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An unknown input interval observer for LPV systems under L_2-gain and L_∞-gain criteria

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Abstract

This paper investigates the design of an unknown input interval observer, for linear parameter varying systems. The proposed method provides a solution to the decoupling conditions under L_2-gain and L_∞-gain performance level. It is shown that the design of the unknown input interval observer parameters can be formulated as a SDP problem. The major advantages with regards to other solutions is that first, it does not require the explicit knowledge of a (time-varying) state transformation, for the estimation error dynamics to satisfy the Meztler property. Rather, it is shown how such a property can be formulated as a part of the SDP optimisation process; Second, decoupling and L_2/L_∞ constraints are formulated on the thickness of the interval length in spite of half the interval length. The proposed methodology is illustrated on a numerical academic example.

Key words: Interval observer; monotone systems; unknown input observer; LPV systems ; H_∞ performance.

1 Introduction

Several set membership/interval observer-based solutions have been proposed in the literature to solve the so-called "guaranteed" state estimation problem for Linear Parameter Varying (LPV)/quasi-LPV systems Efimov et al. (2012); Chebotarev et al. (2015); Wang et al. (2015).The design of an interval observer is generally based on the monotone system theory Farina & Rinaldi (2000) to get the cooperativity property of the observation error dynamics. The difficulty to obtain this property was relaxed for a large class of systems using a time invariant state transformation Raissi et al. (2012) and a time varying state transformation Mazenc & Bernard (2011); Thabet et al. (2014). Another approach consists in applying a transformation to put the system into a positive form before designing an interval observer Cacace et al. (2015). However, it has been proposed only for Linear Time Invariant (LTI) and Linear Time Variant (LTV) systems. A last approach has been proposed based on Müller's theorem and interval analysis for nonlinear system Kieffer & Walter (2006); Meslem & Ramdani (2011). This approach also exploits the monotone property.
The main limitation of the previous mentionned approaches is their ability to simultaneously satisfy the conditions of the existence of the interval observer (especially cooperativity property) and a certain level of *a priori* given performance like the interval length. The work reported in Chebotarev et al. (2015) aims at proposing a solution to this problem within the L_1/L_2 framework. Unfortunately, the problem of considering the Meztler property as a part of the design process, is not addressed. Furthermore, it should be pointed out that the proposed approach is based on a LTI interval observer (to deal with LPV systems) and this leads, obviously, to conservative results. The work introduced in Briat & Khammash (2016) proposes an interval observer design method based on L_∞ performance criterion, but it is limited to LTI systems.
The contribution of this paper must be thought in this context. The goal is to develop an interval observer for a general class of LPV systems, under unknown input decoupling, L_2-gain and L_∞-gain criteria (and then a mixed L_2/L_∞-gain criterion), considering the cooperativity property of the observation error dynamics as a part of the design process. To this end, a new structure of an interval observer is proposed. The major advantages with regards to existing solutions is that it does not require the explicit knowledge of a (time-varying) state transformation, for the estimation error dynamics to satisfy the Meztler property.

Notations: In this paper, we adopt the notations that are commonly used in the robust control and

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interval communities: $\mathbb{R}$ and $\mathbb{R}^+ := [0, \infty) \subset \mathbb{R}$ are the set of real numbers and the set of nonnegative real numbers, respectively. $\mathbb{R}^n$ is a n-dimensional real space and $\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices. $\mathbf{I}$, $\mathbf{0}$ denote respectively the identity and the null matrices of appropriate dimensions. For a matrix $\mathcal{A} \in \mathbb{R}^{n\times m}$, $\leq, \geq, <, >$ refer to component-wise. Denote $\mathcal{A}^+ = max(\mathcal{A}, 0)$, $\mathcal{A}^- = \mathcal{A}^+ - \mathcal{A}$, $|\mathcal{A}| = \mathcal{A}^+ + \mathcal{A}^-$ and $\left[\underline{\mathcal{A}}, \overline{\mathcal{A}}\right] = \left\{\mathcal{A} \in \mathbb{R}^{n\times m} \mid \underline{\mathcal{A}} \leq \mathcal{A} \leq \overline{\mathcal{A}}\right\}$. The same notations are adopted for vectors. For a vector $v \in \mathbb{R}^n$, the notation v^* refers to $v^* = \left[(\overline{v}^+)^T \ (\overline{v}^-)^T \ (\underline{v}^+)^T \ (\underline{v}^-)^T\right]^T$, $\tilde{v} = \overline{v} - \underline{v}$, $\hat{v} = \overline{v}^- + \underline{v}^+$. For a matrix $\mathcal{A}$, $\mathcal{A} \succ 0$ ($\mathcal{A} \prec 0$) defines a strictly positive (negative) definite matrix. $\mathcal{A}^\dagger$ refers to the Moore-Penrose inverse of a full column matrix rank $\mathcal{A}$ and $\mathcal{H}e\{\mathcal{A}\} = \mathcal{A} + \mathcal{A}^T$. In large symmetric matrix expressions, terms denoted $\star$ refer to terms induced by symmetry.

2 Preliminaries

This section is devoted to definitions and lemmas that will be later used in the paper.

Definition 1 *Farina & Rinaldi (2000) (Theorem 2 p.14). A square matrix is said to be Metzler if all its off-diagonal elements are non-negative.*

Definition 2 *Poole & Boullion (1974) (Theorem 2.1 p.420). Let a square matrix $W = [w_{i,j}]$ satisfying $w_{ij} \leq 0$, $\forall i \neq j$ and $w_{ij} > 0$ $\forall i = j$ then W is said to be a M-matrix if W^{-1} exists and $W^{-1} \geq 0$.*

The following lemmas provide useful results for Metzler matrices and cooperative systems.

Lemma 3 *Consider a (known) parameter-dependent matrix $L(\rho) \in \mathbb{R}^{n\times n}$ where $\rho \in \Psi \subset \mathbb{R}^s$, an invertible M-matrix $W \in \mathbb{R}^{n\times n}$ and a real scalar $\mu \geq 0$. Then $L(\rho)$ is a Metzler matrix if $WL(\rho) + \mu W \geq 0, \forall \rho \in \Psi$.*

PROOF. Since W is a M-matrix then $W^{-1} \geq 0$. Multiplying $WL(\rho) + \mu W$ by W^{-1} leads to $L(\rho) + \mu I_n \geq 0$, which terminates the proof. □.

Lemma 4 *Farina & Rinaldi (2000) (Theorem 2 p.14). Let a non-autonomous system given by $\dot{e}(t) = Ae(t) + b(t)$ where $e \in \mathbb{R}^n$, the matrix $A \in \mathbb{R}^{n\times n}$ is Metzler and $b(t) \in \mathbb{R}^n \geq 0$, $\forall t \geq 0$. If $e(0) \geq 0$ then $e(t) \geq 0$, $\forall t \geq 0$. Such systems are called cooperative.*

Definition 5 *(L_2-norm, L_∞-norm) Let $u : \mathbb{R}^+ \to \mathbb{R}^n$. The L_2-norm of u (often called simply the 2-norm of $u(t)$) is defined as $\|u\|_2 := \left(\int_0^\infty u(t)^T u(t) dt\right)^{1/2}$. The L_∞-norm of u is defined as $\|u\|_\infty := \sup_{t\geq 0} \left\{\left(u(t)^T u(t)\right)^{1/2}\right\}$.*

The shorthand $u \in L_2$ ($u \in L_\infty$) where L_2 (L_∞) refers to the space of signals mapping $\mathbb{R}^+$ to $\mathbb{R}^n$ with finite L_2-norm (L_∞-norm), is equivalent to $||u||_2 < \infty$ ($||u||_\infty < \infty$).

Definition 6 *(L_2-gain, L_∞-gain) Consider an LPV system where x, u, y denote respectively its state, input and output vectors respectively. Let us define the operator Σ_ρ such that $y(t) = (\Sigma_\rho u)(t)$ for some parameter trajectory ρ. For zeros initial state condition (i.e. $x(0) = 0$), $\sup_{\rho\in\Psi, ||u||_2=1} ||\Sigma_\rho u||_2$ is the L_2-gain of the LPV system, where Ψ is the set of all possible parameter trajectories. Similarly, $\sup_{\rho\in\Psi, ||u||_\infty=1} ||\Sigma_\rho u||_\infty$ defines the L_∞-gain.*

3 Problem statement

Consider the following class of LPV systems, driven by unknown inputs $w(t) \in \mathbb{R}^m$ and $d(t) \in \mathbb{R}^q$

$$\begin{cases} \dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))w(t) + E(\theta(t))d(t) & \text{(1a)} \\ y(t) = C(\theta(t))x(t) + D(\theta(t))w(t) & \text{(1b)} \end{cases}$$

Here, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ denote state and measured output vectors, respectively. Both $d(t)$ and $w(t)$ are unknown input signals. However, $w(t)$ is assumed to be bounded with known bounds $\underline{w}(t), \overline{w}(t)$, whereas there is no assumption on $d(t)$. The scheduling parameter vector $\theta(t) \in \Theta \subset \mathbb{R}^l$ and its time derivative $\dot{\theta}(t) \in \Gamma_\theta$ are assumed to be bounded with *a priori* known values.

Assumption 7 *It is assumed that $\theta_m(t) = [\theta_1(t) \ \theta_2(t) \ \cdots \ \theta_s(t)] \in \Theta_m$, $0 \leq s \leq l$, is measured through $\rho(t) = [\rho_1(t) \ \rho_2(t) \ \cdots \ \rho_s(t)] \in \Psi$ where $\rho_i(t) = \lambda_i(t)\theta_i(t) + \delta_i(t)$ with $\lambda_i \neq 0, \delta_i$ are elements of $\lambda \in \Delta_\lambda \subset \mathbb{R}^s$, $\delta \in \Delta_\delta \subset \mathbb{R}^s$. Each vector λ, δ are continuously differentiable such that $\dot{\lambda} \in \Gamma_\lambda \subset \mathbb{R}^s$, $\dot{\delta} \in \Gamma_\delta \subset \mathbb{R}^s$. It is also assumed that ρ and $\dot{\rho}$ evolve in hyper-rectangles so that*

$$\mathscr{V} = \left\{(\alpha_1, \alpha_2, \cdots, \alpha_s) \mid \alpha_i \in \left\{\underline{\rho_i}, \overline{\rho_i}\right\}\right\} \quad (2)$$

$$\mathscr{R} = \left\{(\beta_1, \beta_2, \cdots, \beta_s) \mid \beta_i \in \left\{\underline{\dot{\rho}_i}, \overline{\dot{\rho}_i}\right\}\right\} \quad (3)$$

Here, $\underline{\rho_i}, \overline{\rho_i}, \underline{\dot{\rho}_i}, \overline{\dot{\rho}_i}$ are the (known) extremal values. The domain of $\dot{\rho}$ is denoted Γ_ρ.

