Weak resilience of the chemostat model to a species invasion with non-autonomous removal rates

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Abstract

In this paper, we study how resilience in the chemostat model with two species can be guaranteed in a weak sense in presence of a species invader. Doing so, we construct a time varying removal rate that allows the resident species to return an infinite number of times to its original density level, even though the invader can never be totally eradicated. Moreover, we prove that the time spent by the system with the resident density above or equal to its original level is of infinite measure introducing that way the concept of "weak resilience". Finally, under the conjecture that there exists an unique periodic solution of the system associated with such a time-varying removal rate, we show that every solution converges asymptotically to this periodic solution.

Keywords. Non-autonomous Dynamics, Chemostat Model, Periodic systems, Asymptotically Periodic Solutions.

1 Introduction

The chemostat model describes microbial ecosystems which are continuously fed by nutrients, as it can be found in natural environments, such as lakes, lagoons, wetlands... and in experimental or industrial bioreactors. Being open systems, chemostats are naturally subject to external perturbations such as species invasion.

In this paper, we focus on the classical chemostat model, for which the Competitive Exclusion Principle (CEP) holds [1]. For a single limiting resource, the CEP states that no more than one species (generically) survives in the long term under constant fed conditions (input substrate concentration and flow rate of the incoming resource), see, *e.g.*, [14, 24].

For dynamical systems, *resilience* is often described as the ability of a system to return to an original state (typically a steady state) after a transient perturbation [17]. In the present work, we wish to study the resilience of the chemostat model to invasions by other species considered as disturbances. In particular, we are interested in the possibility of extending the resilience domain using time-varying input conditions.

For the chemostat model, it has already been pointed out that non-constant removal rates could allow the coexistence of two species, under some precise integral conditions, see for instance [22, 9, 15, 20, 19, 25] for periodic removal rates or [16] for slow varying environments. The idea is to create a time-varying growth environment which alternates the favored species. Recently, the question of quantifying the excursions of the state variables in the chemostat model under periodic removal rates, has been studied in [7, 8], but for the mono-specific case only. To our knowledge, it has not been investigated how to synthesize time-varying removal rates allowing resilience of a mono-specific chemostat system in presence of an invasion by a new species, in such a way that the resident species returns to the same density level than before invasion an infinite number of times. The design of such time-varying removal rate is precisely the matter of the present work.

The paper is structured as follows. In Section 2, we state the resilience problem in the context of the chemostat model with two species. In particular, we show the existence of a threshold on the level of the resident species above which resilience is lost that allows us to introduce a concept of *weak resilience* in a time varying context. In Section 3, we provide a construction of a time-varying removal rate which guarantees

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weak resilience in the sense given in Section 2. In Section 4, we show that there exist weakly resilient periodic solutions, and conjecture that there exists an unique periodic solution associated to the time-varying removal rate that we construct. Then, we show that any solution of the system associated with this time-varying removal rate converges to this periodic solution.

2 Assumptions and definition of weak resilience

We start by recalling the chemostat model with two microbial species of concentrations x_1, x_2, respectively, that compete for a single resource of concentration s:

$$\left\{\begin{array}{l} \dot{s} = -\mu_1(s)x_1 - \mu_2(s)x_2 + D(s_{in} - s), \\ \dot{x_1} = (\mu_1(s) - D)x_1, \\ \dot{x_2} = (\mu_2(s) - D)x_2, \end{array}\right. \tag{2.1}$$

in which the yield coefficients are equal to one. Parameters D and s_{in} are respectively the removal rate (imposed by the input flow rate) and the input concentration of the resource. Here, species 1 and 2 play the respective roles of the *resident* and *invasive* species, as described in the introduction: the chemostat is considered first with the species 1 (the resident) alone and at some time t_0 (chosen equal to 0 for simplicity), the invasive species 2 appears. In the sequel, we consider that the following assumption on the growth functions $\mu_i(\cdot)$, $i = 1, 2$ in (2.1) is fulfilled.

Assumption 2.1. *The functions $\mu_i(\cdot)$ are of class C^1, monotone increasing with $\mu_i(0) = 0$, $i = 1, 2$.*

For $i = 1, 2$, the *break-even concentration* for species i (related to the parameter s_{in}) is defined as

$$\lambda_i(D) := \sup\{s \in [0, s_{in}] \; ; \; \mu_i(s) < D\} \in [0, +\infty], \quad i = 1, 2.$$

When $\lambda_i(D) < +\infty$, and because μ_i are strictly monotone, one has

$$\mu_i(\lambda_i(D)) = D \quad \Longleftrightarrow \quad \lambda_i(D) = \mu_i^{-1}(D).$$

Consider now the competition between the two species. The CEP states that only the species that realizes the minimum of the numbers $\lambda_i(D)$, $i = 1, 2$ (with $D > 0$) has a non null concentration at steady state. For a given constant D with $0 < D < \mu_1(s_{in})$, this means that species 1 is excluded by an invasion by a species 2 when $\lambda_2(D) < \lambda_1(D)$, or equivalently that the state of (2.1) converges asymptotically to $(\lambda_2(D), 0, s_{in} - \lambda_2(D))$. Let us now consider a situation for which the dominance of one growth function over the other one is alternated with respect to the level of the resource, in the following way.

Assumption 2.2. *There exists $\bar{s} \in (0, s_{in})$ such that*

$$(\mu_1(s) - \mu_2(s))(s - \bar{s}) > 0, \quad s \in [0, s_{in}] \text{ and } s \neq \bar{s}. \tag{2.2}$$

In the rest of the paper, we assume that this assumption holds true. Note that for Monod's growth functions [18] (which are quite popular in microbiology and bio-processes), one has

$$\mu_i(s) = \frac{\bar{\mu}_i s}{K_i + s}, \quad i = 1, 2.$$

Assumption 2.2 then amounts to have the following condition

$$0 < \bar{\mu}_2 K_1 - \bar{\mu}_1 K_2 < (\bar{\mu}_1 - \bar{\mu}_2)s_{in}$$

to be fulfilled. From (2.2), one must have $\mu_1(\bar{s}) - \mu_2(\bar{s}) = 0$, therefore we set

$$\bar{D} := \mu_1(\bar{s}) = \mu_2(\bar{s}).$$

We formulate the problem of invasion of species 1 by species 2 as follows. Suppose that only species 1 is present in a bioreactor, at the initial time (*i.e.*, $x_2 = 0$ in (2.1)). For a constant value of D, a straightforward analysis shows that a necessary condition for species 1 to set up at steady state is to have $\lambda_1(D) < s_{in}$. Then, the corresponding steady state is given by $(\lambda_1(D), s_{in} - \lambda_1(D))$, which is globally asymptotically stable in the (s, x_1) positive plane (see for instance [14]). Moreover, if D is less than $\bar{D}$, then the concentration of species 1 at steady state, x_1^{eq}, is necessarily such that $x_1^{eq} \geq \bar{x}_1$ with

$$\bar{x}_1 := s_{in} - \mu_1^{-1}(\bar{D}) = s_{in} - \bar{s} > 0.$$

Suppose now that a new species (species 2) that fulfills Assumption 2.2, invades the growth vessel and that the removal rate D is less than $\bar{D}$. Then, the new system (2.1) (with $x_2 \neq 0$) will not be able to return to the original state (from the CEP cited before, the state of (2.1) converges asymptotically to $(\lambda_2(D), 0, s_{in} - \lambda_2(D))$), after this perturbation. This is why, we say that the dynamics is *not resilient* when the concentration x_1 at steady state, before the invasion, is above the threshold $\bar{x}_1$. The question of interest is to investigate if, even so, the system can be resilient with x_1 above the threshold $\bar{x}_1$, when considering time-varying removal rate $D(\cdot)$. First, one can easily check that the domain

