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# Further investigation on ”A multiplicative regularization for force reconstruction”

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## Abstract

We have recently proposed a multiplicative regularization to reconstruct mechanical forces acting on a structure from vibration measurements. This method does not require any selection procedure for choosing the regularization parameter, since the amount of regularization is automatically adjusted throughout an iterative resolution process. The proposed iterative algorithm has been developed with performance and efficiency in mind, but it is actually a simplified version of a full iterative procedure not described in the original paper. The present paper aims at introducing the full resolution algorithm and comparing it with its simplified version in terms of computational efficiency and solution accuracy. In particular, it is shown that both algorithms lead to very similar identified solutions.

*Keywords:* Inverse problem, Force reconstruction, Multiplicative regularization

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## 1. Introduction

In a paper, recently published in MSSP, we have introduced a multiplicative regularization to tackle source reconstruction problems [1]. The proposed formulation aims at properly exploiting one's prior knowledge on the sources to identify. To this end, it is assumed that the structure is excited in  $N$  different regions by local excitation fields  $\mathbf{F}_i$  of different natures (localized or distributed), while the measured vibration field  $\mathbf{X}$  is supposed to be corrupted by an additive Gaussian white noise. Under these assumptions, the reconstructed excitation field  $\mathbf{F}_m$  is sought as a stationary point of the functional:

$$J_m(\mathbf{F}) = \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 \cdot \sum_{i=1}^N \|\mathbf{L}_i \mathbf{F}_i\|_{q_i}^{q_i}, \quad (1)$$

where:

- $\mathbf{H}$  is the transfer functions matrix of the structure, which describes its dynamic behavior;
- $\mathbf{L}_i$  is a smoothing operator controlling the regularity of the solution in region  $i$ ;
- $q_i$  is a tuning parameter defined in the interval  $]0, +\infty[$  and  $\|\bullet\|_{q_i}$  is the  $\ell_{q_i}$ -norm. Practically,  $q_i \leq 1$  if the solution vector  $\mathbf{L}_i \mathbf{F}_i$  tends to be a priori sparse. On the contrary,  $q_i = 2$  if the solution vector is a priori rather distributed.

By construction, the resolution of the reconstruction problem from the multiplicative regularization defined in Eq. (1) requires the implementation of an iterative procedure. In the original paper, we have implemented an

adapted Iteratively Reweighted Least Squares (IRLS) algorithm [2, 3]. Basically, it consists in computing iteratively the solution of the problem by recasting the  $\ell_{q_i}$ -norm into a weighted  $\ell_2$ -norm. As a result, the estimated force vector  $\mathbf{F}_m^{(k+1)}$  at iteration  $k+1$  of the IRLS algorithm is sought as the stationary point of the functional (see Ref. [1]):

$$J_m^{(k+1)}(\mathbf{F}) = \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 \cdot \left\| \mathbf{W}^{(k)1/2} \mathbf{L}\mathbf{F} \right\|_2^2, \quad (2)$$

where  $\mathbf{W}^{(k)}$  is a global diagonal weighting matrix defined from the solution computed at iteration  $k$ , namely  $\mathbf{F}_m^{(k)}$ , and  $\mathbf{L}$  is the global smoothing operator.

In the original paper, it is indicated that after some calculations  $\mathbf{F}_m^{(k+1)}$  is finally expressed as:

$$\mathbf{F}_m^{(k+1)} = \left( \mathbf{H}^H \mathbf{H} + \alpha^{(k+1)} \mathbf{L}^H \mathbf{W}^{(k)} \mathbf{L} \right)^{-1} \mathbf{H}^H \mathbf{X}, \quad (3)$$

where  $\alpha^{(k+1)}$  is the adaptive regularization parameter, defined such that:

$$\alpha^{(k+1)} := \frac{\left\| \mathbf{X} - \mathbf{H}\mathbf{F}_m^{(k)} \right\|_2^2}{\left\| \mathbf{W}^{(k)1/2} \mathbf{L}\mathbf{F}_m^{(k)} \right\|_2^2}. \quad (4)$$

