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A comparison of harmonic modeling methods with application to control of switched systems with active filtering

Nicolas Blin^{1,2}, Pierre Riedinger¹, Jamal Daafouz¹, Louis Grimaud² and Philippe Feyel²

Abstract—Many harmonic modeling approaches have been introduced in the literature, such as generalized state-space averaging, dynamic phasors, extended harmonic domain, and harmonic state-space. They are capable of capturing both the transient evolution and the steady-state of harmonics. They model the frequency coupling nature of a system and can expose the frequency couplings within interconnected components. By these modeling techniques, a linear time-periodic system can be converted into a linear time-invariant system, which allows the use of traditional analysis and control methods. This paper presents a state of the art of harmonic modeling approaches. Its contribution is to clearly establish the links between the different approaches, in particular through the specification of the decomposition of non-periodic signals in generalized Fourier series with time-varying coefficients. This paper also shows the advantages of harmonic modeling to analyse the frequency couplings within associated systems and to the control with active filtering.

I. INTRODUCTION

Transmission and conversion of electric power is an essential element of modern society. To meet the new needs the power systems are more and more complex, which leads to several challenges. The non-linear components introduce couplings between frequencies. In particular, high-frequency converters inherently generate harmonics and can interact with each other. These interactions may cause unpredicted instability problems involving many different frequencies, like in the Swiss locomotives case [14]. This has motivated the researches for an analysis approach capable of characterizing the frequency interactions in both transient and steady state.

A review of the literature in this area shows that various frequency domain based approaches have been introduced. Generalized State-Space Averaging (GSSA) [18] and Dynamic Phasors (DP) [12] are generalized averaging approaches to model the harmonic content of general linear systems. They employ Fourier series of time-varying coefficients and have been widely used in power system modeling and control, e.g. [9], [12], [19]. Extended Harmonic Domain (EHD) [11] and Dynamic Harmonic Domain (DHD) [3] have been introduced to analyse Linear Time-Periodic (LTP) systems. They convert LTP systems into Linear Time-Invariant (LTI) systems in the harmonic domain.

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Most LTI systems are easy to analyse since they can be characterized entirely by their impulse response, allowing traditional analysis methods. These approaches have also been widely used, e.g. [10], [17]. Meanwhile, Harmonic State Space (HSS) has been introduced for LTP system analysis [20]. The HSS approach makes it possible to represent a LTP system as a LTI state space of infinite dimension in the harmonic domain by the use of the concept of Exponentially Modulated Periodic (EMP) signals.

The main contribution of this paper is to clearly establish the links between the various harmonic modeling techniques, in particular through the specification of the decomposition in generalized Fourier series with time-varying coefficients. Especially, it is important to determinate the links between HSS and averaging approaches. As a result, these methods are different in their writing but they can be derived from a unique state-space representation which is introduced in this article. One more contribution of the paper is to incorporate harmonic modeling techniques into the control of a switched affine system to act on its state harmonics.

II. HARMONIC MODELING

In this section, we discuss the various harmonic modeling approaches that have been introduced in the literature. To clearly state the links between them, we present a harmonic model from which they can all be derived. We also propose a clear proof for the use of time-varying Fourier coefficients. In this context, let us consider a linear time-varying system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned}\quad (1)$$

with $x(t) \in \mathbb{R}^{n_x}$, $y(t) \in \mathbb{R}^{n_y}$, $u(t) \in \mathbb{R}^{n_u}$, and where $A(t), B(t), C(t), D(t)$ are some real-valued matrices of appropriate dimension. If the matrices $A(t), B(t), C(t), D(t)$ are T -periodic, the system is a LTP system. If they are constant, the system is a LTI system.

A. State-space averaging

The SSA approach is a signal average over a specified period T , first introduced by Middlebrook and Cuk to model switching converters [13]. They have chosen the period T equal to the switching period of the converter to obtain a simple state-space whose matrices only depend on the duty cycle. To capture the transient evolution of the state variables, this average is computed on a sliding window

$$\bar{x}(t) = \frac{1}{T} \int_{t-T}^t x(p) dp \quad (2)$$

The SSA approach can describe the signals accurately under the linear ripple assumption and the small ripple assumption.

In other words, the signal waveforms should be linear functions of time over the observation intervals and the signals should be dominated by their dc term [13]. The model needs higher order harmonics to describe large-signals accurately [4]. By extension, the SSA approach has resulted in the GSSA.

B. Generalized averaging

The GSSA approach is a general Fourier series development with time-varying coefficients, first introduced by Sanders [18]. The DP (Dynamic Phasors) approach has been introduced by Mattavelli, Verghese, and Stankovic [12] to emphasize the time-varying nature of the Fourier coefficients, but GSSA and DP approaches essentially share the same formulation. This section provides a clear statement of this averaging methods and the decomposition in time-varying Fourier coefficients is specified for non-periodic signals. The links with EHD and DHD approaches are also discussed and a more general model is introduced from which any previous model can be easily derived.