$$\{(s, x_1, x_2) \in \mathbb{R}_+^3 \; ; \; s + x_1 + x_2 = s_{in}\},$$

is an invariant and attractive set for the dynamics (2.1) when $D(\cdot)$ is persistently exciting[1]. Assuming that system (2.1) is already at steady state before invasion, we shall consider in the sequel the reduced dynamics on this domain, that is,

$$\dot{x} = f(x, D) := \begin{bmatrix} (\mu_1(s_{in} - x_1 - x_2) - D)x_1 \\ (\mu_2(s_{in} - x_1 - x_2) - D)x_2 \end{bmatrix}, \tag{2.3}$$

defined on the invariant set

$$\mathcal{S} := \{x \in \mathbb{R}_+^2 \; ; \; x_1 + x_2 \leq s_{in}\}.$$

From now on, we consider $D(\cdot)$ as a control variable, *i.e.*, as a measurable function of time taking values within some interval $[D_m, D_M]$ where the minimum and maximum dilution rates D_m and D_M satisfy the inequality:

$$0 < D_m < \bar{D} < \mu_1(s_{in}) < D_M. \tag{2.4}$$

Note that D_M is large enough to have the possibility to drive solutions of (2.3) to the washout[2] of both species. To introduce resilience, we consider a threshold

$$x_1^r \in (\bar{x}_1, s_{in}),$$

for species 1 aiming at keeping x_1 above x_1^r as much as possible. It is then natural to introduce the subset of $\mathcal{S}$, $\mathcal{K}(x_1^r)$, defined as:

$$\mathcal{K}(x_1^r) := \{x \in \mathcal{S} \; ; \; x_1 \geq x_1^r \text{ and } x_2 > 0\},$$

and to ask about weak invariance properties of $\mathcal{K}(x_1^r)$ for the dynamics (2.3) in the context of viability theory [2]. Recall that given a controlled system $\dot{x} = g(x, u)$ (with $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$) and given a closed subset $K \subset \mathbb{R}^n$, the *viability kernel*, denoted by $Viab(K)$, is defined as the largest subset of initial states $x_0 \in K$ for which there is an admissible control $u(\cdot)$ such that the unique solution $x(\cdot)$ of the dynamics associated with u and such that $x(0) = x_0$, satisfies $x(t) \in K$ for any time $t \geq 0$ (see [2]). We then say that the viability kernel is *weakly invariant*. Going back to (2.3), we assume in the rest of the paper (in addition to Assumptions 2.1, 2.2) that the following assumption is fulfilled:

Assumption 2.3. *The threshold x_1^r satisfies*

$$0 < D_m < \mu_1(s_{in} - x_1^r). \tag{2.5}$$

Remark 2.1. *The choice of the three parameters D_m, D_M, and x_1^r is crucial throughout this work. In this approach, note that we first chose D_m, D_M satisfying (2.4), and then we supposed that the threshold x_1^r satisfies (2.5). It is worth to mention that we could alternatively fix $x_1^r \in (\bar{x}_1, s_{in})$ and then choose the minimal and maximal dilution rates in such a way that (2.5) and the inequality $\mu_1(s_{in}) < D_M$ are verified.*

One has the following property, in terms of viability analysis (hereafter $cl\ S$ is the closure of a set $S \subset \mathbb{R}^2$).

[1] By persistently exciting, we mean that the non-negative function $D(\cdot)$ is such that $\int_0^{+\infty} D(t)dt = +\infty$, see [3].

[2] This means that for D_M sufficiently large ($D_M > \mu_1(s_{in})$), solutions of (2.3) converge to the origin.

Lemma 2.4. *The viability kernel $Viab(cl\,\mathcal{K}(x_1^r))$ of the set $\mathcal{K}(x_1^r)$ for* (2.3) *satisfies*

$$Viab(cl\,\mathcal{K}(x_1^r)) = [x_1^r, s_{in}] \times \{0\}.$$

Proof. Recall that x_1^r is such that $x_1^r \in (\bar{x}_1, s_{in} - \lambda_1(D_m))$ (by (2.5)). Assumptions 2.1, 2.2 and (2.4) also imply the inequality $0 < \lambda_2(D_m) < \lambda_1(D_m) < \bar{s}$. Take an initial condition $(x_{1,0}, x_{2,0})$ in $cl\,\mathcal{K}(x_1^r)$. If $x_{2,0} = 0$, the solution of (2.3) verifies $x_2(t) = 0$ for any $t > 0$ and any time-varying $D(\cdot)$. From (2.5), we can choose a constant D such that $0 < D_m \leq D < \mu_1(s_{in} - x_1^r)$, and we observe that (2.3) with $D(t) = D$ satisfies

$$x_1 = x_1^r \Rightarrow \dot{x}_1 = (\mu_1(s_{in} - x_1^r) - D)x_1^r > 0 \quad \text{and} \quad x_1 = s_{in} \Rightarrow \dot{x}_1 = -Ds_{in} < 0.$$

Thus, this constant D prevents $x_1(\cdot)$ to leave the interval $[x_1^r, s_{in}]$ over $[0, +\infty)$.
Assume now that $x_{2,0} > 0$. Then, a solution $x(\cdot)$ of (2.3) associated with an admissible time-varying function $D(\cdot)$ verifies $x_2(t) > 0$ for any time $t \geq 0$. Suppose by contradiction that $x(\cdot)$ stays in $\mathcal{K}(x_1^r)$ for any time $t \geq 0$. Then one has $s(t) < s_{in} - x_1^r < \bar{s}$ for any time $t \geq 0$. By Assumption 2.2, one has for any $t \geq 0$, $\mu_1(s(t)) - \mu_2(s(t)) < 0$ and thus, we deduce the inequality

$$\dot{s}(t) > -\mu_2(s(t))x_1(t) - \mu_2(s(t))x_2(t) + D_m(s_{in} - s(t)) = (D_m - \mu_2(s(t)))(s_{in} - s(t)), \quad t \geq 0.$$

Notice that any positive solution $\zeta(\cdot)$ of $\dot{\zeta} = (D_m - \mu_2(\zeta(t)))(s_{in} - \zeta(t))$ converges to $\lambda_2(D_m)$ when $t \to +\infty$. From (2.5) one has $\lambda_2(D_m) < s_{in} - x_1^r$, hence there exist $t_1 \geq 0$ and $\underline{s} \in (0, \lambda_2(D_m))$ such that one has $\zeta(t) > \underline{s}$ for any $t \geq t_1$. From the preceding inequality, we deduce that $s(\cdot)$ satisfies $s(t) \geq \zeta(t)$ for any time $t \geq 0$. We thus deduce that for any time $t \geq t_1$, one has

$$\mu_1(s(t)) - \mu_2(s(t)) \leq c := \min\{\mu_1(\sigma) - \mu_2(\sigma)\ ;\ \sigma \in [\underline{s}, s_{in} - x_1^r]\} < 0.$$

If we differentiate the function $q_1 := x_1/x_2$ w.r.t. t, we find that

$$\dot{q}_1 = \Big(\mu_1(s(t)) - \mu_2(s(t))\Big)q_1,$$

with $s(t) = s_{in} - x_1(t) - x_2(t)$. One then obtains $\dot{q}_1 < c\ q_1$. Therefore q_1 decreases to zero and x_1 as well, leading to a contradiction. We conclude that the only solutions of (2.3) that stay in $cl\,\mathcal{K}(x_1^r)$ for any time are the ones starting with $x_{2,0} = 0$ as was to be proved. □

This lemma shows that for any given threshold x_1^r satisfying (2.5), *i.e.*, such that

$$\bar{x}_1 < x_1^r < s_{in} - \lambda_1(D_m),$$

(or equivalently $\lambda_1(D_m) < s_{in} - x_1^r < \bar{s}$), the dynamics (2.3) is not resilient for the domain $\mathcal{K}(x_1^r)$ in presence of species 2. This is precisely our starting point to introduce the concept of weak resilience.