In order to improve the readability of the following developments, the time dependency of signals and parameters is omitted from now on. The following assumption is made:

Assumption 8 *$E(\theta)$ is assumed to be of full column rank and $rank(C(\theta)E(\theta)) = rank(E(\theta)) = q$, $\forall \theta \in \Theta$ with $q \leq p$.*

Note that assumption 8 can be done without loss of generality since, in (1a), d refers to the maxiumum unknwon

inputs the interval unknown input observer can decouple. That's why it is a classical assumption in Unknown Input Observer (UIO) theory, Darouach et al. (1994).

The objective is to design the unknown input interval observer (UIIO) that admits the following structure

$$\begin{cases} \dot{\overline{z}} = L(\rho)\overline{z} + (M(\rho) + L(\rho)N)y + \overline{\eta}(\overline{r}, \underline{r}, \overline{w}, \underline{w}, \rho) & (4a) \\ \overline{r} = \overline{z} + Ny + \overline{\varphi}(\overline{w}, \underline{w}, \rho) & (4b) \\ \overline{x} = \overline{g}(\overline{r}, \underline{r}, \rho) & (4c) \\ \dot{\underline{z}} = L(\rho)\underline{z} + (M(\rho) + L(\rho)N)y + \underline{\eta}(\overline{r}, \underline{r}, \overline{w}, \underline{w}, \rho) & (4d) \\ \underline{r} = \underline{z} + Ny + \underline{\varphi}(\overline{w}, \underline{w}, \rho) & (4e) \\ \underline{x} = \underline{g}(\overline{r}, \underline{r}, \rho) & (4f) \end{cases}$$

so that its dynamics is decoupled from the unknown inputs d and the effect of w on the size of the total state estimate envelop $\tilde{x} = \overline{x} - \underline{x} : \tilde{x} \in \mathbb{R}^n$ is minimized, in the L_2-gain and L_∞-gain sense.
In Eqs. (4a)-(4f), $\overline{\eta}, \underline{\eta} : \mathbb{R}^{2n+2m+s} \to \mathbb{R}^n$, $\overline{\varphi}, \underline{\varphi} : \mathbb{R}^{2m+s} \to \mathbb{R}^n$, $\overline{g}, \underline{g} : \mathbb{R}^{2n+s} \to \mathbb{R}^n$, $L(\rho) \in \mathbb{R}^{n \times n}$, $M(\rho), N \in \mathbb{R}^{n \times p}$. $\overline{z}, \underline{z} \in \mathbb{R}^n$ are, respectively, the upper and lower bounds of the state $z \in \mathbb{R}^n$ that we define as a parameter dependent linear combination of x, so that:

$$z = R(\theta)x \quad \text{s.t.} \quad R(\theta)E(\theta) = 0 \tag{5}$$

Eq. (5) guarantees that z is decoupled from the unknown inputs d, and so the envelope tight $\tilde{x}$ is.
By extending the results presented in Darouach et al. (1994) to Eq. (5), it can be shown that Eq. (5) is fulfilled, if and only if $R(\theta) = K(\theta)\left(I_n - S(\theta)C(\theta)\right)$ where $S(\theta) = E(\theta)(C(\theta)E(\theta))^\dagger$ and where $K(\theta) \in \mathbb{R}^{n \times n}$ is an arbitrary matrix. Note that the classical choice in the UIO theory for $K(\theta)$ is $K(\theta) = I_n$. Here, $K(\theta)$ is a degree of freedom that will be exploited later.
Additionally, we enforce the following relation

$$x = Q(\theta)r \quad Q(\theta) \in \mathbb{R}^{n \times n} \tag{6}$$

so that $\overline{r}, \underline{r} \in \mathbb{R}^n$, $\overline{x}, \underline{x} \in \mathbb{R}^n$ in (4b),(4e),(4c),(4f) are respectively the upper and lower bounds of r and x.

Assumption 9 *It is assumed that the matrices $K(\theta)$ and N are known matrices such that $Q(\theta)$ in (6) is invertible and satisfies $Q(\theta)^{-1} = R(\theta) + NC(\theta)$.*

Since $C(\theta)$ is of full row rank $\forall \theta \in \Theta$, it is always possible to find a matrix N that compensates the rank deficiency of $S(\theta)$ since it is guaranteed by assumption 2 that $q \leq p$. Note that the particular case $p = q$ is covered by the definition of $R(\theta)$ since it corresponds to particular parametrization of $K(\theta)$, see Darouach et al. (1994).

The objective is now to design the functions $\overline{\eta}$,$\underline{\eta}$,$\overline{\varphi}$,$\underline{\varphi}$,$\overline{g}$,$\underline{g}$ and the matrices $K(\theta)$, N, $L(\rho)$, $M(\rho)$ so that the effect of w on the size of the total state estimate envelop $\tilde{x}$ is minimized, in the L_2-gain and L_∞-gain sense, $\tilde{x}$ being decoupled from d, thanks to (5). The following section states the solution to this problem. Section 4.2 is devoted to the derivation of necessary and sufficient conditions for (4a)-(4f) to be an UIIO for the system (1a)-(1b). Section 4.3.1 proposes a method to derive $\overline{\eta}$,$\underline{\eta}$,$\overline{\varphi}$,$\underline{\varphi}$,$\overline{g}$,$\underline{g}$ in a non conservative way by keeping the dependency of the variables entering in η and φ. while section 4.3.2 proposed a Semi Definite Programming (SDP) formulation to optimally compute $L(\rho), M(\rho)$. With regards to $K(\theta)$ and N, guidelines are also provided in section 5.

4 Solution to the UIIO design problem

4.1 Preliminary results

The following lemma provides a constructive method to design the lower and upper bounds of a product of a vector by a parameter dependent matrix in the set-membership context. The core element is that any matrix can always be decomposed into two (non unique) matrices which elements are positives.

Lemma 10 *Let a vector $v \in \left[\underline{v}, \overline{v}\right] \subset \mathbb{R}^n$ and some matrices $\mathcal{X}_k(\theta) \in \mathbb{R}^{m \times n}$, $\theta \in \Theta$, $k \in \{a, b\}$, such that $\mathcal{X}(\theta) = \mathcal{X}_a(\theta) - \mathcal{X}_b(\theta)$. Then:*
1) $\forall \theta \in \Theta$, assume that $\exists \mathcal{X}_k(\theta) \in \left[\underline{\mathcal{X}_k}, \overline{\mathcal{X}_k}\right] \subset \mathbb{R}^{m \times n}$ with $\underline{\mathcal{X}_k} \geq 0$. Then, with $\overline{\mathcal{X}}^\bullet = \begin{bmatrix} \overline{\mathcal{X}_a} & -\underline{\mathcal{X}_a} & -\underline{\mathcal{X}_b} & \overline{\mathcal{X}_b} \end{bmatrix}$, $\underline{\mathcal{X}}^\bullet = \begin{bmatrix} -\overline{\mathcal{X}_b} & \underline{\mathcal{X}_b} & \underline{\mathcal{X}_a} & -\overline{\mathcal{X}_a} \end{bmatrix}$, it follows

$$\underline{\mathcal{X}}^\bullet v^* \leq \mathcal{X}(\theta)v \leq \overline{\mathcal{X}}^\bullet v^* \tag{7}$$

2) Assume that a subset of $\theta \in \Theta$ is known through $\rho \in \Psi$, $0 < s \leq l$, suppose there exist $\mathcal{X}_k(\theta) \in \left[\underline{\mathcal{X}_k}(\rho), \overline{\mathcal{X}_k}(\rho)\right]$ with $\underline{\mathcal{X}_k}(\rho) \geq 0$, it follows

$$\underline{\mathcal{X}}^\bullet(\rho)v^* \leq \mathcal{X}(\theta)v \leq \overline{\mathcal{X}}^\bullet(\rho)v^* \tag{8}$$

PROOF. The proof for *1) is addressed. Given that $v \in [\underline{v}, \overline{v}]$ then $0 \leq \underline{v}^+ \leq v^+ \leq \overline{v}^+$, $0 \leq \overline{v}^- \leq v^- \leq \underline{v}^-$. Now considering that $0 \leq \underline{\mathcal{X}_k} \leq \mathcal{X}_k(\theta) \leq \overline{\mathcal{X}_k}$ then $\underline{\mathcal{X}_k}\underline{v}^+ \leq \mathcal{X}_k(\theta)v^+ \leq \overline{\mathcal{X}_k}\overline{v}^+$ and $\underline{\mathcal{X}_k}\overline{v}^- \leq \mathcal{X}_k(\theta)v^- \leq \overline{\mathcal{X}_k}\underline{v}^-$. Since $\mathcal{X}(\theta)v = \left(\mathcal{X}_a(\theta) - \mathcal{X}_b(\theta)\right)(v^+ - v^-)$, we deduce that $\underline{\mathcal{X}}^\bullet v^* \leq \mathcal{X}(\theta)v \leq \overline{\mathcal{X}}^\bullet v^*$. The proof for 2) is similar to the proof for 1) by a slight adaptation considering the parameter dependence on ρ.* □

Note that, Lemma 10 generalizes Lemma 2 in Efimov et al. (2012). Indeed, in the particular case where $\mathcal{X}_a(\theta) = \mathcal{X}(\theta)^+$, $\mathcal{X}_b(\theta) = \mathcal{X}(\theta)^-$ and $\mathcal{X}(\theta) \in \left[\underline{\mathcal{X}}, \overline{\mathcal{X}}\right]$ then Lemma 2 in Efimov et al. (2012) is equivalent to Lemma 10. It allows that the bounds structure of the

matrix part is not anymore unique and do not necessarily depend on the non-linear operators $(.)^+$ and $(.)^-$ for the matrix $\mathcal{X}(\theta)$.

