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Decay of solutions to a water wave model with a nonlocal viscous term

S. Dumont¹, O. Goubet², I. Manoubi³

Abstract

We update here the results on the decay of solutions to a nonlocal water wave equation that reads

$$u_t + u_x + \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds + uu_x = u_{xx},$$

where $\frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds$ is the Riemann–Liouville half-order derivative.

Keywords Nonlocal water wave equation · Decay rate of solutions

Mathematics Subject Classification 35B40 · 35Q35

1 Introduction

1.1 The modeling

This article addresses some issues related to the decay rate of solutions to a one-direction asymptotical water wave model with a nonlocal viscous term.

Without viscosity, it is now standard to derive asymptotical models as the Korteweg-de Vries equation or the Boussinesq system from the Euler equations [2]. Taking into account the viscosity is challenging. In the last decades, many studies partakes of this modeling: the first model is due to T. Kakutani and K. Matsuuchi in their article [16], more recently, Liu and Orfila [20], and Dutykh and Dias [8] have independently derived viscous asymptotic models for transient long-wave propagation on viscous shallow water.

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In this article we focus on a single equation for a one-way wave equation that was investigated in [22], where fractional term is described by the Riemann–Liouville half derivative. This equation reads

$$u_t + u_x + \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds + uu_x = u_{xx}. \quad (1.1)$$

Here, for the sake of convenience, the physical constants are set equal to 1. The nonlocal term provides both diffusion and dispersion to the equation [5,22].

On the other hand, diffusion equations with non local operators are of great interest for mathematicians and also for other scientific communities (see [7,12,13,15,23–25] and the references therein).

The initial value problem for (1.1) was previously addressed in [11,22]; we know that for initial data u_0 in $L^2(\mathbb{R})$, there exists a unique solution $u \in C(\mathbb{R}_+; L_x^2(\mathbb{R})) \cap C^{1/2}(\mathbb{R}_+; H_x^{-2}(\mathbb{R}))$. Besides, for more regular initial data $u_0 \in \mathcal{S}(\mathbb{R})$ in the Schwartz class, the solution belongs to $C^{1/2}(\mathbb{R}_+; \mathcal{S}(\mathbb{R}))$. The solution map $t \mapsto u(t)$ is C^1 but at $t = 0^+$. In this article we are interested in the decay rate of solutions towards 0. We prove that any solution converges towards 0 with the expected decay rate; this is an improvement over [22] where a smallness assumption on the initial data was required.

On the other hand, the numerical investigation of this decay rate has also received interest also in the last decade. Many authors have provided different numerical methods to approximate the nonlocal fractional operator [5,9,10,17,22]. Some have proposed the so-called diffusive realization of fractional operators. This alternative provides a local representation of the fractional operator as a system of linear differential equation. Several applications of this approach can be found in [1,13,23,26]. We use here this diffusive approach to prove theoretical results. Due to uniqueness, we will use in the sequel that a solution of the original equation with u_0 in $L^2(\mathbb{R})$ is solution to the diffusive representation and vice-versa.

This article is organized as follows. We introduce a suitable diffusive representation for the equation, we state the main theoretical result and we proceed to the proof. We complete this article by numerical computations that illustrate our theoretical results.

1.2 Statement of the result

The aim of this article is to answer a question left open in [22]. Concerning the decay rate of solutions, the third author proved that if $\|u_0\|_{L^1(\mathbb{R})}$ is small enough, then the solution u satisfies the decay estimate

$$\max(1, t^{3/4}) \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} + \max(t^{1/2}, t) \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C.$$

However, this result is proved assuming a smallness condition on the initial data. We relax here this assumption *first* for *non negative* initial data $u_0 \geq 0$, and then conclude by a comparison result. Our main result states as follows

Theorem 1.1 *Assume u_0 in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then there exists $C(u_0)$ such that for any $t \geq 0$,*

$$\max(1, t^{3/4}) \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} + \max(t^{1/2}, t) \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C(u_0). \quad (1.2)$$

We provide here a complete proof that relies on an ad hoc representation of the solution.

1.3 A diffusive representation of the solution

Introducing a new variable $\sigma \geq 0$, for a given function $u(t, x)$ that depends on t and on x (the variable x will be omitted in the remaining of this section) we introduce

$$\varphi(t, \sigma) = \int_0^t u(s) \exp(-\sigma^2(t-s)) ds. \quad (1.3)$$

Then the half-integral of a function u reads

$$Iu(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds = \Gamma(\varphi)(t) = \frac{2}{\pi} \int_0^{+\infty} \varphi(t, \sigma) d\sigma. \quad (1.4)$$

Hence equation (1.1) reads also

$$\begin{aligned} u_t + u_x + \Gamma(\varphi_t) + uu_x &= u_{xx}, \\ \varphi_t + \sigma^2 \varphi &= u, \end{aligned} \quad (1.5)$$

supplemented with initial data $(u_0, 0)$.

