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A STATIONARY ITERATED MULTIPLICATIVE REGULARIZATION FOR FORCE RECONSTRUCTION PROBLEMS

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Abstract

Iterated Tikhonov regularization can be seen as an iterative refinement of an initial solution obtained by applying the standard Tikhonov regularization. This kind of approach is particularly suited when noisy data with low signal-to-noise ratio (SNR) are to be used. However, to apply such an approach, an initial regularization parameter has to be estimated. Generally, it is obtained numerically through automatic selection procedures that can require intensive computations. In this respect, the multiplicative regularization is an interesting alternative, since it does not require any selection procedure for choosing the regularization parameter. In return, the amount of regularization is automatically adjusted throughout an iterative resolution process. To combine the best of both worlds for solving force reconstruction problems, we propose a stationary iterated multiplicative regularization that allows using prior information on the type of sources acting on the structure. The validity and the suitability of the proposed approach are illustrated numerically on a free-free beam excited by a point force.

1 Introduction

Iterated Tikhonov regularization has been extensively studied by the inverse problems community [1, 2, 3, 4, 5]. It can practically be viewed as an iterative refinement of an initial solution obtained by applying the standard Tikhonov regularization. Surprisingly, this regularization strategy seems to be seldom applied in mechanics [6, 7] despite its ability to compute improved identified solutions. In the context of force reconstruction problems, iterated Tikhonov regularization has some drawbacks. First, it requires the estimation of a regularization parameter, that can be numerically computed through automatic selection procedures such as the L-curve principle [8] or the Generalized Cross-Validation [9]. However, when the reconstruction problem is large, this calculation can be very expensive. Second, the regularization term promotes distributed solutions, which is not desirable when the structure is actually excited by localized sources. In this respect, the multiplicative regularization [10] is an interesting alternative, since it eliminates the need for the selection of the regularization parameter while taking into account one's prior knowledge on the sources to identify. However, when this strategy is applied to very noisy data, obtained reconstructions are generally disappointing. In such a situation, an iterated approach is particularly well suited. That is why, to combine the best of both worlds, a stationary iterated multiplicative regularization is proposed. In this contribution, the main features of the proposed regularization strategy are set out, while its validity is assessed through a numerical experiment performed on a free-free beam.

2 The need for another regularization strategy

Classically, the stationary iterated Tikhonov regularization is defined as follows:

$$\begin{cases} \widehat{\mathbf{F}}^{(0)} = \mathbf{0} \\ \widehat{\mathbf{F}}^{(k)} = \underset{\mathbf{F}}{\operatorname{argmin}} \ \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 + \lambda \left\| \mathbf{F} - \widehat{\mathbf{F}}^{(k-1)} \right\|_2^2 \quad \text{for } k \geq 1 \end{cases}, \tag{1}$$

where in the context of force reconstruction:

- $\mathbf{X}$ is the measured vibration field;
- $\mathbf{H}$ is the transfer functions matrix of the structure describing its dynamic behavior;
- $\widehat{\mathbf{F}}^{(k)}$ is the force vector identified at iteration k;
- λ is the regularization parameter.

When applying this formulation to the identification of a point force exciting a free-free steel beam at 350 Hz from a measured vibration field having an SNR equal to 30 dB [see section 4 for a detailed description of the test case], it is clear that the reconstructions resulting from the application of the standard Tikhonov regularization or its stationary iterated version are too smooth compared to the target excitation fields [see Figure 1].