Definition 2.5. *Let $x_1^r \in (\bar{x}_1, s_{in} - \lambda_1(D_m))$. The system* (2.3) *is said to be weakly resilient for the set $\mathcal{K}(x_1^r)$ if for any initial condition in $\mathcal{K}(x_1^r)$, there exists a time-varying function $D(\cdot)$ with values in $[D_m, D_M]$ such that the corresponding solution of* (2.3) *satisfies*

$$\text{meas}\,\{t \geq 0\ ;\ x(t) \in \mathcal{K}(x_1^r)\} = +\infty.$$

Such a function $D(\cdot)$ will be called a weakly resilient removal rate.

This definition is related to the *minimal time crisis* of controlled dynamics, studied in [4, 5, 6, 13], although we do not look in this paper for control functions minimizing the time spent outside the set $\mathcal{K}(x_1^r)$ over a given time period $[0, T]$.

3 Construction of a weakly resilient removal rate

The aim of this section is to propose a robust and systematic way to build a time-varying $D(\cdot)$ taking alternatively the values D_m and D_M, and that is weakly resilient for (2.3), without requiring a precise knowledge of the expressions of the growth functions $\mu_i(\cdot)$. Recall that we suppose Assumptions 2.1, 2.2, and 2.3 to be fulfilled and that the parameter x_1^r satisfies

$$x_1^r \in (\bar{x}_1, s_{in} - \lambda_1(D_m)).$$

We begin by giving the main result (in Section 3.1) which gives a construction of a weakly resilient $D(\cdot)$. Next, we provide some properties of the dynamical system(2.3) in the domain $\mathcal{S}$ for a constant D ($D = D_M$ and $D = D_m$). Finally, we give the proof of Proposition 3.1 at the end of this section.

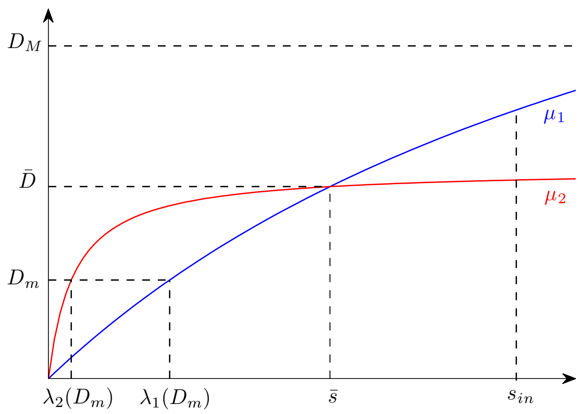

Figure 1: Illustration of Assumption 2.3.

3.1 Synthesis of a weakly resilient removal rate

In the following Proposition, we propose a time-varying $D(\cdot)$ allowing the dynamics (2.3) to be weakly resilient for the set $\mathcal{K}(x_1^r)$. For any $\varepsilon > 0$, we define the set $\mathcal{E}$

$$\mathcal{E} := (0, \bar{x}_1] \times (0, \varepsilon].$$

Proposition 3.1. *For ε small enough and any initial condition $x_0 := (x_{1,0}, x_{2,0}) \in \mathcal{K}(x_1^r)$ there exists a piecewise constant function $D(\cdot)$ which alternates the values D_M, D_m on time intervals $[T_i, T_{i+1})$, $i \in \mathbb{N}$ satisfying:*

$$T_0 = 0 < T_1 < \cdots < T_i < T_{i+1} < \cdots \quad \text{and} \quad \lim_{i\to\infty} T_i = +\infty, \tag{3.1}$$

where $x(\cdot)$ is the unique solution of (2.3) associated with the time-varying $D(\cdot)$ such that

(i) *if $x(T_i) \notin \mathcal{E}$, one has $D(t) = D_M$ for $t \in [T_i, T_{i+1})$ with T_{i+1} is defined as the first next entry time in $\mathcal{E}$.*

(ii) *if $x(T_i) \in \mathcal{E}$, one has $D(t) = D_m$ for $t \in [T_i, T_{i+1})$, the trajectory $x(\cdot)$ enters to the set $\mathcal{K}(x_1^r)$ in finite time and T_{i+1} is defined as the first next exit time from $\mathcal{K}(x_1^r)$.*

Finally, the time-varying $D(\cdot)$ is a weakly resilient removal rate.

We begin by a lemma which describes the asymptotic behavior of (2.3) when D is constant.

Lemma 3.1. *Any solution of (2.3) in $\mathcal{S}$ with a constant removal rate D converges asymptotically to an equilibrium.*

Proof. First, consider an initial condition on the axes that are invariant by (2.3). Then, the variable x_i (i equal to 1 or 2) is solution of a scalar autonomous dynamics on the x_i-axis. Therefore, either it converges to an equilibrium point on the axis, or it tends to infinity, which is not possible as the domain $\mathcal{S}$ is bounded.

Consider now a positive initial condition in the set $\mathcal{S}$. The corresponding solution then remains in the positive orthant, and one can consider the variables $\xi_i = \ln(x_i)$, $i = 1, 2$, whose dynamics is

$$\dot{\xi}_i = F_i(\xi) := \mu_i \left(s_{in} - e^{\xi_1} - e^{\xi_2}\right) - D, \quad i = 1, 2.$$

For a constant D, one has

$$\mathrm{div} F(\xi) = \sum_{i=1,2} \partial_{\xi_i} F_i(\xi) = -\sum_{i=1,2} \mu_i \left(s_{in} - e^{\xi_1} - e^{\xi_2}\right) e^{\xi_i} < 0.$$

By Dulac's criterion, the system has no closed orbit and by Poincaré-Bendixon Theorem (see [21]). We can then conclude that solutions of (2.3) converge asymptotically to an equilibrium, since trajectories are bounded. $\square$

In the sequel, we denote by $z(\cdot, \zeta, D)$ the unique solution of (2.3) (over $\mathbb{R}$) for an initial condition $z(0) = \zeta \in \mathcal{S}$ and a constant $D \in \{D_m, D_M\}$.

3.2 Properties of the reduced dynamics with constant $D = D_M$

We now provide asymptotic properties of (2.3) when D is constant equal to D_M, that are illustrated on Fig. 2.