However, attentive readers will notice that the force vector  $\mathbf{F}_m^{(k+1)}$  given by Eq. (3) with  $\alpha^{(k+1)}$  defined by Eq. (4) is not exactly a stationary point of  $J_m^{(k+1)}(\mathbf{F})$ , since the latter is obtained from Eq. (3) with  $\alpha^{(k+1)}$  defined such that:

$$\alpha^{(k+1)} = \frac{\left\| \mathbf{X} - \mathbf{H}\mathbf{F}_m^{(k+1)} \right\|_2^2}{\left\| \mathbf{W}^{(k)1/2} \mathbf{L}\mathbf{F}_m^{(k+1)} \right\|_2^2}. \quad (5)$$

However, because  $\alpha^{(k+1)}$  depends explicitly on  $\mathbf{F}_m^{(k+1)}$ , finding a stationary point of  $J_m^{(k+1)}(\mathbf{F})$  requires the implementation of an iterative procedure

[4, 5, 6, 7]. Consequently, replacing Eq. (3) by an ad hoc iterative resolution allows defining the full resolution algorithm.

In the following sections, we will show that the resolution algorithm presented in the original paper is actually a simplified version of the full resolution algorithm described below. More specifically, it will be shown through numerical and experimental validations that both algorithms lead to very similar reconstructed excitation fields, while exhibiting different performances.

## 2. Full resolution algorithm

As explained in the introduction, the full resolution algorithm consists in replacing the calculation of  $\mathbf{F}_{\mathbf{m}}^{(k+1)}$  from Eq. (3) by an adapted iterative process. In other words, the full resolution algorithm is composed of a main (outer) iteration corresponding to the initialization step, the calculation of the global weighting matrix  $\mathbf{W}^{(k)}$  and the evaluation of the stopping criterion as defined in Ref. [1] and a nested (inner) iterative procedure to compute  $\mathbf{F}_{\mathbf{m}}^{(k+1)}$  and  $\alpha^{(k+1)}$ . Consequently, this section focuses on the implementation of the nested iterative algorithm only, since the rest of the overall resolution procedure remains unchanged compared to the original paper.

### 2.1. Fixed point iteration

As explained previously, the aim of the nested iterative procedure is to compute  $\mathbf{F}_{\mathbf{m}}^{(k+1)}$  so that it be a stationary point of  $J_m^{(k+1)}(\mathbf{F})$ . The idea here is to implement a fixed point algorithm, for which the fixed point  $\mathbf{F}_{\mathbf{m}}^{(k+1,j+1)}$

at (inner) iteration  $j+1$  of the nested process and main (outer) iteration  $k+1$  is expressed as:

$$\mathbf{F}_m^{(k+1,j+1)} = \left( \mathbf{H}^H \mathbf{H} + \alpha^{(k+1,j+1)} \mathbf{L}^H \mathbf{W}^{(k)} \mathbf{L} \right)^{-1} \mathbf{H}^H \mathbf{X}, \quad (6)$$

where the adaptive regularization parameter  $\alpha^{(k+1,j+1)}$  writes:

$$\alpha^{(k+1,j+1)} = \frac{\left\| \mathbf{X} - \mathbf{H} \mathbf{F}_m^{(k+1,j)} \right\|_2^2}{\left\| \mathbf{W}^{(k)1/2} \mathbf{L} \mathbf{F}_m^{(k+1,j)} \right\|_2^2}. \quad (7)$$

## 2.2. Initial solution and stopping criterion

As every iterative procedure, the nested algorithm requires the definition of an initial solution and a stopping criterion. To properly define all the necessary notations, we introduce the stopping criterion before the initial solution.

### 2.2.1. Stopping criterion

To determine when stopping the nested iterative algorithm, the relative variation  $\delta$  of the related adaptive regularization parameter between two successive iterations is studied. Practically, the relative variation  $\delta$  is defined such that:

$$\delta = \frac{|\alpha^{(k+1,j+1)} - \alpha^{(k+1,j)}|}{\alpha^{(k+1,j)}}. \quad (8)$$

As classically done in the literature, the iterative process is stopped when the relative variation  $\delta$  is less than or equal to some tolerance. In the present letter, the tolerance is set to  $10^{-8}$ , which corresponds to the tolerance of the main iteration loop (see Ref. [1]).

