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Philippe Jaming, Vilmos Komornik

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MOVING AND OBLIQUE OBSERVATIONS OF BEAMS AND PLATES

PHILIPPE JAMING AND VILMOS KOMORNIK

Abstract. We study the observability of the one-dimensional Schrödinger equation and of the beam and plate equations by moving or oblique observations. Applying different versions and adaptations of Ingham's theorem on nonharmonic Fourier series, we obtain various observability and non-observability theorems. Several open problems are also formulated at the end of the paper.

1. Introduction

Fourier series methods have been applied for a long time in control theory [5, 19, 20, 3]. Since Haraux [6] recognized the usefulness of a classical theorem of Ingham [7] in this context, many new results have been obtained by applying multiple variants of Ingham's theorem [1, 2, 4, 11, 17].

The purpose of this paper is to investigate the observability of beams and plates by moving or oblique observations.

Moving point observability theorems for parabolic and hyperbolic equations have been obtained earlier by Khapalov by different methods [12, 13].

Another motivation for this paper was the following recent result of the first author with K. Kellay [10]:

Theorem 1.1. *Let μ be a bounded measure on $\mathbb{R}^2$ and let $u = \widehat{\mu}$ be its Fourier transform. Assume that u is a solution of the Schrödinger equation $\partial_t u(t,x) + i\partial_x^2 u(t,x) = 0$ on $\mathbb{R}_+ \times \mathbb{R}$ and assume that, for some $a \neq b \in \mathbb{R}$, $u(t,at) = u(t,bt) = 0$ for every $t > 0$ then $u = 0$.*

In other words, a solution of the Schrödinger equation is uniquely determined by its value in two moving points $x = at$ and $x = bt$, $t > 0$. The proof however does not provide any quantitative estimate on u from its values on these points.

We first consider the one-dimensional Schrödinger equation $u_t + iu_{xx} = 0$ in a bounded interval I with periodic boundary conditions and initial data $u_0 \in L^2(I)$. We prove among other things the observability relations

$$\int_0^T |u(t,at)|^2 \, \mathrm{d}t \asymp \|u_0\|^2_{L^2(I)}$$

for all non-integer real numbers a and for all $T > 0$. Here and in the sequel the notation $A \asymp B$, where A and B are functions of u, means that the quantities A, B that depend on the inital data u_0 satisfy an inequality of the form $c_1 A \le B \le c_2 A$ with some positive constants c_1, c_2 that do not depend on the particular choice of the initial data.

On the other hand, the relations

$$\sum_{i=1}^m \int_0^T |u(t,a_i t)|^2 \, \mathrm{d}t \asymp \|u_0\|^2_{L^2(I)}$$

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fail for any choice of finitely many integers a_i and for any $T > 0$. More precisely, this means that for every c_1, c_2 there exists initial data such that one of the inequalities $c_1 A \leq B$ or $B \leq c_2 A$ fails.

Next we carry over a similar study for the one-dimensional beam equation $u_{tt} + u_{xxxx} = 0$ in a bounded interval I with periodic boundary conditions and initial data $u_0 \in L^2(I)$, $u_1 \in H^{-2}(I)$. For example, we have

$$\int_0^T |u(t, at)|^2 \, \mathrm{d}t \asymp \|u_0\|^2_{L^2(I)} + \|u_1\|^2_{H^{-2}(I)}$$

if and only if the circle centered in $(\frac{-a}{2}, \frac{a}{2})$ and passing through the origin contains no other points with integer coordinates. This is the case whenever a is irrational. Furthermore, given two numbers a_1, a_2, we give a necessary and sufficient geometric condition determined by a_1, a_2 for the validity of the estimates

$$\int_0^T |u(t, a_1 t)|^2 \, \mathrm{d}t + \int_0^T |u(t, a_2 t)|^2 \, \mathrm{d}t \asymp \|u_0\|^2_{L^2(I)} + \|u_1\|^2_{H^{-2}(I)}$$

More precisely, these estimates fail if and only if an associated graph $G(a_1, a_2)$ has cycles (see Theorem 4.8 for more details). However, we have no examples where such cycles exist, and it remains an open question whether there exist exceptional parameters a_1, a_2 for which the above estimates fail.

Finally we consider vibrating rectangular plates. Improving several earlier theorems given in [6, 8, 9, 14], it was shown in [17] that these plates may be observed on an arbitrarily small segment which is parallel to one of the sides of the rectangle. Using a different tool we prove that the observability still holds for oblique segments.

The paper is organized as follows. In Section 2 we recall some Ingham type theorems that we need in the subsequent proofs. Section 3 is then devoted to the one-dimensional Schrödinger equation while Section 4 is devoted to the one-dimensional beam equation and Section 5 to vibrating rectangular plates. We end the paper with a list of open questions related to the problems studied here.

2. A review of Ingham type inequalities

Ingham type inequalities play a central role is this study. We therefore devote this section to summarize the results we use.

If I is an interval of length $|I| = 2\pi$, then Parseval's equality

$$\frac{1}{|I|} \int_I \left| \sum_{k \in \mathbb{Z}} c_k e^{ikx} \right|^2 \mathrm{d}x = \sum_{k \in \mathbb{Z}} |c_k|^2$$

holds for all square summable sequences (c_k) of complex numbers. This equality remains valid if the length of I is a positive multiple of 2π. It follows by an elementary argument that if $2k\pi < |I| < (2k+2)\pi$ for some nonnegative integer k, then

$$2k\pi \sum_{k \in \mathbb{Z}} |c_k|^2 \leq \int_I \left| \sum_{k \in \mathbb{Z}} c_k e^{ikx} \right|^2 \mathrm{d}x \leq (2k+2)\pi \sum_{k \in \mathbb{Z}} |c_k|^2$$

for all square summable sequences (c_k), and the constants $2k\pi, (2k+2)\pi$ are the best possible here. Hence

$$\int_I \left| \sum_{k \in \mathbb{Z}} c_k e^{ikx} \right|^2 \mathrm{d}x \asymp \sum_{k \in \mathbb{Z}} |c_k|^2$$

for every bounded interval I of length $\geq 2\pi$. Here and in the sequel we use the notation $A \ll B$ if there exists a positive constant α such that $A \leq \alpha B$ for all sequences (c_k), and $A \asymp B$ if $A \ll B$ and $B \ll A$.

Ingham [7] proved an important generalization of the last relation. A set Λ of real numbers (or more generally a set of vectors in $\mathbb{R}^d$) is called *uniformly separated* if

$$\gamma(\Lambda) := \inf\{|\lambda_1 - \lambda_2| \ : \ \lambda_1, \lambda_2 \in \Lambda \quad \text{and} \quad \lambda_1 \neq \lambda_2\} > 0; \tag{2.1}$$

then $\gamma(\Lambda)$ is called the *uniform gap* of Λ. For example, $\mathbb{Z}$ is uniformly separated with $\gamma(\mathbb{Z}) = 1$. Note that the empty set and the one-point sets are uniformly separated with $\gamma(\Lambda) = \infty$.

Theorem A (Ingham)**.** *Let $\Lambda \subset \mathbb{R}$ be a uniformly separated set.*

(i) $\sum_{\lambda\in\Lambda} c_\lambda e^{i\lambda x}$ *is a well-defined locally square summable function on $\mathbb{R}$ for every square summable sequence (c_λ).*

(ii) *The direct inequality*

$$\int_I \left|\sum_{\lambda\in\Lambda} c_\lambda e^{i\lambda x}\right|^2 dx \ll \sum_{\lambda\in\Lambda} |c_\lambda|^2$$

holds for every bounded interval I.

(iii) *The inverse inequality*

$$\sum_{\lambda\in\Lambda} |c_\lambda|^2 \ll \int_I \left|\sum_{\lambda\in\Lambda} c_k e^{i\lambda x}\right|^2 dx$$

holds for every bounded interval I of length $> \frac{2\pi}{\gamma(\Lambda)}$.

Remark 2.1. If Λ is not uniformly separated, but it is the union of finitely many, say m uniformly separated sets, then a simple application of the inequality

$$(x_1 + \cdots + x_m)^2 \leq m(x_1^2 + \cdots + x_m^2)$$

shows that the direct inequality still holds.

Definition 2.2. Let $\Lambda \subset \mathbb{R}^d$ be a uniformly separated set and $U \subset \mathbb{R}^d$ a bounded open set. We say that Λ is *associated* with U if $\left\|\sum_{\lambda\in\Lambda} a_\lambda e^{i\langle\lambda,x\rangle}\right\|_{L^2(U)} \asymp \|a_\lambda\|_{\ell^2(\Lambda)}$ for all $(a_\lambda) \in \ell^2(\Lambda)$.