Lemma 3.2. *Any solution $z(\cdot, \zeta, D_M)$ of (2.3) with $\zeta \in \mathcal{S}$ converges asymptotically to the origin. Moreover, for any positive initial condition ζ in $\mathcal{S}$, $z_1(\cdot, \zeta, D_M)$ and $z_2(\cdot, \zeta, D_M)$ are decreasing functions and the trajectory converges to the origin tangentially to the x_1-axis.*

Proof. From Assumption 2.3, the system (2.3) has a unique steady state $(0, 0)$ in $\mathcal{S}$. From Lemma 3.1, we deduce that it is globally asymptotically stable on $\mathcal{S}$. Consider a positive initial condition. From the expression of the dynamics (2.3), the solution $x(\cdot) = z(\cdot, \zeta, D_M)$ is clearly positive for any $t \geq 0$, and by Assumption 2.3, one gets $\dot{x}_i(t) < 0$ for any $t \geq 0$, thus $z_i(\cdot, \zeta, D_M)$ is decreasing for $i = 1, 2$. Consider then the function $q_2 := x_2/x_1$. A straightforward computation of its derivative yields

$$\dot{q}_2 = \Big(\mu_2(s_{in} - x_1(t) - x_2(t)) - \mu_1(s_{in} - x_1(t) - x_2(t))\Big) q_2.$$

By Assumption 2.2, one has $\mu_2(s_{in}) - \mu_1(s_{in}) < 0$. Since $x(\cdot)$ converges to 0, we deduce that there exist $\eta > 0$ and $t_M > 0$ such that $\dot{q}_2(t) < -\eta q_2(t)$ for any time $t > t_M$. This proves that $q_2(\cdot)$ converges asymptotically to 0 and that trajectories are tangent to the x_1-axis at $(0, 0)$. $\square$

We are now in a position to introduce the following notation that will be used hereafter (see Fig. 2):

- Consider the point $\hat{P}^r := (x_1^r, s_{in} - x_1^r)$ on the boundary of $\mathcal{S}$ and set

$$\hat{x}(\cdot) := z(\cdot, \hat{P}^r, D_M).$$

- The forward semi-orbit of (2.3) with $D = D_M$ passing through P^r is denoted by (see Fig. 2):

$$\hat{\gamma}^+ := \{\hat{x}(t)\,;\, t \geq 0\}.$$

- In view of Lemma 3.2, $\hat{x}_1(\cdot)$ is decreasing and thus reaches $\bar{x}_1$ in finite time. Hence, there are $\hat{t} > 0$ and $\delta > 0$ satisfying:

$$\hat{t} := \inf\{t > 0;\ \hat{x}_1(t) < \bar{x}_1\} \quad \text{and} \quad \delta := \hat{x}_2(\hat{t}).$$

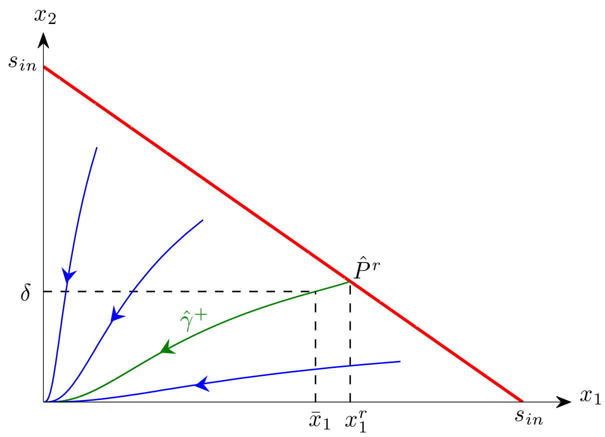

Figure 2: Phase portrait of (2.3) with constant $D = D_M$ and plot of the points $\hat{P}^r$, $(\bar{x}_1, \delta)$, and the semi-orbit $\hat{\gamma}^+$.

3.3 Properties of the reduced dynamics with constant $D = D_m$

We now turn to asymptotic properties of (2.3) with $D = D_m$. In the sequel, we shall use the notation

$$x_i^m := s_{in} - \lambda_i(D_m), \quad i = 1, 2.$$

The scalar product in $\mathbb{R}^2$ is written $a \cdot b$ with $a, b \in \mathbb{R}^2$ and $\|a\|$ denotes the euclidean norm of a vector $a \in \mathbb{R}^2$.

Lemma 3.3. *The system (2.3) with $D = D_m$ possesses the following properties.*

(i) *It admits exactly three equilibria in $\mathcal{S}$: $E_0 := (0, 0)$, $E_1^m := (x_1^m, 0)$ and $E_2^m := (0, x_2^m)$ which are respectively an unstable node, a saddle point, and a stable node.*

(ii) *One has $x_1^m < x_2^m$ and the "strip" $\mathcal{S}_{1,2}^m$ defined as*

$$\mathcal{S}_{1,2}^m := \{x \in \mathcal{S} \; ; \; x_1 + x_2 \in [x_1^m, x_2^m]\},$$

is invariant (by (2.3) with $D = D_m$).

(iii) *The edge $(0, s_{in}] \times \{0\}$ is the stable manifold of (2.3) with $D = D_m$ at E_1^m on $\mathcal{S}$. The unstable manifold at E_1^m in the domain $\mathcal{S}$ is denoted by $W^u(E_1^m)$: it connects E_1^m to E_2^m and satisfies $W^u(E_1^m) \subset \mathcal{S}_{1,2}^m$.*

Proof. Inequality (2.5) and the monotonicity of the functions $\mu_i(\cdot)$ (Assumption 2.1) imply that there are two equilibria of (2.3) in $\mathcal{S}$ that are distinct of the origin and on the axes. They are uniquely defined by E_1^m and E_2^m. The Jacobian matrix at E_1^m and E_2^m are respectively given by

$$J(E_1^m) := \begin{bmatrix} -\mu_1'(s_{in} - x_1^m)x_1^m & -\mu_1'(s_{in} - x_1^m)x_1^m \\ 0 & \mu_2(s_{in} - x_1^m) - D_m \end{bmatrix}, \; J(E_2^m) := \begin{bmatrix} \mu_1(s_{in} - x_2^m) - D_m & 0 \\ -\mu_2'(s_{in} - x_2^m)x_2^m & -\mu_2'(s_{in} - x_2^m)x_2^m \end{bmatrix}.$$

The point E_1^m is a saddle point because the eigenvalues of $J(E_1^m)$ are of opposite sign, E_2^m is a stable node because the eigenvalues of $J(E_2^m)$ are negative, and E_0 is clearly an unstable node which proves (i). It is worth noting that the unstable manifold $W^u(E_1^m)$ necessarily connects E_1^m to E_2^m by Lemma 3.1.

Let us now prove (ii). From Assumption 2.2, one has $x_1^m < x_2^m$. When $x \in \mathcal{S}_{1,2}^m$ is such that $x_1 + x_2 = x_1^m$, one has $\dot{x}_1 = 0$ and $\dot{x}_2 > 0$ whereas if $x_1 + x_2 = x_2^m$, one has $\dot{x}_1 < 0$ and $\dot{x}_2 = 0$. Hence $\mathcal{S}_{1,2}^m$ is invariant.

