quad x \in (0, 1). \quad (5.10)$$

Here Ψ denotes a given applied potential, $\Delta \mathcal{V}_i$ are the voltage drop parameters and $(\mathcal{A}_0, \mathcal{A}_1) \in \mathbb{R}^2$. Furthermore, for any $i \in \{1, 2, 3\}$, $u_0^i \geq 0$ belongs to $L^2(0, 1)$. For further explanations on this model, the reader is referred to [1] as well as to the references therein. Appealing to Theorem 4.2 with $\rho = 0$, $k^i = u_{\max}^i$ and $\Lambda_T = 0$ (see also Subsection 5.1.1), we infer that (5.8)–(5.10) possesses exactly one solution $u = (u^1, u^2, u^3)$ belonging to $L^2(0, T; \mathbf{H}^1(0, 1)) \cap L^\infty(0, T; \mathbf{L}^2(0, 1))$ for any $T > 0$. In contrast with a former existence result given in [8] ($n = 2$), our result holds true for $n = 3$ and even for an arbitrary number of species ($n \in \mathbb{N}^*$). Furthermore, the conditions on the *voltage drops* and *other structural coefficients* in Theorem 1.1 in [8] have been removed.

Finally, as quoted above, we have assumed that the boundary conditions on u^1 , u^2 and u^3 have the same form. Nevertheless, in the original corrosion system depicted in [1], this is not the case. As easily verified, the second boundary condition given therein does not meet our assumption (A–3). In consequence, it is unclear to us whether this boundary condition is mathematically sound or not.

5.2 The self-gravitational system

We consider the self-gravitational system described in [5]. Let $u(t, x)$ be the evolution density of identical attracting particles and $\mathcal{V}(t, x)$ be the gravitational potential. The mathematical problem can be written as follows:

$$\partial_t u = \nabla \cdot (\nabla u + u \nabla \mathcal{V}), \quad (t, x) \in (0, T) \times \Omega, \quad (5.11a)$$

$$\Delta \mathcal{V} = u, \quad (t, x) \in (0, T) \times \Omega, \quad (5.11b)$$

together with the boundary conditions on $(0, T) \times \partial\Omega$

$$\frac{\partial u}{\partial \zeta} + u \frac{\partial \mathcal{V}}{\partial \zeta} = 0, \quad (t, x) \in (0, T) \times \partial\Omega, \quad (5.12a)$$

$$\mathcal{V} = 0, \quad (t, x) \in (0, T) \times \partial\Omega, \quad (5.12b)$$

and with initial data

$$u(0) = u_0. \quad (5.13)$$

Observe that the above system corresponds to $n = 1$, $\sigma = 0$, $k^i = 0$ and $\Lambda_T = 0$. In the sequel, we restrict to the case $d = 2$, and derive as in [5] an existence result for a small L^2 initial data. The proof relies on a L^2 estimate on the function u . Since we can use the Theorem 4.2 in order to get a global existence result for a mollified system, it is enough to prove that the crucial L^2 estimate holds uniformly true for the family of approximate solutions.

Remark that the resolvent $\mathcal{B} = \Delta_D^{-1}$ of the Poisson-Dirichlet problem on $(0, T) \times \Omega$, namely $\Delta(\mathcal{B}(f)) = f$ and $\mathcal{B}(f)|_{\partial\Omega} = 0$, do not fulfill the L^1 or the $W^{1,\infty}$ condition in (A–1). Nevertheless,

\mathcal{B} defines a continuous operator in $L^p(\Omega) \rightarrow W^{2,p}(\Omega)$ for any $1 < p < \infty$. We regularize the operator \mathcal{B} in the following way. Let $\phi \in \mathcal{D}(\mathbb{R}^2)$ be a density probability. For any $p \in \mathbb{N}^*$, set $\phi_p(x) = p\phi(px)$, $x \in \mathbb{R}^2$. Let also E be the extension by zero operator. Notice that $E : L^1(\Omega) \rightarrow L^1(\mathbb{R}^2)$ is continuous. Moreover, $Ev \geq 0$ for any positive $v \in L^1(\Omega)$. For $p \in \mathbb{N}^*$, define \mathcal{B}_p on $v \in L^1(\Omega)$ by $\mathcal{B}_p(v) = \mathcal{B}((Ev \star \phi_p)|_\Omega)$. As easily checked, for any $1 < q < \infty$, $\mathcal{B}_p : L^1(\Omega) \rightarrow W^{2,q}(\Omega)$ continuously. In fact, for any $v \in L^1(\Omega)$, we have

$$\begin{aligned} \|\mathcal{B}_p(v)\|_{W^{2,q}(\Omega)} &\leq C_q \|(Ev \star \phi_p)|_\Omega\|_{L^q(\Omega)} \leq C_q \|Ev \star \phi_p\|_{L^q(\mathbb{R}^2)} \\ &\leq C_q \|Ev\|_{L^1(\mathbb{R}^2)} \|\phi_p\|_{L^q(\mathbb{R}^2)} \leq C_{p,q} \|v\|_{L^1(\Omega)}. \end{aligned}$$