Ingham's Theorem can then be restated as follows: *a uniformly separated set $\Lambda \subset \mathbb{R}$ is associated with every interval I of length $> \frac{2\pi}{\gamma(\Lambda)}$.* Kahane [11] extended this result to higher dimensions:

Theorem B (Kahane)**.** *There is a constant χ_d such that a uniformly separated set $\Lambda \subset \mathbb{R}^d$ is associated with every bounded set containing a ball B of radius $R > \frac{\chi_d}{\gamma(\Lambda)}$.*

For $d > 2$ his constants χ_d were improved by a different proof in [1]; see also [16].

The condition on the radius of the ball B was further improved by Kahane by partitioning the set Λ:

Theorem C (Kahane)**.** *Let $\Lambda \subset \mathbb{R}^d$ be a uniformly separated set and consider a partition $\Lambda = \bigcup_{j=1}^n \Lambda_j$. If each Λ_j is associated with a bounded open set U_j, then Λ is associated with*

$$U_1 + \cdots + U_n := \{x_1 + \cdots + x_n \ : \ x_j \in U_j, \quad j = 1, \ldots, n\}.$$

In particular, Λ is associated with every bounded set containing a ball of radius

$$R > \sum_{j=1}^{n} \frac{\chi_d}{\gamma(\Lambda_j)}.$$

The quantity $\sum_{j=1}^{n} \frac{\chi_d}{\gamma(\Lambda_j)}$ may be much smaller than $\frac{\chi_d}{\gamma(\Lambda)}$. For example, in the one-dimensional case Theorem C implies Beurling's [4] optimal condition on the length of the interval I for the inverse inequality. See also [2] for a generalization of Beurling's theorem for weakly separated sets.

Theorem C also implies a theorem of Haraux [6]:

Corollary D (Haraux)**.** *If $\Lambda \subset \mathbb{R}$ is a uniformly separated set and $F \subset \Lambda$ is a finite subset, then the inverse inequality of Theorem A holds under the condition $|I| > \frac{2\pi}{\gamma(\Lambda \setminus F)}$.*

Proof. Writing $F = \{\lambda_1, \dots, \lambda_{n-1}\}$, we may apply Theorem C with the one-point sets $\Lambda_j := \{\lambda_j\}$ for $j = 1, \dots, n-1$, and with $\Lambda_n := \Lambda \setminus F$. □

We mention, however, that Haraux's original proof was much more elementary and constructive.

Example 2.3. The square numbers $0, 1, 4, 9, \dots$ form a uniformly separated set Λ with $\gamma(\Lambda) = 1$. If $F = \{0, 1, \dots, m-1\}$ for some positive integer m, then $\gamma(\Lambda \setminus F) = 2m+1$, so that the inverse inequality holds under the condition $|I| > \frac{2\pi}{2m+1}$. Since m may be chosen arbitrarily large, hence the inverse inequality holds for all non-degenerated bounded intervals.

Remark 2.4. It was shown in [16, Chapter 9] that the above direct and inverse inequalities remain valid under the same assumptions for more general sums of the form

$$\sum_{\lambda \in \Lambda'} \sum_{\alpha \in N_\lambda} c_{\lambda',\alpha} x^\alpha e^{i\langle \lambda', x\rangle} + \sum_{\lambda \in \Lambda} c_\lambda e^{i\langle \lambda, x\rangle}$$

where Λ' is some finite subset of $\mathbb{R}^d \setminus \Lambda$, and $N_\lambda \subset \mathbf{N}^d$ is some finite set for each $\lambda \in \Lambda'$.

3. One-dimensional Schrödinger equation

We consider the one-dimensional Schrödinger equation on a bounded interval with periodic boundary condition. Up to an affine change of variable, we may assume that the interval is $(0, 2\pi)$. Thus we consider the following system

$$\begin{cases} u_t + iu_{xx} = 0 & \text{in } \mathbb{R} \times (0, 2\pi), \\ u(t,0) = u(t, 2\pi) & \text{for } t \in \mathbb{R}, \\ u_x(t,0) = u_x(t, 2\pi) & \text{for } t \in \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for } x \in (0, 2\pi). \end{cases} \tag{3.1}$$

Setting $L^2 := L^2(0, 2\pi)$ for brevity and introducing the Sobolev space

$$H^2_p := \left\{ v \in H^2(0, 2\pi) \ : \ v(0) = v(2\pi) \quad \text{and} \quad v_x(0) = v_x(2\pi) \right\},$$

for each initial datum $u_0 \in H^2_p$ there is a unique weak solution

$$u \in C(\mathbb{R}, H^2_p) \cap C^1(\mathbb{R}, L^2).$$

Furthermore, u has a Fourier series representation

$$u(t,x) = \sum_{k \in \mathbb{Z}} c_k e^{i(k^2 t + kx)} \tag{3.2}$$

where the c_k's are the Fourier coefficients of u_0:

$$u_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}.$$

In particular, the c_k satisfy Parseval's identity

$$\sum_{k\in\mathbb{Z}} |c_k|^2 = \frac{1}{2\pi} \|u_0\|_{L^2}^2 .$$

Using (3.2) we extend the solutions to $\mathbb{R}^2$ by 2π-periodicity in x and t.

First we ask whether the observability of the solutions on a fixed line segment of $\mathbb{R}^2$ allows us to identify the unknown initial datum.

The case of vertical segments is easy: since the exponential functions e^{ikx} form an orthogonal basis in $L^2(I)$ on every interval I of length 2π, we infer from the formula

$$u(t_1, x) = \sum_{k\in\mathbb{Z}} \left(c_k e^{ik^2 t_1}\right) e^{ikx}$$

that the knowledge of u on a segment $\{t_1\}\times I$ determines u_0 if and only if $|I| \geq 2\pi$. Indeed, if $(d_k)_{k\in\mathbb{Z}}$ is a square summable sequence of complex numbers, then the formula $f(x) := \sum_{k\in\mathbb{Z}} d_k e^{ikx}$ defines a 2π-periodic, locally square summable function. The restriction of f to any interval of length $\geq 2\pi$ allows us to determine the coefficients d_k. On the other hand, the restriction of f to an interval I of length $< 2\pi$ does not allows us to determine the coefficients d_k because there are infinitely many different 2π-periodic, locally square summable functions that coincide with f on I, and they do not have the same Fourier coefficients. Since t_1 is given, our assertion follows by taking $d_k := c_k e^{ik^2 t_1}$.

In the case $|I| \geq 2\pi$ we also have the quantitative relation

$$\int_I |u(t_1, x)| \, \mathrm{d}x \asymp \sum_{k\in\mathbb{Z}} |c_k|^2 .$$

The case of horizontal segments (pointwise observability) is different: we infer from the equality

$$u(t, x_1) = \sum_{k\in\mathbb{Z}} \left(c_k e^{ikx_1}\right) e^{ik^2 t} = c_0 + \sum_{k=1}^{\infty} e^{ik^2 t} \left(c_k e^{ikx_1} + c_{-k} e^{-ikx_1}\right)$$

that the knowledge of u even on the line on a segment $\mathbb{R} \times \{x_1\}$ does not determine u_0. For example, if $u_0(x) = e^{-ix_1} e^{ix} - e^{ix_1} e^{-ix}$, that is $c_1 = e^{-ix_1}$, $c_{-1} = -e^{ix_1}$ and $c_k = 0$ for all other k's, then $u(t, x_1) = 0$ for all $t \in \mathbb{R}$, although $u(t, x)$ is not the zero solution.

The situation is much better for most *oblique* segments:

Theorem 3.1. *Fix $(t_1, x_1) \in \mathbb{R}^2$, $a \in \mathbb{R}$ and $T > 0$ arbitrarily, and consider the solutions of* (3.1).

(i) *The direct inequality*

$$\int_0^T |u(t_1 + t, x_1 - at)|^2 \, dt \ll \sum_{k\in\mathbb{Z}} |c_k|^2$$

always holds.

(ii) *If $a \notin \mathbb{Z}$, then the inverse inequality*

$$\sum_{k\in\mathbb{Z}} |c_k|^2 \ll \int_0^T |u(t_1 + t, x_1 - at)|^2 \, dt \tag{3.3}$$

also holds.

(iii) *If $a \in \mathbb{Z}$, then*

$$\int_0^T |u(t_1+t, x_1-at)|^2 \, dt \asymp \sum_{k\in\mathbb{Z}} |d_k + d_{a-k}|^2 , \tag{3.4}$$

where we use the notations

$$d_k := c_k e^{i(k^2 t_1 + k x_1)}, \quad k \in \mathbb{Z}.$$

In particular, then there exist non-trivial solutions satisfying

$$u(t_1+t, x_1-at) = 0 \quad \textit{for all} \quad t \in \mathbb{R}, \tag{3.5}$$

and therefore the inverse inequality (3.3) *fails.*

Changing $u(t,x)$ to $v(t,x) := u(-t,x)$ we see that analogous results hold if we change the equation in (3.1) to $u_t - iu_{xx} = 0$.

