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## Bayesian structural source identification using local generalized Gaussian priors

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### ABSTRACT

The reconstruction of mechanical sources from vibration measurements is known to be an ill-posed inverse problem. A classical solution to overcome this difficulty consists in including prior information on the spatial distribution of the sources to constrain the space of solutions. Among all the methods developed to this end, the Tikhonov regularization is certainly the most popular. However, it assumes a global a priori on the spatial distribution of sources. Incidentally, poor results can be obtained if a structure is subjected to localized and distributed sources. This paper aims at providing an identification methodology able to take advantage of prior local information on both the nature and location of excitation sources. For this purpose, the Bayesian framework is well adapted, since it offers a rigorous probabilistic approach to exploit our a priori knowledge on the sources to identify. The proposed Bayesian formulation is based on the use of generalized Gaussian priors, which provide a flexible way to introduce local a priori information. Practically, the resulting optimization problem is solved from a Generalized Iteratively Reweighted Least-Squares algorithm. The validity of the proposed methodology is illustrated numerically. It is especially shown that local information improves drastically the quality of the source identification.

Keywords: Source identification, Vibration, Regularization

### 1. INTRODUCTION

The source identification of mechanical sources from vibration measurements is an ill-posed inverse problem, meaning that the existence of a unique stable solution is not guaranteed. A classical approach to bypass this difficulty consists in including in the formulation of the inverse problem some prior information on the measurement noise and the spatial distribution of sources to constrain the space of solutions. This idea is at core of Tikhonov-like regularization methods [1]. In such a regularization procedure, however, the a priori on the spatial distribution of sources is global. Incidentally, this can lead to poor identification if actual sources combine both localized and distributed sources, since the a priori has to reflect a compromise between two contradictory distributions.

The present paper aims at remedying this problem using available local information on the distribution and the nature of sources to identify. The proposed approach relies on the Bayesian inference, which offers a rigorous mathematical framework allowing combining both probabilistic and mechanical data [2]. To exploit our a priori knowledge of excitation sources, the structure is divided into different zones in which it is assumed that local priors on the sources to identify follow generalized Gaussian distributions. In doing so,

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prior information can be accurately tuned in each zone of the structure. From a mathematical point of view, the solution of the problem is defined as the maximum a posteriori estimate. Practically, one seeks the solution of the dual minimization problem, which is solved from a Generalized Iteratively Reweighted Least-Squares algorithm [3]. To assess the validity of such an approach, a numerical validation is proposed. Obtained results clearly show that using prior local information significantly improves the quality of the identification.

## 2. BAYESIAN SOURCE IDENTIFICATION

### 2.1 Problem description

Let consider the practical situation where the vibration field  $\mathbf{X}$ , measured over the surface of a structure, is caused by an unknown excitation field  $\mathbf{F}$ . If the structure is linear, its dynamic behavior is completely determined by the transfer functions matrix  $\mathbf{H}$ , relating the vibration field  $\mathbf{X}$  to the unknown field  $\mathbf{F}$ . Now, let suppose that the measured vibration field is corrupted by a measurement noise  $\mathbf{N}$ . In such a situation, the measured vibration field  $\mathbf{X}$  is obtained from the following direct formulation:

$$\mathbf{X} = \mathbf{H}\mathbf{F} + \mathbf{N}. \quad (1)$$

The structural source identification problem consists in estimating the unknown excitation field  $\mathbf{F}$  acting on a structure from the vibration field  $\mathbf{X}$  measured on its surface. To this end, the Bayesian framework is adopted. The Bayesian paradigm consists in considering all the parameters of the problem as random variables. Consequently, the uncertainty on each parameter is modeled by a probability distribution, describing the state of knowledge or the prior on this parameter. The Bayesian source identification formulation relies on the Bayes' rule:

$$p(\mathbf{F}|\mathbf{X}) \propto p(\mathbf{X}|\mathbf{F})p(\mathbf{F}), \quad (2)$$

where:

- $p(\mathbf{F}|\mathbf{X})$  is the posterior probability distribution, representing the probability of observing  $\mathbf{F}$  given a vibration field  $\mathbf{X}$ . In other words, it defines what it is known about the excitation field  $\mathbf{F}$  after making vibration measurements;
- $p(\mathbf{X}|\mathbf{F})$  is the likelihood function, representing the probability of measuring  $\mathbf{X}$  given an excitation field  $\mathbf{F}$ . It reflects the uncertainty related to the measurement of the vibration field  $\mathbf{X}$ ;
- $p(\mathbf{F})$  is the prior probability distribution, representing our knowledge on the unknown excitation field  $\mathbf{F}$  before measuring the vibration field  $\mathbf{X}$ .