Let us finally prove (iii). The positive half axis $x_1 > 0$ is clearly the stable manifold of (2.3) with $D = D_m$ at E_1^m. Consider a non-null eigenvector v^+ of $J(E_1^m)$ associated with the positive eigenvalue $\mu_2(s_{in} - x_1^m) - D_m$, and let n be an outward normal to $\mathcal{S}_{1,2}^m$ at E_1^m. A straightforward calculation yields

$$v^+ = \begin{bmatrix} -1 \\ 1 + \frac{\mu_2(\lambda_1(D_m)) - D_m}{x_1^m \mu_1'(\lambda_1(D_m))} \end{bmatrix}, \quad n = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

implying that

$$v^+ \cdot n = -\frac{\mu_2(\lambda_1(D_m)) - D_m}{x_1^m \mu_1'(\lambda_1(D_m))} < 0.$$

One then concludes that the vector v^+ points inward $\mathcal{S}_{1,2}^m$ at E_1^m. On another hand, from the Theorem of the stable and unstable manifolds [21], we know that $W^u(E_1^m)$ is tangent to v^+ at E_1^m. Therefore, there is a neighborhood $\mathcal{V}$ of E_1^m in $\mathcal{S}_{1,2}^m$ such that $W^u(E_1^m) \cap \mathcal{V} \subset \mathcal{S}_{1,2}^m$, but, as $\mathcal{S}_{1,2}^m$ is invariant, we conclude that $W^u(E_1^m) \subset \mathcal{S}_{1,2}^m$ as was to be proved. □

Recall that the unstable manifold $W^u(E_1^m)$ is a trajectory of (2.3) with $D = D_m$, and that (2.3) satisfies $\dot{x}_1 < 0$ on *int* $\mathcal{S}_{1,2}^m$. So, $W^u(E_1^m)$ can be parametrized as a function $x_1 \mapsto w_u(x_1)$, $x_1 \in [0, x_1^m]$. Hereafter, $\mathrm{hyp}(w_u)$ stands for the hypograph of w_u and let $\mathcal{D} \subset \mathcal{S}$ be defined as (see Fig. 3):

$$\mathcal{D} := \mathrm{hyp}(w_u) \cap \mathcal{S}.$$

Note that the domain $\mathcal{D}$ is necessarily forward and backward invariant (for (2.3) with $D = D_m$) as its boundary is a locus of trajectories. Similarly as with $D = D_M$, let us introduce the following notation (see Fig. 3):

- Since there is a unique intersection point between $W^u(E_1^m)$ and the line $\{x_1 = x_1^r\}$, we set:

$$\bar{x}_2 = w_u(x_1^r), \tag{3.2}$$

  one can also write $(x_1^r, \bar{x}_2) = W^u(E_1^m) \cap \{x_1 = x_1^r\}$.

- From Assumption 2.3, one has $x_1^r < x_1^m$, thus we can fix a point $\check{P}^r := (x_1^r, \check{x}_{2,0}) \in \mathcal{D}$ such that:

$$0 < \check{x}_{2,0} < \min(\delta, x_1^m - x_1^r), \tag{3.3}$$

  where $\check{x}_{2,0}$ is small enough to ensure $\check{P}^r \in \mathcal{D}$.
- The backward semi-orbit of (2.3) with $D = D_m$ passing though $\check{P}^r$ is denoted by:

$$\check{\gamma}^- := \{\check{x}(t)\,;\, t \leq 0\},$$

  where $\check{x}(\cdot) = z(\cdot, \check{P}^r, D_m)$.
- Finally, define a positive parameter $\eta > 0$ as $\eta := x_1^m - x_1^r - \check{x}_{2,0}$.

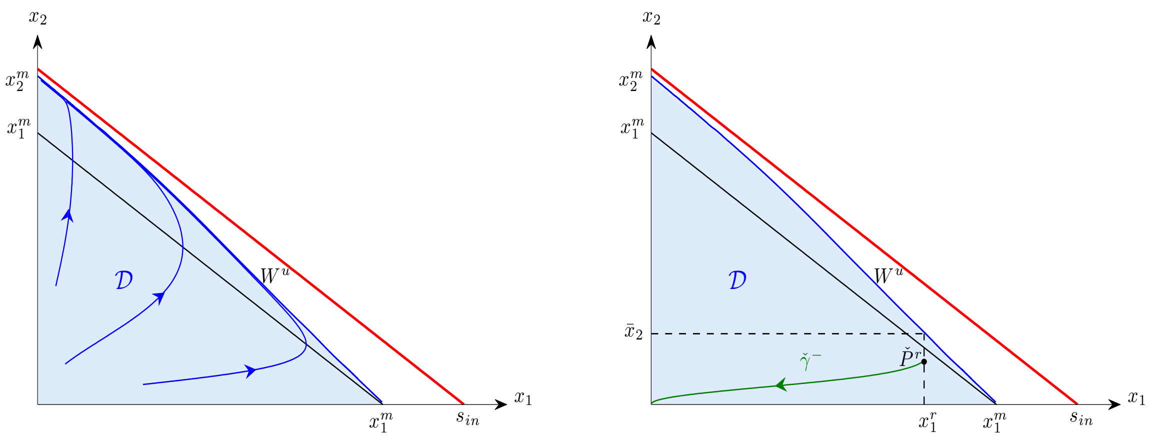

Figure 3: Left: phase portrait of (2.3) with constant $D = D_m$ in the domain $\mathcal{D}$ whose boundary is the unstable manifold $W^u(E_1^m)$. Right: plot of the point $\check{P}^r$ and the semi-orbit $\check{\gamma}^-$.

Remark 3.1. (i) *From Lemma 3.3 (i), any solution of (2.3) with $D = D_m$ in $\mathcal{D} \setminus ([0, x_1^m] \times \{0\})$ converges to E_2^m. Note also that for $(x_1, x_2) \in \mathcal{D} \setminus \{E_2^m\}$, one has $x_1 + x_2 < x_2^m$, which implies the inequality $\mu_2(s_{in} - x_1 - x_2) > D_m$. Hence, any solution $x(\cdot)$ in $\mathcal{D} \setminus \{E_2^m\}$ satisfies $\dot{x}_2 > 0$. Moreover, one has:*

$$\begin{array}{rcl} (x_1, x_2) \in \{x \in \mathcal{D}\,;\; x_1 > 0 \text{ and } x_1 + x_2 < x_1^m\} & \Rightarrow & \dot{x}_1 > 0, \\ (x_1, x_2) \in \{x \in \mathcal{D}\,;\; x_1 > 0 \text{ and } x_1 + x_2 > x_1^m\} & \Rightarrow & \dot{x}_1 < 0. \end{array}$$

(ii) *As $W^u(E_1^m)$ is a trajectory, any solution of (2.3) with $D = D_m$ crosses the line $\{x_1 = x_1^r\}$ at some point (x_1^r, x_2) such that $x_2 < \bar{x}_2$.*
(iii) *Since $\mathcal{D}$ is backward invariant by (2.3) with $D = D_m$ and $\check{P}^r \in \mathcal{D}$, the inclusion $\check{\gamma}^- \subset \mathcal{D}$ is fulfilled.*

The next lemma will be useful to define a small $\varepsilon > 0$ and times $\hat{T}$ and $\check{T}$ (see Remarks 3.2 and 3.3 below).