Proof. (i) For any fixed $a \in \mathbb{R}$ a straightforward computation shows that

$$u(t_1+t, x_1-at) = \sum_{k\in\mathbb{Z}} c_k e^{i(k^2(t_1+t)+k(x_1-at))} = \sum_{k\in\mathbb{Z}} d_k e^{i(k^2-ak)t}. \tag{3.6}$$

Since $\Lambda := \{k^2 - ak \ : \ k \in \mathbb{Z}\}$ is the union of

$$\left\{k^2 - ak \ : \ k \in \mathbb{Z}, \quad k \geq a/2\right\} \quad \text{and} \quad \left\{k^2 - ak \ : \ k \in \mathbb{Z}, \quad k < a/2\right\},$$

it suffices to show that latter two sets are uniformly discrete. (In view of Theorem A (i) this will also show that the restrictions of the solutions for segments are well defined.) This follows from the following inequalities: if $k \geq a/2$

$$\big((k+1)^2 - a(k+1)\big) - (k^2 - ak) = 2k - a + 1 \geq 1$$

while if $k < a/2$,

$$(k^2 - ak) - \big((k-1)^2 - a(k-1)\big) = 2k - a - 1 \leq -1.$$

(ii) If $a \notin \mathbb{Z}$, then the set $\{k^2 - ak \ : \ k \in \mathbb{Z}\}$ itself is uniformly discrete. Indeed, if k and m are different integers, then

$$|(k^2 - ak) - (m^2 - am)| = |k-m||k+m-a| \geq d(a, \mathbb{Z}) := \max(a - [a], [a] + 1 - a)$$

where $[a]$ is the integer part of a. If, for some positive integer N, $k \neq m$ and $k, m \notin \{-N, \dots, N\}$, then, using again the identity

$$(k^2 - ak) - (m^2 - am) = (k-m)(k+m-a),$$

we have

$$\big|(k^2 - ak) - (m^2 - am)\big| \geq \begin{cases} 2N - a & \text{if } k > m \geq N, \\ 2N + a & \text{if } k < m \leq -N, \\ 2Nd(a, \mathbb{Z}) & \text{if } km < 0. \end{cases}$$

It follows that

$$\gamma(\Lambda \setminus \{-N, \dots, N\}) \geq N d(a, \mathbb{Z})$$

for all integers $N > |a|$. Letting $N \to \infty$ and applying Corollary C we get the inverse inequality (3.3).

(iii) If $a \in \mathbb{Z}$, then we may rewrite (3.6) in the form

$$u(t_1+t, x_1-at) = d_{a/2} e^{-i(a^2/4)t} + \sum_{k\in\mathbb{Z}, k>a/2} (d_k + d_{a-k}) e^{i(k^2-ak)t}$$

with the convention $d_{a/2} := 0$ if a is an odd integer.

Since the set

$$\left\{k^2 - ak \ : \ k \in \mathbb{Z}, \quad k \geq a/2\right\}$$

is uniformly separated, and

$$\left|(k^2 - ak) - (m^2 - am)\right| \geq 2N - a$$

whenever $k > m \geq N$, applying Corollary C we get (3.4) because

$$\left|d_{a/2}\right|^2 + \sum_{k\in\mathbb{Z},\ k>a/2}^{\infty} \left|d_k + d_{a-k}\right|^2 \asymp \sum_{k\in\mathbb{Z}}^{\infty} \left|d_k + d_{a-k}\right|^2 .$$

It follows from (3.4) that all solutions satisfying $d_{a/2} = 0$ and $d_k + d_{a-k} = 0$ for all $k \in \mathbb{Z}$ satisfy the equality (3.5). If at least one of these coefficients is different from zero, then the right side of (3.3) vanishes, while the left side is positive.

A concrete nonzero function satisfying (3.5) may be given as follows. We choose an integer $k \neq a/2$ and then two nonzero numbers c_k, c_{a-k} satisfying the equality

$$c_k e^{i(k^2 t_1 + k x_1)} + c_{a-k} e^{i((a-k)^2 t_1 + (a-k) x_1)} = 0.$$

Then the function

$$u(t,x) := c_k e^{i(k^2 t + kx)} + c_{a-k} e^{i((a-k)^2 t + (a-k)x)}$$

has the required properties. □

Next we investigate what happens if we observe the solutions on two or more segments. In view of Theorem 3.1 (i) we only investigate the validity of the inverse inequalities.

Theorem 3.2. *Fix $T > 0$ arbitrarily, and consider the solutions of* (3.1).

(i) *Let $(t_1, x_1), (t_2, x_2) \in \mathbb{R}^2$, and a_1, a_2 two different integers. If*

$$u(t_1 + t, x_1 - a_1 t) = u(t_2 + t, x_2 - a_2 t) = 0 \quad \textit{for all} \quad t \in (0, T),$$

then u is the trivial solution, i.e., $u_0 = 0$. Nevertheless, the inverse inequality

$$\sum_{k\in\mathbb{Z}} |c_k|^2 \ll \int_0^T \left|u(t_1 + t, x_1 - a_1 t)\right|^2 + \left|u(t_2 + t, x_2 - a_2 t)\right|^2 \, dt$$

fails.

(ii) *The inverse inequality*

$$\sum_{k\in\mathbb{Z}} |c_k|^2 \ll \sum_{i=1}^{m} \int_0^T \left|u(t_1 + t, x_1 - a_i t)\right|^2 \, dt$$

also fails for any $(t_1, x_1) \in \mathbb{R}^2$ and any finite number of integers $a_1, \ldots, a_m$.

Proof. (i) Writing $d_k = c_k e^{i(k^2 t_1 + k x_1)}$ again, the equality $u(t_1 + t, x_1 - a_1 t) = 0$ for $t \in (0, T)$ together with (3.4) imply that $d_k = -d_{a_1 - k}$ for every k. It follows that $|c_k| = |c_{a_1 - k}|$ for every k.

Similarly, $u(t_2 + t, x_2 - a_2 t) = 0$ for $t \in (0, T)$ implies $|c_k| = |c_{a_2 - k}|$ for every k. But then $|c_k| = |c_{a_2 - k}| = |c_{a_1 - (a_1 - a_2 + k)}| = |c_{a_1 - a_2 + k}|$, that is $|c_k|$ is $(a_1 - a_2)$-periodic. As $|c_k|$ is square-summable, this can only happen if $c_k = 0$ for every k.

For the second part, we construct a sequence $(u_{0,n}) \subset L^2(0, 2\pi)$ of initial data such that the corresponding solutions u_n satisfy the relation

$$\frac{1}{\|u_{0,n}\|_2^2} \sum_{i=1}^{2} \int_0^T \left|u_n(t_i + t, x_i - a_i t)\right|^2 \, \mathrm{d}t \to 0. \tag{3.7}$$

By the previous theorem the solutions satisfy the relation

$$\sum_{i=1}^{2}\int_0^T |u(t_i+t,x_i-a_it)|^2\,\mathrm{d}t \asymp \sum_{k\in\mathbb{Z}}^{\infty} |d_k+d_{a_1-k}|^2+|e_k+e_{a_2-k}|^2$$

with the notations

$$d_k := c_k e^{i(k^2t_1+kx_1)} \quad \text{and} \quad e_k := c_k e^{i(k^2t_2+kx_2)}.$$

Assuming by symmetry that $a_2>a_1$, and setting $p=a_2-a_1$ the relation may be rewritten in the form

$$\sum_{i=1}^{2}\int_0^T |u(t_i+t,x_i-a_it)|^2\,\mathrm{d}t \asymp \sum_{k\in\mathbb{Z}}^{\infty} |d_k+d_{a_1-k}|^2+|\omega_k d_k+d_{a_1-k+p}|^2$$

with suitable unimodular complex numbers ω_k.

Fix an integer $q>\frac{a_1}{2}$. For any fixed positive integer n we define consecutively the following numbers d_k:

$$\begin{aligned}
&d_q := 1, && d_{a_1-q} := -d_q,\\
&d_{q+p} := -\omega_{a_1-q}d_{a_1-q}, && d_{a_1-q-p} := -d_{q+p},\\
&d_{q+2p} := -\omega_{a_1-q-p}d_{a_1-q-p}, && d_{a_1-q-2p} := -d_{q+2p},\\
&\dots\\
&d_{q+np} := -\omega_{a_1-q-(n-1)p}d_{a_1-q-(n-1)p}, && d_{a_1-q-np} := -d_{q+np}.
\end{aligned}$$

Setting $d_k :=$ for all other indices, we obtain a trigonometric polynomial

$$u_{0,n}(x)=\sum_{k\in\mathbb{Z}} c_k e^{ikx} := \sum_{k\in\mathbb{Z}} d_k e^{-i(k^2t_1+kx_1)}e^{ikx}$$

satisfying

$$\|u_{0,n}\|_2^2 = 2\pi\sum_{k\in\mathbb{Z}}|c_k|^2 = 2n+2$$

and

$$\sum_{k\in\mathbb{Z}}^{\infty} |d_k+d_{a_1-k}|^2+|\omega_k d_k+d_{a_1-k+p}|^2 = 2.$$

This proves (3.7).