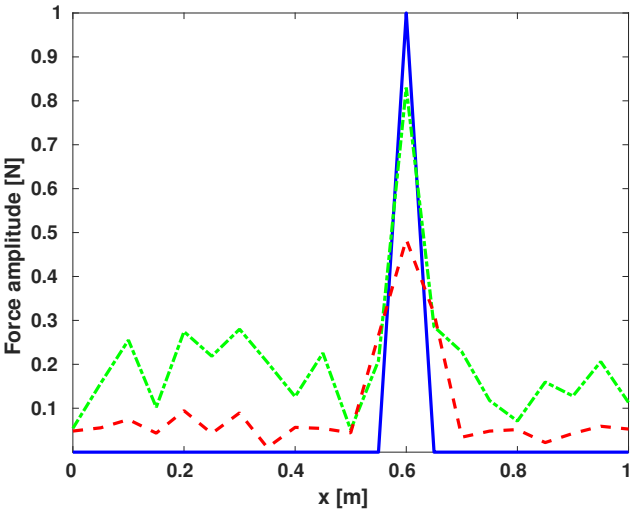


Figure 1: Reconstruction of the excitation field at 350 Hz from a measured vibration field with high SNR (30 dB) - (—) Reference, (– –) Standard Tikhonov regularization and (– · –) Stationary iterated Tikhonov regularization.

As explained in the introduction, the previous formulation has two drawbacks, since it does not exploit one's prior knowledge of the sources to identify and it requires the calculation of a regularization parameter, which can be computationally intensive depending on the size of the reconstruction problem. To bypass these potential difficulties, a multiplicative regularization has been introduced [10, 11]. Formally, it consists in solving the following minimization problem:

$$\widehat{\mathbf{F}} = \underset{\mathbf{F}\setminus\{\mathbf{0}\}}{\operatorname{argmin}} \, \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 \cdot \|\mathbf{F}\|_q^q \,, \tag{2}$$

where q is a tuning parameter included in the interval $]0, +\infty[$, $\| \bullet \|_q$ is the ℓ_q-norm or quasi-norm. Practically, q is set to 2 if the solution is supposed to be distributed, while q can be chosen equal to or less than 1 if the solution is supposed to be localized [12].

The application of the multiplicative regularization from the measured vibration field used previously gives, when $q = 0.5$, the reconstruction field presented in Figure 2. This figure shows that the reference excitation field and the reconstructed one are very close together when the measured data are little noisy (high SNR). Unfortunately, when the data are very noisy (low SNR), a degradation of the regularized solution accuracy is observed [see Figure 2].

In the light of these observations, it seems interesting to develop an iterated multiplicative regularization, combining the benefits of the multiplicative regularization and the iterated Tikhonov regularization. A first approach to this problem consists in developing a stationary iterated multiplicative regularization, which allows obtaining a refined regularized solution, while preserving the computational efficiency of the regularization strategy.

3 Stationary iterated multiplicative regularization

This section aims at introducing the proposed stationary iterated multiplicative regularization. To render the presentation more didactic, it is focused on the main concepts and features of the method.

3.1 Basic principle

The proposed stationary iterated multiplicative regularization is actually a simplified version of the following non-stationary iterated minimization problem:

$$\begin{cases} \widehat{\mathbf{F}}^{(0)} = \mathbf{0} \\ \widehat{\mathbf{F}}^{(k)} = \underset{\mathbf{F}\setminus\{\widehat{\mathbf{F}}^{(k-1)}\}}{\operatorname{argmin}} \, \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 \cdot \left\|\mathbf{F} - \widehat{\mathbf{F}}^{(k-1)}\right\|_q^q \quad \text{for } k \geq 1 \,. \end{cases} \tag{3}$$