Lemma 3.4. *The curves $\hat{\gamma}^+$ and $\check{\gamma}^-$ intersect in the domain $(0, \bar{x}_1) \times (0, \check{x}_{2,0})$.*

Proof. Consider the variable $\check{q}_1 = \check{x}_1/\check{x}_2$ on the positive orthant. As previously, one has

$$\dot{\check{q}}_1 = \Big(\mu_1(s_{in} - \check{x}_1(t) - \check{x}_2(t)) - \mu_2(s_{in} - \check{x}_1(t) - \check{x}_2(t))\Big)\check{q}_1.$$

From Assumption 2.2, $\mu_1(s_{in}) - \mu_2(s_{in}) > 0$. As $\check{x}(t) \to (0, 0)$ when $t \to -\infty$, there exist $\check{t} < 0$ and $c' > 0$ such that

$$\mu_1(s_{in} - \check{x}_1(t) - \check{x}_2(t)) - \mu_2(s_{in} - \check{x}_1(t) - \check{x}_2(t)) > c', \quad t \leq \check{t},$$

which shows that $\check{q}_1(t)$ tends to 0 when t tends to $-\infty$. Therefore $\check{\gamma}^-$ is tangent to the x_2-axis at E_0. As $\hat{\gamma}^+$ is tangent to the x_1-axis at E_0 (Lemma 3.2), we deduce that the curve $\check{\gamma}^-$ is above the curve $\hat{\gamma}^+$, in a neighborhood of the point E_0 in $\mathcal{S}$. Since $\check{x}_1(0) = x_1^r > \bar{x}_1$ and $\check{x}_1(t)$ tends to 0 when t tends to $-\infty$, there exists $\bar{t} < 0$ such that

$$\check{x}_1(\bar{t}) = \bar{x}_1 \quad \text{with} \quad \bar{t} = \sup\{t < 0;\ \check{x}_1(t) = \bar{x}_1\}.$$

This means that $\check{x}_1(t) > \bar{x}_1$ for all $t \in (\bar{t}, 0]$. As $\check{x}_2(\cdot)$ is increasing on $\mathcal{D}$ (*cf.* Remark 3.1), one has $\check{x}_2(\bar{t}) < \check{x}_2(0)$ and from the choice of $\check{x}_{2,0} = \check{x}_2(0) < \delta$ (see condition (3.3)), one gets $\check{x}_2(t) < \delta$, for all $t \leq 0$ and in particular at $t = \bar{t}$. The point $(\bar{x}_1, \check{x}_2(\bar{t}))$ of $\check{\gamma}^-$ is thus below $(\bar{x}_1, \delta)$, which belongs to $\hat{\gamma}^+$. Therefore, $\hat{\gamma}^+$ and $\check{\gamma}^-$ have to cross at some point $\check{x}(\check{T})$ with $\check{T} < \bar{t}$, which verifies $0 < \check{x}_1(\check{T}) < \bar{x}_1$ and $0 < \check{x}_2(\check{T}) < \check{x}_{2,0}$. ☐

Remark 3.2. *Since $\check{\gamma}^- \subset \mathcal{D}$ (cf. Remark 3.1), the intersection between $\check{\gamma}^-$ and $\hat{\gamma}^+$ is also contained in $\mathcal{D}$.*

Lemma 3.4 implies that for each choice of the point $\check{P}^r$, there is an intersection point

$$P_\varepsilon := (\underline{x}_1, \varepsilon) \in \mathcal{D},$$

between $\hat{\gamma}^+$ and $\check{\gamma}^-$ such that

$$0 < \underline{x}_1 < \bar{x}_1 \quad \text{and} \quad 0 < \varepsilon < \check{x}_{2,0},$$

see Fig. 4. By construction, there are $\hat{T} > 0$ and $\check{T} > 0$ such that $P_\varepsilon = \hat{x}(\hat{T}) = \check{x}(-\check{T})$. Recall that $\mathcal{E}$ is by definition

$$\mathcal{E} := (0, \bar{x}_1] \times (0, \varepsilon].$$

Since $\bar{x}_1 < x_1^r$ and $\varepsilon < \check{x}_{2,0}$, the corner point $(\bar{x}_1, \varepsilon)$ of $\mathcal{E}$ is below the curve $\check{\gamma}^-$. Thus, one has also the inclusion $\mathcal{E} \subset \mathcal{D}$ (because $\check{\gamma}^- \subset \mathcal{D}$), see Fig. 4.

Remark 3.3. *Given D_m, D_M and x_1^r that fulfill Assumption 2.3, the parameter ε can be chosen arbitrarily small taking the parameter $\check{x}_{2,0}$ in the definition of $\check{P}^r$ small enough (recall (3.3)).*

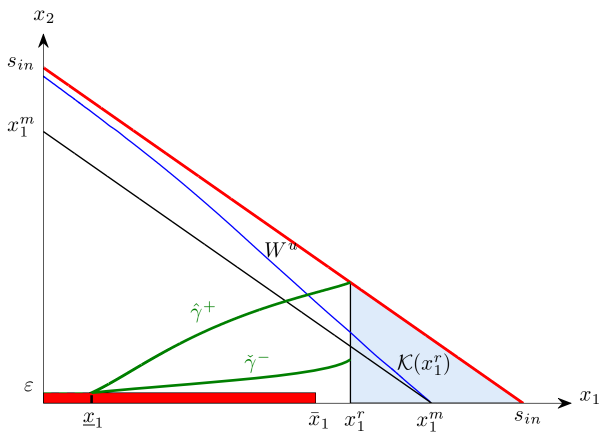

Figure 4: Plot of the set $\mathcal{E} := (0, \bar{x}_1] \times (0, \varepsilon]$ (in red) and the intersection point $P_\varepsilon = (\bar{x}_1, \varepsilon)$ between $\hat{\gamma}^+$ and $\check{\gamma}^-$. The set $\mathcal{K}(x_1^r)$ is depicted in blue.

3.4 Proof of Proposition 3.1

To help the reader, we provide in Appendix 1 a list of the notations used in Section 3.

We start by giving the following definition.

Definition 3.5. *The "southeast" order in $\mathbb{R}^2$ (denoted by $\preccurlyeq$) is defined as*

$$\forall (x, y) \in \mathbb{R}^2, \quad x \preccurlyeq y \Longleftrightarrow \{x_1 \leq y_1,\ x_2 \geq y_2\}.$$

Notice that the dynamics (2.3) is competitive, and therefore (2.3) preserves the order $\preccurlyeq$ (see [23]): for any admissible time-varying function $D(\cdot)$, one has:

$$\forall(\zeta_1,\zeta_2)\in\mathcal{S},\quad \zeta_1\preccurlyeq\zeta_2 \Longrightarrow z(t,\zeta_1,D(t))\preccurlyeq z(t,\zeta_2,D(t)),\quad \forall t\geq 0. \tag{3.4}$$

We have now given the necessary definitions and properties so that we can give to the proof of Proposition 3.1.

Proof. For clarity, we present the proof in several steps.

Step 1. Fix an initial condition $x_0\in\mathcal{K}(x_1^r)$ with $x_{2,0}<s_{in}-x_1^r$. As $x_0\notin\mathcal{E}$, we set $D=D_M$. Accordingly to Lemma 3.2, $x(\cdot)$ converges asymptotically to E_0 and thus $T_1:=\inf\{t>0\ ;\ x(t)\in\mathcal{E}\}$ is well defined. Notice that there are two ways to reach $\mathcal{E}$:

(i) $x_1(T_1)=\bar{x}_1$ and $x_2(T_1)<\varepsilon$,

(ii) $x_1(T_1)\leq\bar{x}_1$ and $x_2(T_1)=\varepsilon$.

