



HAL
open science

On a space-frequency regularization for source reconstruction

Mathieu Aucejo, Olivier de Smet

► **To cite this version:**

Mathieu Aucejo, Olivier de Smet. On a space-frequency regularization for source reconstruction. 13th International Conference on Motion and Vibration Control and of the 12th International Conference on Recent Advances in Structural Dynamics, MOVIC RASD 2016, Jul 2016, Southampton, United Kingdom. pp.012191, 10.1088/1742-6596/744/1/012191 . hal-02068543

HAL Id: hal-02068543

<https://hal.science/hal-02068543>

Submitted on 15 Mar 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On a space-frequency regularization for source reconstruction

Mathieu Aucejo and Olivier De Smet

Structural Mechanics and Coupled Systems Laboratory, Conservatoire National des Arts et Métiers, 2 rue Conté, 75003 Paris, France

E-mail: mathieu.aucejo@cnam.fr and olivier.de_smet@cnam.fr

Abstract. To identify mechanical sources acting on a structure, Tikhonov-like regularizations are generally used. These approaches, referred to as additive regularizations, require the calculation of a regularization parameter from adapted selection procedures such as the L-curve method. However, such selection procedures can be computationally intensive. In this contribution, a space-frequency multiplicative regularization is introduced. The proposed strategy has the merit of avoiding the need for the determination of a regularization parameter beforehand, while taking advantage of one's prior knowledge of the type of the sources as well as the nature of the excitation signal. By construction, the regularized solution is computed in an iterative manner, which allows adapting the importance of the regularization term all along the resolution process. The validity of the proposed approach is illustrated numerically on a simply supported beam.

1. Introduction

The knowledge of external loads is essential to properly design a mechanical system or control the vibration levels. The direct measurement of excitation sources is generally the first-line approach. However, such strategy is intrusive and potentially time-consuming to implement and calibrate. Furthermore, direct measurements can be impossible in practical situations. To circumvent these undesirable features, a possible alternative is to perform indirect measurements using accessible quantities such as displacement or acceleration fields and a model of the dynamic behaviour of the studied structure. Unfortunately, the reconstruction of mechanical sources from vibration measurements is an ill-posed inverse problem. A classical approach to bypass this difficulty consists in including in the formulation of the inverse problem a regularization term reflecting one's prior knowledge on the spatial distribution of sources or the type of excitation signal to constrain the space of solutions. This idea is at core of Tikhonov-like regularization methods [1, 2, 3], that are widely used to tackle the inverse problem in the frequency domain. However, the reconstruction problem is generally solved frequency by frequency, meaning that only the spatial prior information on the sources to identify is exploited. From a spectral point of view, this is equivalent to suppose that the frequency spectrum of the identified sources is discontinuous. This lack of continuity can induce potential inaccuracies in the reconstructed excitation fields if the sources are broadband [4]. Consequently, it appears that the vast majority of the frequency methods proposed in the literature are unable to simultaneously exploit both the spatial and spectral features at the same time. When one seeks to localize excitation

sources and determine their frequency content, it is crucial to take advantage of both information to aid the reconstruction process in converging to a consistent regularized solution. These requirements are actually satisfied by regularization terms derived from a mixed $\ell_{p,q}$ -norm. In the context of force reconstruction, Rezayat et al. first derive a Tikhonov-like approach using a regularization term based on a mixed $\ell_{2,1}$ -norm in order to identify broadband point forces [4]. From a technical standpoint, the problem is then solved by a generalization of the Fast Iterative Shrinkage-Thresholding Algorithm (FISTA) proposed by Beck and Teboulle [5].

In the present contribution, an original regularization strategy is proposed to solve both the localization and the spectral reconstruction problem within a unique framework. The proposed approach first relies on the definition of a regularization term constructed from a mixed $\ell_{2,q}$ -norm. Such a space-frequency regularization term has the merit to be flexible enough to reflect one's prior knowledge on the type of the excitation sources and the nature of the excitation signal. Then, this regularization term is included in the formulation of the inverse problem as a multiplicative constraint. This regularization strategy, originally developed by Van den Berg et al. [6], has several advantages compared to the more classical Tikhonov-like regularizations (a.k.a. additive regularizations). In particular, it avoids the use of expensive selection procedures such as the L-curve principle [7], since, by construction, the regularized solution has to be computed in an iterative manner. This specific feature allows adapting the importance of the regularization term all along the resolution process. Practically, the proposed inverse formulation is solved from an adapted Iteratively Reweighted Least-Squares (IRLS) algorithm. The ability of the proposed space-frequency approach in providing consistent reconstructions is illustrated on a simply supported beam excited by a broadband point force using synthetic data.