We emphasize (see [22]) that there is no uniqueness for the diffusive representation. The representation above is convenient for our purpose.

1.4 Outline of the proof and mathematical framework

The proof of Theorem 1.1 relies on two arguments. The first one is a comparison principle between solutions. The second one is the proof of the main theorem for non negative solutions.

Throughout the article, the variable x is omitted for the functions u and φ . We set

$$\|u\|_{L^2}^2 = \int_{\mathbb{R}} u^2(x) dx,$$

for functions of one variable and

$$\|\varphi\|_{L^2}^2 = \int_{\mathbb{R} \times [0, +\infty)} \varphi^2(\sigma, x) dx d\sigma,$$

for functions of two variables. Here the third variable t is omitted.

2 Proof of the main theorem

2.1 A comparison result

Our first milestone is to prove a comparison result between solutions.

Proposition 2.1 *Consider $u_0 \leq v_0$ two ordered initial data that satisfy the assumptions of Theorem 1.1, i.e. u_0, v_0 in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. We have that $u(t, x) \leq v(t, x)$ for any $t > 0$ and a.e. $x \in \mathbb{R}$.*

Proof We prove the comparison result first for initial data in the Schwartz's class $S(\mathbb{R})$ and then the final result for general initial data by a limiting argument. This limiting argument is standard and then omitted for the sake of conciseness.

We now consider $u_0(x) < v_0(x)$ two (strictly) ordered initial data in the Schwartz class. Consider u, v the corresponding solutions and φ^u, φ^v the associated diffusive representation of solutions. Consider $w = v - u$ and $\psi = \varphi^v - \varphi^u$. Introduce

$$T = \inf \left\{ t > 0; \inf_x w(t, x) < 0 \right\}. \quad (2.1)$$

We prove that $T = +\infty$ arguing by contradiction. Assume that $T < +\infty$. There exists x_0 in \mathbb{R} such that $\inf_x w(T, x) = w(T, x_0) = 0$ and then $w_x(T, x_0) = 0, w_{xx}(T, x_0) \geq 0$. Since the solution is smooth away from the boundary $t = 0$ we also have $w_t(T, x_0) \leq 0$.

On the other hand we infer from the first equation in (1.5) that

$$\begin{aligned} \Gamma(\psi_t)(T, x_0) &= -w_t(T, x_0) + w_{xx}(T, x_0) \\ &+ w_x(T, x_0) \left(1 + \frac{u(T, x_0) + v(T, x_0)}{2} \right) \\ &+ w(T, x_0) \left(\frac{u_x(T, x_0) + v_x(T, x_0)}{2} \right) \geq 0. \end{aligned} \quad (2.2)$$

Besides, since $w(t, x) \geq 0$ for $t \leq T$, then $\psi(T, x, \sigma) \geq 0$. Therefore the second equation in (1.5) yields $\psi_t(T, x_0, \sigma) = -\sigma^2 \psi(T, x_0, \sigma) \leq 0$ for any $\sigma \geq 0$. Gathering this inequality with (2.2) we have that $\Gamma(\psi_t)(T, x_0) = 0$. Therefore $\Gamma(\sigma^2 \psi)(T, x_0) = 0$, and then $\psi(T, x_0, \sigma) = 0$ for any σ . This yields $w(t, x_0) = 0$ for any $t \leq T$ then the contradiction. \square

Choosing $0 \leq u_0$ in Proposition 2.1 yields

Corollary 2.2 *Assume u_0 in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $u_0 \geq 0$ then $u(t, x) \geq 0$ for almost every $x \in \mathbb{R}$ and $t > 0$.*

2.2 Decay of the L^1 norm

Proposition 2.3 *Assume u_0 in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. We have*

$$|u(t)|_{L^1} \leq \min \left(1, \frac{1}{\sqrt{t}} \right) |u_0|_{L^1}. \quad (2.3)$$

Proof In fact we just have to prove the result for $u_0 \geq 0$ Since $\frac{u_0 - |u_0|}{2} \leq u_0 \leq \frac{u_0 + |u_0|}{2}$, then the corresponding solutions starting from these three initial data remain ordered, say $v \leq u \leq w$. We prove the decay estimate for $-v$ and w that are non negative, and conclude.