Let us show that in both cases one has $x_1(T_1)>\underline{x}_1$. As $\bar{x}_1>\underline{x}_1$ (recall Lemma 3.4), we get $x_1(T_1)>\underline{x}_1$ in case (i). In case (ii), one can also write $T_1=\inf\{t>0\ ;\ z_2(t,x_0,D_M)\leq\varepsilon\}$. The order property (3.4) then implies that

$$z(t,\hat{P}^r,D_M)\preccurlyeq z(t,x_0,D_M),\quad t\geq 0.$$

As $z(\hat{T},\hat{P}^r,D_M)=\hat{x}(\hat{T})=(\underline{x}_1,\varepsilon)$, we deduce that

$$z_1(\hat{T},x_0,D_M)\geq\underline{x}_1,\quad z_2(\hat{T},x_0,D_M)\leq\varepsilon.$$

From the definition of T_1, one has $T_1<\hat{T}$ (one gets $T_1=\hat{T}$ if $x_0=\hat{P}^r$) and thus $x_1(T_1)=z_1(T_1,x_0,D_M)\geq\underline{x}_1$. If $x_1(T_1)=\underline{x}_1$, one should then have $x_2(T_1)=\varepsilon$, that is, $z(\cdot,x_0,D_M)=z(\cdot,\hat{P}^r,D_M)$ which is not true. Hence, we obtain as well $x_1(T_1)>\underline{x}_1$ as was to be proved. In addition, notice that $P_\varepsilon\preccurlyeq x(T_1)$. Since $\mathcal{E}\subset\mathcal{D}$, the point $x(T_1)$ necessarily satisfies $x(T_1)\in\mathcal{D}$.

Step 2. At $t=T_1$, we set $D=D_m$. We use again the order property (3.4):

$$P_\varepsilon\preccurlyeq x(T_1)\Longrightarrow z(t,P_\varepsilon,D_m)\preccurlyeq z(t,x(T_1),D_m),\quad \forall t\geq 0, \tag{3.5}$$

and, as we have shown that $x_1(T_1)>\underline{x}_1$, we obtain the inequalities

$$z_1(\check{T},x(T_1),D_m)>z_1(\check{T},P_\varepsilon,D_m)=x_1^r,\quad z_2(\check{T},x(T_1),D_m)\leq z_2(\check{T},P_\varepsilon,D_m)=\check{x}_{2,0}.$$

Therefore one has $x(T_1+\check{T})=z(\check{T},x(T_1),D_m)\in int\ \mathcal{K}(x_1^r)$. One can then define a time $\bar{T}_1$ as:

$$\bar{T}_1:=\inf\{t>T_1\ ;\ x_1(t)>x_1^r\},$$

which is such that $\bar{T}_1\in(T_1,T_1+\check{T})$. From the monotonicity of $x_2(\cdot)$ in the set $\mathcal{D}$ (*cf.* Remark 3.1), one obtains

$$x_2(\bar{T}_1)<x_2(T_1+\check{T})=z_2(\check{T},x(T_1),D_m)\leq\check{x}_{2,0}.$$

As $x(T_1)$ belongs to the set $\mathcal{D}$ with $x_1(T_1)>0$, $z(\cdot,x(T_1),D_m)$ converges asymptotically to the equilibrium E_2^m that lies on the x_2-axis (*cf.* Remark 3.1). The time $T_2>T_1+\check{T}$ such that $x_1(T_2)=x_1^r$, where $x(t):=z(t-T_1,x(T_1),D_m)$, $t\geq T_1$, is thus well defined. Moreover one has $x(T_2)\in\mathcal{K}(x_1^r)$ and $x(T_2)\in\mathcal{D}$ as $\mathcal{D}$ is invariant by (2.3) (for $D=D_m$).

Step 3. At time T_2, we have shown that $x(T_2)$ belongs to the set $\mathcal{K}(x_1^r)$, and also to the set $\mathcal{D}$ which implies that $x_2(T_2)<s_{in}-x_1^r$. Therefore, we can consider $x(T_2)$ as a new initial condition and apply iteratively the results of steps 1 and 2, defining an increasing sequence of times $(T_i)_{i\in\mathbb{N}}$. For $i=2k+1$ (with $k\in\mathbb{N}$), one has $x(T_{2k+1})\in\mathcal{E}$ and, as in step 2, we can define $\bar{T}_{2k+1}\in(T_{2k+1},T_{2k+1}+\check{T})$ such that

$$x_1(\bar{T}_{2k+1})=x_1^r\quad\text{with}\quad \bar{T}_{2k+1}:=\inf\{t>T_{2k+1}\ ;\ x_1(t)>x_1^r\}. \tag{3.6}$$

As shown in step 2, we necessarily have

$$x_2(\bar{T}_{2k+1})<\check{x}_{2,0}. \tag{3.7}$$

Because $T_{2k+2} > T_{2k+1} + \check{T}$, we get that $\lim_{i\to+\infty} T_i = +\infty$ which concludes that property (3.1) is fulfilled.

Note that if we chose $x_0 = \hat{P}^r$ then one obtains $T_1 = \hat{T}$ (and then by the unicity of the solution $x(T_1) = P_\varepsilon$). Furthermore, in this case, the time $\bar{T}_1$ is such that $\bar{T}_1 = T_1 + \check{T}$ with $x(\bar{T}) \in \mathcal{K}(x_1^r)$, since $x(T_1) = \check{x}(-\check{T})$. The time $T_2 > \bar{T}_1$ that is the first exit time from $\mathcal{K}(x_1^r)$ is well defined with $x_2(T_2) < s_{in} - x_1^r$.

Step 4. We now show that the time spent by $x_1(\cdot)$ above the threshold x_1^r is of infinite measure. For each $k \in \mathbb{N}$, one has, from the definition of T_{2k+2}:

$$\text{meas}\{t \in [T_{2k+1}, T_{2k+2}]\ ;\ x_1(t) > x_1^r\} = T_{2k+2} - \bar{T}_{2k+1}.$$

From Remark 3.1, one has $\dot{x}_1(t) > 0$ when $s(t) = s_{in} - x_1(t) - x_2(t) > \lambda_1(D_m)$. At time $\bar{T}_{2k+1}$, the inequality (3.7) implies that $s(\bar{T}_{2k+1}) > s_{in} - x_1^r - \check{x}_{2,0}$. We deduce that

$$s(\bar{T}_{2k+1}) - \lambda_1(D_m) = s(\bar{T}_{2k+1}) - (s_{in} - x_1^r - \check{x}_{2,0}) + \eta > \eta > 0. \tag{3.8}$$

Next, let us define

$$\tau_k := \sup\{\theta > 0\ ;\ s(\bar{T}_{2k+1} + \theta) > \lambda_1(D_m)\}.$$

Then, one has necessarily $x_1(t) > x_1^r$ for any $t \in [\bar{T}_{2k+1}, \bar{T}_{2k+1} + \tau_k]$ and one obtains the inequality

$$T_{2k+2} - \bar{T}_{2k+1} > \tau_k.$$

Let us now give a lower bound on the value of τ_k. From equations (2.3), the following properties hold true:

$$\left\{\begin{array}{lcl} x_1 & > & x_1^r, \\ s & > & \lambda_1(D_m) \end{array}\right. \Rightarrow \left\{\begin{array}{lclcl} s & = & s_{in} - x_1 - x_2 & < & s_{in} - x_1^r, \\ x_1 & = & s_{in} - s - x_2 & < & s_{in} - \lambda_1(D_m) = x_1^m, \\ x_2 & = & s_{in} - s - x_1 & < & s_{in} - \lambda_1(D_m) - x_1^r = x_1^m - x_1^r, \end{array}\right.$$

from which one can obtain a lower bound on the speed at which the variable s decreases (as long as s is above $\lambda_1(D_m)$ and x_1 above x_1^r):

$$\begin{array}{rcl} \dot{s} & = & -\mu_1(s)x_1 - \mu_2(x)x_2 + D_m(s_{in} - s) \\ & \geq & -\mu_1(s_{in} - x_1^r)x_1^m - \mu_2(s_{in} - x_1^r)(x_1^m - x_1^r) := -c''. \end{array}$$