When the nested iterative process has converged after  $N_j$  iterations, the solution vector  $\mathbf{F}_m^{(k+1)}$  and adaptive regularization parameter  $\alpha^{(k+1)}$  are returned by the algorithm and are defined such that:

$$\mathbf{F}_m^{(k+1)} := \mathbf{F}_m^{(k+1, N_j)} \quad \text{and} \quad \alpha^{(k+1)} := \alpha^{(k+1, N_j)}. \quad (9)$$

### 2.2.2. Initial solution

When entering in the nested iterative step of the overall algorithm, the solution  $\mathbf{F}_m^{(k)}$  at the previous outer iteration  $k$  is known and is used as initial solution. Formally, this means that the initial solution of the inner iterative process  $\mathbf{F}_m^{(k+1, 0)}$  is expressed as:

$$\mathbf{F}_m^{(k+1, 0)} := \mathbf{F}_m^{(k)}. \quad (10)$$

### 2.3. Comments

The previous sections reveals the relation between the full resolution algorithm and its simplified version introduced in the original paper. Indeed, the simplified resolution algorithm is obtained by stopping the nested iterative process after only one iteration. In addition, to make the differences between both versions clearer, an overview of both resolution procedures is given in Fig. 1. One clearly observes that the simplified resolution algorithm requires less computations than the full version. Consequently, when using the simplified algorithm, one can expect a decrease in the overall computation time and memory allocations requirements.