Assume then $u_0 \geq 0$. We now know that the solution remains non negative for all time. Integrating in x (and in t) the Eq. (1.1) we have, setting $m(t) = \int_{\mathbb{R}} u(t, x) dx$ that is the L^1 norm

$$m(t) + Im(t) = m(0). \quad (2.4)$$

Appealing Proposition 1 in [22] we know that the solution of (2.4) reads

$$m(t) = \left(\int_t^{+\infty} \frac{e^{t-s}}{\sqrt{s}} ds \right) m(0).$$

This completes the proof of the Proposition. \square

2.3 Revisiting the Schonbek's splitting method

Consider a solution (u, φ) to (1.5). Introduce the energy functional

$$E(t) = |u(t)|_{L^2}^2 + \frac{2}{\pi} \|\sigma\varphi\|_{L^2}^2. \quad (2.5)$$

Multiplying the first equation in (1.5) by u and the second one by $\frac{2}{\pi}\varphi_t$ and summing the resulting equations lead to the energy dissipation mechanism

$$\frac{d}{dt}E(t) + 2|u_x(t)|_{L^2}^2 + \frac{4}{\pi} \|\varphi_t\|_{L^2}^2 = 0. \quad (2.6)$$

This implies that $t \mapsto |u(t)|_{L^2}$ remains bounded by $|u_0|_{L^2}$. We now state and prove a first result

Lemma 2.4 *There exists a numerical constant c such that for any $t \geq 0$*

$$tE(t) \leq c \left(|u_0|_{L^1}^2 + \left(\int_0^t |u(s)|_{L^2}^{\frac{4}{3}} ds \right)^{\frac{3}{2}} \right). \quad (2.7)$$

Proof We begin with the identity

$$\frac{d}{dt}(tE(t)) = E(t) - 2t|u_x|_{L^2}^2 - \frac{4t}{\pi} \|\varphi_t\|_{L^2}^2. \quad (2.8)$$

Dropping the last term, we infer from this

$$\frac{d}{dt}(tE(t)) \leq (|u(t)|_{L^2}^2 - 2t|u_x|_{L^2}^2) + \frac{2}{\pi} \|\sigma\varphi\|_{L^2}^2. \quad (2.9)$$

We now bound by above the last term in the right hand side of (2.9). Due to Minkowski's inequality, equality (1.3), and Fubini-Tonelli theorem we have

$$\|\sigma\varphi\|_{L^2} \leq \int_0^t |u(s)|_{L^2} \left(\int_0^\infty \sigma^2 \exp(-2\sigma^2(t-s)) d\sigma \right)^{\frac{1}{2}} ds \quad (2.10)$$

using that $\int_0^\infty x^2 \exp(-ax^2) dx = \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}}$, we deduce from (2.10)

$$\|\sigma\varphi\|_{L^2} \leq \pi^{\frac{1}{4}} 2^{-\frac{7}{4}} \int_0^t \frac{|u(s)|_{L^2}}{(t-s)^{\frac{3}{4}}} ds. \quad (2.11)$$

We now tackle the first term in the right hand side of (2.9). Proceeding as in the classical Schonbek's argument we have, appealing Plancherel's theorem and the inverse Fourier transform

$$\begin{aligned} |u(t)|_{L^2}^2 - 2t|u_x|_{L^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{R}} (1 - 2t|\xi|^2) |\hat{u}(t, \xi)|^2 d\xi \\ &\leq \frac{1}{2\pi} \int_{2t|\xi|^2 \leq 1} |\hat{u}(t, \xi)|^2 d\xi \leq 4\pi |u(t)|_{L^1}^2 \int_0^{\frac{1}{\sqrt{2t}}} d\xi. \end{aligned} \quad (2.12)$$

Combining this with the L^1 estimate (2.3) leads to

$$|u(t)|_{L^2}^2 - 2t|u_x|_{L^2}^2 \leq \frac{2\sqrt{2\pi}|u_0|_{L^1}^2}{\sqrt{t}} \min\left(1, \frac{1}{t}\right). \quad (2.13)$$

It is worth to point out that the right hand side of (2.13) is integrable for $t \in [0, +\infty)$. Gathering these inequalities and integrating in time (2.9) yields

$$tE(t) \leq 8\sqrt{2}\pi|u_0|_{L^1}^2 + \pi^{-\frac{1}{2}}2^{-\frac{5}{2}} \int_0^t \left(\int_0^\tau \frac{|u(s)|_{L^2} ds}{(\tau-s)^{\frac{3}{4}}} \right)^2 d\tau. \quad (2.14)$$