Proposition 1: Any piecewise C^0 signal $x(\cdot)$ with bounded variations can be represented using a general Fourier series of arbitrary period $T = \frac{2\pi}{w}$ by

$$x(t) = 2 \sum_{k=-\infty}^{\infty} X_k(t) e^{jkwt} - x(t-T) \quad a.e. \quad (3)$$

and for a real number $\delta \in]0, T[$, by the non-causal relation

$$x(t-\delta) = \sum_{k=-\infty}^{\infty} X_k(t) e^{jkw(t-\delta)} \quad a.e. \quad (4)$$

with time-varying coefficients

$$X_k(t) = \frac{1}{T} \int_{t-T}^t x(p) e^{-jkwp} dp \quad (5)$$

Proof: Let $x(\cdot)$ be a piecewise continuous signal with bounded variations. This signal is locally integrable in the sense of Lebesgue. Let us introduce the intermediate signal

$$\tilde{x}_T(t, \tau) = \sum_{k=-\infty}^{\infty} X_k(t) e^{jkw\tau} \quad (6)$$

with time-varying coefficients

$$X_k(t) = \frac{1}{T} \int_{t-T}^t x(p) e^{-jkwp} dp \quad (7)$$

This signal depends on two time scales expressed by the two variables t and τ and is T -periodic with respect to τ . It is plotted for two different values of t in Fig. 1.

By application of Dirichlet's theorem on Fourier series convergence: for any fixed real number t and for any $\tau \in]t-T, t[$,

$$\tilde{x}_T(t, \tau) = \frac{x(\tau_-) + x(\tau_+)}{2} \quad (8)$$

where $x(\tau_-)$ and $x(\tau_+)$ stands for the left and right limits of $x(\cdot)$ in τ . Otherwise, for $\tau = t$,

$$\tilde{x}_T(t, t) = \frac{x(t_-) + x((t-T)_+)}{2} \quad (9)$$

So that (3) holds almost everywhere, except at the discontinuities of $x(\cdot)$.

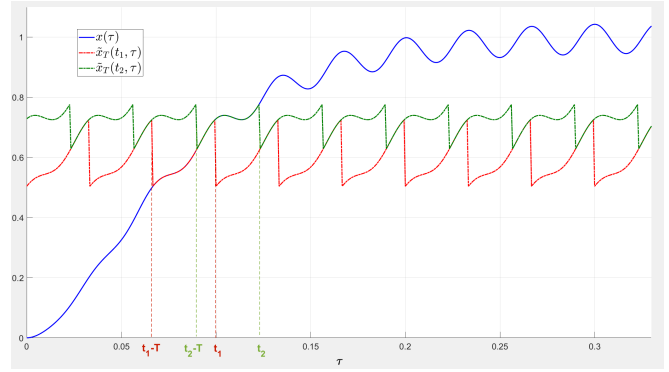


Fig. 1. The signal $x(\tau)$ is plotted in blue and the signal $\tilde{x}_T(t, \tau)$ is plotted for $t = t_1$ in red and for $t = t_2$ in green versus its second time scale τ .

For a fixed real number $\delta \in]0, T[$, by writing (8) for $\tau = t - \delta$,

$$\tilde{x}_T(t, t - \delta) = \frac{x((t-\delta)_-) + x((t-\delta)_+)}{2} \quad (10)$$

So that (4) holds almost everywhere, except at the discontinuities of $x(\cdot)$. ■

The coefficients $X_k(t)$ are called *generalized Fourier coefficients* in GSSA but also referred to as *phasors* in DP. They are functions of time since the interval under consideration slides as a function of time. $X_k(t)$ is the k^{th} coefficient (or k -phasor) at time t . If the signal $x(\cdot)$ is T -periodic after a certain time t_r , then we can see by Proposition 1 that the general Fourier coefficients are constant: $X_k(t) = X_k(t_r + T)$ for $t \geq t_r + T$. So the signal is directly defined by its Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} X_k(t_r + T) e^{jkwt}$ for $t \geq t_r + T$. This is the reason why a good suggestion to analyse the harmonic content of DC-DC switching converters is to choose the period T equal to the switching period. However, it is important to highlight that the proposition holds for any period T .

Proposition 2: The harmonic dynamics of a C^0 , piecewise C^1 signal $x(\cdot)$ of derivative of bounded variations can be described as

$$\dot{X}_k(t) = \langle \dot{x} \rangle_k(t) - jkwX_k(t) \quad a.e. \quad (11)$$

where $\dot{X}_k(\cdot)$ stands for the derivative of the k^{th} general Fourier coefficient of $x(\cdot)$ (the k^{th} harmonic dynamic) and $\langle \dot{x} \rangle_k(\cdot)$ stands for the k^{th} general Fourier coefficient of the derivative of $x(\cdot)$.