The following notations are now adopted in the following

$$\begin{array}{ll} \overline{\mathcal{X}}(\rho) = \overline{\mathcal{X}_a}(\rho) - \underline{\mathcal{X}_b}(\rho) & \underline{\mathcal{X}}(\rho) = \underline{\mathcal{X}_a}(\rho) - \overline{\mathcal{X}_b}(\rho) \\ \widetilde{\mathcal{X}}(\rho) = \overline{\mathcal{X}}(\rho) - \underline{\mathcal{X}}(\rho) & \breve{\mathcal{X}}(\rho) = \overline{\mathcal{X}_a}(\rho) + \overline{\mathcal{X}_b}(\rho) \end{array} \quad (9)$$

4.2 Necessary and sufficient conditions

The following theorem gives the existence conditions for Eqs. (4a) -(4f) to be an UIIO for the system (1a)-(1b).

Theorem 11 *Under the assumptions 8 and 9, the state x involved in Eqs.* (1a)-(1b) *is bounded by $\overline{x}$, $\underline{x}$ defined by Eqs.* (4c), (4f) *i.e.*

$$x(t) \in [\underline{x}(t), \overline{x}(t)] \ \forall t \geq 0 \quad (10)$$

iff the following conditions hold for the system (4a)-(4f)*:*

(i) $\forall \rho \in \Psi$, there exists a matrix $L(\rho)$ such that

$$L(\rho) \text{ is Metzler} \quad (11a)$$

(ii) $\forall \rho \in \Psi$, $\forall \theta \in \Theta$, $\forall \dot{\theta} \in \Gamma_\theta$, $\exists\ \overline{\eta}$, $\underline{\eta}$, $\overline{\varphi}$, $\underline{\varphi}$, $\overline{g}$, $\underline{g}$ so that

$$\underline{\eta}(\overline{r}, \underline{r}, \overline{w}, \underline{w}, \rho) \leq \eta(r, w, \rho, \theta, \dot{\theta}) \leq \overline{\eta}(\overline{r}, \underline{r}, \overline{w}, \underline{w}, \rho) \quad (11b)$$

$$\underline{\varphi}(\overline{w}, \underline{w}, \rho) \leq \varphi(w, \theta) \leq \overline{\varphi}(\overline{w}, \underline{w}, \rho) \quad (11c)$$

$$\underline{g}(\overline{r}, \underline{r}, \rho) \leq g(r, \theta) \leq \overline{g}(\overline{r}, \underline{r}, \rho) \quad (11d)$$

where $\eta(r, w, \rho, \theta, \dot{\theta}) = F(\rho, \theta, \dot{\theta})r + G(\rho, \theta)w$,
$\varphi(w, \theta) = H(\theta)w, g(r, \theta) = Q(\theta)r$,
with $F(\rho, \theta, \dot{\theta}) = T(\rho)Y(\theta, \dot{\theta}) - L(\rho)$, $G(\rho, \theta) = T(\rho)Z(\theta)$, $H(\theta) = -ND(\theta)$ and $T(\rho) = \Big[L(\rho)\ M(\rho)\ \mathbf{I}\Big]$,

$$Y(\theta, \dot{\theta}) = \begin{bmatrix} \mathbf{0} \\ -C(\theta)Q(\theta) \\ \big(R(\theta)A(\theta) + \dot{R}(\theta)\big)\, Q(\theta) \end{bmatrix},\ Z(\theta) = \begin{bmatrix} -ND(\theta) \\ -D(\theta) \\ R(\theta)B(\theta) \end{bmatrix}$$

(iii) the initial conditions for $\underline{z}$, $\overline{z}$ are given by

$$\underline{z}(0) \leq \underline{R}^{\bullet}(\rho(0))x(0)^* \quad \overline{z}(0) \geq \overline{R}^{\bullet}(\rho(0))x(0)^* \quad (11e)$$

where the notations $\underline{(\cdot)}^{\bullet}, \overline{(\cdot)}^{\bullet}$ are defined in lemma 10.

PROOF. Consider the two following observation errors

$$\overline{e}_z = \overline{z} - z \quad \underline{e}_z = z - \underline{z} \quad (12)$$

Computing the time derivative of $\overline{e}_z$, $\underline{e}_z$ and considering the decoupling condition (5) (i.e. $R(\theta)E(\theta) = 0$) then :

$$\dot{\overline{e}}_z = L(\rho)\overline{e}_z + \overline{\eta}(\overline{r}, \underline{r}, \overline{w}, \underline{w}, \rho) - \eta(r, w, \rho, \theta, \dot{\theta}) \quad (13a)$$

$$\dot{\underline{e}}_z = L(\rho)\underline{e}_z + \eta(r, w, \rho, \theta, \dot{\theta}) - \underline{\eta}(\overline{r}, \underline{r}, \overline{w}, \underline{w}, \rho) \quad (13b)$$

Under the condition (11b), the Metzler property (11a) and the initial conditions (11e), it can be verified that (13a)-(13b) fulfil the cooperativity property and then

$$\overline{e}_z(t), \underline{e}_z(t) \geq 0\ \forall t \geq 0 \quad (14)$$

Now, combining (5) and (12) , it is deduced that $\overline{z}$, $\underline{z}$ are equivalent to

$$\overline{z} = \overline{e}_z + R(\theta)x \quad \underline{z} = R(\theta)x - \underline{e}_z \quad (15)$$

Using (15) and (1b) to replace $\overline{z}$, $\underline{z}$, y into (4b),(4e) and considering the inverse of (6) (i.e. $r = (R(\theta) + NC(\theta))\, x$) then

$$\overline{r} - r = \overline{e}_z + \overline{\varphi}(\overline{w}, \underline{w}, \rho) - \varphi(w, \theta) \quad (16a)$$

$$r - \underline{r} = \underline{e}_z + \varphi(w, \theta) - \underline{\varphi}(\overline{w}, \underline{w}, \rho) \quad (16b)$$

Under the conditions (11c) and (14), it is deduced that $r(t) \in [\underline{r}(t), \overline{r}(t)]\,,\ \forall t \geq 0$. Finally, considering (6) and recalling that $g(r, \theta) = Q(\theta)r$ in (11d) leads to $x(t) \in [\underline{x}(t), \overline{x}(t)]\,,\ \forall t \geq 0$ where the bounds are defined in (4c),(4f), which concludes the proof. □

4.3 SDP formulation of L_2/L_∞-gain performance

By virtue of (16a)-(16b), the following equation yields

$$\tilde{r} = \tilde{z} + \overline{\varphi}(\overline{w}, \underline{w}, \rho) - \underline{\varphi}(\overline{w}, \underline{w}, \rho) \quad (17)$$

where $\tilde{r} = \overline{r} - \underline{r}$ and $\tilde{z} = \overline{e}_z + \underline{e}_z$. The following lemma gives the solution to determine the functions $\overline{g}$, $\underline{g}$.

Lemma 12 *Consider $Q(\theta)$ defined in* (6) *satisfying $Q(\theta) = Q_a(\theta) - Q_b(\theta)$ such that, for $k \in \{a, b\}$, $Q_k(\theta) \in \Big[\underline{Q_k}(\rho), \overline{Q_k}(\rho)\Big]$ with $\underline{Q_k}(\rho) \geq 0$, $\forall \rho \in \Psi$. Then, for $r \in [\underline{r},\ \overline{r}]$, the inequality* (11d) *of Theorem 11 holds with $\overline{g}(\overline{r}, \underline{r}, \rho) = \overline{Q}^{\bullet}(\rho)r^*$ and $\underline{g}(\overline{r}, \underline{r}, \rho) = \underline{Q}^{\bullet}(\rho)r^*$.*

PROOF. Direct application of lemma 10 on (6).

With lemma 12, (4c) and (4f), it can be verified that the following equation yields,

$$\tilde{x} = \breve{Q}(\rho)\tilde{r} + \widetilde{Q}(\rho)\hat{r} \quad (18)$$

where the notations $\breve{Q}(\rho)$, $\widetilde{Q}(\rho)$ are defined in (9).
Then, using (13a)-(13b), it follows that the time derivative of $\tilde{z} = \overline{e}_z + \underline{e}_z$ is given by

$$\dot{\tilde{z}} = L(\rho)\tilde{z} + \overline{\eta}(\overline{r}, \underline{r}, \overline{w}, \underline{w}, \rho) - \underline{\eta}(\overline{r}, \underline{r}, \overline{w}, \underline{w}, \rho) \quad (19)$$

Finally, injecting (17) into (18) and combining the result with (19), it follows that:

$$\begin{cases} \dot{\tilde{z}} = L(\rho)\tilde{z} + \overline{\eta}(\overline{r}, \underline{r}, \overline{w}, \underline{w}, \rho) - \underline{\eta}(\overline{r}, \underline{r}, \overline{w}, \underline{w}, \rho) & (20a) \\ \tilde{x} = \breve{Q}(\rho)\big(\tilde{z} + \overline{\varphi}(\overline{w}, \underline{w}, \rho) - \underline{\varphi}(\overline{w}, \underline{w}, \rho)\big) + \widetilde{Q}(\rho)\hat{r} & (20b) \end{cases}$$

Thus, the problem turns out to be the determination of $\overline{\eta}, \underline{\eta}, \overline{\varphi}, \underline{\varphi}$. Note that it is always possible to derive bounds for η and φ ignoring the coupling of the variables they depend on, but it seems obvious that it may lead to a conservative solution and thus may increase the tight of the interval $\tilde{x}$. Towards this end, it is preferred here to keep the dependency of variables entering in η and φ.