2. Reconstruction model

The implementation of the space-frequency regularization requires the definition of a reconstruction model describing the relation between the measured vibration field and the excitation field to identify. If the structure is linear and time-invariant, its dynamic behaviour at a particular frequency f_j is completely defined by the transfer functions matrix $\mathbf{H}(f_j)$ relating the vibration field $\mathbf{X}(f_j)$ to the unknown excitation field $\mathbf{F}(f_j)$. Now, let us suppose that the vibration field is corrupted by a measurement noise $\mathbf{N}(f_j)$. In such a situation, the reconstruction model simply writes:

$$\mathbf{X}(f_j) = \mathbf{H}(f_j)\mathbf{F}(f_j) + \mathbf{N}(f_j), \quad \forall j = 1, \dots, N, \quad (1)$$

where N is the number of studied frequencies.

From the previous equation, it is possible to construct a global model by collecting all the studied frequencies together. In doing so, the global reconstruction model is:

$$\bar{\mathbf{X}} = \bar{\mathbf{H}}\bar{\mathbf{F}} + \bar{\mathbf{N}}, \quad (2)$$

where the global quantities are given by:

$$\bar{\mathbf{H}} = \begin{bmatrix} \mathbf{H}(f_1) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{H}(f_N) \end{bmatrix}, \quad \bar{\mathbf{X}} = \begin{bmatrix} \mathbf{X}(f_1) \\ \vdots \\ \mathbf{X}(f_N) \end{bmatrix}, \quad \bar{\mathbf{F}} = \begin{bmatrix} \mathbf{F}(f_1) \\ \vdots \\ \mathbf{F}(f_N) \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{N}} = \begin{bmatrix} \mathbf{N}(f_1) \\ \vdots \\ \mathbf{N}(f_N) \end{bmatrix}. \quad (3)$$

3. Proposed regularization strategy

In the present contribution, it is proposed to seek the unknown global excitation field $\bar{\mathbf{F}}$ as the solution of the following minimization problem:

$$\bar{\mathbf{F}}_{\mathbf{m}} = \underset{\bar{\mathbf{F}}}{\operatorname{argmin}} \mathcal{F}(\bar{\mathbf{X}} - \bar{\mathbf{H}}\bar{\mathbf{F}}) \cdot \mathcal{R}(\bar{\mathbf{F}}), \quad (4)$$

where:

- $\mathcal{F}(\bar{\mathbf{X}} - \bar{\mathbf{H}}\bar{\mathbf{F}})$ is the data-fidelity term that encodes prior information on the noise $\bar{\mathbf{N}}$ corrupting the data [8];
- $\mathcal{R}(\bar{\mathbf{F}})$ is the regularization term that controls the a priori on the unknown excitation field $\bar{\mathbf{F}}$ [9].

To obtain consistent reconstructions, a proper choice of the data-fidelity and regularization terms is crucial, since the more prior information on the noise and the sources is meaningful, the more the confidence in the reconstruction is high [3].

3.1. Definition of the data-fidelity term

As evoked previously, the data-fidelity term reflects prior information on the noise corrupting the data. A common assumption consists in considering that the vibration field is corrupted at each frequency by an additive Gaussian white noise. In this case, the data-fidelity term can be expressed as:

$$\mathcal{F}(\bar{\mathbf{X}} - \bar{\mathbf{H}}\bar{\mathbf{F}}) = \left\| \bar{\mathbf{X}} - \bar{\mathbf{H}}\bar{\mathbf{F}} \right\|_2^2. \quad (5)$$