Proof: Let the signal $x(\cdot)$ be continuous and differentiable with piecewise continuous derivative and let its derivative have bounded variations. Since $x(\cdot)$ is C^0 , for a real positive number T and a given $\delta \in]0, T[$, (4) holds for any time. Differentiation of (4) with respect to time gives the state dynamic according to all its harmonic dynamics

$$\dot{x}(t - \delta) = \sum_{k=-\infty}^{\infty} (\dot{X}_k(t) + jkwX_k(t)) e^{jkw(t-\delta)} \quad (12)$$

By Proposition 1, since $\dot{x}(\cdot)$ is piecewise C^0 with bounded variations:

$$\dot{x}(t - \delta) = \sum_{k=-\infty}^{\infty} \langle \dot{x} \rangle_k(t) e^{jk\omega(t-\delta)} \quad a.e. \quad (13)$$

Then, by identifying (12) and (13):

$$\langle \dot{x} \rangle_k(t) = \dot{X}_k(t) + jk\omega X_k(t) \quad a.e. \quad (14)$$

Proposition 3: The harmonics of a product of two piecewise C^0 signals $y(\cdot)$ and $x(\cdot)$ can be expressed using the following convolution-based procedure

$$\langle yx \rangle_k(t) = \sum_{i=-\infty}^{\infty} Y_{k-i}(t)X_i(t) \quad a.e. \quad (15)$$

where $Y_k(t), X_k(t)$ are the k^{th} general Fourier coefficients of $y(t), x(t)$.

Proof: It can be easily proved using (4) for both signals. ■

In the literature, GSSA has been applied by taking few harmonics into account, although it was acknowledged that higher order harmonics could also be included and must be incorporated for a more accurate model of a large-signal switching converter. This is because GSSA equations does not allow to add harmonics easily, which has led to EHD approach that systematically describes a LTP system by its state space in the harmonic domain with an arbitrary number of harmonics, possibly infinite [11], [17]. The term DHD has also been introduced in [3] but it shares an identical formulation. However, it is proposed on the following to take into account an infinite number of harmonics but not to be limited to LTP systems. Basically, GSSA equations are used to generate systematically the state-space of a general linear system in a harmonic basis. This requires to introduce the concept of Toeplitz form presented in [20] and generalized here to time-varying Fourier coefficients.

Definition 1: The Toeplitz form of a piecewise C^0 matrix $\mathcal{A}(t)$ is the infinite block matrix of the form

$$\mathcal{A}(t) = \begin{bmatrix} A_0(t) & A_{-1}(t) & A_{-2}(t) & \dots \\ A_1(t) & A_0(t) & A_{-1}(t) & \dots \\ A_2(t) & A_1(t) & A_0(t) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (16)$$

that contains all its generalized Fourier coefficients $A_k(t)$.

Proposition 4: A linear time-varying system (1) with piecewise C^0 input signal and matrices can be represented in the harmonic basis of period T as a state space model of the form

$$\begin{aligned} \dot{X}(t) &= (\mathcal{A}(t) - \mathcal{N})X(t) + \mathcal{B}(t)U(t) \\ Y(t) &= \mathcal{C}(t)X(t) + \mathcal{D}(t)U(t) \end{aligned} \quad (17)$$

with

$$\begin{aligned} X(t) &= [\dots X_{-k}(t) \dots X_0(t) \dots X_k(t) \dots]^T \\ Y(t) &= [\dots Y_{-k}(t) \dots Y_0(t) \dots Y_k(t) \dots]^T \\ U(t) &= [\dots U_{-k}(t) \dots U_0(t) \dots U_k(t) \dots]^T \\ \mathcal{N} &= \text{diag}(\dots -jk\omega \dots 0 \dots jk\omega \dots) \end{aligned} \quad (18)$$

where $X_k(t), Y_k(t), U_k(t)$ are the generalized Fourier coefficients of $x(t), y(t), u(t)$ and $\mathcal{A}(t), \mathcal{B}(t), \mathcal{C}(t), \mathcal{D}(t)$ are the Toeplitz form of $A(t), B(t), C(t), D(t)$.

Proof: Let us consider (1) with piecewise C^0 input signal and matrices. The k^{th} coefficient of $\dot{x}(\cdot)$ and $y(\cdot)$ are expressed as

$$\begin{aligned} \langle \dot{x} \rangle_k(t) &= \langle Ax \rangle_k(t) + \langle Bu \rangle_k(t) \\ Y_k(t) &= \langle Cx \rangle_k(t) + \langle Du \rangle_k(t) \end{aligned} \quad (19)$$

By the GSSA convolution-based procedure (15),

$$\begin{aligned} \langle \dot{x} \rangle_k(t) &= \sum_{i=-\infty}^{\infty} A_{k-i}(t)X_i(t) + \sum_{i=-\infty}^{\infty} B_{k-i}(t)U_i(t) \\ Y_k(t) &= \sum_{i=-\infty}^{\infty} C_{k-i}(t)X_i(t) + \sum_{i=-\infty}^{\infty} D_{k-i}(t)U_i(t) \end{aligned} \quad (20)$$

Then, the infinite vectors of the harmonics of $\dot{x}(t)$ and $y(t)$ can be written with the Toeplitz form of matrices as

$$\begin{aligned} \langle \dot{x} \rangle(t) &= \mathcal{A}(t)X(t) + \mathcal{B}(t)U(t) \\ Y(t) &= \mathcal{C}(t)X(t) + \mathcal{D}(t)U(t) \end{aligned} \quad (21)$$

By (11), we can write

$$\langle \dot{x} \rangle(t) = \dot{X}(t) - \mathcal{N}X(t) \quad (22)$$

so that the derivative of the infinite vector $X(t)$ of the harmonics of $x(t)$ can be expressed as

$$\dot{X}(t) = (\mathcal{A}(t) - \mathcal{N})X(t) + \mathcal{B}(t)U(t) \quad (23)$$