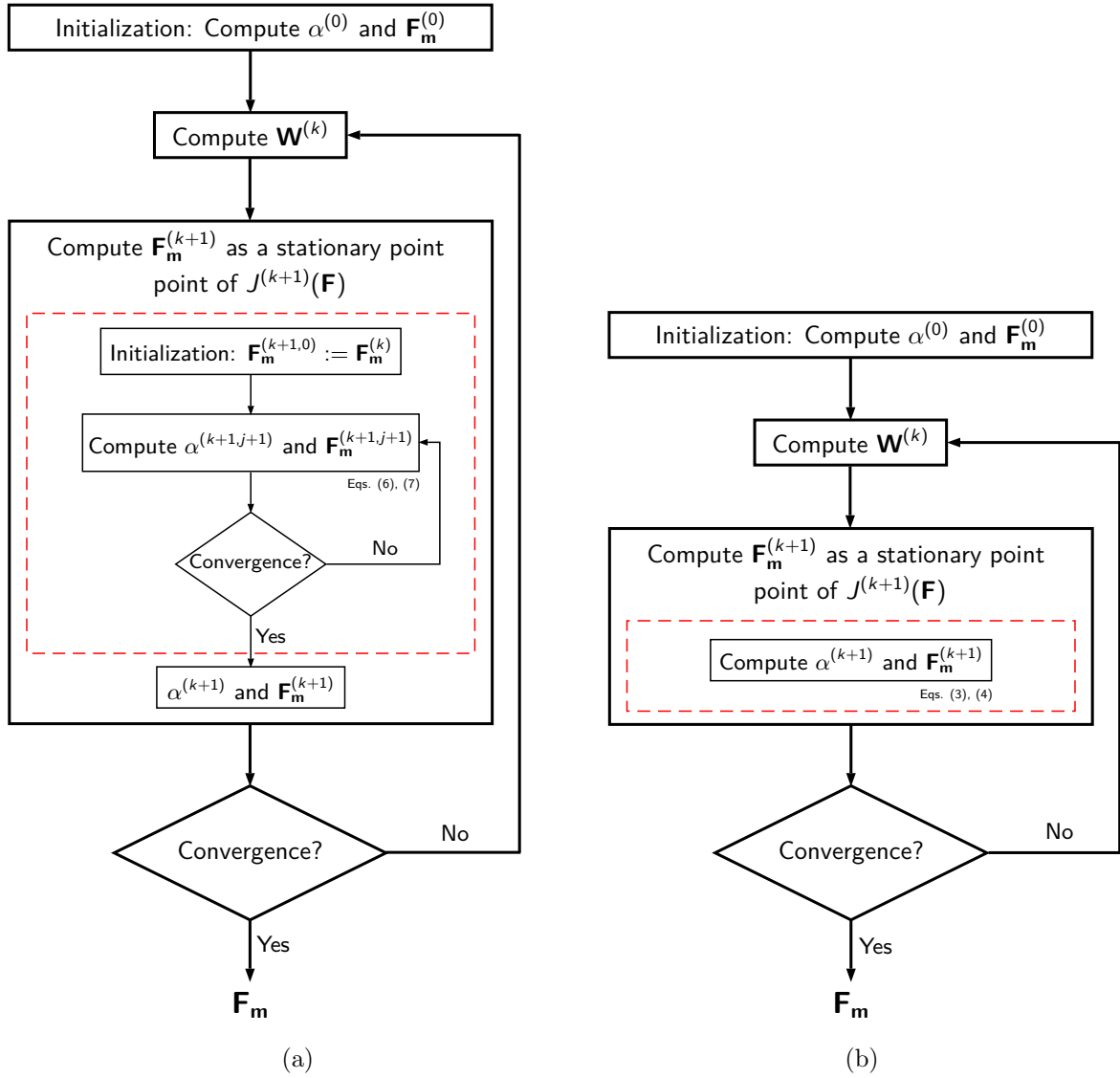


Figure 1: Overview of the resolution procedure – (a) Full resolution algorithm and (b) Simplified version



### 3. Numerical comparison of the full resolution algorithm with its simplified version

To compare the full resolution algorithm with its simplified version, we propose to use the test case that has served for the numerical validation in the original paper. It consists of a simply supported steel plate excited by a unit point force at 500 Hz. It is also important to note that the exact vibration field, computed using the finite element method, has been corrupted by an additive Gaussian white noise with a signal-to-noise ratio equal to 30 dB to simulate the measurement noise. For a complete description of the numerical test case, the readers is invited to refer to Ref. [1].

To obtain consistent reconstructions, it is also important to properly exploit the characteristics of the excitation field. From Fig. 2a, one can notice that the force vector to identify is sparse except at the boundaries of the plate. This observation leads to define two reconstruction zones: (i) a central region associated to the tuning parameter  $q_1$  and containing the point force only, in which a sparsity-promoting prior has to be employed, and (ii) a region associated to the tuning parameter  $q_2$  and corresponding to the boundaries of the plate, in which a prior promoting smooth solutions is required (see Fig. 5 of Ref. [1]). From the explanations given above, one sets  $(q_1, q_2) = (0.5, 2)$ .

After application of the full and simplified resolution algorithms to the problem described previously, it seems clear from Figs. 2b and 2c that both algorithms lead to very similar identified solutions. This qualitative obser-

vation is confirmed by Fig. 2d, presenting the absolute difference  $\Delta \mathbf{F}_m$  of the excitation fields  $\mathbf{F}_m^{\text{full}}$  and  $\mathbf{F}_m^{\text{simp}}$  reconstructed respectively from the full resolution algorithm and its simplified version. Indeed, one can notice that  $\Delta \mathbf{F}_m$  is at most of the order of  $10^{-15}$  N in the central region and  $10^{-10}$  N at boundaries. To further compare both algorithms in terms of performances and accuracy, three indicators, namely, the identified point force amplitude  $F_{m0}$ , the elapsed time  $t_e$  (i.e. the time taken by each algorithm to compute the regularized solution) and the global reconstruction error  $E$  are studied. Here, the reconstruction error is defined from the following relation:

$$E = \frac{\|\mathbf{F}_m - \mathbf{F}_{\text{ref}}\|_2^2}{\|\mathbf{F}_{\text{ref}}\|_2^2} \times 100, \quad (11)$$

where  $\mathbf{F}_{\text{ref}}$  is the reference force vector. The values of the indicators, gathered in Table 1, show that the values of the resulting point force amplitude  $F_{m0}$  and the reconstruction error  $E$  are similar for both algorithms, which is consistent with the results presented in Fig. 2. However, it is worth noting that the comparison of the elapsed times evidences that the simplified algorithm has a better computational efficiency than the full version. This result was actually expected because the inner iterative step is stopped after only one iteration in the simplified resolution algorithm. To have better a better insight into the performances of full and simplified resolution processes, it is interesting to analyze their behavior in terms of number of iterations. Fig. 3a presents the evolution of the adaptive regularization parameter  $\alpha^{(k)}$  with respect to the main iteration number. This figure shows that the full resolution algorithm tends to converge faster than the simplified version regarding the main iteration loop. Consequently, the performance difference between the two algorithms is related to the inner iteration loop of the full version [see