From Hölder inequality and Sobolev embeddings, it follows from this $L^1 - W^{2,q}$ continuity that the operator \mathcal{B}_p satisfies the condition (A-1). Therefore (see Theorem 4.2), for any $p \in \mathbb{N}^*$, we can define $u_p \in E_T \cap C^0([0, T]; L^2(\Omega))$ as the solution of (\mathcal{P}_T) . It means that, $\frac{du_p}{dt} \in L^2(0, T; (\mathbf{H}^1(\Omega))')$, $u_p(0) = u_0$ and, for any $w \in L^2(0, T; \mathbf{H}^1(\Omega))$

$$\int_0^T \left\langle \frac{du_p}{d\tau}(\tau), w(\tau) \right\rangle d\tau + \int_{\mathcal{Q}_T} (\nabla u_p + u_p \nabla \mathcal{V}_p)(\tau, x) \cdot \nabla w(\tau, x) dx d\tau \quad (5.14)$$

with $\mathcal{V}_p(t) = \mathcal{B}_p(t, u_p(t))$ for a.e $t \in (0, T)$.

Theorem 5.1. *Let Ω be a smooth, bounded domain of \mathbb{R}^2 . There exists $\eta > 0$ such that, for any $T > 0$, any $u_0 \in L^2(\Omega; \mathbb{R})$ with $u_0 \geq 0$ and $\|u_0\|_{L^1(\Omega)} \leq \eta$, the problem*

$$\begin{cases} \text{Find } u \in C^0([0, T], L^2(\Omega)) \cap L^2(0, T; \mathbf{H}^1(\Omega)) \text{ with } \frac{du}{dt} \in L^2(0, T; (\mathbf{H}^1(\Omega))') \\ \text{such that, for any } w \in L^2(0, T; \mathbf{H}^1(\Omega)) : \\ \int_0^T \left\langle \frac{du}{d\tau}(\tau), w(\tau) \right\rangle d\tau + \int_{\mathcal{Q}_T} (\nabla u + u \nabla \mathcal{V}) \cdot \nabla w dx d\tau = 0 \\ \text{with } \mathcal{V} \stackrel{\text{def}}{=} \Delta_D^{-1} u \text{ and } u(0) = u_0, \end{cases}$$

admits at least one solution.

Proof. We mainly have to prove that the above sequence $\{u_p\}_{p \in \mathbb{N}^*}$ is bounded in E_T . Since u_p satisfies the formulation (\mathcal{P}_T) , taking $w = u_p$ as a test function ($p \in \mathbb{N}^*$), we get

$$\frac{1}{2} \|u_p(t)\|_{L^2(\Omega)}^2 + \|\nabla u_p\|_{L^2(0, T; L^2(\Omega))}^2 \leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 - \int_{\mathcal{Q}_t} u_p \nabla \mathcal{V}_p \cdot \nabla u_p dx d\tau. \quad (5.15)$$

Notice that

$$\int_{\mathcal{Q}_t} u_p \nabla \mathcal{V}_p \cdot \nabla u_p dx d\tau = -\frac{1}{2} \int_{\mathcal{Q}_t} u_p^2 \Delta \mathcal{V}_p dx d\tau + \frac{1}{2} \int_{\mathcal{Q}_t} u_p^2 \frac{\partial \mathcal{V}_p}{\partial \zeta} d\mu d\tau. \quad (5.16)$$

In order to remove the second term in the right hand side of equality (5.16), remark that $u_p \geq 0$. Hence, $Eu_p \geq 0$ so that $\Delta \mathcal{V}_p = (Eu_p \star \phi_p)|_\Omega \geq 0$. Recalling the equality $\mathcal{V}_p|_{\partial\Omega} = 0$, we therefore obtain $\frac{\partial \mathcal{V}_p}{\partial \zeta} \geq 0$. Finally, according to (5.15) and (5.16) and the definition of \mathcal{V}_p , this leads to

$$\frac{1}{2} \|u_p(t)\|_{L^2(\Omega)}^2 + \|\nabla u_p\|_{L^2(0, t; L^2(\Omega))}^2 \leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 + \int_{\mathcal{Q}_t} u_p^2 (Eu_p \star \phi_p) dx d\tau. \quad (5.17)$$