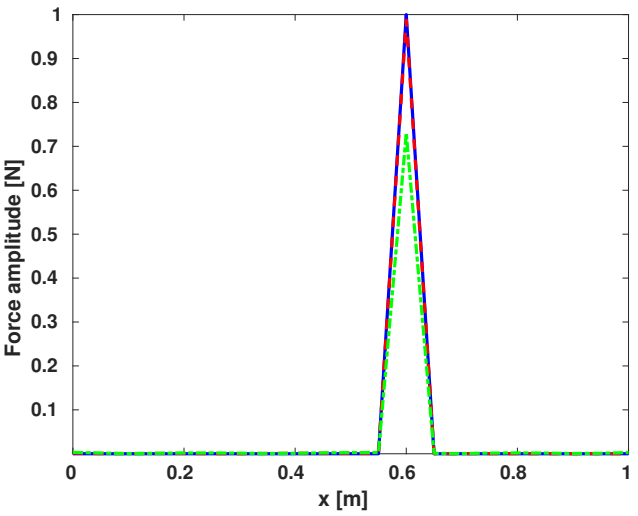


Figure 2: Reconstruction of the excitation field at 350 Hz from a measured vibration field - (—) Reference, (– –) Multiplicative regularization for $q = 0.5$ and SNR = 30 dB and (– · –) Multiplicative regularization for $q = 0.5$ and SNR = 13 dB.

Practically, the resolution of the minimization problem at iteration $k \geq 1$ requires the implementation of an iterative procedure. In the present contribution, an Iteratively Reweighted Least Squares (IRLS) algorithm has been implemented. Basically, it consists in iteratively computing the solution of the problem by recasting the ℓ_q-norm into a weighted ℓ_2-norm. As a result, at iteration k of the proposed iterated regularization, the estimated force vector $\widehat{\mathbf{F}}^{(k,j)}$ at iteration j of the IRLS algorithm is explicitly given by [10]:

$$\widehat{\mathbf{F}}^{(k,j)} = \left[\mathbf{H}^H\mathbf{H} + \alpha^{(k,j)}\mathbf{W}^{(k,j-1)}\right]^{-1}\left[\mathbf{H}^H\mathbf{X} + \alpha^{(k,j)}\mathbf{W}^{(k,j-1)}\widehat{\mathbf{F}}^{(k-1)}\right], \tag{4}$$

where:

- $\mathbf{W}^{(k,j-1)}$ is a diagonal weighting matrix depending on the tuning parameter q and the solution at iteration $j-1$, namely $\widehat{\mathbf{F}}^{(k,j-1)}$ [10, 12, 13, 14];
- $\alpha^{(k,j)}$ is the adaptive regularization parameter[1], defined such that:

$$\alpha^{(k,j)} := \frac{\left\|\mathbf{X} - \mathbf{H}\widehat{\mathbf{F}}^{(k,j-1)}\right\|_2^2}{\left\|\mathbf{W}^{(k,j-1)\,1/2}\left(\widehat{\mathbf{F}}^{(k,j-1)} - \widehat{\mathbf{F}}^{(k-1)}\right)\right\|_2^2}. \tag{5}$$

When the IRLS algorithm has converged after N_j iterations, the solution vector $\widehat{\mathbf{F}}^{(k)}$, the weighting matrix $\mathbf{W}^{(k)}$ and the adaptive regularization parameter $\alpha^{(k)}$ are returned by the algorithm and are defined such that:

$$\widehat{\mathbf{F}}^{(k)} := \widehat{\mathbf{F}}^{(k,N_j)}, \quad \mathbf{W}^{(k)} := \mathbf{W}^{(k,N_j)} \quad \text{and} \quad \alpha^{(k)} := \alpha^{(k,N_j)}. \tag{6}$$

Consequently, the previous non-stationary iterated multiplicative algorithm is composed of a main (outer) iteration corresponding to the refinement stage and a nested (inner) iterative procedure to compute $\widehat{\mathbf{F}}^{(k)}$. From there, the stationary iterated multiplicative algorithm is derived by fixing once for all the values of the weighting matrix $\mathbf{W}^{(k)}$ and the adaptive regularization parameter $\alpha^{(k)}$ to their values obtained after the first main iteration (i.e. $k = 1$) of the non-stationary algorithm, that is:

$$\overline{\mathbf{W}} = \mathbf{W}^{(k)} := \mathbf{W}^{(1,N_j)} \quad \text{and} \quad \overline{\alpha} = \alpha^{(k)} := \alpha^{(1,N_j)}. \tag{7}$$