■
The state-space formulation (17) is an extension of GSSA approach to an infinite number of harmonics. It provides the state space of a general linear system in a harmonic basis of chosen period T . The Fourier series must be truncated for computation and the choice of period T becomes a crucial criterion for model accuracy. If the investigated system is LTP of period T , then by choosing T or a natural integer multiple of T as the harmonic basis period, the Fourier coefficients of the matrices appear to be time-invariant, so are their Toeplitz form. In other words, a LTP system of period T is converted into a LTI system of the form (17) with $\mathcal{A}(t) = \mathcal{A}$, $\mathcal{B}(t) = \mathcal{B}$, $\mathcal{C}(t) = \mathcal{C}$, $\mathcal{D}(t) = \mathcal{D}$ in such a harmonic basis. Similar results are presented in EHD and DHD approaches for which the harmonic basis period is almost always equal to the system period. The modeling approach introduced in this article gives the same results than EHD for the analysis of LTP systems but it is more general since it can also model non-periodic systems.

C. Harmonic state-space

The fundamental notion behind the development of the traditional analysis tools like Bode diagram or pole placement for LTI systems is that a sinusoidal input at a given frequency is mapped by the LTI transfer function into a sinusoidal output of the same frequency, but with possibly different amplitude and phase. In contrast, if a sinusoid is input to a LTP system, several sinusoids may appear in the output signal at the frequency of the input sinusoid plus integer multiples of the LTP system fundamental frequency

[21]. As a result, the Laplace domain used for LTI system analysis cannot be used as it stands in LTP analysis context and an analogous approach is required to define a LTP transfer function and use traditional analysis tools. In that context, HSS approach has been introduced by Wereley [20].

Definition 2: A geometrically periodic (GP) signal $x(\cdot)$, with fundamental period T , has the property that

$$x(t + nT) = z^n x(t) \quad (24)$$

where $z \in \mathbb{C}$ and $n \in \mathbb{N}$.

Definition 3: An exponentially modulated periodic (EMP) signal $x(\cdot)$ can be expressed as the complex Fourier series of a periodic signal of fundamental period $T = \frac{2\pi}{w}$ modulated by a complex exponential

$$x(t) = e^{st} \sum_{k=-\infty}^{\infty} X_k e^{jkwt} \quad (25)$$

where $s \in \mathbb{C}$. It can be written on compact form

$$x(t) = e^{st} \Gamma(t) X \quad (26)$$

where

$$\begin{aligned} \Gamma(t) &= [\dots e^{-jkwt} \dots 1 \dots e^{jkwt} \dots] \\ X &= [\dots X_{-k} \dots X_0 \dots X_k \dots]^T \end{aligned} \quad (27)$$

Proposition 5: GP and EMP signal spaces defined for a period T are fundamental signal spaces for LTP systems of period T .

Proof: The proof is essentially based on the study of the state transition matrix over one period called *monodromy matrix* in Floquet theory [5] and can be found in [20]. ■

Proposition 5 means that a LTP system maps a GP (or EMP) signal to a GP (or EMP) signal of same fundamental frequency in steady state. This allows the definition of a LTP transfer function. EMP signals have been introduced by Hill [7] through periodic extension to kernel function of Laplace to make the connection with the assumed Floquet solution. A strong analogy proposed is that EMP signals are to LTP systems as complex exponential signals are to LTI systems. More details about Floquet theory and EMP signals can be found in [2].

Proposition 6: A LTP system (1) of matrices $A(t), B(t), C(t), D(t)$ of period T can be written in the harmonic basis of period T as an infinite LTI system of the form

$$\begin{aligned} sX &= (\mathcal{A} - \mathcal{N})X + \mathcal{B}U \\ Y &= \mathcal{C}X + \mathcal{D}U \end{aligned} \quad (28)$$

with

$$\begin{aligned} X &= [\dots X_{-k} \dots X_0 \dots X_k \dots]^T \\ Y &= [\dots Y_{-k} \dots Y_0 \dots Y_k \dots]^T \\ U &= [\dots U_{-k} \dots U_0 \dots U_k \dots]^T \\ \mathcal{N} &= \text{diag}(\dots -jkw \dots 0 \dots jkw \dots) \end{aligned} \quad (29)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are the Toeplitz matrices of $A(t), B(t), C(t), D(t)$ and $s \in \mathbb{C}$.