(ii) Note that in this part, (x_1,t_1) is the same for each $a_1,\dots,a_m$. We will take advantage of this to construct the sequence d_k.

Now we are looking for a sequence $(u_{0,n})\subset L^2(0,2\pi)$ of initial data such that the corresponding solutions u_n satisfy the relation

$$\frac{1}{\|u_{0,n}\|_2^2}\sum_{i=1}^{m}\int_0^T |u_n(t_1+t,x_1-a_it)|^2\,\mathrm{d}t \to 0.$$

For any fixed positive integer n we consider the numbers

$$d_k := \begin{cases} \operatorname{sgn} k & \text{if} \quad |k|\le n,\\ 0 & \text{if} \quad |k|>n,\end{cases}$$

and we define

$$u_n(x) := \sum_{k=-n}^{n} c_k e^{ikx} \quad \text{with} \quad c_k := d_k e^{-i(k^2t_1+kx_1)}.$$

Then

$$\|u_{0,n}\|_2^2 = 2\pi \sum_{k\in\mathbb{Z}} |c_k|^2 = 2\pi \sum_{k\in\mathbb{Z}} |d_k|^2 = 4n\pi.$$

On the other hand, using (3.4),

$$\sum_{i=1}^{m} \int_0^T |u(t_1+t, x_1 - a_i t)|^2 \, \mathrm{d}t \asymp \sum_{i=1}^{m} \sum_{k\in\mathbb{Z}}^{\infty} |d_k + d_{a_i - k}|^2 .$$

Therefore we will reach a contradiction if we bound $\sum_{k\in\mathbb{Z}}^{\infty} |d_k + d_{a_i-k}|^2$ independently of n.

Fix i arbitrarily and write $a := a_i$ for brevity. The sequence $(d_k + d_{a-k})$ takes only the values $-2, -1, 0, 1, 2$. It suffices to show that the number of k's for which $d_k + d_{a-k} \neq 0$ is $\ll 1 + |a|$. By the symmetry of the sequence (d_k) it suffices to consider the values 1 and 2.

We have

$$d_k + d_{a-k} = 2 \Longleftrightarrow 1 \le k \le n \quad \text{and} \quad 1 \le a - k \le n \Longrightarrow 1 \le k \le 1 + a,$$

so that we have either no such k if $a < 0$ or at most $1 + a$ such indices k if $a > 0$.

Next, we have $d_k + d_{a-k} = 1$ in the following three cases:

$$\begin{aligned} &1 \le k \le n \quad \text{and} \quad a - k \ge n + 1 \Longrightarrow 1 \le k \le a - 1 - n; \\ &1 \le k \le n \quad \text{and} \quad a - k = 0; \\ &1 \le k \le n \quad \text{and} \quad a - k \le -n - 1 \Longrightarrow a + 1 + n \le k \le n, \end{aligned}$$

and in three other symmetric cases by exchanging k and $a - k$.

Since the first two cases above may only occur for $a > 0$, while the third case only for $a < 0$, at most

$$\max\{(a - 1 - n) + 1, -a\} \le |a|$$

indices k satisfy one of them. We have the same upper bound for the three symmetric cases, so that there are at most $2|a|$ indices k for which $d_k + d_{a-k} = 1$. □

We may also consider other boundary conditions. Let us consider for example the Dirichlet condition:

$$\begin{cases} u_t + i u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \pi), \\ u(t, 0) = u(t, \pi) = 0 & \text{for } t \in \mathbb{R}, \\ u(0, x) = u_0(x) & \text{for } x \in (0, \pi). \end{cases} \tag{3.8}$$

The problem is well posed for every $u_0 \in H_0^2(0, \pi)$. Let us observe that extending an arbitrary solution of (3.8) to a 2π-periodic odd function in the x variable we obtain a solution of (3.1). Therefore Theorem 3.1 (i), (ii), (iii) and Theorem 3.2 (i) remain valid for the solutions of (3.8).

The remaining parts were based on the construction of special solutions, so we need some additional arguments. We have the following

Proposition 3.3. *Fix $T > 0$ arbitrarily, and consider the solutions of* (3.8).

(i) *For any given $(t_1, x_1) \in \mathbb{R}^2$ and $a \in \mathbb{Z}$ there exist non-trivial solutions of* (3.8) *satisfying*

$$u(t_1 + t, x_1 - at) = 0 \quad \textit{for all} \quad t \in \mathbb{R}.$$

(ii) *The inverse inequality*

$$\sum_{k\in\mathbb{Z}} |c_k|^2 \ll \sum_{i=1}^{m} \int_0^T |u(t,-a_i t)|^2 \, dt$$

fails for any $T>0$ and for any finite number of integers $a_1,\dots,a_m$.

Proof. The solutions of (3.8) are given by the series

$$u(t,x) = \sum_{k\in\mathbb{Z}} c_k e^{i(k^2 t + kx)}$$

with suitable square summable complex coefficients c_k satisfying the relations $c_k + c_{-k} = 0$.

(i) We choose an integer k for which the four numbers $k, a-k, -k, k-a$ are different that and then two nonzero numbers c_k, c_{a-k} satisfying the equality

$$c_k e^{i(k^2 t_1 + k x_1)} + c_{a-k} e^{i((a-k)^2 t_1 + (a-k)x_1)} = 0.$$

Then the function

$$\begin{aligned} u(t,x) :&= c_k \sin(k^2 t + kx) + c_{a-k}\sin((a-k)^2 t + (a-k)x) \\ &= \Big(\frac{c_k}{2i} e^{i(k^2 t + kx)} + \frac{c_{a-k}}{2i} e^{i((a-k)^2 t + (a-k)x)}\Big) - \Big(\frac{c_k}{2i} e^{i(k^2 t - kx)} + \frac{c_{a-k}}{2i} e^{i((a-k)^2 t - (a-k)x)}\Big) \end{aligned}$$

has the required properties by the same arguments as in the proof of Theorem 3.1 (iv).

(ii) Since $(t_1, x_1) = (0,0)$ and therefore $d_k = c_k$ for all k, the sequences constructed in the proof of Theorem 3.2 (ii) define solutions of not only (3.1), but also of (3.8). □

4. Beam equation

We consider the one-dimensional linear beam equation with periodic boundary conditions:

$$\begin{cases} u_{tt} + u_{xxxx} = 0 & \text{in } \mathbb{R}\times(0,2\pi), \\ u(t,0) = u(t,2\pi) & \text{for } t\in\mathbb{R}, \\ u_x(t,0) = u_x(t,2\pi) & \text{for } t\in\mathbb{R}, \\ u(0,x) = u_0(x) & \text{for } x\in(0,2\pi), \\ u_t(0,x) = u_1(x) & \text{for } x\in(0,2\pi). \end{cases} \tag{4.1}$$

For any given initial data $u_0 \in H^2_p$ and $u_1 \in L^2$ there is a unique weak solution

$$u \in C(\mathbb{R}, H^2_p) \cap C^1(\mathbb{R}, L^2).$$

Furthermore, u has a Fourier series representation

$$u(t,x) = c_0^+ + c_0^- t + \sum_{k\in\mathbb{Z}^*} \Big(c_k^+ e^{i(k^2 t + kx)} + c_k^- e^{i(-k^2 t + kx)}\Big) \tag{4.2}$$

with suitable square summable complex coefficients c_k^+, c_k^- satisfying the relations

$$\sum_{k\in\mathbb{Z}} (1+k^4)(|c_k^+|^2 + |c_k^-|^2) \asymp \|u_0\|^2_{H^2_p} + \|u_1\|^2_{L^2}.$$

Using (4.2) we extend the solutions to $\mathbb{R}^2$ by 2π-periodicity in x.

Remark 4.1. Observe that (4.2) is no longer a trigonometric series if $c_0^- \neq 0$. However, the results and proofs of this section remain valid in the general case by Remark 2.4.

First we consider the observation of the solutions on vertical line segments. (Analogous theorems have been proved in [21] for the Klein–Gordon equations by a different approach.)

Theorem 4.2. *Fix two distinct nonzero real numbers t_1, t_2, a number $X > 0$, and consider the solutions of* (4.1).