Fig. 3b]. However, the increase in the computation time resulting from the full resolution algorithm is limited, since the latter is globally as fast as the corresponding algorithm used to solve the updated additive regularization defined in Ref. [1], which solved the reconstruction problem in  $t_e = 0.35$  s with  $F_{m0} = 0.986$  N and  $E = 6.22$  %.

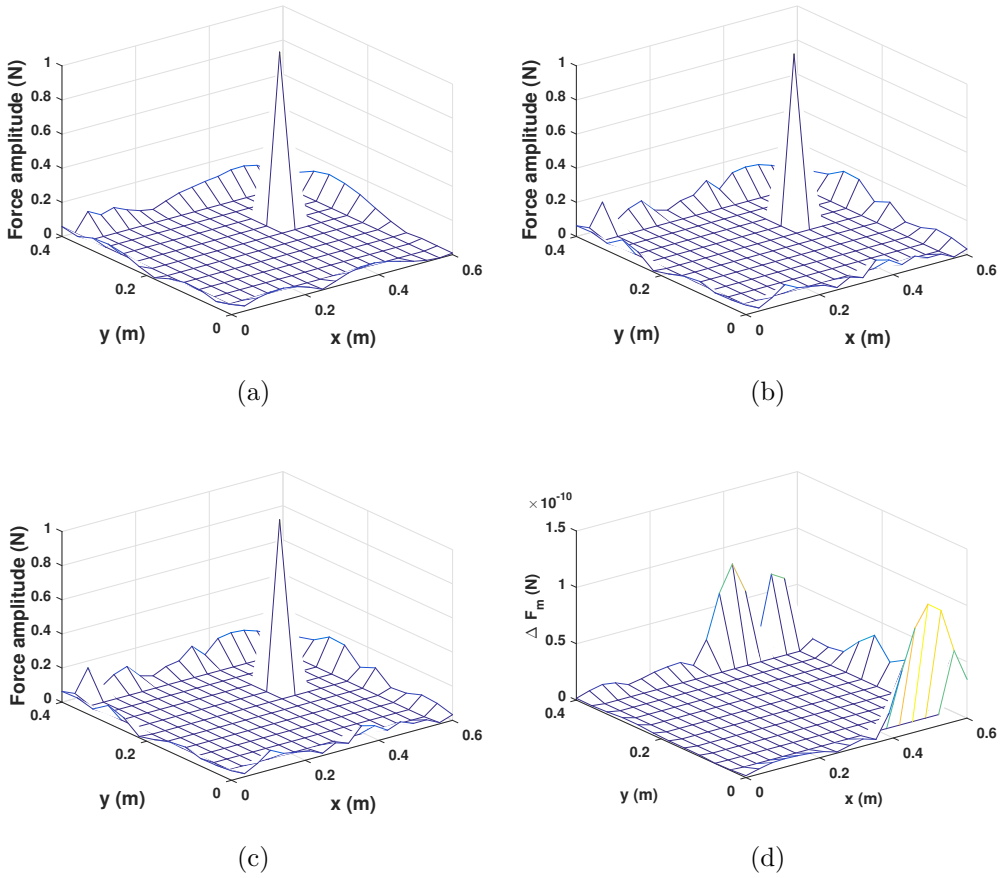


Figure 2: Identification of the force vector  $\mathbf{F}_m$  at 500 Hz – (a) Reference force vector, (b) Reconstruction from the simplified resolution algorithm, (c) Reconstruction from the full resolution algorithm and (d)  $\Delta \mathbf{F}_m = \left| \mathbf{F}_m^{\text{full}} - \mathbf{F}_m^{\text{simp}} \right| - (q_1, q_2) = (0.5, 2)$  and  $\mathbf{L} = \mathbf{I}$