Proof: As EMP signal space is a fundamental signal space for LTP systems, we consider

$$\begin{aligned} x(t) &= e^{st} \Gamma(t) X \\ y(t) &= e^{st} \Gamma(t) Y \\ u(t) &= e^{st} \Gamma(t) U \end{aligned} \quad (30)$$

The new state vector X contains an infinite number of time-invariant coefficients. As the matrices are T -periodic, they are equal to their Fourier series with classical Fourier coefficients

$$A(t) = \sum_{n=-\infty}^{\infty} A_n e^{jnwt} \quad (31)$$

so that

$$\begin{aligned} A(t)x(t) &= \left(\sum_{n=-\infty}^{\infty} A_n e^{jnwt} \right) \left(e^{st} \sum_{m=-\infty}^{\infty} X_m e^{jmwt} \right) \\ &= e^{st} \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} X_m A_{k-m} \right) e^{jkwt} \end{aligned} \quad (32)$$

which can be written in compact form

$$A(t)x(t) = e^{st} \Gamma(t) \mathcal{A} X \quad (33)$$

where \mathcal{A} is the Toeplitz matrix of $A(t)$. Similar results are obtained:

$$\begin{aligned} B(t)u(t) &= e^{st} \Gamma(t) \mathcal{B} U \\ C(t)x(t) &= e^{st} \Gamma(t) \mathcal{C} X \\ D(t)u(t) &= e^{st} \Gamma(t) \mathcal{D} U \end{aligned} \quad (34)$$

Otherwise, let us note that

$$\dot{\Gamma}(t) = \mathcal{N} \Gamma(t) \quad (35)$$

So, on the one hand,

$$\begin{aligned} \dot{x}(t) &= s e^{st} \Gamma(t) X + e^{st} \dot{\Gamma}(t) X \\ &= e^{st} \Gamma(t) (sX + \mathcal{N} X) \\ y(t) &= e^{st} \Gamma(t) Y \end{aligned} \quad (36)$$

And on the other hand, from (33) and (34) we write

$$\begin{aligned} A(t)x(t) + B(t)u(t) &= e^{st} \Gamma(t) (\mathcal{A} X + \mathcal{B} U) \\ y(t) &= e^{st} \Gamma(t) (\mathcal{C} X + \mathcal{D} U) \end{aligned} \quad (37)$$

Finally, we can identify (36) and (37) and let the exponential terms e^{st} and $\Gamma(t)$ as they never go to zero, so that we get to the state-space form (28). ■

Definition 4: The harmonic transfer function of a LTP system (28) of period T is the infinite matrix

$$G(s) = \mathcal{C}((s\mathcal{I} - (\mathcal{A} - \mathcal{N}))^{-1} \mathcal{B} + \mathcal{D}) \quad (38)$$

The state-space form (28) allows the use of traditional LTI analysis techniques, as pole-zero map [15]. The main contribution is that it is possible to analyze harmonic couplings by using the harmonic transfer function (38). This time-invariant matrix is a linear operator of the harmonic domain. Another strong analogy is given: HSS domain is to LTP systems as Laplace domain is for LTI systems. As opposed to the Laplace domain of unique reference frame e^{st} , this representation contains an infinite range of reference frames $e^{(s+jkw)t}$. The term " $-\mathcal{N}X$ " in (28) can be interpreted as a constrained rotation of the k^{th} component of the harmonic state variable at e^{-jkw} . The complex exponential e^{st} of EMP signals is related to their non-periodic evolution on direction s , i.e. the transient evolution of harmonics. Their derivative is se^{st} so that the multiplication by s denotes differentiation over time on direction s . The product sX stands for the dynamic of the state harmonics in direction s . The links between averaging techniques and HSS are discussed on the following part.

D. Discussion

Many publications have focused on harmonic content, resulting in several methods. The literature has not described precisely the links between these different approaches because of the differences in their writing and in their goals [8]. DP and GSSA are essentially identical, as they are based on same generalized Fourier representation (4) with time-varying coefficients (5) and generate the same equations (11) and (15). In the literature, these approaches have only been developed for few harmonics. Mostly, only the first harmonic at fundamental frequency is incorporated [18], [19], or some known harmonics of the system are selected to write the state-space and design a controller [6]. However, it could have been developed for an arbitrary large number of harmonics [9]. What is not clearly stated in the literature is that EHD and DHD approaches are equivalent and that they are a systematical writing of the LTI state space model that one can get by applying GSSA equations to analyze a LTP system with an arbitrary, possibly infinite number of harmonics. The link becomes apparent by means of the model proposed in Proposition 4 which is a systematical writing of the state space model that one can get by applying GSSA equations with an infinite number of harmonics. HSS is a purely frequential release of EHD state space (28). It is a frequential analysis domain for LTP systems that is analogous to the Laplace domain for LTI system analysis. Let us remark that HSS model can be derived from EHD model by Laplace transform, which makes sense since EHD model is LTI.

The model presented in this article in Proposition 4 is a generalization of all the harmonic modeling approaches as we can not only choose the number of harmonics to take into account but also consider general linear (time-varying) systems, including non-periodic systems. From this model can be derived all the averaging models by truncation or selection of harmonics, and HSS by Laplace transform. The links between the different harmonic approaches are summarized on Fig. 2.

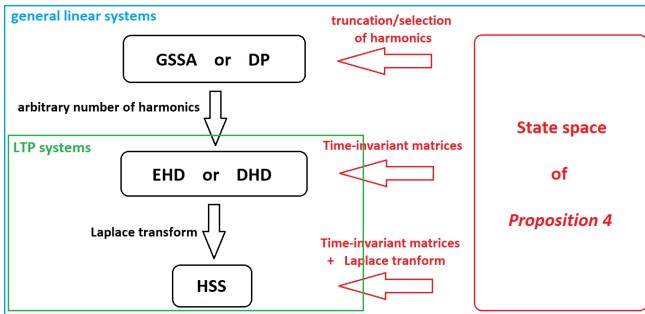


Fig. 2. Links between the different harmonic modeling approaches. All models can be derived from the state space of Proposition 4.