3.2. Definition of the regularization term

The main objective of this contribution is to simultaneously exploit one's prior knowledge of the type of sources (localized or distributed) and the nature of the excitation signal (broadband or narrowband). To this end, it is relevant to define a regularization term based on the mixed $\ell_{p,q}$. Indeed, if one represents the global unknown force vector $\bar{\mathbf{F}}$ as a matrix, where the rows correspond to the force spectrum at a particular location and the columns to the force field at a specific frequency, it comes:

$$\bar{\mathbf{F}} = [\mathbf{F}(f_1) \cdots \mathbf{F}(f_j) \cdots \mathbf{F}(f_N)] = \begin{bmatrix} F_1(f_1) & \cdots & F_1(f_j) & \cdots & F_1(f_N) \\ \vdots & & \vdots & & \vdots \\ F_i(f_1) & \cdots & F_i(f_j) & \cdots & F_i(f_N) \\ \vdots & & \vdots & & \vdots \\ F_M(f_1) & \cdots & F_M(f_j) & \cdots & F_M(f_N) \end{bmatrix}, \quad (6)$$

where M is the number of spatial reconstruction points.

Furthermore, if one reminds that the mixed $\ell_{p,q}$ -norm (or quasinorm) is defined by:

$$\left\| \bar{\mathbf{F}} \right\|_{p,q} = \left[\sum_{i=1}^M \left(\sum_{j=1}^N |F_{ij}|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}, \quad \forall (p, q) \in]0, +\infty[^2, \quad (7)$$

it becomes clear that such a regularization term introduce explicitly a coupling between the coefficients of the matrix. Consequently, a mixed norm allows promoting some structures observed

in real signals [10]. To illustrate this particular property of mixed norms, let us consider the case $(p, q) = (2, 1)$. In this situation, the global force $\bar{\mathbf{F}}$ is supposed to be sparse along the lines (space) and full along the rows (frequency). In other words, one promotes the spatial sparsity of the excitation field (localized sources) and the continuity of its frequency spectrum (broadband source signal).

From the previous considerations, the proposed regularization term is such that:

$$\mathcal{R}(\bar{\mathbf{F}}) = \|\bar{\mathbf{F}}\|_{2,q}^q = \sum_{i=1}^M \|\bar{\mathbf{F}}[i]\|_2^q, \quad (8)$$

where $\bar{\mathbf{F}}[i]$ is the i^{th} row of the global force vector written under the matrix form [see Eq. (6)].

Consequently, we restrict ourselves to localized ($q \leq 1$) or distributed ($q = 2$) broadband ($p = 2$) excitation sources.

3.3. Practical form of the identification problem

From the foregoing, the generic form of the proposed space-frequency multiplicative regularization is given by:

$$\bar{\mathbf{F}}_{\mathbf{m}} = \underset{\bar{\mathbf{F}}}{\operatorname{argmin}} \left\| \bar{\mathbf{X}} - \bar{\mathbf{H}}\bar{\mathbf{F}} \right\|_2^2 \cdot \|\bar{\mathbf{F}}\|_{2,q}^q. \quad (9)$$

4. Resolution of the identification problem

The solution of the minimization problem given in the previous section can only be found in an iterative manner, insofar as no explicit solution exists for Eq. (9). For this particular reason, the regularized solution is computed from an adapted Iteratively Reweighted Least-Squares (IRLS) algorithm [11].

4.1. General principle

The core idea of the IRLS algorithm is to replace the direct resolution of the minimization problem by an equivalent iterative process having an explicit unique solution at each iteration. To this end, the $\ell_{2,q}$ -norm is replaced by a weighted ℓ_2 -norm, that is:

$$\forall q, \quad \|\bar{\mathbf{F}}\|_{2,q}^q = \left\| \bar{\mathbf{W}}(\bar{\mathbf{F}})^{1/2} \bar{\mathbf{F}} \right\|_2^2, \quad (10)$$

where $\bar{\mathbf{W}}(\bar{\mathbf{F}})$ is the global weighting matrix defined as a function of $\bar{\mathbf{F}}$.