It remains to estimate the last term on the right hand side of (5.17). To this aim, we use the Hölder and the convolution inequalities to get

$$\int_{\mathcal{Q}_t} u_p^2 (Eu_p \star \phi_p) dx d\tau \leq \|u_p\|_{L^3(\mathcal{Q}_t)}^2 \|Eu_p \star \phi_p\|_{L^3(\mathcal{Q}_t)} \leq \|u_p\|_{L^3(\mathcal{Q}_t)}^3. \quad (5.18)$$

By using (5.18) and Gagliardo-Nirenberg inequality, it follows that there exists a constant $C_{\text{GN}} > 0$ independent of $p \in \mathbb{N}$ such that

$$\int_{\mathcal{Q}_t} u_p^2(Eu_p \star \phi_p) dx d\tau \leq C_{\text{GN}} \int_0^t \|u_p(\tau)\|_{L^1(\Omega)} \|u_p(\tau)\|_{H^1(\Omega)}^2 d\tau. \quad (5.19)$$

Appealing to (2.11b) with $k^i = 0$ and $\Lambda_T = 0$, we see that

$$\forall \tau \in [0, t] : \|u_p(\tau)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}. \quad (5.20)$$

Hence (5.17)–(5.20) leads to

$$\frac{1}{2} \|u_p(t)\|_{L^2(\Omega)}^2 + \|\nabla u_p\|_{L^2(0,t;L^2(\Omega))}^2 \leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 + C_{\text{GN}} \|u_0\|_{L^1(\Omega)} \int_0^t \|u_p(\tau)\|_{H^1(\Omega)}^2 d\tau. \quad (5.21)$$

For $u_0 \in L^2(\Omega)$ with $\|u_0\|_{L^1(\Omega)} \leq \frac{1}{2C_{\text{GN}}}$, we deduce from (5.21) that

$$\|u_p(t)\|_{L^2(\Omega)}^2 + \|\nabla u_p\|_{L^2(0,t;L^2(\Omega))}^2 \leq \|u_0\|_{L^2(\Omega)}^2 + \int_0^t \|u_p(\tau)\|_{L^2(\Omega)}^2 d\tau. \quad (5.22)$$

From (5.22) and Grönwall's lemma, we conclude that

$$\{u_p\}_{p \in \mathbb{N}^*} \text{ is bounded in } E_T \hookrightarrow L^2(0, T, L^r(\Omega)) \quad (5.23)$$

for any $1 \leq r < \infty$. Extracting if necessary a subsequence, (5.23) gives

$$\nabla u_p \xrightarrow{p \rightarrow \infty} \nabla u \text{ weakly in } L^2(0, T, L^2(\Omega)). \quad (5.24)$$

Since $\mathcal{V}_p(t) = \Delta_D^{-1}((Eu_p \star \phi_p)|_{\partial\Omega})$, (5.23) and Lemma 2.1 leads to

$$\{\nabla \mathcal{V}_p\}_{p \in \mathbb{N}^*} \text{ is bounded in } L^\infty(0, T, H^1(\Omega)) \hookrightarrow L^\infty(0, T, L^r(\Omega)) \quad (5.25)$$

for any $1 \leq r < \infty$. Now, due to (5.14), (5.23), and (5.25) we also obtain that $\{\frac{du_p}{dt}\}_{p \in \mathbb{N}^*}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$. By the Aubin-Lions lemma, extracting if necessary a subsequence, we conclude the existence of $u \in L^2(0, T; L^2(\Omega))$ such that

$$u_p \xrightarrow{p \rightarrow \infty} u \quad \text{in } L^2(\mathcal{Q}_T).$$

and we can moreover assume that $u_p(t) \xrightarrow{p \rightarrow \infty} u(t)$ for a.e $t \in (0, T)$ and $\sup_{p \in \mathbb{N}^*} \|u_p\|_{L^2(\Omega)} \in L^2(0, T)$. It follows easily that

$$\nabla \mathcal{V}_p \xrightarrow{p \rightarrow \infty} \nabla \Delta_D^{-1} u \in L^2(\mathcal{Q}_T). \quad (5.26)$$

From (5.14), (5.23), and (5.25) we also have that $\{\frac{du_p}{dt}\}_{p \in \mathbb{N}^*}$ is bounded in $L^2(0, T; (\mathbf{H}^1(\Omega))')$, hence, up to a subsequence, weakly- \star convergent in $L^2(0, T; (\mathbf{H}^1(\Omega))')$. With (5.23), (5.24), (5.25), (5.26), we may conclude that u satisfies the variational formulation in \mathcal{P}_T with test functions $w \in C^\infty([0, T] \times \bar{\Omega})$. By density, this holds true for $w \in L^2(0, T, \mathbf{H}^1(\Omega))$. The end of the proof is omitted. \square

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