4.3.1 Determination of $\overline{\eta}$, $\underline{\eta}$, $\overline{\varphi}$, $\underline{\varphi}$

The determination of $\overline{\eta}, \underline{\eta}, \overline{\varphi}, \underline{\varphi}$ is based on an adaptation of Lemma 10 applied on η, φ. Its application on each element of η, φ is based on the following decompositions:

$$F(\rho,\theta,\dot{\theta}) = F_a(\rho,\theta,\dot{\theta}) - F_b(\rho,\theta,\dot{\theta}) \quad (21)$$
$$G(\rho,\theta) = G_a(\rho,\theta) - G_b(\rho,\theta) \quad (22)$$
$$H(\rho,\theta) = H_a(\rho,\theta) - H_b(\rho,\theta) \quad (23)$$

Following the materials discussed in lemma 10, the matrices $F_k(\rho,\theta,\dot{\theta}), G_k(\rho,\theta), H_k(\rho,\theta)$, $k \in \{a,b\}$, must be defined so that each component of the lower bound is positive. Possible candidates for these matrices can be proposed as follows: Consider the decomposition

$$T(\rho) = T_a(\rho) - T_b(\rho) \quad (24)$$

with $T_a(\rho) = \begin{bmatrix} L_a(\rho) & M_a(\rho) & \mathbf{I} \end{bmatrix}$, $T_b(\rho) = \begin{bmatrix} L_b(\rho) & M_b(\rho) & \mathbf{0} \end{bmatrix}$. such that $L_a(\rho), L_b(\rho), M_a(\rho), M_b(\rho) \geq 0$, $\forall \rho \in \Psi$.

• With regards to the matrix $F(\rho,\theta,\dot{\theta})$, the following form for $F_a(\rho,\theta,\dot{\theta})$ and $F_b(\rho,\theta,\dot{\theta})$ is retained

$$\begin{array}{l} F_a(\rho,\theta,\dot{\theta}) = L_b(\rho) + (T_a(\rho) - T_b(\rho))\, Y(\theta,\dot{\theta}) \\ F_b(\rho,\theta,\dot{\theta}) = F_b(\rho) = L_a(\rho), \end{array} \quad (25)$$

where $Y(\theta,\dot{\theta})$ satisfies the decomposition $Y(\theta,\dot{\theta}) = Y_a(\theta,\dot{\theta}) - Y_b(\theta,\dot{\theta})$ such that, for $k \in \{a,b\}$, $Y_k(\rho,\theta,\dot{\theta}) \in \left[\underline{Y_k}(\rho), \overline{Y_k}(\rho)\right]$ with $\underline{Y_k}(\rho) \geq 0$, $\forall \rho \in \Psi$, $\forall \dot{\theta} \in \Gamma_\theta$ and the constraint $L_b(\rho) + T_a(\rho)\underline{Y}(\rho) - T_b(\rho)\overline{Y}(\rho) \geq 0$ holds, for all $\rho \in \Psi$. The notations $\underline{Y}(\rho)$, $\overline{Y}(\rho)$ are defined in (9).

• With regards to the matrix $G(\rho,\theta)$, the following form for $G_a(\rho,\theta)$ and $G_b(\rho,\theta)$ is retained

$$\begin{array}{l} G_a(\rho,\theta) = T_a(\rho)Z_a(\theta) + T_b(\rho)Z_b(\theta) \\ G_b(\rho,\theta) = T_a(\rho)Z_b(\theta) + T_b(\rho)Z_a(\theta) \end{array} \quad (26)$$

where the matrices $Z_a(\theta), Z_b(\theta)$ are given by the decomposition $Z(\theta) = Z_a(\theta) - Z_b(\theta)$ such that, for $k \in \{a,b\}$, $Z_k(\theta) \in \left[\underline{Z_k}(\rho), \overline{Z_k}(\rho)\right]$ with $\underline{Z_k}(\rho) \geq 0$ for all $\rho \in \Psi$.

• Finally, with regards to $H(\theta)$, the following conditions for $H_k(\theta)$, $k \in \{a,b\}$ are considered

$$H_k(\theta) \in \left[\underline{H_k}(\rho), \overline{H_k}(\rho)\right] \text{ with } \underline{H_k}(\rho) \geq 0, \forall \rho \in \Psi \quad (27)$$

The following lemma provides the solution to the design of $\overline{\eta}$, $\underline{\eta}$, $\overline{\varphi}$, $\underline{\varphi}$.

Lemma 13 *Let us consider the matrices $F_k(\rho,\theta,\dot{\theta})$, $G_k(\rho,\theta)$, $H_k(\rho,\theta)$, $k \in \{a,b\}$ defined by Eqs. (25), (26), (27). Then, for $r \in [\underline{r},\ \overline{r}]$, the inequality (11b) and (11c) of Theorem 11 holds with*

$$\overline{\eta}(\overline{r},\underline{r},\tau) = \overline{F}^{\bullet}(\rho)r^* + \overline{G}^{\bullet}(\rho)w^* \quad (28)$$
$$\underline{\eta}(\overline{r},\underline{r},\tau) = \underline{F}^{\bullet}(\rho)r^* + \underline{G}^{\bullet}(\rho)w^* \quad (29)$$
$$\overline{\varphi}(\tau) = \overline{H}^{\bullet}(\rho)w^* \quad (30)$$
$$\underline{\varphi}(\tau) = \underline{H}^{\bullet}(\rho)w^* \quad (31)$$

where $\underline{F_a}(\rho) = L_b(\rho) + T_a(\rho)\underline{Y}(\rho) - T_b(\rho)\overline{Y}(\rho)$, $\overline{F_a}(\rho) = L_b(\rho) + T_a(\rho)\overline{Y}(\rho) - T_b(\rho)\underline{Y}(\rho)$, $\underline{F_b}(\rho) = \overline{F_b}(\rho) = L_a(\rho)$, $\underline{G_a}(\rho) = T_a(\rho)\underline{Z_a}(\rho) + T_b(\rho)\underline{Z_b}(\rho)$, $\overline{G_a}(\rho) = T_a(\rho)\overline{Z_a}(\rho) + T_b(\rho)\overline{Z_b}(\rho)$, $\underline{G_b}(\rho) = T_a(\rho)\underline{Z_b}(\rho) + T_b(\rho)\underline{Z_a}(\rho)$, $\overline{G_b}(\rho) = T_a(\rho)\overline{Z_b}(\rho) + T_b(\rho)\overline{Z_a}(\rho)$.

PROOF. First, note that the bounds of $F_k(\cdot)$, $G_k(\cdot)$, $H_k(\cdot)$, $k \in \{a,b\}$ satisfy the conditions of lemma 10. Then, applying lemma 10 on the parts $F(\rho,\theta,\dot{\theta})r$, $G(\rho,\theta)w$ and $H(\theta)w$, see the inequalities (11b) and (11c) respectively, leads to (28)-(29) and (30)-(31). □

4.3.2 Sufficient LMIs for $L(\rho), M(\rho)$

Using the above derived expressions of $\overline{\eta}$, $\underline{\eta}$, $\overline{\varphi}$, $\underline{\varphi}$, (20a)-(20b) can be rewritten as equations depending linearly on $L(\rho), M(\rho)$. To proceed, and similarly to (18), we have

$$\begin{array}{l} \overline{\eta}(\overline{r},\underline{r},\tau) - \underline{\eta}(\overline{r},\underline{r},\tau) = \breve{F}(\rho)\tilde{r} + (T_a(\rho) + T_b(\rho))\,\widetilde{Y}(\rho)\hat{r} \\ + (T_a(\rho) + T_b(\rho))\,\breve{Z}(\rho)\tilde{w} + (T_a(\rho) + T_b(\rho))\,\widetilde{Z}(\rho)\hat{w} \end{array} \quad (32)$$
$$\overline{\varphi}(\tau) - \underline{\varphi}(\tau) = \breve{H}(\rho)\tilde{w} + \widetilde{H}(\rho)\hat{w} \quad (33)$$

with the notations $\widetilde{(\cdot)}, \breve{(\cdot)}$ defined in (9) and where $\tau = [\overline{w}^T\ \underline{w}^T\ \rho^T]$.
Injecting (32)-(33) into the system (20a)-(20b) with $\tilde{r}$ defined in (17), it follows that

$$\begin{cases} \dot{\tilde{z}} = \mathcal{A}(\rho)\tilde{z} + f(\tilde{z}) + \breve{F}(\rho)\left(\breve{H}(\rho)\tilde{w} + \widetilde{H}(\rho)\hat{w}\right) \\ \qquad + (T_a(\rho) + T_b(\rho))\left(\breve{Z}(\rho)\tilde{w} + \widetilde{Z}(\rho)\hat{w}\right) & \text{(34a)} \\ \tilde{x} = \breve{Q}(\rho)\tilde{z} + g(\tilde{z}) + \breve{Q}(\rho)\left(\breve{H}(\rho)\tilde{w} + \widetilde{H}(\rho)\hat{w}\right) & \text{(34b)} \\ \tilde{z}(0) = \overline{R}^{\bullet}(\rho(0))x(0)^* - \underline{R}^{\bullet}(\rho(0))x(0)^* & \text{(34c)} \end{cases}$$

where $f(\tilde{z}) = (T_a(\rho) + T_b(\rho))\,\widetilde{Y}(\rho)\hat{r}(\overline{z},\underline{z})$, $g(\tilde{z}) = \widetilde{Q}(\rho)\hat{r}(\overline{z},\underline{z})$ and $\mathcal{A}(\rho) = L(\rho) + \breve{F}(\rho)$.

The problem remains now to design $L(\rho)$ and $M(\rho)$ so that the effect of the envelop size $\tilde{w} = \overline{w} - \underline{w}$ and the magnitude $\hat{w}$ on the length of the total envelop $\tilde{x} = \overline{x} - \underline{x}$ is minimized, in the L_2-gain and/or in the L_∞-gain sense. The used method is based on the cooperative property to demonstrate that both the state $\tilde{z}$ and the output $\tilde{x}$ of the system (34a)-(34c) can be bounded by the state and the output of a fictitious stable (pure) LPV system, scheduled by the parameter vector ρ and driven by an augmented vector that contains $\tilde{w}$ and $\hat{w}$. Then, by proving that the L_2-gain (L_∞-gain) of the fictitious LPV system is an upper bound for the L_2-gain (L_∞-gain) of the system (34a)-(34c), it follows that the system (34a)-(34c) is stable with a finite L_2-gain (L_∞-gain) performance. For that purpose, the proposed theory uses the slack matrix method initially proposed in Bara et al. (2001) with a parameter-dependent Lyapunov matrix through the concept of Affine Quadratic Stability (AQS).
To proceed, we first derive the following lemma that provides a procedure to get the decomposition $\mathcal{K}(\theta,\dot{\theta}) = \mathcal{K}_a(\theta,\dot{\theta}) - \mathcal{K}_b(\theta,\dot{\theta})$ for the matrix $\mathcal{K} \in \{Q, Y, Z, H\}$ defined in lemmas 12 and 13.