Table 1: Comparison of the full resolution algorithm with its simplified version -  $F_{m0}$ : identified point force amplitude (N),  $t_e$ : elapsed time (s),  $E$ : global reconstruction error (%)

Algorithm version	$F_{m0}$	$t_e$	$E$
Simplified	0.988	0.06	5.94
Full	0.988	0.11	5.94

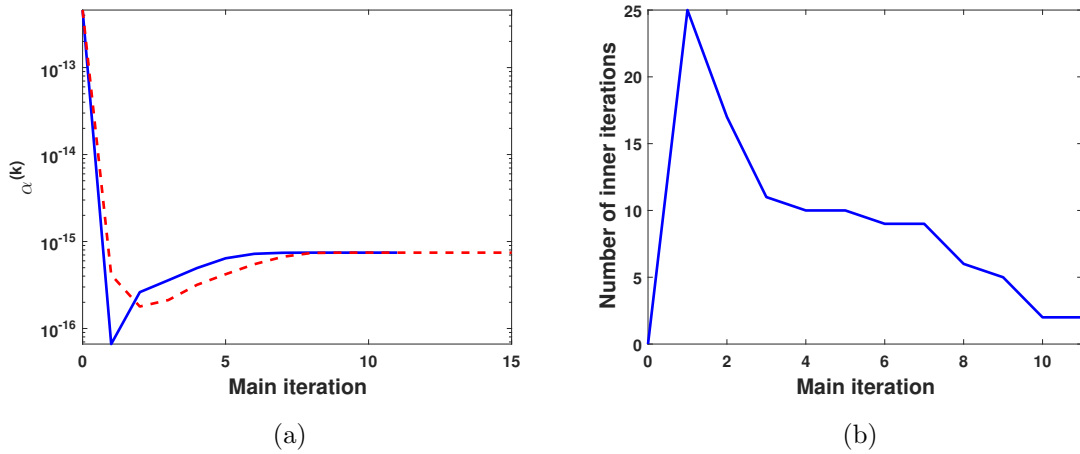


Figure 3: Comparison of the full resolution algorithm with its simplified version – (a) Evolution of the adaptive regularization parameter  $\alpha^{(k)}$  for (—) the full resolution algorithm and (---) its simplified version and (b) Number of inner iterations of the full resolution algorithm w.r.t. the main iterations

#### 4. Experimental comparison of the full resolution algorithm with its simplified version

The previous section tends to indicate that both resolution algorithms perform equally well. It is however important to confirm this observation by extending the analysis to a real-world application. The considered structure is a suspended aluminum plate excited at  $(x_0, y_0) = (0.405 \text{ m}, 0.255 \text{ m})$  by a shaker fed by a white noise and equipped with a force sensor [see Fig. 4]. For a complete description of the experimental test case, the reader is referred to Ref. [1].

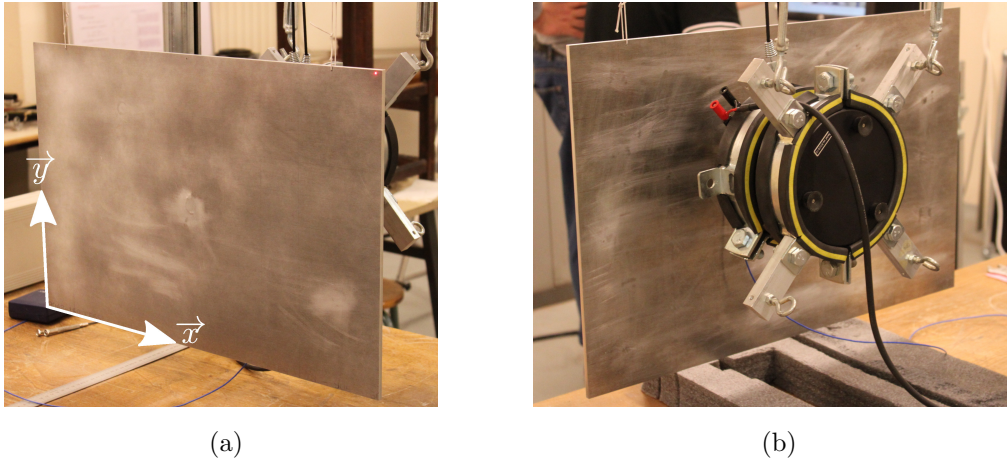


Figure 4: Experimental set-up - (a) Suspended plate and (b) Excitation device

As in the numerical validation, it is important to exploit our prior knowledge on the excitation field to reconstruct. The analysis of the experimental set-up suggests the definition of a sole reconstruction region corresponding to the whole plate. In addition, the target excitation field being very sparse, the

tuning parameter  $q$  is set to 0.5 according to the guidelines given in section 3 of Ref. [1].