(i) *The direct inequality*

$$\int_0^X |u(t_1,x)|^2 + |u(t_2,x)|^2\,dx \ll \sum_{k\in\mathbb{Z}} (|c_k^+|^2 + |c_k^-|^2) \tag{4.3}$$

and the weakened inverse inequality

$$|c_0^+|^2 + |c_0^-|^2 + \sum_{k\in\mathbb{Z}^*} \sin^2 k^2(t_1-t_2)(|c_k^+|^2 + |c_k^-|^2) \ll \int_0^X |u(t_1,x)|^2 + |u(t_2,x)|^2\,dx \tag{4.4}$$

always holds.

(ii) *If $(t_2-t_1)/\pi$ is irrational, then the right hand side of* (4.4) *does not vanish for any non-trivial solution.*

(iii) *If $(t_2-t_1)/\pi$ is rational, there exist non-trivial solutions for which the right hand side of* (4.4) *vanishes.*

(iv) *The inverse inequality*

$$\sum_{k\in\mathbb{Z}} \left(\left|c_k^+\right|^2 + \left|c_k^-\right|^2\right) \ll \int_0^X |u(t_1,x)|^2 + |u(t_2,x)|^2\,dx \tag{4.5}$$

always fails.

Proof. (i) Since

$$u(t_j,x) = c_0^+ + c_0^- t_j + \sum_{k\in\mathbb{Z}^*} \left(c_k^+ e^{ik^2t_j} + c_k^- e^{-ik^2t_j}\right) e^{ikx},$$

for $j = 1, 2$, applying Corollary C we get the relations

$$\left|c_0^+ + c_0^- t_j\right|^2 + \sum_{k\in\mathbb{Z}^*} \left|c_k^+ e^{ik^2t_j} + c_k^- e^{-ik^2t_j}\right|^2 \asymp \int_0^X |u(t_j,x)|^2\,\mathrm{d}x. \tag{4.6}$$

They imply (4.3) by using the elementary inequality $|a+b|^2 \le 2\,|a|^2 + 2\,|b|^2$.

The relations (4.4) follows by adding (4.6) for $j = 1, 2$, and using for each $k \in \mathbb{Z}^*$ the following estimates with $a = k^2t_1$ and $b = k^2t_2$:

$$\begin{aligned}
\left|xe^{ia} + ye^{-ia}\right|^2 + \left|xe^{ib} + ye^{-ib}\right|^2 &= \left|xe^{2ia} + y\right|^2 + \left|xe^{2ib} + y\right|^2 \\
&= 2(|x|^2 + |y|^2) + 2\Re\left(x\overline{y}\left(e^{2ia} + e^{2ib}\right)\right) \\
&\ge (|x|^2 + |y|^2)\left(2 - \left|e^{2ia} + e^{2ib}\right|\right) \\
&= (|x|^2 + |y|^2)\frac{4 - \left|e^{2ia} + e^{2ib}\right|^2}{2 + |e^{2ia} + e^{2ib}|} \\
&= (|x|^2 + |y|^2)\frac{4\sin^2(a-b)}{2 + |e^{2ia} + e^{2ib}|} \\
&\ge (|x|^2 + |y|^2)\sin^2(a-b).
\end{aligned}$$

(ii) If $(t_2 - t_1)/\pi$ is irrational and the right side of (4.4) vanishes for some solution u, then we infer from (4.4) that all coefficients $c_k^{\pm}$ are equal to zero because $\sin^2 k^2(t_1 - t_2) \neq 0$ for all $k \in \mathbb{Z}^*$, so that u is the trivial solution.

(iii) If $(t_2 - t_1)/\pi$ is rational, there exists a nonzero integer k such that $k^2(t_1 - t_2)$ is a multiple of 2π. Then the formula

$$u(t,x) = \left(e^{-ik^2t_1}e^{ik^2t} - e^{ik^2t_1}e^{-ik^2t}\right)e^{ikx}$$

defines a non-trivial solution of (4.1) such that $u(t_1, x) = u(t_2, x) = 0$ for all $x \in \mathbb{R}$.

(iv) It follows from (4.6) that the inverse inequality (4.5) holds if and only if the matrices

$$A_k := \begin{pmatrix} e^{ik^2t_1} & e^{-ik^2t_1} \\ e^{ik^2t_2} & e^{-ik^2t_2} \end{pmatrix}$$

are invertible, and the norms of their inverses are bounded by some uniform constant.

If $(t_2 - t_1)/\pi$ is rational, then not all matrices A_k are invertible by (iii). Otherwise, by the irrationality there exists a sequence (k_j) of positive integers such that $k_j^2(t_1 - t_2) \to 0$ mod 2π, and then the above norms tend to ∞ as $j \to \infty$. □

Now we turn to the case of oblique segments. Given a real number a, if u is a solution of (4.1), then a straightforward computation shows that

$$u(t_0 + t, x_0 - at) = d_0^+ + d_0^- t + \sum_{k\in\mathbb{Z}^*} \left(d_k^+ e^{i(k^2-ak)t} + d_k^- e^{i(-k^2-ak)t}\right), \tag{4.7}$$

where we use the notations

$$d_0^+ := c_0^+ + c_0^- t_0, \quad d_0^- := c_0^-$$

and

$$d_k^+ := c_k^+ e^{i(k^2t_0+kx_0)}, \quad d_k^- := c_k^- e^{i(-k^2t_0+kx_0)} \quad \text{for} \quad k \in \mathbb{Z}^*.$$

Observe that

$$\sum_{k\in\mathbb{Z}} \left(\left|d_k^+\right|^2 + \left|d_k^-\right|^2\right) \asymp \sum_{k\in\mathbb{Z}} \left(\left|c_k^+\right|^2 + \left|c_k^-\right|^2\right).$$

In order to state our results we introduce the circle $S_a \subset \mathbb{R}^2$ centered in $(a/2, -a/2)$ and passing through the origin. Its cartesian equation is

$$\left(x - \frac{a}{2}\right)^2 + \left(y + \frac{a}{2}\right)^2 = \frac{a^2}{2} \quad \text{or equivalently} \quad x^2 - ax + y^2 + ay = 0.$$

Furthermore, we introduce the set $A_a = S_a \cap \mathbb{Z}^2 \setminus \{(0,0)\}$.

Remarks 4.3.

(i) Since the distance between distinct elements of A_a is at least one, A_a cannot have more elements than the perimeter of the circle S_a: $|A_a| \leq \sqrt{2}\pi a$.

(ii) If a is irrational, then A_a is empty. Indeed, if $(k,m) \in A_a$, then $a = \frac{k^2+m^2}{k-m} \in \mathbb{Q}$.

(iii) If a is not an integer, then no element of A_a has any zero coordinate, and hence $A_a = (\mathbb{Z}^*)^2 \cap S_a$. Indeed, if $(k,0) \in A_a$, then $a = k \in \mathbb{Z}$ from the above equation of S_a.

Theorem 4.4. *Fix $(t_1, x_1) \in \mathbb{R}^2$, $a \in \mathbb{R}$ and $T > 0$ arbitrarily, and consider the solutions of* (4.1).

(i) *The direct inequality*

$$\int_0^T |u(t_1+t, x_1-at)|^2 \, dt \ll \sum_{k\in\mathbb{Z}} \left(\left|c_k^+\right|^2 + \left|c_k^-\right|^2 \right)$$

always holds.

(ii) *If $a \neq 0$ and $A_a = \varnothing$, then the inverse inequality*

$$\sum_{k\in\mathbb{Z}} \left(\left|c_k^+\right|^2 + \left|c_k^-\right|^2 \right) \ll \int_0^T |u(t_1+t, x_1-at)|^2 \, dt$$

also holds. In particular, the inverse inequality holds whenever a is irrational.

(iii) *If $a \in \mathbb{Z}$ or if $A_a \neq \varnothing$, then there exist non-trivial solutions satisfying*

$$u(t_1+t, x_1-at) = 0 \quad \textit{for all} \quad t \in \mathbb{R},$$

so that the inverse inequality in (ii) fails.

Remark 4.5. Similarly to the Schrödinger equation, analogous results may be obtained for other boundary conditions; the details are left to the reader.

Proof of Theorem 4.4. (i) The proof of Theorem 3.1 (i) shows that the exponents in (4.7) form a finite union of uniformly separated sets. Hence the direct inequality holds by Theorem A and the remarks following Theorems A and B.

(ii) Since $(a, -a) \in A_a$ for all nonzero integers, $a \notin \mathbb{Z}$ by our assumptions, and therefore both sets

$$\left\{k^2 - ak \ : \ k \in \mathbb{Z}\right\} \quad \text{and} \quad \left\{-k^2 - ak \ : \ k \in \mathbb{Z}\right\}$$

are uniformly discrete by the proof of Theorem 3.1 (ii). In view of (4.7) we have to show that their union is also uniformly discrete.