Lemma 14 *Considering that a part of the vector θ, denoted as $\theta_m \in \mathbb{R}^s$ is measured through $\rho \in \mathbb{R}^s$ (see section 3) and for $i = \{1, \cdots, s\}$, let some functions n_i such that $h_i = \frac{\rho_i}{\lambda_i} - n_i \geq 0$, $\forall \rho_i \in \Psi_i$. For $\mathcal{K}(\theta,\dot{\theta})$ such that*

$$\mathcal{K}(\theta,\dot{\theta}) = \mathcal{K}_0(\theta,\dot{\theta}) + \sum_{i=1}^{s} \theta_i \mathcal{K}_i(\theta,\dot{\theta}) \tag{35}$$

then, under assumption 7 i.e $\theta_i = \frac{\rho_i - \delta_i}{\lambda_i}$, the decomposition $\mathcal{K}(\theta,\dot{\theta}) = \mathcal{K}_a(\theta,\dot{\theta}) - \mathcal{K}_b(\theta,\dot{\theta})$ is ensured for (35) by adopting, for $k \in \{a, b\}$,

$$\mathcal{K}_k(\theta,\dot{\theta}) = \mathcal{K}_{k0}(\theta,\dot{\theta}) + \sum_{i=1}^{s} h_i \mathcal{K}_{ki}(\theta,\dot{\theta}) \tag{36}$$

where $\mathcal{K}_{a0}(\theta,\dot{\theta}) = \Sigma(\theta,\dot{\theta})^+$, $\mathcal{K}_{ai}(\theta,\dot{\theta}) = \mathcal{K}_i(\theta,\dot{\theta})^+$, $\mathcal{K}_{b0}(\theta,\dot{\theta}) = \Sigma(\theta,\dot{\theta})^-$, $\mathcal{K}_{bi}(\theta,\dot{\theta}) = \mathcal{K}_i(\theta,\dot{\theta})^-$, with $\Sigma(\theta,\dot{\theta}) = \mathcal{K}_0(\theta,\dot{\theta}) + \sum_{i=1}^{s} \varepsilon_i \mathcal{K}_i(\theta,\dot{\theta})$ and $h_i = \frac{\rho_i}{\lambda_i} - n_i$, $\varepsilon_i = n_i - \frac{\delta_i}{\lambda_i}$. For $\mathcal{K}_j(\theta,\dot{\theta}) \in \left[\underline{\mathcal{K}_j}, \overline{\mathcal{K}_j}\right] \subset \mathbb{R}^{m\times n}$, $j \in \{0, \cdots, s\}$, upper and lower bounds of $\mathcal{K}_k(\theta,\dot{\theta})$ are

$$\begin{bmatrix} \underline{\mathcal{K}_a}(\rho) & \overline{\mathcal{K}_a}(\rho) \\ \underline{\mathcal{K}_b}(\rho) & \overline{\mathcal{K}_b}(\rho) \end{bmatrix} = \begin{bmatrix} \underline{\Sigma}^+ & \overline{\Sigma}^+ \\ \overline{\Sigma}^- & \underline{\Sigma}^- \end{bmatrix} + \sum_{i=1}^{s} h_i \begin{bmatrix} \underline{\mathcal{K}_i}^+ & \overline{\mathcal{K}_i}^+ \\ \overline{\mathcal{K}_i}^- & \underline{\mathcal{K}_i}^- \end{bmatrix} \tag{37}$$

where $\underline{\Sigma} = \underline{\mathcal{K}_0} + \sum_{i=1}^{s} \underline{\mathcal{K}_i}^\bullet \varepsilon_i^*$, $\overline{\Sigma} = \overline{\mathcal{K}_0} + \sum_{i=1}^{s} \overline{\mathcal{K}_i}^\bullet \varepsilon_i^*$.

PROOF. It is obvious that (35) is equivalent to the form $\mathcal{K}(\theta,\dot{\theta}) = \mathcal{K}_a(\theta,\dot{\theta}) - \mathcal{K}_b(\theta,\dot{\theta})$ where, for $k \in \{a, b\}$, the matrices $\mathcal{K}_k(\theta,\dot{\theta})$ are defined by Eq. (36). Designing term by term the bounds of Eq. (36) leads to (37) and $\underline{\mathcal{K}_k}(\rho) \geq 0$, $\forall \rho \in \Psi$. □

Note that the construction of the bounds $\underline{\mathcal{K}_j}$, $\overline{\mathcal{K}_j}$ for $j \in \{0, \cdots, s\}$ is possible since $K(\theta)$ and N are assumed to be *a priori* known, see assumption 9.

The following theorems, i.e. theorem 15 and theorem 17, give the solutions to the L_∞-gain and L_2-gain performance problems, respectively.

Theorem 15 *Consider the assumptions 7-9 and the following LPV model*

$$\begin{cases} \dot{\nu} = \mathcal{A}(\rho)\nu + \mathcal{B}(\rho)\psi & \text{(38a)} \\ \xi = \mathcal{C}(\rho)\nu + \mathcal{D}(\rho)\psi & \text{(38b)} \\ \nu(0) \geq \tilde{z}(0). & \text{(38c)} \end{cases}$$

where $\nu, \xi \in \mathbb{R}^n$, $\psi = \left[\tilde{w}^T \ \hat{w}^T \ |r|^T\right]^T$ and the matrices $\mathcal{C}(\rho) = \breve{Q}(\rho)$, $\mathcal{D}(\rho) = \left[\breve{Q}(\rho)\breve{H}(\rho) \ \breve{Q}(\rho)\widetilde{H}(\rho) \ \widetilde{Q}(\rho)\right]$, $\mathcal{A}(\rho) = L(\rho) + \breve{F}(\rho)$, $\mathcal{B}(\rho) = (T_a(\rho) + T_b(\rho))\,\Omega(\rho) + \breve{F}(\rho)\Phi(\rho)$ with $\Omega(\rho) = \left[\breve{Z}(\rho) \ \widetilde{Z}(\rho) \ \widetilde{Y}(\rho)\right]$, $\Phi(\rho) = \left[\breve{H}(\rho) \ \widetilde{H}(\rho) \ 0_{n,n}\right]$, and the notations $\widetilde{(\cdot)}, \breve{(\cdot)}$ are defined in (9). Consider also matrices $P_j = P_j^T \in \mathbb{R}^{n\times n}$ and $\mathscr{I}_j = \left[\mathcal{L}_{aj} \ \mathcal{L}_{bj} \ \mathcal{M}_{aj} \ \mathcal{M}_{bj}\right]$ where $\mathcal{L}_{aj}, \mathcal{L}_{bj} \in \mathbb{R}^{n\times n}$, $\mathcal{M}_{aj}, \mathcal{M}_{bj} \in \mathbb{R}^{n\times p}$, for $j \in \{0, \cdots, s\}$, such that

$$P(\alpha) = P_0 + \alpha_1 P_1 + \cdots + \alpha_s P_s \succ 0, \forall \alpha \in \mathscr{V} \tag{39}$$
$$\mathscr{I}(\alpha) = \mathscr{I}_0 + \alpha_1 \mathscr{I}_1 + \cdots + \alpha_s \mathscr{I}_s > 0, \forall \alpha \in \mathscr{V} \tag{40}$$

and matrices $\mathcal{E}_{ii} = \mathcal{E}_{ii}^T \succeq 0$, $\mathcal{F}_{ii} = \mathcal{F}_{ii}^T \succeq 0$, $\mathcal{H}_{ii} > 0$ so that

$$\mathcal{E}(\alpha) = \alpha_1^2 \mathcal{E}_{11} + \cdots + \alpha_s^2 \mathcal{E}_{ss} \succeq 0, \forall \alpha \in \mathscr{V} \tag{41}$$
$$\mathcal{F}(\alpha) = \alpha_1^2 \mathcal{F}_{11} + \cdots + \alpha_s^2 \mathcal{F}_{ss} \succeq 0, \forall \alpha \in \mathscr{V} \tag{42}$$
$$\mathcal{H}(\alpha) = \alpha_1^2 \mathcal{H}_{11} + \cdots + \alpha_s^2 \mathcal{H}_{ss} > 0, \forall \alpha \in \mathscr{V}. \tag{43}$$

If there exist some real scalars $\mu \geq 0$, $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$ and a M-matrix $W \in \mathbb{R}^{n\times n}$ such that the following constraints hold:

(i) $\forall (\alpha, \beta) \in \mathscr{V} \times \mathscr{R}$,

$$\Upsilon(\alpha,\beta,\gamma_1,\gamma_2) + \mathcal{E}(\alpha) \prec 0 \tag{44}$$
$$\Xi(\alpha,\gamma_3,\gamma_4) + \mathcal{F}(\alpha) \prec 0 \tag{45}$$
$$-\left(\mathcal{L}_a(\alpha) - \mathcal{L}_b(\alpha)\right) - \mu W < 0 \tag{46}$$
$$-\left(\mathcal{L}_b(\alpha) + \mathcal{T}_a(\alpha)\underline{Y}(\alpha) - \mathcal{T}_b(\alpha)\overline{Y}(\alpha)\right) + \mathcal{H}(\alpha) < 0 \tag{47}$$

with $\Upsilon(\alpha,\beta,\gamma_1,\gamma_2) =$

$$\begin{bmatrix} -\mathcal{H}e\{W\} & \star & \star & \star \\ \Upsilon_{2,1}(\alpha)^T & \Upsilon_{2,2}(\alpha,\gamma_2) & \star & \star \\ \Upsilon_{3,1}(\alpha)^T & \mathbf{0} & \Upsilon_{3,3}(\alpha,\beta,\gamma_1) & \star \\ W^T & \mathbf{0} & \mathbf{0} & -P(\alpha) \end{bmatrix}, \Upsilon_{2,1}(\alpha) =$$

$$\left[\left(\mathcal{T}_a(\alpha) + \mathcal{T}_b(\alpha)\right)\Omega(\alpha) \ \mathcal{J}(\alpha)\right], \quad \Upsilon_{3,1}(\alpha) = P(\alpha) +$$