To perform a thorough and more detailed analysis of the proposed algorithms, they are applied at two resonance frequencies (196.25 Hz and 1161.25 Hz) and two non-resonant frequencies (725 Hz and 1500 Hz). As observed in the numerical validation, both algorithms provide similar results, but, in all cases, the simplified resolution algorithm is obviously computationally more effective than the full version [see Table 2]. As an example, the excitation field reconstructed at 196.25 Hz and 725 Hz are presented in Fig. 5. The analysis of this figure shows that in each case the point force location is properly identified, since  $(\hat{x}_0, \hat{y}_0) = (0.404 \text{ m}, 0.256 \text{ m})$ . Regarding the reconstruction of the point force amplitude, obtained results must be compared with the value measured by the force sensor. This analysis can be performed from Table 2. This table is interesting for several reasons. First, it shows that both algorithms provide a consistent identification of the point force amplitude. Second, as noticed in the numerical validation, it is confirmed that both versions of the resolution algorithm lead to very similar identified solutions. Third, the behavior of the full resolution algorithm is hard to predict (or erratic) at resonance frequencies in terms of main iteration number. Indeed, the result obtained at 196.25 Hz corresponds to the behavior observed in the numerical validation, since the full resolution algorithm converges faster than its simplified version regarding the main iteration loop. However, a very different behavior is observed at 1161.25 Hz, where the convergence of the main loop of the full resolution algorithm is significantly slower than that of

the simplified version. This can be possibly explained by the non-uniqueness of the solution at resonance frequencies. Finally, it should be noticed that the full resolution algorithm generally exhibits performances of the same order of magnitude as the related updated additive regularization in terms of computation time [see. Ref. [1] for details].

Table 2: Comparison of the multiplicative regularization and the additive regularizations -  $F_0$ : Target point force amplitude (mN),  $F_{m0}$ : identified point force amplitude (mN),  $N_{it}$ : number of iterations,  $t_e$ : elapsed time (s)

Frequency	$F_0$	Simplified version			Full version		
		$F_{m0}$	$N_{it}$	$t_e$	$F_{m0}$	$N_{it}$	$t_e$
196.25 Hz	38	39	191	84	39	46	164
725 Hz	42	42	21	11	42	12	110
1161.25 Hz	52	55	40	17	55	52	124
1500 Hz	19	19	30	14	19	13	105

## 5. Conclusion

This paper can be seen as an extension to the original research paper entitled "A multiplicative regularization for force reconstruction", which aims at clarifying the derivation of the resolution algorithm introduced in it. In particular, it has been shown that the resolution algorithm presented in the original paper is actually a simplified version of the full resolution algorithm described in this letter. Furthermore, it has been demonstrated through numerical and experimental validations that both algorithms lead to very