In practise, the Fourier series have to be truncated so that they can be computed. Then, the accuracy of the truncated model highly relies on the choice of the harmonic basis period and the number of harmonics. The impact of the

basis period has been weakly developed so far. The bigger the period of a harmonic basis is, the more precise it is, as the frequencies represented in the basis are separated from $\frac{1}{T}$. Practically speaking, we should choose the period large enough to capture the low frequencies and include enough harmonics to capture the high frequencies. As an example, let's consider the waveform $x(t) = a_1 \sin(\omega t) + a_3 \sin(3\omega t)$. By choosing $T = \frac{2\pi}{\omega}$, only three constant harmonics at ω , 2ω , and 3ω would be needed to model perfectly the signal. More generally, by choosing $T = \frac{2\pi N}{\omega}$, $N \in \mathbb{N}^*$, we need $3N$ harmonics. However, by choosing T arbitrarily (maybe if we don't know ω), we may need much more harmonics, possibly an infinite number, and the harmonics have no reason to be constant. The better choice to model and analyze a DC-DC switched system seems to be a harmonic basis of period equal to the switching period, so that the resulting state-space model is LTI in steady state and only a few harmonics are needed for accuracy.

Harmonic modeling has proved to be an interesting tool to analyse the harmonic content of a system or associated systems. It is valuable to make good choices when associating components by analysing their frequency interactions. In a control context, harmonic modeling allows to directly control the harmonics and include harmonic control specifications. This point has not been much explored yet and it is considered in the next part.

III. CONTROL AND ACTIVE FILTERING OF SWITCHED SYSTEMS

The harmonic modeling can be employed to improve the control of industrial systems. Not only does it allow the analysis of frequency couplings, but it also extends the notion of equilibrium points. Indeed, a T -periodic signal is constant in the harmonic basis of period T , so that it can be viewed as an equilibrium point. It also enables to act separately on these harmonics, to analyse their interactions and to control them. These advantages have been used to design a LQ controller for a full-bridge type rectifier in [6]. The harmonic state-space model has been linearized and a LQR feedback control synthesis in the harmonic domain has given harmonic rejection criteria. We develop on the following a globally asymptotically stable control law for switched affine systems with periodic input perturbation rejection. This control takes advantage of harmonic modeling without linearization and we show that we can use it to eliminate input periodic perturbations on the output voltage of a Boost converter.

A. Switched affine system

A switched system can be described by a state-space model of the form

$$\dot{x}(t) = A(d(t))x(t) + B(d(t))u(t) \quad (39)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the input vector and $d(t)$ is the switching (piecewise continuous) signal. The time-varying matrices can be decomposed as

$$\begin{aligned} A(d(t)) &= A_{ind} + A_{dep} \otimes d(t) \\ B(d(t)) &= B_{ind} + B_{dep} \otimes d(t) \end{aligned} \quad (40)$$

where \otimes stands for the kronecker product. Our interest is about time-varying input signals with some unwanted harmonics. This input signal is assumed to be T -periodic, so that it is constant in the harmonic basis of period T . By Proposition 4, the corresponding state space in the harmonic basis of period T is

$$\dot{X}(t) = \mathcal{A}(D(t))X(t) + \mathcal{B}(D(t))U \quad (41)$$

with

$$\begin{aligned} X(t) &= [X_{\{1\}}(t) \dots X_{\{n_x\}}(t)]^T \\ U &= [U_{\{1\}} \dots U_{\{n_u\}}]^T \\ \mathcal{A}(D(t)) &= A_{ind} \otimes \mathcal{I} - \tilde{\mathcal{N}} + A_{dep} \otimes D_T(t) \\ \mathcal{B}(D(t)) &= B_{ind} \otimes \mathcal{I} + B_{dep} \otimes D_T(t) \\ \tilde{\mathcal{N}} &= \text{diag}(\mathcal{N}, \dots, \mathcal{N}) \end{aligned} \quad (42)$$

where \mathcal{I} is the identity matrix, $X_{\{j\}}(t)$ is the harmonic vector of the j^{th} state variable and $U_{\{l\}}$ is the harmonic vector of the l^{th} input variable. n_x is the number of states and n_u is the number of inputs.

Definition 5: The set of harmonic equilibrium points is

$$\mathcal{E} = \{(X^e, D^e) \mid X^e = -\mathcal{A}(D^e)^{-1}\mathcal{B}(D^e)U\} \quad (43)$$

Harmonic equilibrium points are not necessarily classical equilibrium points as they contain harmonics of fundamental period T . We are interested in accessible harmonic equilibrium points for which $\mathcal{A}(D^e)$ is Hurwitz. Their set is noted \mathcal{E}_a .