$\mathcal{L}_a(\alpha) - \mathcal{L}_b(\alpha) + \mathcal{J}(\alpha)$, $\mathcal{J}(\alpha) = \mathcal{L}_b(\alpha) + \mathcal{T}_a(\alpha)\overline{Y}(\alpha) - \mathcal{T}_b(\alpha)\underline{Y}(\alpha) + \mathcal{L}_a(\alpha)$, $\mathcal{T}_a(\alpha) = \begin{bmatrix}\mathcal{L}_a(\alpha)\,\mathcal{M}_a(\alpha)\,W\end{bmatrix}$, $\mathcal{T}_b(\alpha) = \begin{bmatrix}\mathcal{L}_b(\alpha)\,\mathcal{M}_b(\alpha)\,\mathbf{0}\end{bmatrix}$, $\Upsilon_{2,2}(\alpha,\gamma_2) = \begin{bmatrix}-\gamma_2\mathbf{I} & \star \\ \Phi(\alpha) & -\mathbf{I}\end{bmatrix}$, $\Upsilon_{3,3}(\alpha,\beta,\gamma_1) = (\gamma_1-1)P(\alpha)+P(\beta)-P_0$, $\Xi(\alpha,\gamma_3,\gamma_4) = \begin{bmatrix}-\gamma_3 P(\alpha) & \star & \star \\ \mathbf{0} & -\gamma_4\mathbf{I} & \star \\ \mathcal{C}(\alpha) & \mathcal{D}(\alpha) & -\mathbf{I}\end{bmatrix}$.
(ii) for $i \in \{1,\cdots,s\}$,

$$\Upsilon_{ii} + \mathcal{E}_{ii} \succeq 0 \quad (48)$$
$$\Xi_{ii} + \mathcal{F}_{ii} \prec 0 \quad (49)$$
$$-\left(\mathcal{T}_{ai}\underline{Y}_i - \mathcal{T}_{bi}\overline{Y}_i\right) + \mathcal{H}_{ii} > 0 \quad (50)$$

where $\Upsilon_{ii} = \begin{bmatrix}\mathbf{0} & \star\ \star\ \star \\ (\Upsilon_{ii})_{2,1}{}^T & \mathbf{0}\ \star\ \star \\ \mathcal{J}_{ii} & \mathbf{0}\ \mathbf{0}\ \star \\ \mathbf{0} & \mathbf{0}\ \mathbf{0}\ \mathbf{0}\end{bmatrix}$, $\Xi(\alpha) = \begin{bmatrix}\mathbf{0} & \star & \star \\ \mathbf{0} & \mathbf{0} & \star \\ \mathbf{0} & \mathcal{D}_{ii} & \mathbf{0}\end{bmatrix}$, $(\Upsilon_{ii})_{2,1} = \begin{bmatrix}(\mathcal{T}_{ai}+\mathcal{T}_{bi})\,\Omega_i & \mathcal{J}_{ii}\end{bmatrix}$, $\mathcal{J}_{ii} = \mathcal{T}_{ai}\overline{Y}_i - \mathcal{T}_{bi}\underline{Y}_i$, $\Omega_i = \begin{bmatrix}\breve{Z}_i & \widetilde{Z}_i & \widetilde{Y}_i\end{bmatrix}$, $\mathcal{T}_{ai} = \begin{bmatrix}\mathcal{L}_{ai}\ \mathcal{M}_{ai}\ \mathbf{0}\end{bmatrix}$, $\mathcal{T}_{bi} = \begin{bmatrix}\mathcal{L}_{bi}\ \mathcal{M}_{bi}\ \mathbf{0}\end{bmatrix}$, $\mathcal{D}_{ii} = \begin{bmatrix}\breve{Q}_i\breve{H}_i\ \breve{Q}_i\widetilde{H}_i\ \mathbf{0}\end{bmatrix}$, *where* $\widetilde{\mathcal{K}}_i$, $\breve{\mathcal{K}}_i$, $\underline{\mathcal{K}}_i$, $\underline{\mathcal{K}}_i$ *for* $\mathcal{K} \in \{Q,Y,Z,H\}$ *deduced from the decompositions* $\begin{bmatrix}\widetilde{\mathcal{K}}(\rho) & \breve{\mathcal{K}}(\rho) \\ \underline{\mathcal{K}}(\rho) & \overline{\mathcal{K}}(\rho)\end{bmatrix} = \begin{bmatrix}\widetilde{\mathcal{K}}_0 & \breve{\mathcal{K}}_0 \\ \underline{\mathcal{K}}_0 & \overline{\mathcal{K}}_0\end{bmatrix} + \sum_{i=1}^{s}\rho_i\begin{bmatrix}\widetilde{\mathcal{K}}_i & \breve{\mathcal{K}}_i \\ \underline{\mathcal{K}}_i & \overline{\mathcal{K}}_i\end{bmatrix}$.

Then, with the UIIO matrices $L(\rho) = L_a(\rho) - L_b(\rho)$ *and* $M(\rho) = M_a(\rho) - M_b(\rho)$ *defined according to*

$$\begin{bmatrix}L_k(\rho)\ M_k(\rho)\end{bmatrix} = W^{-1}\begin{bmatrix}\mathcal{L}_k(\rho)\ \mathcal{M}_k(\rho)\end{bmatrix}\ k \in \{a,b\} \quad (51)$$

the system (34a)-(34c) *is stable with its asymptotic* L_∞*-gain performance level, less to* $\gamma_{L_\infty} = \sqrt{\frac{\gamma_3\gamma_2}{\gamma_1} + \gamma_4}$.

PROOF. The proof is organized into two main steps:

a) First, it is shown that the LPV model (38a)-(38c) is stable $\forall\rho \in \Psi$ and $\forall\dot\rho \in \Gamma_\rho$ with asymptotic L_∞-gain less or equal to γ_{L_∞} (sufficient condition).
b) Second, it is shown that this is a sufficient condition for the system (34a)-(34c) to be stable with an asymptotic L_∞-gain performance level less to γ_{L_∞}.

Step a): Let us first consider $\mathcal{E}(\alpha) = 0, \mathcal{F}(\alpha) = 0, \mathcal{E}_{ii} = 0$ and $\mathcal{F}_{ii} = 0$ in Eqs. (44), (45), (48) and (49). This will be justified later, see remark 16. Since in theorem 15, $\mathcal{L}_k(\rho) = WL_k(\rho)$ and $\mathcal{M}_k(\rho) = WM_k(\rho)$ for $k \in \{a,b\}$ are LMI variables, it follows that the set of constraints (39)-(50) implies $\forall\rho \in \Psi$, $\forall\dot\rho \in \Gamma_\rho$ (see Gahinet et al. (1996))

$$\Upsilon(\rho,\dot\rho) \prec 0 \quad (52)$$
$$\Xi(\rho) \prec 0 \quad (53)$$
$$W\left(L_a(\rho) - L_b(\rho)\right) + \mu W > 0 \quad (54)$$
$$W\begin{bmatrix}L_a(\rho)\ L_b(\rho)\ M_a(\rho)\ M_b(\rho)\end{bmatrix} > 0 \quad (55)$$
$$W\left(L_b(\rho) + T_a(\rho)\underline{Y}(\rho) - T_b(\rho)\overline{Y}(\rho)\right) - \mathcal{H}(\rho) > 0 \quad (56)$$

Since $W^{-1} \geq 0$, Eqs. (54)-(55) ensure that $L(\rho)$ is Metzler and that $L(\rho)$, $M(\rho)$ satisfy (24). Eq. (56) guarantees the decomposition (25) and the conditions (52)-(53) guarantee L_∞-gain performance for (38a)-(38c).
Now, consider the condition (52). Since by definition, $\Upsilon_{2,1}(\rho) = W\mathcal{G}(\rho)$ and $\Upsilon_{3,1}(\rho) = P(\rho) + W\mathcal{A}(\rho)$ with $\mathcal{G}(\rho) = \begin{bmatrix}(T_a(\rho)+T_b(\rho))\,\Omega(\rho)\ \breve{F}(\rho)\end{bmatrix}$ then applying the Schur complement to (52) leads to

$$\Lambda(\rho,\dot\rho) + \mathcal{Q}(\rho)^T W^T\mathcal{I} + \mathcal{I}^T W\mathcal{Q}(\rho) \prec 0 \quad (57)$$

where $\mathcal{I} = \begin{bmatrix}\mathbf{I}\ \mathbf{0}\end{bmatrix}$ $\mathcal{Q}(\rho) = \begin{bmatrix}-\mathbf{I}\ \mathcal{G}(\rho)\ \mathcal{A}(\rho)\ \mathbf{I}\end{bmatrix}$ and $\Lambda(\rho,\dot\rho) = \begin{bmatrix}\mathbf{0} & \star & \star & \star \\ \mathbf{0} & \Upsilon_{2,2}(\rho) & \star & \star \\ P(\rho) & \mathbf{0} & \Upsilon_{3,3}(\rho,\dot\rho) & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -P(\rho)\end{bmatrix}$. By virtue of the projection lemma, (57) has a solution for some matrix W if and only if

$$N_{\mathcal{Q}}(\rho)^T\Lambda(\rho)N_{\mathcal{Q}}(\rho) \prec 0 \quad N_{\mathcal{I}}(\rho)^T\Lambda(\rho)N_{\mathcal{I}}(\rho) \prec 0 \quad (58)$$

where the null-spaces $N_{\mathcal{Q}}(\rho)$ and $N_{\mathcal{I}}(\rho)$ are given by $N_{\mathcal{Q}}(\rho) = \begin{bmatrix}\mathcal{A}(\rho)^T & \mathbf{0}\ \mathbf{I}\ \mathbf{0} \\ \mathcal{G}(\rho)^T & \mathbf{I}\ \mathbf{0}\ \mathbf{0} \\ \mathbf{I} & \mathbf{0}\ \mathbf{0}\ \mathbf{I}\end{bmatrix}^T$, $N_{\mathcal{I}}(\rho) = \begin{bmatrix}\mathbf{0}\ \mathbf{I}\ \mathbf{0}\ \mathbf{0} \\ \mathbf{0}\ \mathbf{0}\ \mathbf{I}\ \mathbf{0} \\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{I}\end{bmatrix}^T$.
Applying the Schur complement on the left inequality of (58) leads to

$$\mathscr{A}(\rho,\dot\rho) + \mathscr{B}(\rho) \prec 0 \quad (59)$$

$$\mathscr{A}(\rho,\dot\rho) = \begin{bmatrix}\mathcal{H}e\{P(\rho)\mathcal{A}(\rho)\} + \dot P(\rho) + \gamma_1 P(\rho) & \star \\ \mathcal{B}^T P(\rho) & -\gamma_2\mathbf{I}\end{bmatrix}$$

$$\mathscr{B}(\rho) = \begin{bmatrix}P(\rho)\breve{F}(\rho)\breve{F}(\rho)^T P(\rho)^T & 0 \\ \mathbf{0} & \Phi(\rho)^T\Phi(\rho)\end{bmatrix}$$