In this paper, one will seek the most probable excitation field  $\hat{\mathbf{F}}$  given a measured vibration field  $\mathbf{X}$ . From the Bayesian point of view, it consists in finding a point estimate of  $\hat{\mathbf{F}}$  corresponding to a mode of the posterior probability distribution. Mathematically speaking, the solution of the identification problem is sought as the maximum a posteriori estimate, that is:

$$\hat{\mathbf{F}} = \underset{\mathbf{F}}{\operatorname{argmax}} p(\mathbf{F}|\mathbf{X}) = \underset{\mathbf{F}}{\operatorname{argmax}} p(\mathbf{X}|\mathbf{F})p(\mathbf{F}). \quad (3)$$

Practically, it is generally easier to find a solution of the following dual minimization problem:

$$\hat{\mathbf{F}} = \underset{\mathbf{F}}{\operatorname{argmin}} -\log p(\mathbf{F}|\mathbf{X}) = \underset{\mathbf{F}}{\operatorname{argmin}} -\log p(\mathbf{X}|\mathbf{F}) - \log p(\mathbf{F}). \quad (4)$$

### 2.2 Choice of the likelihood function and the prior probability distribution

#### 2.2.1 Choice of the likelihood function

The likelihood function reflects the uncertainty related to the measurement of the vibration field  $\mathbf{X}$ . By definition, this uncertainty is mainly related to the measurement noise  $\mathbf{N}$ . Consequently, the likelihood function  $p(\mathbf{X}|\mathbf{F})$  can be rewritten under the following form:

$$p(\mathbf{X}|\mathbf{F}) = p(\mathbf{X} - \mathbf{H}\mathbf{F}|\mathbf{N}), \quad (5)$$

representing the probability of obtaining  $\mathbf{X} - \mathbf{H}\mathbf{F} = \mathbf{0}$  given the measurement noise  $\mathbf{N}$ .

If the noise is supposed to be due to multiple independent causes, then the likelihood function  $p(\mathbf{X}|\mathbf{F})$  can be represented by a normal distribution with mean  $\mathbf{H}\mathbf{F}$  and variance  $\alpha^2$ :

$$p(\mathbf{X}|\mathbf{F}) \propto \exp\left[-\frac{1}{2} \frac{\|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2}{\alpha^2}\right], \quad (6)$$

where  $\|\cdot\|_2$  is the  $L_2$ -norm. It can be noticed that the form of the probability distribution implicitly assumes that all the elements of the vector  $\mathbf{X} - \mathbf{H}\mathbf{F}$  are independent identically distributed variables with same standard deviation  $\alpha$ .

### 2.2.2 Choice of the prior probability distribution

The prior probability distribution reflects the uncertainty related to the unknown excitation field  $\mathbf{F}$ . It can be seen as a measure of the a priori knowledge of the experimenter on the sources to identify. If one supposes that the structure is excited in different regions by uncorrelated excitations of various types (localized or distributed), then local excitation fields  $\mathbf{F}_i$  can be considered as independent identically distributed variables. As a result, the prior probability distribution is written:

$$p(\mathbf{F}) = \prod_{i=1}^N p(\mathbf{F}_i), \quad (7)$$

where  $N$  is the number of selected zones and  $p(\mathbf{F}_i)$  is the local prior probability distribution reflecting the prior knowledge of the expert on the nature of the sources in the zone  $i$ .

For practical reasons, one assumes that the each excitation fields  $\mathbf{F}_i$  follows a generalized Gaussian distribution with zero mean, that is:

$$p(\mathbf{F}_i) \propto \exp\left[-\frac{1}{q_i} \frac{\|\mathbf{L}_i \mathbf{F}_i\|_{q_i}^{q_i}}{\beta_i^{q_i}}\right], \quad (8)$$

where:

- $q_i$  is the shape parameter controlling the shape of the probability distribution in the zone  $i$ . The shape parameter is defined in the interval  $]0, +\infty[$ ;
- $\|\cdot\|_{q_i}$  is  $L_{q_i}$ -norm or quasi-norm;
- $\beta_i$  is a scaling factor which defines the dispersion of the distribution around the mean. It is therefore a generalized measure of the variance in the zone  $i$ ;
- $\mathbf{L}_i$  is a differentiation operator that allows controlling the regularity of the solution.