In other words, the stationary multiplicative regularization can be expressed as:

$$\begin{cases} \widehat{\mathbf{F}}^{(0)} = \mathbf{0} \\ \left(\widehat{\mathbf{F}}^{(1)}, \overline{\mathbf{W}}, \overline{\alpha}\right) = \underset{\mathbf{F}\setminus\{\widehat{\mathbf{F}}^{(0)}\}}{\operatorname{argmin}} \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 \cdot \left\|\mathbf{F} - \widehat{\mathbf{F}}^{(0)}\right\|_q^q & \text{for } k = 1 \\ \widehat{\mathbf{F}}^{(k)} = \underset{\mathbf{F}}{\operatorname{argmin}} \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 + \overline{\alpha}\left\|\overline{\mathbf{W}}^{1/2}\left(\mathbf{F} - \widehat{\mathbf{F}}^{(k-1)}\right)\right\|_2^2 & \text{for } k \geq 2 \end{cases}. \tag{8}$$

Finally, it is important to note that from $k = 2$, the explicit expression of the force vector $\widehat{\mathbf{F}}^{(k)}$ is:

$$\widehat{\mathbf{F}}^{(k)} = \left[\mathbf{H}^H\mathbf{H} + \overline{\alpha}\,\overline{\mathbf{W}}\right]^{-1}\left[\mathbf{H}^H\mathbf{X} + \overline{\alpha}\,\overline{\mathbf{W}}\,\widehat{\mathbf{F}}^{(k-1)}\right]. \tag{9}$$

[1] because its value is updated at each iteration of the IRLS

Consequently, only the matrix $\mathbf{A} = \mathbf{H}^H\mathbf{H} + \overline{\alpha}\,\overline{\mathbf{W}}$ has to be inverted to compute $\widehat{\mathbf{F}}^{(k)}$ for any $k \geq 2$. This allows deriving a computationally efficient algorithm, since if $\mathbf{A}$ is factorized using a LU decomposition for instance, the overall computation of $\widehat{\mathbf{F}}^{(k)}$ for $k \geq 2$ is almost as expensive as than the computation of $\widehat{\mathbf{F}}^{(1)}$.

3.2 Stopping criterion

As explained in the previous section, the proposed stationary iterated multiplicative regularization first requires the resolution of a multiplicative regularization at iteration $k = 1$ from the IRLS procedure described in Ref. [10], defining thoroughly the initial solution and the stopping criterion. Here, the question is when to stop the main iteration loop. In this contribution, the stopping criterion is related to the relative variation of the functional:

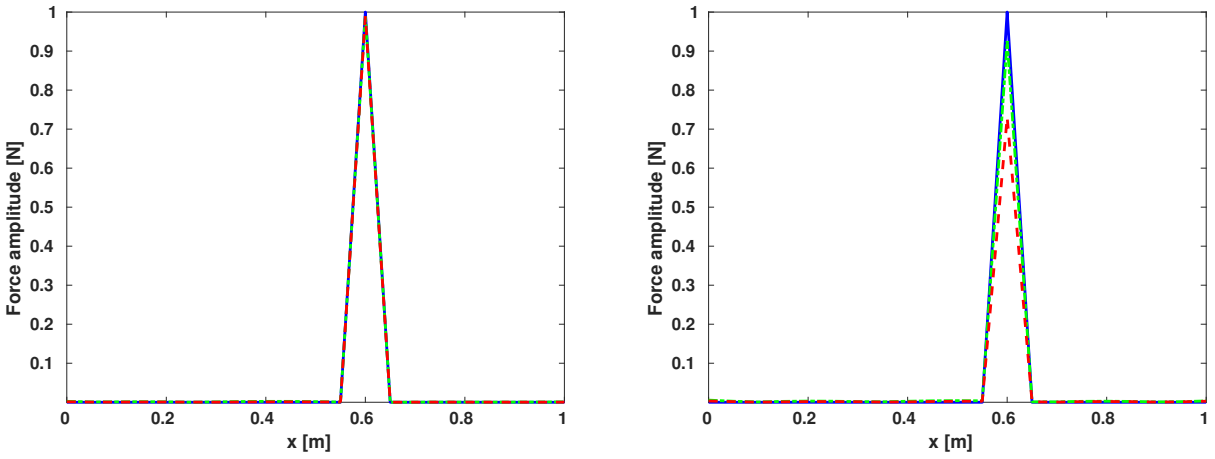
$$J(\widehat{\mathbf{F}}^{(k)}) = \|\mathbf{X} - \mathbf{H}\widehat{\mathbf{F}}^{(k)}\|_2^2 + \overline{\alpha}\left\|\overline{\mathbf{W}}^{1/2}\left(\widehat{\mathbf{F}}^{(k)} - \widehat{\mathbf{F}}^{(k-1)}\right)\right\|_2^2 \tag{10}$$