Besides, we recall the Hardy–Littlewood–Sobolev inequality (see Sect. 4.3 in [19] or Theorem 9.13 in [27]) that reads: there exists a constant c_{HLS} (see [19] for the precise value of this constant) such that

$$\| \int_{\mathbb{R}} |g(s)||s-t|^{-\frac{3}{4}} ds \|_{L^2_t} \leq c_{HLS}^2 \|g\|_{L^{\frac{4}{3}}}^2. \quad (2.15)$$

Applying this to the function that is $g(s) = |u(s)|_{L^2}$ if $s \in (0, t)$, and $g(s) = 0$ elsewhere leads to

$$\int_0^t \left(\int_0^\tau \frac{|u(s)|_{L^2} ds}{(\tau-s)^{\frac{3}{4}}} \right)^2 d\tau \leq c_{HLS}^2 \left(\int_0^t |u(s)|_{L^2}^{\frac{4}{3}} ds \right)^{\frac{3}{2}}. \quad (2.16)$$

This completes the proof of the Lemma. \square

We now prove the next milestone

Corollary 2.5 *Assume that u_0 in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. There exists a numerical constant c such that*

$$\sqrt{t}E(t) \leq c|u_0|_{L^1 \cap L^2}^2.$$

Proof Here and in the sequel $|u_0|_{L^1 \cap L^2} = |u_0|_{L^1} + |u_0|_{L^2}$. We begin with an a priori estimate on the L^2 norm of the solution. Appealing Gagliardo–Nirenberg inequality (see [3])

$$|u|_{L^2} \leq |u|_{L^1}^{\frac{2}{3}} |u_x|_{L^2}^{\frac{1}{3}}. \quad (2.17)$$

Then using the Hölder's inequality

$$\int_0^t |u(s)|_{L^2}^{\frac{4}{3}} ds \leq \left(\int_0^t |u(s)|_{L^1}^{\frac{8}{3}} ds \right)^{\frac{7}{9}} \left(\int_0^t |u_x|_{L^2}^2 ds \right)^{\frac{2}{9}}. \quad (2.18)$$

Therefore, since the last term in the right hand side is bounded by $C(u_0) = \left(\frac{|u_0|_{L^2}^2}{2}\right)^{\frac{2}{9}}$ thanks to (2.6), using (2.3)

$$\int_0^t |u|_{L^2}^{\frac{4}{3}} \leq 2^{-\frac{2}{9}} |u_0|_{L^2}^{\frac{4}{9}} |u_0|_{L^1}^{\frac{8}{9}} \left(\int_0^t \min\left(1, \frac{1}{s^{\frac{4}{7}}}\right) ds \right)^{\frac{7}{9}}. \quad (2.19)$$

since $(\int_0^t \min(1, \frac{1}{s^{\frac{4}{7}}}) ds)^{\frac{7}{9}} \leq (\frac{7}{3})^{\frac{7}{9}} t^{\frac{1}{3}}$, plugging this into Lemma 2.4 leads to

$$tE(t) \leq c|u_0|_{L^1 \cap L^2}^2 (1 + \sqrt{t}); \quad (2.20)$$

Since we know by (2.6) that $E(t) \leq |u_0|_{L^2}^2$, gathering this with (2.20) completes the proof of the Corollary. \square

2.4 Completing the proof of the theorem

We already know that $t \mapsto |u(t)|_{L^2}^2$ is bounded by $|u_0|_{L^2}^2$. We know from Corollary 2.5 that for large t this function decays as $t^{-\frac{1}{4}}$. Before improving this decay rate, we prove a first decay estimate for the L^∞ norm of the solution.

Using the results in [22] we know that there exists two functions $K(t)(x)$ and $N(t)(x)$ such that, where $*$ stands for the convolution in the x variable

$$u(t) = K(t) * u_0 + \int_0^t N(t-s) * u^2(s) ds. \quad (2.21)$$

We also have that there exists c a numerical constant such that (see [22])

$$\max(1, \sqrt{t}) \left(\sqrt{t} |K(t)|_{L^\infty} + |K(t)|_{L^1} \right) + \left(\sqrt{t} |N(t)|_{L^1} + t |N(t)|_{L^\infty} \right) \leq c. \quad (2.22)$$

Therefore

$$\begin{aligned} |u(t)|_{L^\infty} &\leq |K(t)|_{L^\infty} |u_0|_{L^1} + \int_0^t |N(t)|_{L^3} |u(s)|_{L^3}^2 ds \\ &\leq c |u_0|_{L^1} \min\left(\frac{1}{\sqrt{t}}, \frac{1}{t}\right) + c \int_0^t (t-s)^{-\frac{5}{6}} |u(s)|_{L^2}^{\frac{4}{3}} |u(s)|_{L^\infty}^{\frac{2}{3}} ds. \end{aligned} \quad (2.23)$$