As part of an iterative scheme, one tries to find the solution $\bar{\mathbf{F}}_{\mathbf{m}}^{(k+1)}$ at iteration $k+1$ from the solution $\bar{\mathbf{F}}_{\mathbf{m}}^{(k)}$ at iteration k by setting $\bar{\mathbf{W}}^{(k)} = \bar{\mathbf{W}}(\bar{\mathbf{F}}^{(k)})$, so as to recover the equality (10) when the convergence is reached. Here, the excitation field $\bar{\mathbf{F}}_{\mathbf{m}}^{(k+1)}$ at iteration $k+1$ is the solution of the following minimization problem:

$$\bar{\mathbf{F}}_{\mathbf{m}}^{(k+1)} = \underset{\bar{\mathbf{F}}}{\operatorname{argmin}} \left\| \bar{\mathbf{X}} - \bar{\mathbf{H}}\bar{\mathbf{F}} \right\|_2^2 \cdot \left\| \bar{\mathbf{W}}^{(k)1/2} \bar{\mathbf{F}} \right\|_2^2. \quad (11)$$

In the previous relation, the global weighting matrix $\bar{\mathbf{W}}^{(k)}$ is such that:

$$\bar{\mathbf{W}}^{(k)} = \mathbf{I}_N \otimes \mathbf{W}^{(k)}, \quad (12)$$

where \mathbf{I}_N is the identity matrix of dimension N and $\mathbf{W}^{(k)}$ is the reduced weighting matrix.

By construction, the reduced weighting matrix writes:

$$\mathbf{W}^{(k)} = \text{diag} \left(W_1^{(k)}, \dots, W_i^{(k)}, \dots, W_M^{(k)} \right), \quad (13)$$

where:

$$W_i^{(k)} = \left[\max \left(\epsilon, \left\| \overline{\mathbf{F}}_{\mathbf{m}}^{(k)}[i] \right\|_2 \right) \right]^{q-2}. \quad (14)$$

In the previous relation, ϵ is a damping parameter that allows avoiding infinite weights when $\left\| \overline{\mathbf{F}}_{\mathbf{m}}^{(k)}[i] \right\|_2 \rightarrow 0$ and $q < 2$. Practically, the damping parameter is automatically selected once for all during the initialization of the iterative process so as to 5% of the values of $\left| \overline{\mathbf{F}}_{\mathbf{m}}^{(0)} \right|$ are less than or equal to ϵ .

At this stage, the explicit form of Eq. (11) should be given in order to highlight the regularization properties of the proposed multiplicative strategy. After some simple calculations, it readily comes:

$$\overline{\mathbf{F}}_{\mathbf{m}}^{(k+1)} = \left(\overline{\mathbf{H}}^H \overline{\mathbf{H}} + \alpha^{(k+1)} \overline{\mathbf{W}}^{(k)} \right)^{-1} \overline{\mathbf{H}}^H \overline{\mathbf{X}}, \quad (15)$$

where $\alpha^{(k+1)}$ is an adaptive regularization parameter, defined such that:

$$\alpha^{(k+1)} = \frac{\left\| \overline{\mathbf{X}} - \overline{\mathbf{H}} \overline{\mathbf{F}}_{\mathbf{m}}^{(k)} \right\|_2^2}{\left\| \overline{\mathbf{W}}^{(k)1/2} \overline{\mathbf{F}}_{\mathbf{m}}^{(k)} \right\|_2^2}. \quad (16)$$

The previous equation clearly shows that the proposed multiplicative strategy avoids the selection of an optimal regularization parameter at preliminary stages of the iterative process, since the adaptive regularization parameter automatically adjusts the amount of regularization throughout the resolution procedure. This particular feature constitutes the definite advantage of the proposed approach over its additive counterpart (i.e. Tikhonov-like regularization).

4.2. Choice of the initial guess

Since the resolution algorithm is iterative, another critical issue is the choice of the initial solution. Choosing a good initial guess is a key point of the convergence of the algorithm, since the functional to minimize is non-convex when $q \leq 1$. The question that arises here is: What is a good initial guess? Actually, it is a coarse solution of the problem, easy to calculate, but sufficiently close to the final solution to ensure the convergence of the iterative process. Such a requirement is fulfilled by the solution of a standard Tikhonov-like regularization, that is:

$$\overline{\mathbf{F}}_{\mathbf{m}}^{(0)} = \left(\overline{\mathbf{H}}^H \overline{\mathbf{H}} + \alpha^{(0)} \overline{\mathbf{I}} \right)^{-1} \overline{\mathbf{H}}^H \overline{\mathbf{X}}, \quad (17)$$

where $\overline{\mathbf{I}}$ is the identity matrix of dimension $M \cdot N$ and $\alpha^{(0)}$ is a rough estimate of the converged value of the adaptive regularization parameter.