Proposition 7: Considering the system (41), any accessible equilibrium point $(X^e, D^e) \in \mathcal{E}_a$ is globally asymptotically stable by the control law

$$D(t) = D^e - \mathcal{K}\mathcal{B}^T(X(t), U)\mathcal{P}(X(t) - X^e) \quad (44)$$

with $\mathcal{K} > 0$ and where $\mathcal{P} > 0$ solution of the Lyapunov equation $\mathcal{A}(D^e)^T\mathcal{P} + \mathcal{P}\mathcal{A}(D^e) + \mathcal{Q} = 0$. The matrix $\mathcal{B}(X(t), U)$ depends on all the harmonics of each state and input variables as

$$\begin{aligned} \mathcal{B}(X(t), U) &= \sum_{j=1}^{n_x} (A_{dep\{c_j\}} \otimes X_{\{j\}T}(t)) \\ &+ \sum_{l=1}^{n_u} (B_{dep\{c_l\}} \otimes U_{\{l\}T}) \end{aligned} \quad (45)$$

where $A_{dep\{c_j\}}, B_{dep\{c_l\}}$ are respectively the j^{th} column of A_{dep} and the l^{th} column of B_{dep} , and with $X_{\{j\}T}(t), U_{\{l\}T}$ the Toeplitz matrices of the harmonic vectors of the j^{th} state variable and the l^{th} input variable.

Proof: The model (41) is affine and we overcome the non-linearity issues by a similar approach than in [1]. The equilibrium equation gives

$$0 = \mathcal{A}(D^e)X^e + \mathcal{B}(D^e)U \quad (46)$$

So that (41) can be written as

$$\begin{aligned} \dot{X}(t) &= \mathcal{A}(D^e)(X(t) - X^e) + (\mathcal{A}(D(t)) - \mathcal{A}(D^e))X(t) \\ &+ (\mathcal{B}(D(t)) - \mathcal{B}(D^e))U \end{aligned} \quad (47)$$

On the one hand,

$$\begin{aligned} \mathcal{A}(D(t)) - \mathcal{A}(D^e) &= A_{dep} \otimes (D(t) - D^e)_T \\ \mathcal{B}(D(t)) - \mathcal{B}(D^e) &= B_{dep} \otimes (D(t) - D^e)_T \end{aligned} \quad (48)$$

And on the other hand,

$$\begin{aligned} (D(t) - D^e)_T X_{\{j\}}(t) &= X_{\{j\}T}(t)(D(t) - D^e) \\ (D(t) - D^e)_T U_{\{l\}} &= U_{\{l\}T}(D(t) - D^e) \end{aligned} \quad (49)$$

So that we can write

$$\dot{X}(t) = \mathcal{A}(D^e)(X(t) - X^e) + \mathcal{B}(X(t), U)(D(t) - D^e) \quad (50)$$

Assuming that $\mathcal{A}(D^e)$ is Hurwitz, for any positive definite matrix \mathcal{Q} , there is a definite positive matrix \mathcal{P} solution of the Lyapunov equation $\mathcal{A}(D^e)^T\mathcal{P} + \mathcal{P}\mathcal{A}(D^e) + \mathcal{Q} = 0$. Then, we focus on the Lyapunov candidate

$$V(X(t), X^e) = (X(t) - X^e)^T\mathcal{P}(X(t) - X^e) \quad (51)$$

Let us note $Y(t) = \mathcal{B}^T(X(t), U)\mathcal{P}(X(t) - X^e)$. We can write

$$\begin{aligned} \dot{V}(X(t), X^e) &= -(X(t) - X^e)^T\mathcal{Q}(X(t) - X^e) \\ &+ (D(t) - D^e)^T Y(t) \\ &+ Y(t)^T(D(t) - D^e) \end{aligned} \quad (52)$$

Thus, by choosing $D(t) = D^e - \mathcal{K}Y(t)$,

$$\begin{aligned} \dot{V}(X(t), X^e) &= -(X(t) - X^e)^T\mathcal{Q}(X(t) - X^e) \\ &- 2Y(t)^T\mathcal{K}Y(t) \\ &\leq 0 \end{aligned} \quad (53)$$

This function is equal to zero if and only if $X(t) = X^e$. Thus, the Lyapunov function $V(X(t), X^e)$ proves the asymptotic stability of the equilibrium point X^e . \mathcal{Q} is the Lyapunov gain to spread the control effort and \mathcal{K} is the gain to adjust the control significance. ■

Such a control law, by using harmonic modeling, can be exploited either to solve properly a tracking problem like a traditional regulation problem or to eliminate some unwanted harmonics.

B. Application to a Boost converter

Since we have defined a more general notion of equilibrium points in (43), an input periodic perturbation can now be considered as some constant perturbations on each harmonics. So the idea to integrate these static errors comes easily. In this part, we present some results about a control law using harmonic modeling to eliminate unwanted harmonics on the input. The chosen system to demonstrate is a Boost converter of switching frequency $f_{sw} = 100kHz$ connected to a constant load. Its temporal model is expressed as in (39) with

$$\begin{aligned} x(t) &= \begin{bmatrix} i(t) \\ v(t) \end{bmatrix} \\ u(t) &= v_{in}(t) \\ A(d(t)) &= \begin{bmatrix} 0 & -\frac{1-d(t)}{L} \\ \frac{1-d(t)}{C} & -\frac{1}{RC} \end{bmatrix} \\ B(d(t)) &= \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \end{aligned} \quad (54)$$

where $i(t)$ is the current through inductance, $v(t)$ is the output voltage that we want to regulate, $v_{in}(t)$ is the input voltage, and $d(t)$ is the pwm signal. We know how to control a Boost converter and how to deal with constant input perturbations. But regarding harmonic issues, the influence of periodic input perturbations need to be studied. Fig. 3 shows the influence of a periodic input perturbation $v_{in}(t) = v_{in_c} + v_{in_{var}} \sin(2\pi ft)$ over the output voltage with traditional control methods, choosing $v_{in_{var}} = \frac{v_{in_c}}{10}$. High-frequency periodic input perturbations are not significant as they are viewed as noises and low-frequency periodic input perturbations can be overcome by adaptive control methods.