Introducing the auxiliary function $A(t) = \sqrt{t} |u(t)|_{L^\infty}$ we then infer from (2.23) and from the L^2 estimate in Corollary 2.5

$$A(t) \leq c |u_0|_{L^1} + c |u_0|_{L^1 \cap L^2}^{\frac{4}{3}} \left(\int_0^t \frac{\sqrt{t} ds}{(t-s)^{-\frac{5}{6}} s^{-\frac{2}{3}}} \right) (\sup_{s \leq t} A(s))^{\frac{2}{3}}. \quad (2.24)$$

We easily infer from (2.24) that $A(t) \leq c |u_0|_{L^1 \cap L^2}^2$ that is the good decay estimate for small t , i.e. $|u_0|_{L^1 \cap L^2} t^{-\frac{1}{2}}$.

We now proceed to the decay rate for large t for the L^2 norm of the solution. Interpolating the L^1 and the L^∞ decay rate we have that

$$|u(t)|_{L^2}^2 \leq |u(t)|_{L^1} |u(t)|_{L^\infty} \leq c |u_0|_{L^1 \cap L^2}^2 t^{-1}; \quad (2.25)$$

we improve this new estimate. We infer from (2.21)–(2.22)

$$\begin{aligned} |u(t)|_{L^2} &\leq |K(t)|_{L^2} |u_0|_{L^1} + \int_0^t |N(t)|_{L^2} |u(s)|_{L^2}^2 ds \\ &\leq c |u_0|_{L^1} t^{-\frac{3}{4}} + c \int_0^t (t-s)^{-\frac{3}{4}} |u(s)|_{L^2}^2 ds. \end{aligned} \quad (2.26)$$

The first term in the right hand side of this inequality has the good decay rate. We focus on the second one. Due to Corollary 2.5 and using that $|u(t)|_{L^2}$ remains bounded

$$\int_0^t (t-s)^{-\frac{3}{4}} |u(s)|_{L^2}^2 ds \leq \tilde{c} |u_0|_{L^1 \cap L^2}^2 \int_0^t (t-s)^{-\frac{3}{4}} \min\left(1, \frac{1}{s}\right) ds. \quad (2.27)$$

This last integral splits as (for $t \geq 1$)

$$\int_0^1 (t-s)^{-\frac{3}{4}} ds + \int_1^t (t-s)^{-\frac{3}{4}} \frac{ds}{s} \leq \frac{1}{(t-1)^{\frac{3}{4}}} + t^{-\frac{3}{4}} \int_{\frac{1}{t}}^1 \frac{ds}{s(1-s)^{\frac{3}{4}}}. \quad (2.28)$$

Hence for $t \geq 2$ there exists a numerical constant c such that

$$\int_0^1 (t-s)^{-\frac{3}{4}} ds + \int_1^t (t-s)^{-\frac{3}{4}} \frac{ds}{s} \leq c \frac{1 + \log t}{t^{\frac{3}{4}}}. \quad (2.29)$$

We now get rid of the extra term $\log t$. From the previous estimates we have improved (2.25) as $|u(t)|_{L^2} \leq C(u_0)t^{-\frac{5}{8}}$ for instance. Using this in (2.27) and performing the same computations as in (2.27)–(2.29) leads to the good decay estimate (without the extra $\log t$ term).

It remains to prove the large t estimate for the L^∞ norm of the solution. We set $B(t) = t|u(t)|_{L^\infty}$. We infer from (2.21)–(2.22)

$$\begin{aligned} B(t) \leq & c \left(|u_0|_{L^1} + t \int_0^1 |N(t-s)|_{L^\infty} |u(s)|_{L^2}^2 ds \right. \\ & \left. + t \int_1^t |N(t-s)|_{L^2} |u(s)|_{L^2} |u(s)|_{L^\infty} ds \right). \end{aligned} \quad (2.30)$$

The second term in the right hand side of (2.30) is bounded by above by $c \frac{t}{t-1} |u_0|_{L^2}^2$ and then bounded for large t . The third term in the right hand side of (2.30) is bounded by above by $c |u_0|_{L^1 \cap L^2}^2 \int_1^t \frac{t ds}{(t-s)^{\frac{3}{4}} s^{\frac{5}{4}}} ds$, using the L^2 estimate in Theorem 1.1 and the previous bound in $Ct^{-\frac{1}{2}}$ for the L^∞ norm of u . This completes the proof of the theorem