### 2.3 Practical form of the identification problem

In order to derive the practical form of the identification problem, one just has to introduce Eqs. (6) and (8) in Eq.(4). In doing so, one finally obtains a formulation of the identification problem equivalent to a generalized Tikhonov regularization:

$$\hat{\mathbf{F}} = \arg \min_{\mathbf{F}} \frac{1}{2} \frac{\|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2}{\alpha^2} + \sum_{i=1}^N \frac{1}{q_i} \frac{\|\mathbf{L}_i \mathbf{F}_i\|_{q_i}^{q_i}}{\beta_i^{q_i}}, \quad (9)$$

At this stage, one has  $2N+1$  parameters  $(q_i, \beta_i, \alpha)$  to mathematically transcribe prior information on the measurement noise and the nature of the sources to identify. However, it can be emphasized that parameters  $q_i$  and  $\beta_i$  play a similar role. Therefore, only one set of parameters  $(q_i$  or  $\beta_i)$  can be used. Here, it has been chosen to work with the scale parameters  $q_i$ , while setting the values of the scale parameters  $\beta_i^{q_i}$  to a unique constant value  $\underline{\beta}$ . Taking this observation into account, one finally gets the practical form of the identification problem:

$$\hat{\mathbf{F}} = \arg \min_{\mathbf{F}} \frac{1}{2} \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 + \lambda \sum_{i=1}^N \frac{1}{q_i} \|\mathbf{L}_i \mathbf{F}_i\|_{q_i}^{q_i}, \quad (10)$$

where  $\lambda = \alpha^2 / \underline{\beta}$  is generally known as the regularization parameter. It can be noticed that the proposed formulation is equivalent to the standard Tikhonov regularization if  $q_i = 2$  for all  $i \in [1, N]$ .

### 3. BAYESIAN SOURCE IDENTIFICATION

In this paper, the solution of the minimization problem given by Eq. (10) is computed from a general version of the Iteratively Reweighted Least-Squares (IRLS) algorithm [3].

#### 3.1 General principle

The core idea of the Generalized Iteratively Reweighted Least-Squares (GIRLS) algorithm is to replace, within an iterative scheme, the  $L_{q_i}$ -norm, that appears in Eq. (10), by a weighted  $L_2$ -norm, so that the functional to minimize has an explicit expression at each iteration. For this purpose, one just has to notice that:

$$\forall x_n, \forall q, \frac{1}{q} \|x\|_q^q = \frac{1}{q} \sum_n |x_n|^q = \frac{1}{2} \sum_n w(x_n) |x_n|^2, \quad (11)$$

where  $w(x_n) = \frac{2}{q} |x_n|^{q-2}$  is the weighting coefficient.

As part of an iterative scheme, one tries to find the solution  $x^{(k+1)}$  at iteration  $k+1$  from the solution  $x^{(k)}$  at iteration  $k$  by setting  $w(x_n) = w(x_n^{(k)})$ , in order to find the equality (11) when the process has converged. Here, the direct application of this idea consists in replacing the minimization problem given by Eq.(11) by an equivalent iterative process, for which the excitation field  $\hat{\mathbf{F}}^{(k+1)}$  at iteration  $k+1$  is the solution of the following minimization problem:

$$\hat{\mathbf{F}}^{(k+1)} = \arg \min_{\mathbf{F}} \frac{1}{2} \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 + \frac{\lambda^{(k+1)}}{2} \sum_{i=1}^N \|\mathbf{W}_i^{(k)} \mathbf{L}_i \mathbf{F}_i\|_2^2, \quad (12)$$

where  $\mathbf{W}_i^{(k)}$  is a definite positive matrix of the form:

$$\mathbf{W}_i^{(k)} = \text{diag} \left[ \frac{2}{q_i} T_\varepsilon \left( \mathbf{L}_i \hat{\mathbf{F}}_i^{(k)} \right) \right], \quad (13)$$

with:

$$T_\varepsilon \left( x^{(k)} \right) = \begin{cases} |x^{(k)}|^{q_i-2} & \text{si } x^{(k)} > \varepsilon_i^{(k)} \\ |\varepsilon_i^{(k)}|^{q_i-2} & \text{si } x^{(k)} \leq \varepsilon_i^{(k)} \end{cases}, \quad (14)$$

where  $\varepsilon_i^{(k)}$  is a small real positive number acting as a damped parameter. It allows avoiding infinite weights when  $|x^{(k)}| \rightarrow 0$  and  $q_i < 2$ .