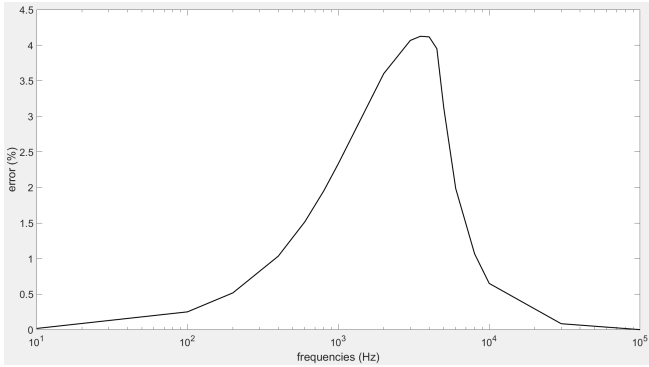


Fig. 3. Influence of periodic input perturbation $v_{in}(t) = v_{in_c} + v_{in_{var}} \sin(2\pi ft)$. A traditional control law is used, improved by adaptive techniques to overcome low-frequency perturbations. The maximal amplitude error on the output voltage is plotted against the perturbation frequency.

Assuming that we know the period T of the input perturbation, the idea is to study the system in the harmonic basis of period T . The corresponding harmonic model is expressed as in (41) with

$$\begin{aligned} X(t) &= \begin{bmatrix} I(t) \\ V(t) \end{bmatrix} \\ U &= V_{in} \\ A(D(t)) &= \begin{bmatrix} -\mathcal{N} & -\frac{1}{L}(\mathcal{I} - D_T(t)) \\ \frac{1}{C}(\mathcal{I} - D_T(t)) & -\frac{1}{RC}\mathcal{I} - \mathcal{N} \end{bmatrix} \\ B(D(t)) &= \begin{bmatrix} \frac{1}{L}\mathcal{I} \\ 0\mathcal{I} \end{bmatrix} \end{aligned} \quad (55)$$

V_{in} is constant since $v_{in}(t)$ is T -periodic. The control objective is to regulate the output voltage to a setpoint. That implies to avoid oscillations on the output voltage. The corresponding reference harmonic vector contains zeros everywhere except at its dc component. The computation of the equilibrium point corresponding to a given reference voltage can bring numerical issues due to the number of variables, but it is overcome by the use of integrators to reject static errors on each harmonics. Fig. 4 shows the better rejection of a periodic perturbation $v_{in}(t) = 30 + 3\sin(2\pi ft)$ of critical frequency $f = 5kHz$ by the use of harmonic modeling techniques.

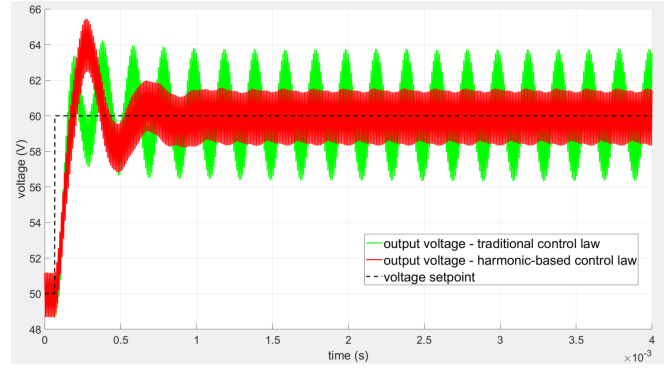


Fig. 4. Output regulation with rejection of a periodic input perturbation of critical frequency $f = 5kHz$. The resulting output voltages are plotted for traditional (temporal-based) control in green and harmonic-based control in red.

IV. CONCLUSION

The harmonic domain can capture the harmonic content of a system and it provides a better understanding of frequency phenomena. Many different approaches have been developed in the harmonic domain, but this article shows that they are based on the same equations. Their differences are only their writing and their objective frameworks for control or analysis. In this paper, a state-space representation is developed to describe the links between the different approaches. From this general modeling can be derived all the models presented in the literature. In practice, the key point of harmonic modeling techniques is the choice of the harmonic basis period and the number of harmonics taken into account. The period has to be large enough to capture the interesting low frequencies and the number of harmonics has to be large enough to capture the interesting high frequencies. Moreover, the period has to be a common multiple of all periods of relevance to obtain a LTI state-space system. One advantage of harmonic modeling is the possibility to control directly the harmonics. A tracking problem can be solved as a simple regulation problem and some harmonic criteria can be included to control or avoid oscillations.

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