3 Numerical results

3.1 The numerical scheme

Here we consider the generalized model

$$u_t + u_x + \frac{\sqrt{\nu}}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds + \gamma u u_x = \alpha u_{xx}, \quad (3.1)$$

where ν , γ and α are non-negative parameters. Using the diffusive realization of the half-order derivative (1.5), the model (3.1) is rewritten as a PDE-ODE coupled system as follows

$$\begin{cases} \partial_t u(t, x) + \partial_x(u + \frac{\gamma}{2} u^2) = -\frac{2\sqrt{\nu}}{\pi} \int_0^\infty (u(t, x) - \sigma^2 \varphi(t, x, \sigma)) d\sigma \\ \quad + \alpha u_{xx}(t, x), \quad t > 0, x \in \mathbb{R} \\ \partial_t \varphi(t, x, \sigma) = -\sigma^2 \varphi(t, x, \sigma) + u(t, x), \quad t > 0, x \in \mathbb{R}, \sigma \geq 0, \end{cases} \quad (3.2)$$

supplemented with the initial conditions

$$\begin{aligned} \forall x \in \mathbb{R}, \forall \sigma \geq 0, \quad \varphi(0, x, \sigma) &= 0, \\ \forall x \in \mathbb{R}, \quad u(0, x) &= u_0(x). \end{aligned}$$

Following [6], we use the Gauss-Jacobi quadrature method with N_m points to approximate the generalized integral in (3.2). We note by w_i the weights and by σ_i the nodes of the

Gauss-Jacobi quadrature method. We get

$$\begin{aligned} D^{1/2}u(t, x) &\simeq \sum_{i=1}^{N_m} w_i \left(\frac{2}{\pi} u(t, x) - \sigma_i^2 \varphi(t, x, \sigma_i) \right) \\ &= \sum_{i=1}^{N_m} w_i \left(\frac{2}{\pi} u(t, x) - \sigma_i^2 \varphi_i(t, x) \right). \end{aligned}$$

We note by $U = (u, \varphi_1, \dots, \varphi_{N_m})^T$ the vector of $(N_m + 1)$ unknowns, by

$$\mathcal{F}(U) = (u + \frac{\gamma}{2}u^2, 0, \dots, 0)^T,$$

and finally

$$\mathbf{S} = \begin{pmatrix} -\frac{2\sqrt{\nu}}{\pi} \sum_{i=1}^{N_m} w_i & \frac{2\sqrt{\nu}}{\pi} w_1 \sigma_1^2 & \dots & \dots & \frac{2\sqrt{\nu}}{\pi} w_{N_m} \sigma_{N_m}^2 \\ 1 & -\sigma_1^2 & 0 & \dots & 0 \\ 1 & 0 & -\sigma_2^2 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \\ 1 & 0 & \dots & \dots & -\sigma_{N_m}^2 \end{pmatrix}.$$

Thus, the problem (3.2) is written in the following form

$$\partial_t U + \partial_x \mathcal{F}(U) = \mathbf{S}(U) + \mathbf{G}_1 \partial_x^2 U, \quad (3.3)$$

where \mathbf{G}_1 is a diagonal matrix of order $N_m + 1$. Following [6], we use a splitting method to provide numerical results. Let $\Delta t > 0$, for all $n \geq 0$, we recall that $t^n = n \Delta t$ and

$$U^n(x) \approx U(n \Delta t, x).$$

From equation (3.3), we consider the propagation equation

$$\partial_t U + \partial_x \mathcal{F}(U) = \mathbf{G}_1 \partial_x^2 U, \quad (3.4)$$

and the diffusive equation

$$\partial_t U = \mathbf{S}(U). \quad (3.5)$$

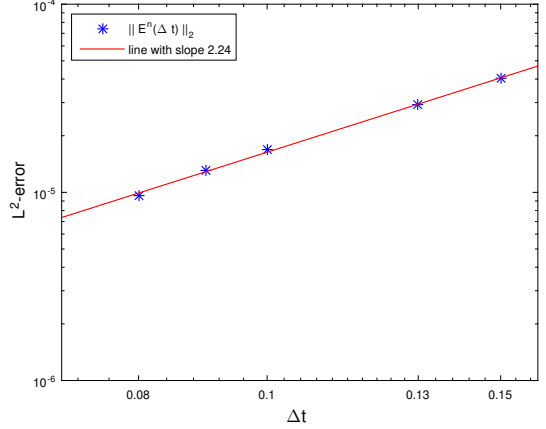
We note by \mathbf{H}_a (respectively, \mathbf{H}_b) the discrete operator of the solution of (3.4) (respectively, the solution of (3.5)). Then a Strang splitting method of order 2 (see for example [14,18]) between t_n and t_{n+1} is used to solve respectively (3.4) and (3.5) as follows

$$\begin{aligned} U^{(1)} &= \mathbf{H}_b \left(\frac{\Delta t}{2} \right) U^n, \\ U^{(2)} &= \mathbf{H}_a (\Delta t) U^{(1)}, \\ U^{n+1} &= \mathbf{H}_b \left(\frac{\Delta t}{2} \right) U^{(2)}. \end{aligned} \quad (3.6)$$