#### 3.2 Choice of the tuning parameters, initial guess and stopping criterion

According to Eq.(12), the tuning parameters of the problem are the shape parameters  $q_i$ , the regularization parameter  $\lambda^{(k+1)}$  and the damping parameter  $\varepsilon_i^{(k)}$ .

To properly choose the shape parameters  $q_i$ , one has to keep in mind that these parameters control the general shape of the distribution. So, using  $q_i \geq 2$  will give priority to smooth solutions, while using  $q_i \leq 1$  will favor sparse solutions.

Regarding the choice of the regularization parameter  $\lambda^{(k+1)}$  and the damping parameter  $\varepsilon_i^{(k)}$ , automatic selection procedures have been implemented. For the regularization parameter, the L-curve principle is used [4], while for the damping parameter, its value is set so that a fixed percentage  $p_i$  of

the smallest values of  $\|\mathbf{L}_i \hat{\mathbf{F}}_i^{(k)}\|$  are below  $\varepsilon_i^{(k)}$ . In the next of the paper,  $p_i = 5\%$ .

Since the algorithm is iterative, another critical issue is the choice of the initial solution as well as that of the stopping criterion. Choosing a good initial guess is a key point of the convergence of the algorithm, since the functional to minimize is non-convex when  $q_i < 1$ . The question that arises here is: What is a good initial guess? Actually, it is a coarse solution of the problem, easy to calculate, but sufficiently close to the final solution to ensure the convergence of the iterative process. Such requirements are fulfilled by the solution of the standard Tikhonov regularization. Finally, it remains to define a stopping criterion for the GIRLS algorithm. In this paper, a stopping criterion related to the variation of the functional  $J(\hat{\mathbf{F}}^{(k)}) = \frac{1}{2} \|\mathbf{X} - \mathbf{H}\hat{\mathbf{F}}^{(k)}\|_2^2 + \frac{\lambda^{(k)}}{2} \sum_{i=1}^N \|\mathbf{W}_i^{(k-1)} \mathbf{L}_i \mathbf{F}_i^{(k)}\|_2^2$  between two successive iterations is used. The algorithm is automatically stopped when a prescribed tolerance defined by the user is reached. Here, the tolerance is set to  $10^{-8}$ .

#### 4. NUMERICAL VALIDATION

In the present numerical validation, one seeks to identify a point force of unit amplitude acting on a thin simply supported steel plate with dimensions  $2 \times 1.5 \times 0.017 \text{ m}^3$ . The coordinates of the point force, measured from the lower left corner of the plate, are  $(x, y) = (1.04 \text{ m}, 0.93 \text{ m})$ . Moreover, to simulate the vibration field  $\mathbf{X}$ , a finite element model of the plate made up with 567 quadrilateral shell elements has been used. It is worth to mention that an additive Gaussian white noise has been added to the data to simulate the measured vibration field. The noise has been computed so as to have a signal-to-noise ratio of 6 dB. Finally, a FE model of the structure with free boundary conditions is used to compute the transfer functions matrix  $\mathbf{H}$ . This has the advantage of enabling the identification of the point force acting on the structure as well as reacting forces at boundaries.

As shown in Figure 1a, in such a configuration, two types of spatial distributions can be distinguished over the structure: a smooth distribution of the reacting forces at boundaries and a singular distribution around the location of the point force.