Since $\mathscr{B}(\rho)$ is strictly positive definite, it follows that $\mathscr{A}(\rho,\dot\rho) \prec 0$ (sufficient condition). Since $\mathscr{A}(\rho,\dot\rho)$ is attached to the Lyapunov function $V(\rho,\nu) = \nu^T P(\rho)\nu$ characterized by $\dot V(\rho,\nu) + \gamma_1 V(\rho,\nu) - \gamma_2\psi^T\psi < 0$, it is deduced that $\dot V(\rho,\nu) < -\gamma_1 V(\rho,\nu) + \gamma_2\|\psi\|_\infty^2$. Then

$$V(\rho(t),\nu(t)) < V(\rho(0),\nu(0))e^{-\gamma_1 t} + \tfrac{\gamma_2}{\gamma_1}\|\psi\|_\infty^2 \quad (60)$$

Noting that (53) is equivalent to $\xi^T\xi + \gamma_3 V(\rho,\nu) - \gamma_4\psi^T\psi < 0$, then $\xi^T\xi < \gamma_3 V(\rho,\nu) + \gamma_4\|\psi\|_\infty^2$, which

leads with (60) to

$$\xi(t)^T\xi(t) < \gamma_{L_\infty}^2\|\psi\|_\infty^2 + \gamma_3 V(\rho(0),\nu(0))e^{-\gamma_1 t} \quad (61)$$

which is the characterisation of the L_∞-gain for the LPV system (38a)-(38c). It follows that the LPV model (38a)-(38c) is stable $\forall\rho \in \Psi$ and $\forall\dot\rho \in \Gamma_\rho$ with asymptotic L_∞-gain performance less or equal to γ_{L_∞}.
Now coming back to the right inequality of (58), since it is nothing else than an additional constraint to the L_∞-gain inequality, it follows that LMIs (39)-(50) provides only a sufficient condition. Unfortunately, there still exists a coupling between the parameter-dependent matrices $P(\rho)$ and $L_a(\rho)$, $L_b(\rho)$, $M_a(\rho)$, $M_b(\rho)$.To overcome this drawback, the slack matrix technique is applied to (59), leading to (52).

Remark 16 *As it is outlined in (Gahinet et al., 1996), it is possible to reduce the conservatism of* (52)-(53) *by somewhat relaxing the multiconvexity requirement* (48)-(49). *This is the role of the matrices* $\mathcal{E}(\alpha)$, $\mathcal{F}(\alpha)$, $\mathcal{E}_{ii}$ *and* $\mathcal{F}_{ii}$ *in Eqs.* (44)-(45) *and* (48)-(49). *The resulting inequalities are generally less conservative since* (52)-(53) *corresponds to* $\mathcal{E}_1 = ... = \mathcal{E}_s = 0$ *and* $\mathcal{F}_1 = ... = \mathcal{F}_s = 0$. *However, this improvement is at the expense of a higher computational overhead, since the number of optimization variables increases.*

Step b): We first demonstrate that

$$0 \le \tilde x(t) \le \xi(t) \quad \forall t \ge 0 \quad (62)$$

Under (54) and Lemma 3, the condition (i) of Theorem 11 holds. Since W is a M-matrix, we have $W^{-1} \ge 0$, then Eqs. (55)-(56) ensure the existence conditions of Lemma 13. Combining the expressions of $\overline{\eta}$, $\underline{\eta}$, $\overline{\varphi}$, $\underline{\varphi}$ given in Lemma 13 and the initial conditions (34c), it is deduced that Theorem 11 holds. A direct consequence is $0 \le \tilde x(t)$, $\forall t \ge 0$. So the left part of the inequality of (62) is proved. Now, let us consider the right part of (62). Subtracting (34a)-(34c) and (38a)-(38c), leads to

$$\begin{cases} \dot\nu - \dot{\tilde z} = \mathcal{A}(\rho)\,(\nu - \tilde z) + \widetilde{F}(\rho)\,(|r| - \hat r) & (63a)\\ \xi - \tilde x = \breve{Q}(\rho)\,(\nu - \tilde z) + \widetilde{Q}(\rho)\,(|r| - \hat r) & (63b)\\ \nu(0) - \tilde z(0) \ge 0 & (63c)\end{cases}$$

where $|r| = r^+ + r^-$, $\hat r = \overline{r}^- + \underline{r}^+$, $\mathcal{A}(\rho) = L(\rho) + \breve{F}(\rho)$. Indeed, for the vector $r \in [\underline{r}, \overline{r}]$, we have $0 \le \underline{r}^+ \le r^+$, $0 \le \overline{r}^- \le r^-$ and, by definition, $\widetilde{F}(\rho) \ge 0$, $\forall\rho \in \Psi$, it is obvious that $\widetilde{F}(\rho)(|r| - \hat r) \ge 0$. Since $L(\rho)$ is Metzler and $\breve{F}(\rho) \ge 0$, $\forall\rho \in \Psi$ then $\mathcal{A}(\rho)$ is Metzler. Then with (63c) and by virtue of the cooperative system theory, it follows that $\nu - \tilde z \ge 0$, $\forall t \ge 0$. Since by definition $\breve{Q}(\rho) \ge 0$,$\widetilde{Q}(\rho) \ge 0$, $\forall\rho \in \Psi$, it follows that (63b) is positive, and then $\tilde x(t) \le \xi(t)$, $\forall t \ge 0$.

Combining the aforementioned results, i.e. $0 \le \tilde x(t) \le \xi(t)$, $\forall t \ge 0$ and $\|\xi\|_\infty \le \gamma_{L_\infty}\|\psi\|_\infty$ $\forall\rho \in \Psi$, $\forall\dot\rho \in \Gamma_\rho$, it follows that γ_{L_∞} is an upper bound of the asymptotic L_∞-gain performance level of the system (34a)-(34c). Furthermore, since $0 \le \tilde z(t) \le \nu(t)$, $\forall t \ge 0$ where $0 \le \tilde z(t)$ is proven by applying the cooperative system theory on (34a) and since it has been proven in the previous step that $\nu(t)$ is stable, (34a)-(34c) is stable. □

The following theorem gives the solution for the L_2-gain performance problem.

Theorem 17 *Consider the assumptions 7-9, the LPV model* (38a)-(38c), *the inequalities* (39)-(40), (43) *and matrices* $\mathcal{G}_{ii} = \mathcal{G}_{ii}^T \succeq 0$, *so that*

$$\mathcal{G}(\alpha) = \alpha_1^2\mathcal{G}_{11} + \cdots + \alpha_s^2\mathcal{G}_{ss} \succeq 0, \forall\alpha \in \mathscr{V} \quad (64)$$

If there exist scalars $\mu \ge 0$, $\gamma > 0$ *and a M-matrix* $W \in \mathbb{R}^{n\times n}$ *such that the following constraints hold:*
(i) $\forall(\alpha,\beta) \in \mathscr{V}\times\mathscr{R}$, *let consider the inequalities* (46)-(47) *in Theorem 15 and*

$$\Pi(\alpha,\beta,\gamma) + \mathcal{G}(\alpha) \prec 0 \quad (65)$$

where $\Pi(\alpha,\beta,\gamma) = \begin{bmatrix} \Upsilon(\alpha,\beta,0,\gamma) & \star \\ \Pi_{2,1}(\alpha) & -\mathbf{I}\end{bmatrix}$ *with* Υ *defined in* (44) *and* $\Pi_{2,1}(\alpha) = \begin{bmatrix}\mathbf{0} & \mathcal{D}(\alpha) & \mathbf{0} & \mathcal{C}(\alpha) & \mathbf{0}\end{bmatrix}$
(ii) for $i \in \{1,\cdots,s\}$, *let consider the inequality* (50) *and*

$$\Pi_{ii} + \mathcal{G}_{ii} \succeq 0 \quad (66)$$

where $\Pi_{ii} = \begin{bmatrix} \Upsilon_{ii} & \star \\ (\Pi_{ii})_{2,1} & \mathbf{0}\end{bmatrix}$, $(\Pi_{ii})_{2,1} = \begin{bmatrix}\mathbf{0} & \mathcal{D}_{ii} & \mathbf{0}\end{bmatrix}$ *with* Υ_{ii}, $\mathcal{D}_{ii}$ *defined in* (48), (49).

Then, with the UIIO matrices $L(\rho) = L_a(\rho) - L_b(\rho)$ *and* $M(\rho) = M_a(\rho) - M_b(\rho)$ *with* $L_k(\rho), M_k(\rho)$ $k \in \{a,b\}$ *defined by* (51), *the system* (34a)-(34c) *is stable with its* L_2*-gain performance level, less or equal to* $\gamma_{L_2} = \sqrt{\gamma}$.

PROOF. The proof is similar to the previous one. □

Theorems 15 and 17 being posed as SDP problems, it is straightforward to combine them into a mixed problem, by using a weighted convex optimization formulation.