Here, the constructed operators \mathbf{H}_a and \mathbf{H}_b are stable and of order 2. Then, the scheme (3.6) provides an approximation of order 2 in time to the problem (3.3). Now, let $t \in [0, T]$, $x \in [0, L]$. We consider periodic boundary conditions i.e $u|_{x=0} = u|_{x=L}$. Let $\Delta t > 0$ be a fixed time step and $h > 0$ be a fixed space step. We set for all $n = 0, \dots, N$, $t_n = n \Delta t$ and for all $j = 1, \dots, K$, $x_j = jh$. We use the following notations:

$$\forall i = 1, \dots, N, \quad u^n(x) \approx u(n \Delta t, x), \quad \psi_i^n(x) \approx \psi(n \Delta t, x, \sigma_i).$$

Fig. 1 Error of the time discretization



For the propagation Eq. (3.4), we use a Crank–Nicolson scheme for the linear part and Adams–Bashforth scheme for the nonlinear part. For the space discretization, we use standard Fourier methods. Hence, the scheme is given by

$$\begin{aligned} \frac{\hat{u}^{n+1} - \hat{u}^n}{\Delta t} &= \frac{\hat{u}^{n+1} + \hat{u}^n}{2} (-\alpha \xi^2 - i \xi) \\ &\quad - \frac{i \gamma \xi}{2} \frac{(\widehat{3(u^n)^2} - \widehat{(u^{n-1})^2})}{4}. \end{aligned} \quad (3.7)$$

Then, for the diffusion equation (3.5), the exact solution is given by

$$\mathbf{H}_b \left(\frac{\Delta t}{2} \right) U = e^{\mathbf{S} \frac{\Delta t}{2}} U. \quad (3.8)$$

We note that the scheme (3.7) as well as the Strang splitting method are of order two. It follows that (3.6) is of order 2.

3.2 Numerical results

In the sequel, we present the numerical results. To this end, we fix the parameters values to $\nu = 1$ and $\alpha = \gamma = 0.5$. This case is theoretically investigated here and in [22]. Let us notice that in this article, contrarily to [6], the dispersive term βu_{xxx} is not taken into account as in the present theoretical study and we have shown in [6] that the magnitude of the constant β has a strong influence on the decay rate of the solution for large times.

Unfortunately, an exact solution of (3.1) is not known, but a numerical study of the convergence of the scheme is provided in [22]. Then, in order to numerically investigate the decay rate of the solutions for large times, we choose an initial data u_0 that can provide an exact KdV soliton for well chosen parameters (see [4]). Moreover, we choose an interval, both in time and in space, such that the solution remains small at the boundary in order to have a numerical solution (on a finite domain) that correctly approximates the exact solution that exists on the whole real line \mathbb{R} . Actually, we have that the initial data u_0 is in the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

We denote by x_0 the middle of the space interval. Then the initial data reads

$$u_0(x) = 0.32 * \operatorname{sech}^2(0.4 * (x - x_0)). \quad (3.9)$$

Fig. 2 Error in terms of grid points N_m

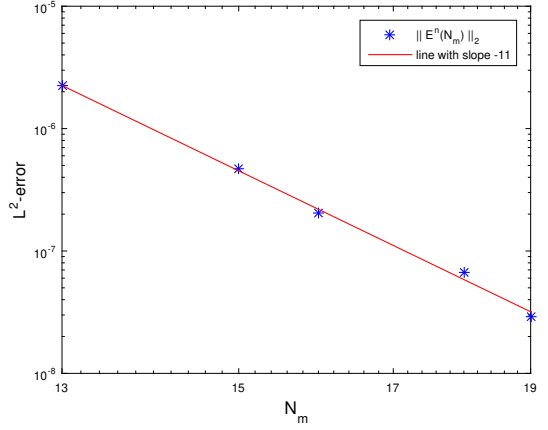
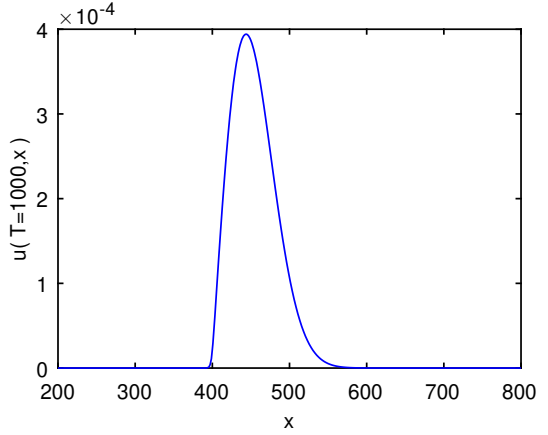