The parameter $\alpha^{(0)}$ has to be ideally determined without using any selection procedures or large computational efforts in order to preserve the advantage of the multiplicative strategy. To this end, one has to notice that the optimal regularization parameter is generally comprised

between the smallest and the largest singular values of $\overline{\mathbf{H}}^H \overline{\mathbf{H}}$. From this observation and a series of numerical experiments, we propose a heuristic rule for determining $\alpha^{(0)}$, that limits the computational efforts and leads to consistent identified solutions. The proposed estimation procedure is divided into three steps:

- (i) Find estimates of the largest and the smallest singular values of $\overline{\mathbf{H}}^H \overline{\mathbf{H}}$, noted $\hat{\sigma}_1$ and $\hat{\sigma}_n$ respectively.

The estimate of the largest singular value is given by the upper bound of σ_1 , namely [12]:

$$\hat{\sigma}_1 \left(\overline{\mathbf{H}}^H \overline{\mathbf{H}} \right) = \sqrt{\|\overline{\mathbf{H}}^H \overline{\mathbf{H}}\|_\infty \|\overline{\mathbf{H}}^H \overline{\mathbf{H}}\|_1}. \quad (18)$$

The estimation of the smallest singular value is obtained from $\hat{\sigma}_1$ and an estimate $\hat{\kappa}$ of the condition number of $\overline{\mathbf{H}}^H \overline{\mathbf{H}}$, namely:

$$\hat{\sigma}_n \left(\overline{\mathbf{H}}^H \overline{\mathbf{H}} \right) = \frac{\hat{\sigma}_1 \left(\overline{\mathbf{H}}^H \overline{\mathbf{H}} \right)}{\hat{\kappa} \left(\overline{\mathbf{H}}^H \overline{\mathbf{H}} \right)}. \quad (19)$$

- (ii) Define a set S_{α_0} of possible values of $\alpha^{(0)} \in [\hat{\sigma}_n, \hat{\sigma}_1]$ using a constant logarithmic spacing to take into account the decrease of the singular values.
- (iii) Choose $\alpha^{(0)} = \text{median}(S_{\alpha_0})$.

Because this estimation procedure is heuristic, it may sometimes fail to give a good starting point for the iterative process. In such a situation, it is always possible to choose $\alpha^{(0)}$ as the regularization parameter picked by the L-curve. Incidentally, the computational efficiency of the overall procedure is affected in proportion to the size of the global transfer functions matrix.

4.3. Choice of the stopping criterion

The proposed iterative algorithm offers a natural definition of the stopping criterion, based on the relative variation of the adaptive regularization parameter between two successive iterations. In the present contribution, the relative variation δ of the adaptive regularization parameter is defined such that:

$$\delta = \frac{|\alpha^{(k+1)} - \alpha^{(k)}|}{\alpha^{(k)}}. \quad (20)$$

As classically done in the literature, the iterative process is stopped when the relative variation δ is less than or equal to some tolerance. Experimentally, it has been found that setting the tolerance to 10^{-8} allows obtaining consistent reconstructions.

5. Numerical validation

In the present numerical validation, the studied structure is a simply supported steel beam with dimensions $1 \times 0.03 \times 0.01 \text{ m}^3$ excited by point force of unit amplitude from 50 Hz to 500 Hz. The coordinate of the point force, measured from the left end of the beam, is $x_0 = 0.6 \text{ m}$. Moreover, to simulate the vibration displacement field \mathbf{X} , a finite element model of the beam made up with 20 plane beam elements has been used. It is worth mentioning that an additive Gaussian white noise has been added to the simulated data in order to synthesize the measured vibration field. The measurement noise has been computed to obtain a signal-to-noise ration equal to 30 dB. Finally, a FE model of the structure with free boundary conditions is used to compute the

transfer functions matrix $\overline{\mathbf{H}}$ by assuming that only bending motions are measurable. In other words, the transfer functions matrices $\mathbf{H}(f_j)$ are dynamically condensed over the measurable dofs. The main interest in using free boundary conditions to model the dynamic behaviour of the beam is to allow the identification of the point force acting on the structure as well as reaction forces at boundaries [3, 13].