5 Academic example

Consider the following model with the same notations than used in the previous sections:

$$A(\theta) = \begin{bmatrix} \theta_3 - 2 & -0.5 & 0 \\ \theta_4 & \theta_1 - 1 & -0.2 \\ 0.1 & 0 & \theta_2 - 2\end{bmatrix},\ C(\theta) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1+\theta_1 & 0\end{bmatrix},$$

$$B(\theta) = \begin{bmatrix} 0 & 1 & 0.2 \\ \theta_1 - 0.1 & 0 & 0 \\ 2 & \theta_2 & 0\end{bmatrix},\ E(\theta) = \begin{bmatrix}\theta_3 + 2\\ 0 \\ \theta_1\end{bmatrix},\ D = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The parameter vector θ and its rate of variations $\dot{\theta}$ are unknown but assumed to be bounded by the sets

$$\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\end{bmatrix} \in \begin{bmatrix}-0.15, 0.15\\ -0.25, 0.25\\ -0.15, 0.15\\ 0.2, 0.8\end{bmatrix}, \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\\ \dot{\theta}_4\end{bmatrix} \in \begin{bmatrix}-0.30, 0.30\\ -0.25, 0.25\\ -0.75, 0.75\\ 0, 0\end{bmatrix}.$$

The parameters θ_1 and θ_2 are respectively measured through $\rho_1 = \theta_1 + \delta_1$, $\rho_2 = \theta_2 + \delta_2$ where $\delta_1 \in \left[-0.03, 0.03\right]$ and $\delta_2 \in \left[-0.03, 0.03\right]$ are bounded noises such that $\dot{\delta}_1 \in \left[-10^3, 10^3\right]$, $\dot{\delta}_2 \in \left[-10^3, 10^3\right]$. It follows that ρ and $\dot{\rho}$ satisfy $\begin{bmatrix}\rho_1\\ \rho_2\end{bmatrix} \in \begin{bmatrix}-0.18, 0.18\\ -0.28, 0.28\end{bmatrix}$, $\begin{bmatrix}\dot{\rho}_1\\ \dot{\rho}_2\end{bmatrix} \in \begin{bmatrix}-1000.30, 1000.30\\ -1000.25, 1000.25\end{bmatrix}$. Hence, assumption 7 is satisfied.

Following the method described in the previous sections, $K(\theta)$ and N must be chosen adequately in order to derive a not too conservative solution for the UIIO. For $K(\theta)$, it is proposed to retain a parameter independent matrix so that $K(\theta) = \kappa I_n$ with $\kappa \neq 0$. Then a valid definition for N is $N = K(\theta)\overline{S} = \kappa \begin{bmatrix}1 & 0\\ 0 & 0\\ 0.081 & 0\end{bmatrix}$ where $\overline{S}$ is the upper bound of the matrix $S(\theta)$, see (5). As $rank(C(\theta)E(\theta)) = rank(E(\theta)) = 1$, $\forall\theta \in \Theta$ then assumption 8 holds, which leads to

$$R(\theta) = \kappa \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ -\frac{\theta_1}{\theta_3+2} & 0 & 1\end{bmatrix}, X(\theta) = \kappa \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0.081 - \frac{\theta_1}{\theta_3+2} & 0 & 1\end{bmatrix}.$$

κ being non null, it is obvious that $X(\theta)$ is invertible $\forall\theta \in \Theta$ and thus, that assumption 9 holds.

The next step is to design $\mathcal{A}(\rho), \mathcal{B}(\rho), \mathcal{C}(\rho), \mathcal{D}(\rho)$ involved in (38a)-(38c) using the method proposed in Lemma 14 with $n_i = -\underline{\rho_i}$, $i \in \{1, 2\}$ for all matrices. Furthermore, and as it is usually done in the L_2-gain setting, the design problem is weighted by introducing a constant weighting function $W_\psi = diag(w_{\psi_i})$, $i = 1, ..., 2m + n$ so that $\psi = W_\psi b_\psi$ where b_ψ is a fictitious vector. The scalars w_{ψ_i}, $i = 1, ..., 2m + n$ are introduced to manage the robustness level of each channel, separately. Noting that γ, γ_2, γ_4 enters linearly in (39)-(50), (64)-(66), we can address the following optimization problem

$$\min \gamma_n = \gamma + \gamma_2 + \gamma_4 \text{ s.t. (39)-(50), (64)-(66)}$$

Since (39)-(50), (64)-(66) depend nonlinearly on κ, γ_1, γ_3 and the LMI variables, this optimization problem is not convex. A simple remedy consists of a grid of κ, γ_1, γ_3 with a sufficiently dense grid and to minimize γ_n at each griding point. Here, by setting $\gamma_1 = 0.7$, $\gamma_3 = 1$, the result, presented in figure 1, is a $\kappa - \gamma_{opt}$ graph where it can be observed the value(s) of κ that lead(s) to a minimal value of γ_n ($\gamma_n \approx 65.7$ for $\kappa \approx \pm 1.4$).

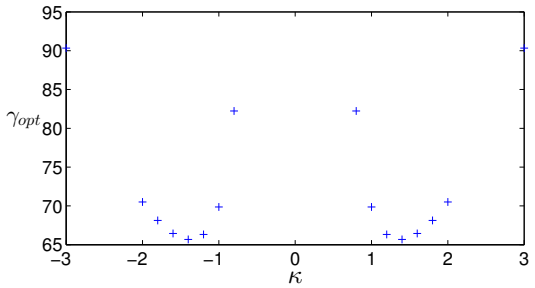

Fig. 1. $\kappa - \gamma$ graph

With $\kappa \approx 1.4$, we obtain $\gamma_{L_\infty} = 7.15$, $\gamma_{L_2} = 5.81$ and the matrices $L_k(\rho)$, $M_k(\rho)$, $k \in \{a, b\}$ follow from (51). The functions $\overline{\eta}$, $\underline{\eta}$, $\overline{\varphi}$, $\underline{\varphi}$ are then deduced from lemma 13, which leads to the complete definition of the UIIO.

Simulation results : Simulations are next performed to appreciate the performance of the designed UIIO. The time behaviour of the parameters are chosen according to $\theta_1(t) = 0.15sin(2t + \frac{\pi}{2})$, $\theta_2(t) = 0.25sin(t + \frac{\pi}{4})$, $\theta_3(t) = 0.15sin(5t + \frac{\pi}{3})$ and $\theta_4 = 0.5$. The unknown input $d(t)$ is fixed to $d(t) = 2cos(2t)$ and the vector $w(t)$ is chosen to be composed of the signal $w_1(t) = -2sin(\frac{t}{2})$ and two noises $w_2(t)$, $w_3(t)$ bounded by known values $\overline{w_2} = \overline{w_3} = 0.1$, $\underline{w_2} = \underline{w_3} = -0.1$. The results of the simulations are presented in figures 2-4. As expected, $\underline{x}(t)$ and $\overline{x}(t)$ are lower and upper bounds of $x(t)\,\forall t \geq 0$ and the distance $\overline{x} - \underline{x}$ converges with a reasonable transient behaviour. Finally, a tight interval can be observed, thanks to the L_2/L_∞-gain performance.

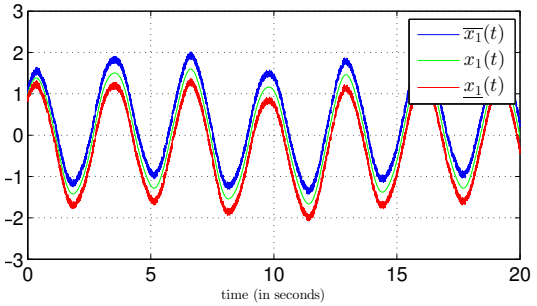

Fig. 2. Interval estimation of the state x_1

Note that it can be observed a high frequency behaviour on $\overline{x}_1(t)/\underline{x}_1(t)$ that does not exist on $\overline{x}_k(t)/\underline{x}_k(t), k = 2, 3$. This is due to the disturbance $w(t)$ and its second component $w_2(t)$ to be more precise. This can be appreciated by looking at the principal gains of the transfers between the different component of the exogenous input w and the estimated upper/lower bounds $\overline{x}_k/\underline{x}_k, i = 1, 2, 3$, for frozen parameters $\theta(t)$. These plots are not presented

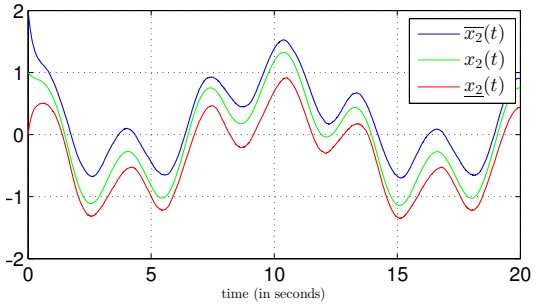

Fig. 3. Interval estimation of the state x_2

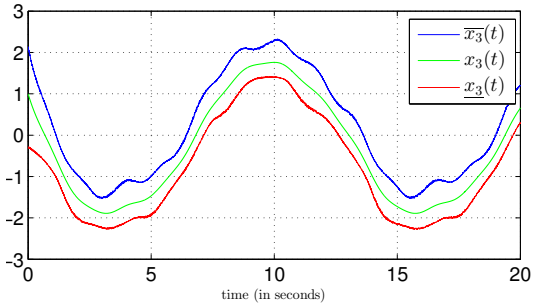

Fig. 4. Interval estimation of the state x_3

here due to page limitation, but they reveal that the high frequency content of $w_2(t)$ is not filtered on $\overline{x}_1(t)/\underline{x}_1(t)$, whereas it is on $\overline{x}_k(t)/\underline{x}_k(t), k = 2, 3$.

6 Conclusion and future works

This paper investigated the design of an unknown input interval observer, for linear parameter varying systems, to robustly estimate the state of linear parameter varying systems, in a guaranteed way under L_2 and L_∞-gain performance. A SDP formulation is derived to design all the observer parameters. The major advantages with regards to existing theories for interval observer design is that it does not require the explicit knowledge of a (time-varying) state transformation, for the estimation error dynamics to satisfy the Meztler property. Furthermore the unknwon input decoupling and L_2/L_∞ constraints are formulated on the thickness of the interval length in spite of half the interval length.

Finally, we would like to conclude by a remark about the conservativeness of the affine parameter dependency of the Lyapunov matrix and the matrices $\mathcal{L}_k, \mathcal{M}_k$, $k \in \{a, b\}$ of theorems 15 and 17. This choice can be relaxed by using the Sum-of-Square (SoS) theory Parrilo (2000). The SoS theory enables to use a general polynomial structure for these matrices and thus offers the potential to derive a solution in the general case but there exist some practical limitations such as the *a priori* choice of the degree and the structure of the polynomials. Nevertheless, its investigation is under current topic.

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