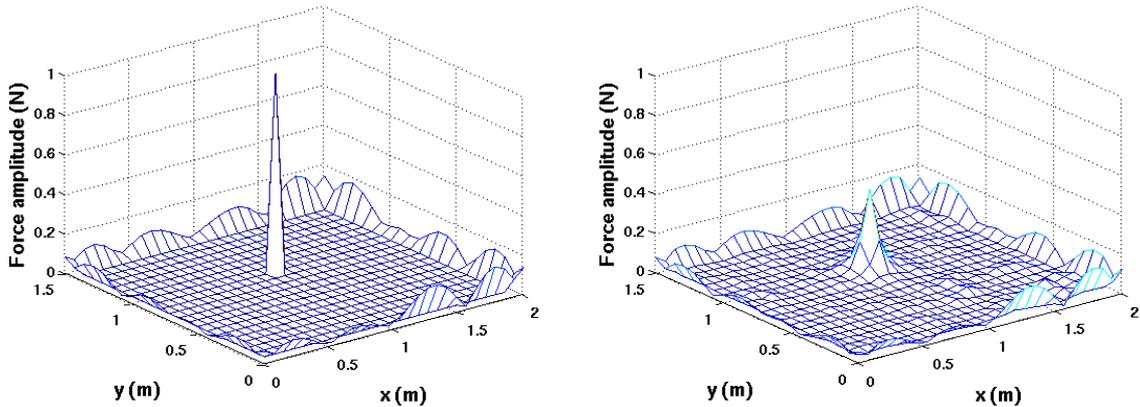


Figure 1 – Numerical validation – (a) Excitation field to identify at 450 Hz and (b) Identified excitation field from standard Tikhonov regularization

Figure 1b presents the excitation field  $\hat{\mathbf{F}}$  from the standard Tikhonov regularization. This Figure clearly shows that the location of the point force is properly estimated while the amplitude is greatly underestimated. On the contrary, reacting forces at boundaries are well identified. This contrasted result can be explained by the fact that the global a priori used in the standard Tikhonov regularization gives priority to smooth solutions. To improve the quality of the identification, the prior knowledge on the nature of sources acting on the source has to be exploited. Indeed, as presented in Figure 1a, one knows that the excitation field to identify is very sparse, except in the vicinity of the boundaries of the plate. Consequently, the structure can be divided into two regions: (i) a region, containing the point

force, in which a sparsity-promoting regularization term is required ( $q_1 = 0.5$ ) and (ii) a region, corresponding to the boundaries of the plate, in which a regularization term that promotes the smoothness of the solution has to be employed ( $q_2 = 2$ ) [see Figure 2a].

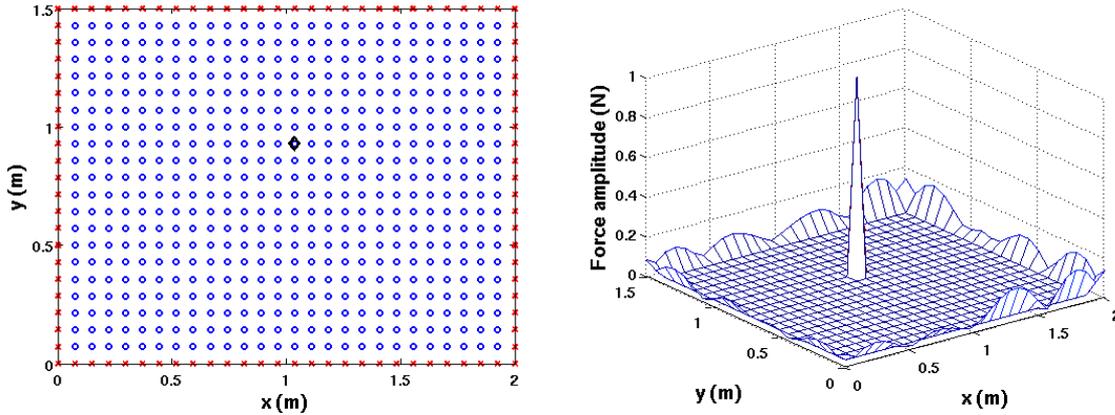


Figure 2 – Numerical validation – (a) Definition of the selected zones: (o) zone 1 (sparse), (x) zone 2 (smooth) and (◇) location of the point force and (b) Excitation field identified using the proposed approach with  $\mathbf{L}_i = \mathbf{Identity}$

Figure 2b presents the excitation field identified from the proposed approach. The comparison of this result with the reference solution clearly shows that the proposed methodology allows identifying properly not only the location and the amplitude of the point force (0.995 instead of 1), but also the reacting forces at boundaries.

## 5. CONCLUSIONS

In the present study, the motivation was to propose a Bayesian formulation of the structural source identification problem able to fully exploit spatial information available a priori on the nature and the location of the sources. To this end, the structure is divided into several regions, in which it is supposed that local priors follow generalized Gaussian laws. This formulation is very flexible, insofar as the tuning parameters of these laws can be precisely adjusted to mathematically transcribe the a priori knowledge of the experimenter on the studied system. Formally, the solution of the problem is defined as the maximum a posteriori estimate, but practically, one seeks the solution of the dual minimization problem, which is solved from a Generalized Iteratively Reweighted Least-Squares algorithm. Numerical results show that exploiting local information substantially improves the quality of the source identification.

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