This amounts to show that

$$\inf \left\{ \left|(k^2 - ak) - (-m^2 - am)\right| \ : \ (k,m) \in \mathbb{Z}^2 \setminus \{(0,0)\} \right\} > 0.$$

Since

$$(k^2 - ak) - (-m^2 - am) = \left(k - \frac{a}{2}\right)^2 + \left(m + \frac{a}{2}\right)^2 - \frac{a^2}{2},$$

this means that the circle S_a has a positive distance from the set $\mathbb{Z}^2 \setminus \{(0,0)\}$. Since the latter set is discrete, this is satisfied by our assumption $A_a = \varnothing$.

(iii) If $a \in \mathbb{Z}$, then the function given in the proof of Theorem 3.1 (iii) also solves (4.1). Otherwise choose $(k,m) \in S_a \cap \mathbb{Z}^2 \setminus \{(0,0)\}$, and set

$$u(t,x) := e^{i(-m^2 t_1 + m x_1)} e^{i(k^2 t + kx)} - e^{i(k^2 t_1 + k x_1)} e^{i(-m^2 t + mx)}.$$

Then $d_k^+ + d_m^- = 0$ and therefore

$$u(t_0+t, x_0-at) = (d_k^+ + d_m^-) e^{i(k^2 - ak)t} = 0$$

for all $t \in \mathbb{R}$. □

Now let us investigate the inverse inequality

$$\sum_{k\in\mathbb{Z}} \left(\left|c_k^+\right|^2 + \left|c_k^-\right|^2 \right) \ll \int_0^T |u(t_1+t, x_1-a_1 t)|^2 + |u(t_2+t, x_2-a_2 t)|^2 \, \mathrm{d}t \tag{4.8}$$

when we observe the solutions on two segments.

We start with some simple observations. We write A_j instead of A_{a_j} for brevity, and we denote by A_j^+, A_j^- its projection on the first and second coordinate axis, respectively.

Remarks 4.6.

(i) If $a_1 \neq 0$ and $A_1 = \varnothing$, then (4.8) holds by the preceding theorem. The same conclusion holds by symmetry if $a_2 \neq 0$ and $A_2 = \varnothing$.

(ii) The proof of Theorem 4.4 (ii) shows that

$$\sum_{k\in\mathbb{Z}\setminus A_j^+} \left|d_k^+\right|^2 + \sum_{m\in\mathbb{Z}\setminus A_j^-} \left|d_m^-\right|^2 + \sum_{(k,m)\in A_j} \left|d_k^+ + d_m^-\right|^2 \asymp \int_0^T |u(t_j+t, x_j - a_j t)|^2 \, \mathrm{d}t \tag{4.9}$$

for $j = 1, 2$. If $A_1^+ \cap A_2^+ = A_1^- \cap A_2^- = \varnothing$, then adding these estimates for $j = 1, 2$ the inequality (4.8) follows.

(iii) On the other hand, if $a_1 = a_2$ and $A_1 \neq \varnothing$, then (4.8) fails. Indeed, if $(k, m) \in A_1$ and $c \in \mathbb{R}$, then changing d_k^+ and $+d_m^-$ to $d_k^+ + c$ and $+d_m^- - c$ the right side of (4.9) remains unchanged.

Lemma 4.7. *If a_1, a_2 are different nonzero integers, then*

(i) $A_1 \cap A_2 = \varnothing$;

(ii) *if $(k, m), (k', m')$ are two different points in A_1 or A_2, then $k \neq k'$ and $m \neq m'$.*

Proof. (i) Since all circles S_a have the same tangent line in the origin, $S_1 \cap S_2 = \{(0, 0)\}$, and hence $A_1 \cap A_2 = \varnothing$.

(ii) If for example $(k, m), (k', m) \in A_1$ with $k \neq k'$, then both k and k' solve the equation $x^2 - a_1 x + m^2 + a_1 m = 0$, and hence $a_1 = k + k' \in \mathbb{Z}$. The other cases are similar. □

At this stage it is convenient to associate a graph $G(a_1, a_2)$ to a pair of distinct integers a_1, a_2. The vertices of this graph form the set $A_1 \cup A_2$. Two vertices are adjacent if they have a common coordinate. The previous lemma then states that this graph is bipartite, namely a vertex in A_1 can only be adjacent to a vertex in A_2 and vice versa. A direct consequence of this is that a vertex has at most two neighbours.

A (simple) path is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ where v_j, v_{j+1} are adjacent for every $j = 1, \ldots, n-1$. In particular, if the first coordinate is common in v_j, v_{j+1} then v_{j+1}, v_{j+2} have the second coordinate in common, and vice versa.

A simple path has at most $|A_1| + |A_2| \leq \sqrt{2}\pi(a_1 + a_2)$ elements. Furthermore, every $v \in A_1 \cup A_2$ belongs to a unique maximal simple path $v_{-\ell_1}, \ldots, v_0, \ldots, v_{\ell_2}$ (see the figure). This maximal path is called a *cycle* if $v_{-\ell_1}$ and v_{ℓ_2} are adjacent, that is, they have a common component. Note that a cycle has necessarily an even number of points.

Theorem 4.8. *Fix $(t_1, x_1), (t_2, x_2) \in \mathbb{R}^2$, two different nonzero integers a_1, a_2 and $T > 0$. The inverse inequality* (4.8) *fails if and only if $G(a_1, a_2)$ has a cycle.*

Proof. Adding the relations (4.9) for $j = 1, 2$ we see that the right hand side of (4.8) is

$$\asymp \sum_{k\in(\mathbb{Z}\setminus A_1^+)\cup(\mathbb{Z}\setminus A_2^+)} \left|d_k^+\right|^2 + \sum_{m\in(\mathbb{Z}\setminus A_1^-)\cup(\mathbb{Z}\setminus A_2^-)} \left|d_m^-\right|^2 + \sum_{(k,m)\in A_1\cup A_2} \left|d_k^+ + d_m^-\right|^2. \tag{4.10}$$

Since $A_1 \cup A_2$ is finite, the difference between (4.10) and the left hand side of (4.8) is a quadratic form in a finite number of variables. Therefore (4.8) is equivalent to the following uniqueness property: if the expression in (4.10) is zero, then all coefficients $d_k^\pm$ vanish.

Assume first that $G(a_1, a_2)$ has a cycle and write it as

$$(k_1, m_1), (k_2, m_1), (k_2, m_2), \ldots, (k_n, m_{n-1}), (k_n, m_n), (k_1, m_n).$$

Up to exchanging A_1 and A_2 we may assume that $(k_1, m_1) \in A_1$, so that $(k_1, m_n) \in A_2$. Note that $k_1, \dots, k_{n-1} \in A_1^+ \cup A_2^+$ and $m_1, \dots, m_n \in A_1^- \cup A_2^-$.

Then setting $d^+_{k_i} = 1$ and $d^-_{m_i} = -1$ for $i = 1, \dots, n$ and $d^+_k = d^-_m = 0$ for all other indices k, m, the expression in (4.10) vanishes.

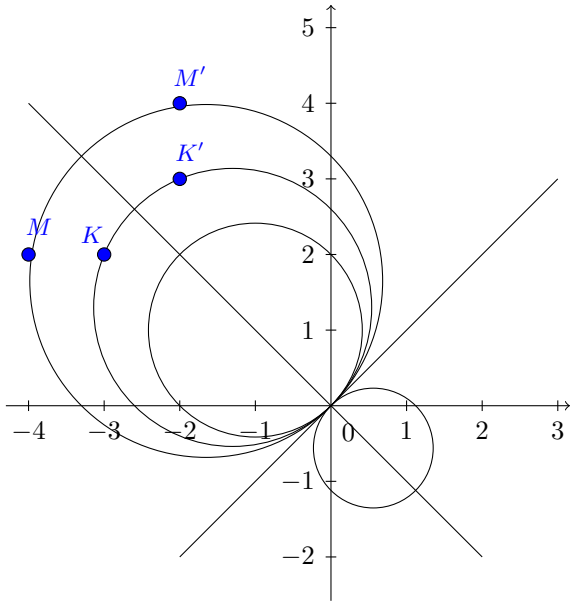
To prove the other direction, assume that $G(a_1, a_2)$ has no cycle, and consider an arbitrary maximal simple path. By symmetry between a_1 and a_2, we may assume that this path starts in A_1. Depending on whether the first move is horizontal or vertical, and whether the path ends in A_1 or A_2, there are four possibilities:

- The first move is horizontal and the path ends in A_2, so that the path has the form
$$(k_1, m_1), (k_2, m_1), (k_2, m_2), \dots, (k_n, m_{n-1})$$
with $(k_1, m_1) \in A_1$ and $(k_n, m_{n-1}) \in A_2$. Since the path is maximal, the only element of A_2 adjacent to (k_1, m_1) is (k_2, m_1); hence $k_1 \notin A_2^+$. Similarly, $k_n \notin A_1^+$.
- The first move is horizontal and the path ends in A_1: we have
$$(k_1, m_1), (k_2, m_1), (k_2, m_2), \dots, (k_n, m_{n-1}), (k_n, m_n)$$
with $(k_1, m_1) \in A_1$ and $(k_n, m_n) \in A_1$. The maximality implies that $k_1 \notin A_2^+$ and $m_n \notin A_2^-$.
- The first move is vertical and the path ends in A_2: we have
$$(k_1, m_1), (k_1, m_2), (k_2, m_2), \dots, (k_{n-1}, m_n)$$
with $(k_1, m_1) \in A_1$ and $(k_{n-1}, m_n) \in A_2$. The maximality implies that $m_1 \notin A_2^-$ and $m_n \notin A_1^-$
- The first move is vertical and the path ends in A_1: we have
$$(k_1, m_1), (k_1, m_2), (k_2, m_2), \dots, (k_{n-1}, m_n), (k_n, m_n)$$
with $(k_1, m_1) \in A_1$ and $(k_n, m_n) \in A_1$. The maximality implies that $m_1 \notin A_2^-$ and $k_n \notin A_2^+$.