between two successive iterations. Practically, the algorithm is automatically stopped when a prescribed tolerance defined by the user is reached. Here, the tolerance is set to 10^{-2}.

4 Numerical validation

In the present numerical validation, the studied structure is a free-free steel beam with dimensions $1\times0.03\times0.01$ m^3 excited by a unit point force at 350 Hz. The coordinate of the point force, measured from the left end of the beam, is $x_0 = 0.6$ m. Moreover, to simulate the vibration displacement field $\mathbf{X}$, a finite element model of the beam made up with 20 plane beam elements has been used. It is also worth mentioning that an additive Gaussian white noise with a prescribed SNR has been added to the simulated data in order to synthesize the measured vibration field. Finally, the FE model of the structure is used to compute the transfer functions matrix $\mathbf{H}$ by assuming that only bending motions are measurable. In other words, the transfer functions matrix is dynamically condensed over the measurable dofs.

To assess the validity of the proposed iterated multiplicative regularization strategy, it is compared to the ordinary multiplicative regularization. For this purpose, the choice of the tuning parameter q is crucial. The analysis of the numerical test case shows that the beam is only excited by a point force. In this context, it is reasonable to set $q = 0.5$ [10]. The resulting reconstructions proposed in Figure 3 point out that for measured data with a high SNR, both regularization strategy provide, as expected, similar results. Here, the overall computation time is equal to 8 ms (14 iterations) for the ordinary multiplicative regularization and 12 ms (14 inner iterations for $k = 1$ and 4 outer iterations) for the stationary iterated multiplicative regularization. But more interestingly, for measured data with a low SNR, the iterated multiplicative regularization allows obtaining a far more better estimation of the point force amplitude than the ordinary multiplicative regularization, while preserving the quality of the reconstructed excitation field. In this situation, the overall computation time is equal to 34 ms (132 iterations) for the ordinary multiplicative regularization and 43 ms (132 inner iterations for $k = 1$ and 5 outer iterations) for the stationary iterated multiplicative regularization.



(a) SNR = 30 dB (b) SNR = 13 dB

Figure 3: Spatial reconstruction of the excitation field at 350 Hz from a vibration field corrupted by an additive Gaussian white noise with (a) a SNR of 30 dB and (b) a SNR of 13 dB - (—) Reference, (– –) Multiplicative regularization for $q = 0.5$ and (– · –) Stationary iterated multiplicative regularization for $q = 0.5$.

5 Conclusion

In the present contribution, a stationary iterated multiplicative regularization has been presented for identifying mechanical sources acting on a structure. The underlying idea behind this regularization strategy was to combine the advantages of the iterated Tikhonov regularization, which can be thought as an iterative refinement of an initial solution obtained using the standard Tikhonov regularization, and the multiplicative regularization that eliminates the need for the selection of the regularization parameter while taking into account one's prior knowledge on the sources to identify. The resulting regularization strategy is consequently flexible and computationally efficient. The main conclusion of this study is that the proposed iterated multiplicative regularization should be used only when the measured data are very noisy. In all other cases, the application of the ordinary multiplicative regularization is generally sufficient to obtain consistent reconstructions.

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