To better highlight the advantage of the proposed regularization term [see Eq. (8)], the space-frequency (SF) regularization is compared with the corresponding frequency-by-frequency (FbF) regularization defined such that:

$$\mathbf{F}_m(f_j) = \underset{\mathbf{F}(f_j)}{\operatorname{argmin}} \|\mathbf{X}(f_j) - \mathbf{H}(f_j)\mathbf{F}(f_j)\|_2^2 \cdot \|\mathbf{F}(f_j)\|_q^q, \quad \forall j = 1, \dots, N. \quad (21)$$

To compare both approaches properly, one has to assess their ability in identifying the mechanical sources acting on the structure. For this purpose, the definition of the regularization term is crucial. The analysis of the numerical test case shows that the beam is only excited by broadband point forces. In this context, it is reasonable to set $q = 0.5$ [see section 3.2]. The reconstructions proposed in Fig. 1 point out that both FbF and SF approaches provide a consistent identification of the reaction forces at boundaries. However, it should be noted that the FbF regularization fails in localizing the point force at one of the resonance frequencies of the beam, which is not the case with the SF strategy [see Fig. 1b].

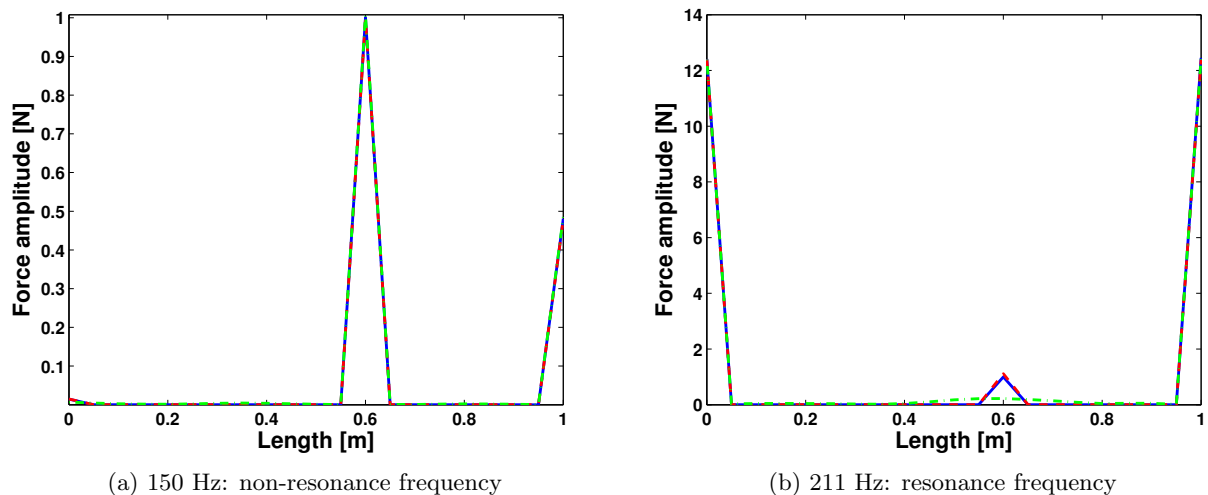


Figure 1: Spatial reconstructions of the excitation field at (a) a non-resonant frequency and (b) a resonant frequency of the beam - (—) Reference, (---) Space-frequency regularization and (- · -) Frequency-by-frequency regularization

This observation is confirmed by the analysis of the spectrum of the point force reconstructed by the FbF or the SF approach and presented in Fig. 2. Indeed, the force spectrum obtained using the FbF strategy exhibits large reconstruction errors around the resonance frequencies (94 Hz, 211 Hz and 375 Hz in the frequency range of interest). On the contrary, the reconstruction error is limited to 1.6 dB at most over all the frequency range with the SF strategy.