Let us consider the first case; the others are similar. If the expression in (4.10) vanishes, then $d^\pm_k = 0$ whenever $k \notin A_1^\pm$ or $k \notin A_2^\pm$, and
$$\sum_{(k,m) \in A_1 \cup A_2} \left| d^+_k + d^-_m \right|^2 = 0.$$
In particular, $d^+_{k_1} = 0$, $d^+_{k_n} = 0$ and
$$|d^+_{k_1} + d^-_{m_1}|^2 + |d^+_{k_2} + d^-_{m_1}|^2 + \cdots + |d^+_{k_n} + d^-_{m_{n-1}}|^2 = 0,$$
whence $d^+_{k_j} + d^-_{m_j} = d^+_{k_{j+1}} + d^-_{m_j} = 0$ for $j = 1, \dots, n-1$. A direct induction then shows that $d^+_{k_j} = d^-_{m_j} = 0$ for $j = 1, \dots, n-1$.

Finally, since every $(k, j) \in A_1 \cup A_2$ belongs to a maximal simple path, we conclude that $d^+_k = d^-_m = 0$ for all k, m as claimed. □

Example 4.9. The figure below shows the situation where $a_1 = -\frac{13}{5}$ and $a_2 = -\frac{10}{3}$. There are two maximal simple paths KM and $K'M'$ having more than one element, and there are no cycles.



We have no concrete example in which $G(a_1, a_2)$ has a cycle. The following proposition indicates that if there exist such examples, they are rare.

Proposition 4.10. *The graph $G(a_1, a_2)$ has no cycle in the following cases:*

(i) *a_1 and a_2 have opposite nonzero signs;*
(ii) *a_1 and a_2 have equal signs, and $a_1/a_2 \geq 3/2$.*

Proof. (i) Without loss of generality, we may assume that $a \geq b > 0$ and $a_1 = -a$, $a_2 = b$.

Let us start with the observation that, to belong to a cycle, a point $(k, m) \in S_{-a}$ must have two neighbours $(k, m'), (k', m) \in S_b$.

Set

$$\tilde{b} = \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) b = \frac{\sqrt{2}-1}{2} b,$$

and denote by $\tilde{a}_+$ and $\tilde{a}_-$ the positive and negative root of $x^2 + ax + \tilde{b}^2 - a\tilde{b} = 0$, that is

$$\tilde{a}_+ = \frac{-a + \sqrt{a^2 + 4a\tilde{b} - 4\tilde{b}^2}}{2}, \quad \tilde{a}_- = \frac{-a - \sqrt{a^2 - 4a\tilde{b} - 4\tilde{b}^2}}{2}.$$

Finally, let $P = (-\tilde{b}, \tilde{a}_-)$ (resp. $Q = (-\tilde{a}_-, \tilde{b})$) be the point on S_{-a} such that the (vertical) (resp. horizontal) line through P and $(-\tilde{b}, -b/2)$ (resp. Q and $(b/2, \tilde{b})$) is tangent to S_b.

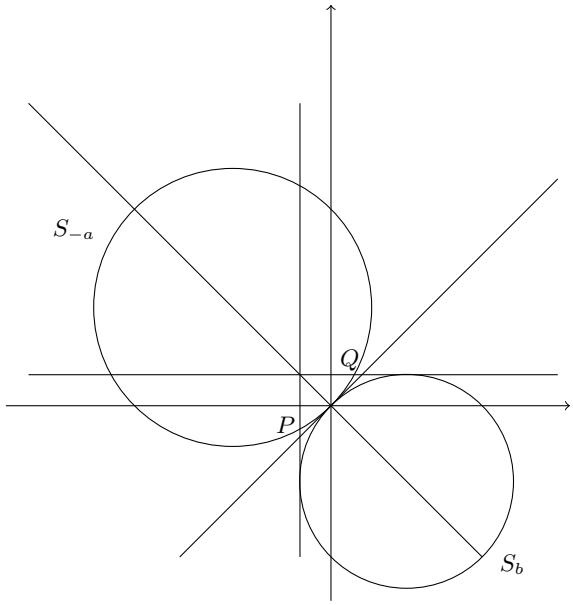


FIGURE: Case (i)

Note that P and Q divide S_{-a} into two arcs. Denote by Γ^0 the one through $(0,0)$ and set

$$\Gamma^0_+ := \Gamma^0 \cap \{(x,y), x \geq 0, y \geq 0\} \quad \text{and} \quad \Gamma^0_- := \Gamma^0 \cap \{(x,y), x \leq 0, y \leq 0\}.$$

Now observe that a vertex of $G(-a,b)$ on $S_{-a} \setminus \Gamma^0$ has at most one neighbour, so that it cannot belong to a cycle. Furthermore, a vertex of $G(-a,b)$ on Γ^0_+ has no neighbour on Γ^0_-. Therefore, if there exists a cycle, then its points on S_{-a} should all belong to Γ^0_+, or should all belong to Γ^0_-. Since the reflection of a cycle with respect to the anti-diagonal is also a cycle, it remains to prove that there is no cycle all of whose points on S_{-a} belong to Γ^0_+.

The geometric property of Γ^0_+ that we use is the following: consider the arc $\tilde{\Gamma}$ of S_b joining $(0,0)$ to $(b/2, \tilde{b})$. Then if we start at $(x,y) \in \Gamma^0_+$, draw a vertical line till we reach $\tilde{\Gamma}$ at some point (x,y') and then draw an horizontal line ℓ. Then ℓ will intersect S_{-a} at a point $(x',y') \in \Gamma^0_+$ with $0 < x' < x$ (and at a second point $(x'',y') \notin \Gamma^0$).

Now it is easy to see that Γ^0_+ contains no cycle. Indeed, if $A_0 = (k,l) \in G(-a,b) \cap \Gamma^0_+$ belonged to a cycle, then it would have a neighbour of the form $A_1 = (k,l') \in \tilde{\Gamma}$. Then the second neighbour of A_1 would be of the form $(k',l') \in \Gamma^0_+$ with $0 < k' < k$. Thus this path cannot return to A_0, a contradiction.

(ii) The condition $a \geq \frac{3}{2}b > 0$ was chosen so that the situation is exactly the same as previously. Again, S_a splits into 2 arcs. On one of them the vertices have no neighbours, and on the other arc after two steps we always get strictly closer to the origin. $\square$

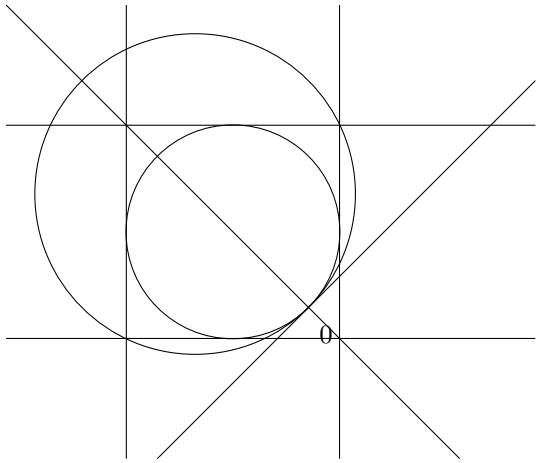


FIGURE: Case (ii)

Example 4.11. On the figure below none of the conditions (i) or (ii) is satisfied. We do not know whether cycles can exist in this case. The larger cicle has now four arcs that could meet cycles.