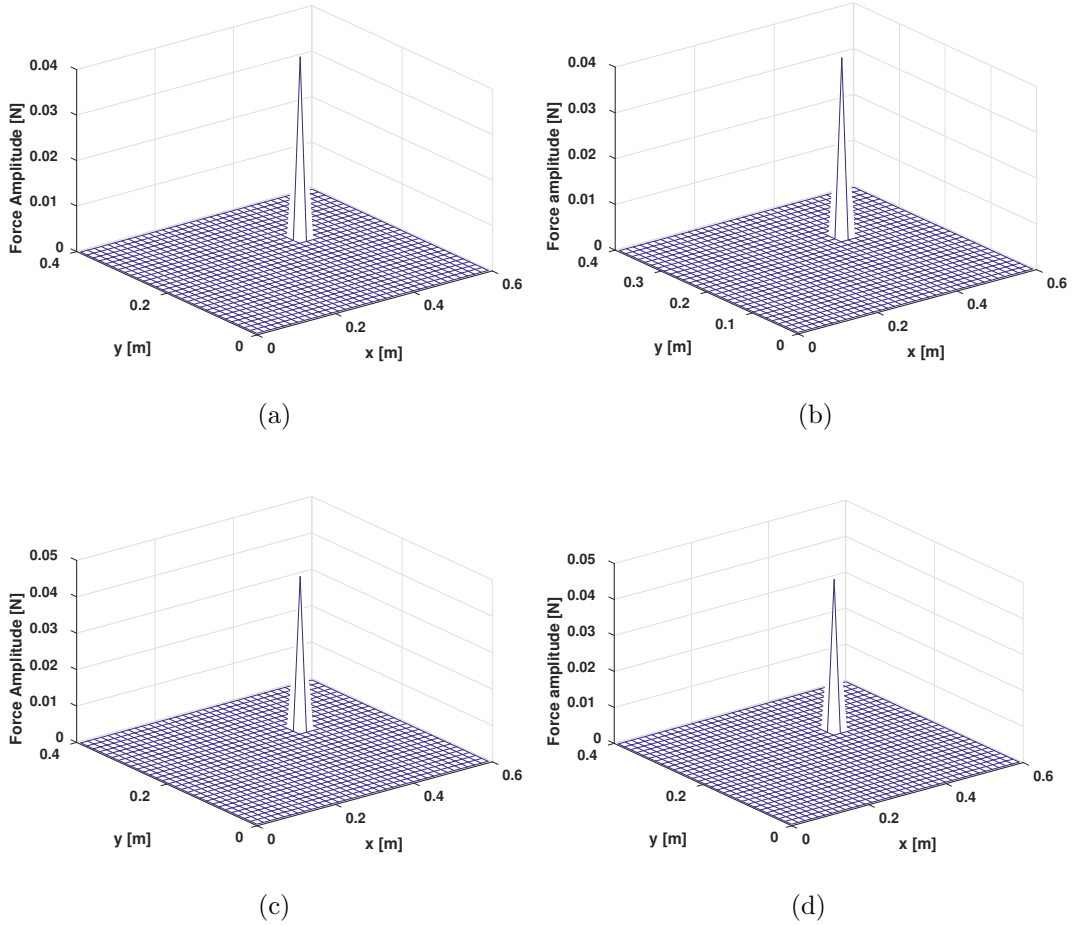


Figure 5: Comparison of the full resolution algorithm with its simplified version at (a,b) 196.25 Hz (resonance frequency) and (c,d) 725 Hz (non-resonant frequency) – (a) and (c) Simplified version, (b) and (d) Full version



similar reconstructed excitation fields, while exhibiting rather different performances in terms of computational efficiency. This consequently explains the choice made in the original paper to introduce the simplified version of the full resolution algorithm.

## References

- [1] M. Aucejo and O. De Smet. A multiplicative regularization for force reconstruction. *Mechanical Systems and Signal Processing*, 85:730–745, 2017.
- [2] P. Rodriguez and B. Wohlberg. An iteratively weighted norm algorithm for total variation regularization. In *Proceedings of the 2006 Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, USA, 2006.
- [3] B. Wohlberg and P. Rodriguez. An iteratively reweighted norm algorithm for minimization of total variation functionals. *Signal Processing, IEEE*, 14 (12):958–951, 2007.
- [4] J. A. Orozco Rodriguez. *Regularization for inverse problems*. PhD thesis, University of Minnesota, 2011.
- [5] E. Gorgin. *An analysis of multiplicative regularization*. PhD thesis, Michigan Technological University, 2015.
- [6] A. Abubakar, P. M. van den Berg, T. M. Habashy, and H. Braunish. A multiplicative regularization approach for deblurring problems. *IEEE Transactions on Image Processing*, 13:1524–1532, 2004.

- [7] F. O. Alpak, T. M. Habashy, A. Abubakar, C. Torres-Verdín, and K. Sepehrnoori. A multiplicative regularized gauss-newton algorithm and its application to the joint inversion of induction logging and near-borehole pressure measurements. *Progress In Electromagnetics Research B*, 29:105–138, 2011.