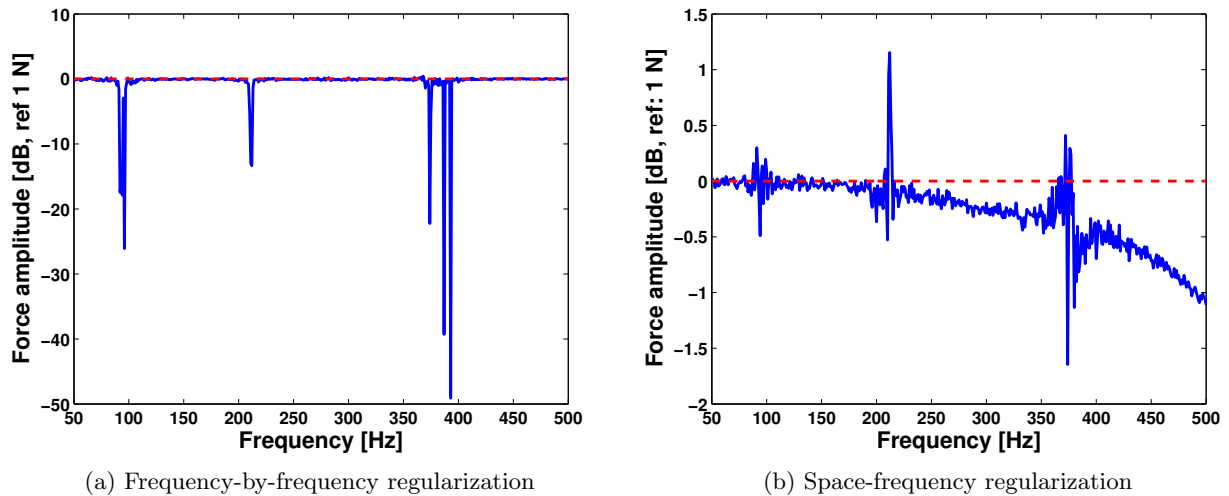


Figure 2: Spectrum of the reconstructed point force - (—) Reconstructed force spectrum and (---) Reference force spectrum

Finally, one has to notice that the proposed regularization strategy is well adapted to solve large reconstruction problem, since the calculation of a SVD, that is generally used to compute the optimal regularization parameter, is avoided. This is all the more interesting than the calculation of the SVD can be time-consuming or even impossible for large-scale systems on a personal computer.

6. Conclusion

In the present study, the initial motivation was to propose a formulation of the force reconstruction problem able to fully exploit information available a priori on the type of the sources and the nature of the excitation signals. To this end, a space-frequency multiplicative regularization has been introduced. This formulation is quite flexible, since it can be used to identify localized or distributed broadband sources. Practically, the regularization problem is solved from an adapted IRLS algorithm. The numerical results are encouraging, since the proposed strategy allows obtaining consistent reconstructions even at structural resonance frequencies.

References

- [1] Tikhonov A N 1963 *Soviet Mathematics* **4** 1624
- [2] Thite A N and Thompson D J 2003 *J. Sound Vib.* **264**(2) 433
- [3] Aucejo M and De Smet O 2016 *Mech. Syst. Signal Pr.* **66-67** 120
- [4] Rezaayat A, Nassiri V, De Pauw B, Ertveldt J, Vanlanduit S and Guillaume P 2016 *Mech. Syst. Signal Pr.* **70-71** 756
- [5] Beck A and Teboulle M 2009 *SIAM J. Imaging Sci.* **2**(1) 183
- [6] van den Berg P M, Abubakar A and Fokkema J T 2003 *Radio Sci.* **38** 8022
- [7] Hansen P C 1998 *Rank-Deficient and Discrete Ill-Posed Problems: Numerical Aspects of Linear Inversion* (Philadelphia: SIAM)
- [8] Nikolova M 2004 *J. Mat. Imaging Vis.* **20**(1-2) 99
- [9] Tibshirani R 1996 *J. Royal. Statist. Soc B.* **58**(1) 267
- [10] Gramfort A, Kowalski M and Hämaläinen M 2012 *Phys. Med. Biol* **57** 1937
- [11] Rodriguez P and Wohlberg B 2006 *Proc. of the 2006 Asilomar Conference on Signals, Systems and Computers*
- [12] Horn R A and Johnson C R 1991 *Topics in Matrix Analysis* (New York: Cambridge University Press)
- [13] Renzi C, Pezerat C and Guyader J L 2013 *J. Sound Vib.* **332** 545