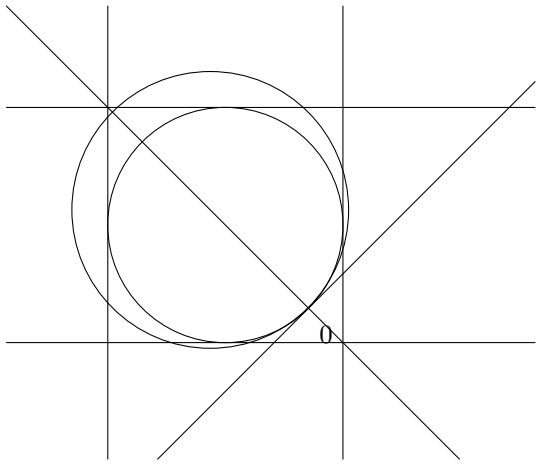


FIGURE: A case where none of (i) and (ii) is satisfied

5. Rectangular plates

We consider the vibrations of a rectangular plate with periodic boundary conditions. More precisely, we consider the following system in $\Omega = (0, 2\pi) \times (0, 2\pi)$:

$$\begin{cases} u_{tt} + \Delta^2 u = 0 & \text{in } \mathbb{R} \times \Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \\ u_t(0, \cdot) = u_1 & \text{in } \Omega, \\ u(t, x, 0) = u(t, x, 2\pi) & \text{for } t \in \mathbb{R} \text{ and } x \in (0, 2\pi), \\ u_y(t, x, 0) = u_y(t, x, 2\pi) & \text{for } t \in \mathbb{R} \text{ and } x \in (0, 2\pi), \\ u(t, 0, y) = u(t, 2\pi, y) & \text{for } t \in \mathbb{R} \text{ and } y \in (0, 2\pi), \\ u_y(t, 0, y) = u_y(t, 2\pi, y) & \text{for } t \in \mathbb{R} \text{ and } y \in (0, 2\pi). \end{cases} \tag{5.1}$$

The results of this section may be extended to general rectangular domains by a simple linear change of variables.

Let us consider the orthonormal basis

$$e_{k,\ell}(x, y) = \frac{1}{2\pi} e^{i(kx+\ell y)}, \quad (k, \ell) \in \mathbb{Z}^2$$

of $L^2(\Omega)$, and for any fixed real number s let D^s be the Hilbert space obtained by completion of the linear span of the functions $e_{k,\ell}$ with respect to the Euclidean norm

$$\left\| \sum_{k,\ell} c_{k,\ell} e_{k,\ell} \right\|_s := \left(\sum_{k,\ell} (1+k^2+\ell^2)^s \left| c_{k,\ell} \right|^2 \right)^{1/2}.$$

For any given initial data $u_0 \in D^s$ and $u_1 \in D^{s-2}$ there is a unique weak solution

$$u \in C(\mathbb{R}, D^s) \cap C^1(\mathbb{R}, D^{s-2}).$$

Furthermore, u has a Fourier series representation

$$u(t,x) = c^+_{0,0} + c^-_{0,0} t + \sum_{\substack{k,\ell \in \mathbb{Z} \\ (k,\ell) \neq (0,0)}} \left(c^+_{k,\ell} e^{i[(k^2+\ell^2)t + kx + \ell y]} + c^-_{k,\ell} e^{i[-(k^2+\ell^2)t + kx + \ell y]} \right) \tag{5.2}$$

with suitable square summable complex coefficients $c^+_{k,\ell}, c^-_{k,\ell}$ satisfying the relations

$$\sum_{k,\ell \in \mathbb{Z}} (1+k^2+\ell^2)^s \left(\left| c^+_{k,\ell} \right|^2 + \left| c^-_{k,\ell} \right|^2 \right) \asymp \|u_0\|_s^2 + \|u_1\|_{s-2}^2 .$$

Using (4.2) we extend the solutions to $\mathbb{R}^3$ (by 2π-periodicity in x and y).

We are interested in the observability of the solutions on a fixed segment of the form

$$\{(x_1 + as, y_1 + bs) \ : \ s \in (0,S)\}$$

during some time interval $(t_1, t_1 + T)$ with given real numbers x_1, y_1, a, b, t_1, T satisfying $(a,b) \neq (0,0)$ and $T > 0$.

Theorem 5.1. *Given any $(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ and real numbers $S, T > 0$, the solutions of* (5.1) *satisfy the estimates*

$$\|u_0\|_0^2 + \|u_1\|_{-2}^2 \ll \int_0^T \int_0^S \left| u(t_1 + t, x_1 + as, y_1 + bs) \right|^2 \, ds \, dt$$

for all $(u_0, u_1) \in D^1 \times D^{-1}$.

Remark 5.2. If $a = 0$ or $b = 0$, then the theorem holds for any nonzero value of the other coefficient by a reasoning similar to the proof of [17, Theorem 1.3].

Proof. It is classical (see, e.g., [15] and the standard trace theorems for Sobolev spaces) that if $(u_0, u_1) \in D^1 \times D^{-1}$, then the right hand side is well defined. We may write $u(t_1 + t, x_1 + as, y_1 + bs)$ in the form

$$\begin{aligned} u(t_1 + t, x_1 + as, y_1 + bs) &= d^+_{0,0} + d^-_{0,0} t \\ &\quad + \sum_{\substack{k,\ell \in \mathbb{Z} \\ (k,\ell) \neq (0,0)}} \left(d^+_{k,\ell} e^{i[(k^2+\ell^2)t + (ak+b\ell)s]} + d^-_{k,\ell} e^{i[-(k^2+\ell^2)t + (ak+b\ell)s]} \right) \end{aligned}$$

with suitable complex coefficients satisfying the relations

$$\left| d^+_{0,0} \right|^2 + \left| d^-_{0,0} \right|^2 \asymp \left| c^+_{0,0} \right|^2 + \left| c^-_{0,0} \right|^2$$

and the equalities $\left| d^\pm_{k,\ell} \right| = \left| c^\pm_{k,\ell} \right|$ for all $(k,\ell) \neq (0,0)$. Setting

$$m := ak + b\ell, \quad n := bk - a\ell$$

we have $(m,n) \in \mathbb{Z}_{a,b} \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ and

$$k^2 + \ell^2 = \frac{m^2 + n^2}{a^2 + b^2}.$$

Noticing that the linear map $A : (k, \ell) \mapsto (m, n)$ is invertible, we may thus write

$$u(t_1 + t, x_1 + as, y_1 + bs) = d^+_{0,0} + d^-_{0,0} t + \sum_{(m,n)\in\mathbb{Z}^2_{a,b}} \left(d^+_{A^{-1}(m,n)} e^{i[\frac{m^2+n^2}{a^2+b^2}t+ms]} + d^-_{A^{-1}(m,n)} e^{i[-\frac{m^2+n^2}{a^2+b^2}t+ms]} \right).$$

It was proved in [22] that the sets

$$\left\{ (m, \frac{m^2+n^2}{a^2+b^2}) \ : \ (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\} \right\}$$

and

$$\left\{ (m, -\frac{m^2+n^2}{a^2+b^2}) \ : \ (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\} \right\}$$

are uniformly separated, and associated with every non-empty bounded open set in $\mathbb{R}^2$. Their union is clearly uniformly separated as well, and Kahane's Theorem C then shows that the union is also associated with every non-empty bounded open set in $\mathbb{R}^2$, in particular with the rectangle $(0, T) \times (0, S)$.

According to Remark 2.4 the presence of the term $d^-_{0,0}t$ does not affect the validity of the direct and inverse inequalities. □

6. Open problems

We end this paper by a list of open questions:

(i) We do not know whether the inverse inequality in Theorem 3.2 (ii) fails for every choice of finitely many segments (x_i, t_i) instead of only one segment.

(ii) Can we consider more general segments in Proposition 3.3 (ii)?

(iii) How to modify Proposition 3.3 for Neumann boundary conditions?

(iv) We have no examples for a cycle in Theorem 4.8. Does the inverse inequality (4.8) of Theorem 4.8 hold for any two different non-integer rationals a_1, a_2?

(v) If there are examples of cycles in the preceding question, then does there exist an integer $m \geq 3$ such that the inverse inequalities

$$\sum_{k\in\mathbb{Z}} \left(\left|c^+_k\right|^2 + \left|c^-_k\right|^2 \right) \ll \sum_{i=1}^{m} \int_0^T |u(t_i + t, x_i - a_i t)|^2 \, \mathrm{d}t$$

hold for all choices of different nonzero integers $a_1, \ldots, a_m$?

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Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France. CNRS, IMB, UMR 5251, F-33400 Talence, France.
E-mail address: `Philippe.Jaming@math.u-bordeaux.fr`

College of Mathematics and Computational Science, Shenzhen Uni- versity, Shenzhen 518060, People's Republic of China, and Département de mathématique, Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg Cedex, France
E-mail address: `komornik@math.unistra.fr`