Fig. 3 Numerical solution when $\nu = 1$, $\alpha = \gamma = 0.5$ and $h = \Delta t = 0.1$



We run the computations for the numerical scheme (3.7)–(3.8) on the space interval $[0, L]$ with $L = 800$. Moreover, we take the space step $h = 0.1$ and $N_m = 20$. In order to numerically confirm the order of the convergence in time of the scheme (3.7)–(3.8), we take $T = 100$, we denote by u_{Ref}^n the reference solution when $\Delta t = 0.05$ and by u^n the numerical solution for different time steps Δt . We denote by $E^n(\Delta t) = \|u_{\text{Ref}}^n - u^n\|_2$ the L^2 -error due to the time discretization of the solution. We illustrate in Fig. 1 the error E^n in terms of Δt in the log-log scale. We observe that, as expected, the error E^n decreases when Δt decreases. Moreover, the measured values are close to a straight line with slope 2.24. This test confirms that the numerical scheme (3.7)–(3.8) is of order 2 in time. In the sequel, we provide our simulations with the time step $\Delta t = 0.1$.

In order to check the convergence of the scheme (3.7)–(3.8) in terms of the Gauss-Jacobi quadrature points N_m , we denote by u_{Ref}^n the reference solution when $N_m = 20$ and by u^n the numerical solution for different values of N_m . We denote by $E^n(N_m) = \|u_{\text{Ref}}^n - u^n\|_2$ the L^2 -error in terms of the quadrature points. We display the results in Fig. 2. As expected, we see that the error decreases when the number of grid points N_m increases. In addition, we observe that the measured values are close to a straight line with slope -11 . Moreover, we observe that the error E^n is less than 10^{-6} from $N_m = 15$. Other numerical simulations are

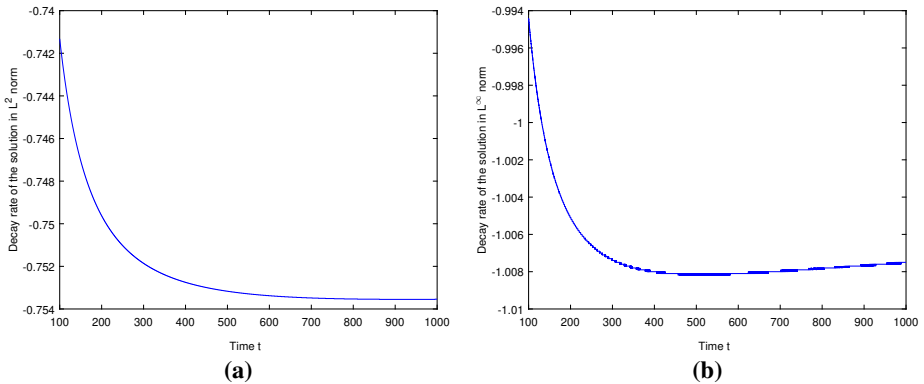


Fig. 4 **a** L^2 -decay rate of the solution, **b** L^∞ -decay rate of the solution when $\nu = 1$, $\alpha = \gamma = 0.5$ and $h = \Delta t = 0.5$

done and we made similar observations (see also [21,22]). Hence, we will use 15 grid points in the remaining of this article.

In the sequel, we use the numerical scheme (3.7)–(3.8) with the previous parameters values on a large time interval $[0, T] = [0, 1000]$. Numerical results are displayed in Figs. 3 and 4. It can be observed that the numerical solution remains non negative. This matches the results of Corollary 2.2. Moreover, the decay rates of the numerical solution provided in Fig. 4 in norm L^2 and in norm L^∞ matches the theoretical results of Theorem 1.1 very precisely. Indeed, one can observe that the relative error between the theoretical and the numerical L^2 decay rate (diffusion) is less than 0.53% whereas it is less than 0.8% the the decay rate in the L^∞ -norm (dispersion).

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