Notice that $c'' > 0$. By integrating the above inequality, one can conclude that s stays above $\lambda_1(D_m)$ for a duration larger than $(s(\bar{T}_{2k+1}) - \lambda_1(D_m))/c''$. Thanks to (3.8), we can thus write

$$\tau_k > M := \frac{\eta}{c''} > 0,$$

where $M > 0$ does not depend on k. Finally, we have shown that for each $k \in \mathbb{N}$, one has

$$\text{meas}\{t \in [0, T_{2k+2}]\ ;\ x_1(t) > x_1^r\} > kM,$$

which shows that the time-varying $D(\cdot)$ is a weaky resilient removal rate as was to be proved. □

Remark 3.4. (i) *In the proof of Proposition 3.1 (step 1), we have seen that $x_1(T_{2k+1}) > \underline{x}_1$ for any $k \in \mathbb{N}$. Therefore, Proposition 3.1 remains valid if the set $\mathcal{E}$ is replaced by*

$$\tilde{\mathcal{E}} := (\underline{x}_1, \bar{x}_1] \times (0, \varepsilon].$$

(ii) *Notice also that the trajectory given by Proposition 3.1 reaches the set $\mathcal{D}$ at some time $t \leq T_1$, and then remains in this set for any future time. Indeed, from Remark 3.1, the trajectory belongs to $\mathcal{D}$ on any time interval $[T_{2k+1}, T_{2k+2}]$ (with $D = D_m$). On a time interval $[T_{2k+2}, T_{2k+3}]$, we have set $D = D_M$, and we have seen in Lemma 3.2 that $x_1(\cdot)$ and $x_2(\cdot)$ are decreasing (with $D = D_M$). So, the trajectory also remains in $\mathcal{D}$ for $t \in [T_{2k+2}, T_{2k+3}]$. One then concludes that the trajectory remains in $\mathcal{D}$ as well as in $\{x_1 > \underline{x}_1\}$ (thanks to point* (i) *above) over $[T_1, +\infty)$*

(iii) *Finally, note that at times $\bar{T}_{2k+1}$ and T_{2k+2}, one has $x_1(\bar{T}_{2k+1}) = x_1(T_{2k+2}) = x_1^r$ and*

$$x_2(\bar{T}_{2k+1}) < x_2(T_{2k+2}) < \bar{x}_2 < x_2^m - x_1^r. \tag{3.9}$$

4 Convergence to periodic solutions

The goal of this section is to show that the weakly resilient removal rate defined by Proposition 3.1 generates asymptotically positive periodic solutions of system (2.3). Before addressing this point, we shall first prove the existence of a periodic solution of (2.3) associated with the time-varying $D(\cdot)$ given by Proposition 3.1. To this end, let us introduce an operator $\mathcal{O}$ as

$$\begin{array}{rccc} \mathcal{O}: & (0, s_{in} - x_1^r] & \to & (0, s_{in} - x_1^r] \\ & x_{2,0} & \mapsto & x_2(T_2) \end{array}$$

where $x(\cdot) = (x_1(\cdot), x_2(\cdot))$ is the unique solution of (2.3) for the initial condition (x_1^r, x_{02}) and the time-varying $D(\cdot)$ given by Proposition 3.1, parameters D_m, D_M, and $\check{P}^r$ being fixed. Notice that times T_i, $i \geq 1$, introduced in Proposition 3.1 depend on $x_{2,0}$ (in particular T_2). Hence, this operator slightly differs from the *Poincaré map* used for instance in [12, 24] for finding periodic solutions of dynamical systems, for which the period is fixed beforehand. We shall next examine properties of the operator $\mathcal{O}$. Doing so, let us introduce the following notation (see Fig. 5):

- Denote by $\tilde{\gamma}$ and $\check{\gamma}$, the orbits of (2.3) with $D = D_m$ passing respectively by $(x_1^r, x_1^m - x_1^r)$ and $\check{P}_r$.
- Observe that $\tilde{\gamma}$ is tangent to the segment $\{x_1 = x_1^r\} \cap \mathcal{S}$ at $(x_1^r, x_1^m - x_1^r)$. Because $\check{x}(\cdot)$ converges to E_2^m (Lemma 3.3), there are exactly two intersection points between $\check{\gamma}$ and $\{x_1 = x_1^r\} \cap \mathcal{S}$, namely $\check{P}_r$ and

$$\check{Q}_r := (x_1^r, \underline{x}_2) \quad \text{with} \quad \underline{x}_2 > x_1^m - x_1^r. \tag{4.1}$$

- In the sequel, we denote by J the interval $J := [\underline{x}_2, \bar{x}_2]$ (recall (3.2)).

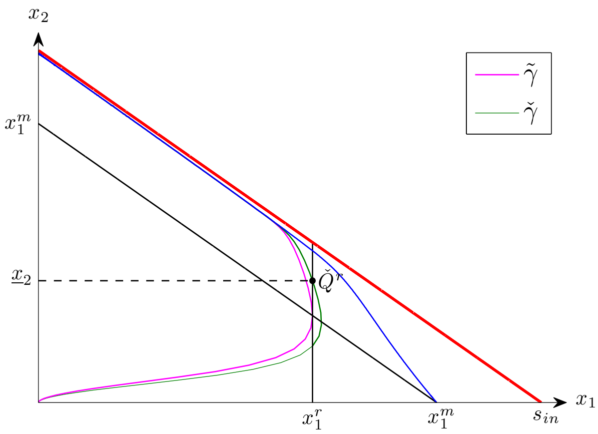

Figure 5: Plot of the orbits $\tilde{\gamma}$ and $\check{\gamma}$ and the point $\check{Q}^r$.

4.1 Properties of the operator $\mathcal{O}$

In this section, we prove that $\mathcal{O}$ is continuous and decreasing. The continuity of $\mathcal{O}$ will follow from the continuity property of the first entry time into a set, that is related to Petrov's condition (see Appendix 2).

For initial condition $(x_1^r, x_{2,0})$ with $x_{2,0} \in (0, S_{in} - x_1^r]$, we denote $T_1(x_{2,0})$, $T_2(x_{2,0})$ the times T_1, T_2 given by Proposition 3.1.

Proposition 4.1. *The time-varying $D(\cdot)$ contructed in Proposition 3.1 fulfills the following continuity properties:*

(i) *Times T_1 and T_2 are Lipschitz continuous functions of initial $x_{2,0}$.*

(ii) *The operator $\mathcal{O}$ is Lipschitz continuous and takes values in J.*

Proof. For a given $x_{2,0} \in (0, s_{in} - x_1^r]$, denote by $x(\cdot)$ the unique solution of (2.3) for the initial condition (x_1^r, x_{02}) and the time-varying $D(\cdot)$ given by Proposition 3.1.

Let us prove (i). For $t \in [0, T_1]$, one has $D(t) = D_M$ and T_1 is defined as the first time that $x(\cdot)$ reaches the set

$$\mathcal{T}_1 := \{x \in \mathcal{S}\ ;\ x_1 \leq \bar{x}_1 \text{ and } x_2 \leq \varepsilon\}.$$

For $t \in (T_1, T_2)$, one has $D(t) = D_m$. We know that there exists $\bar{T}_1 \in (T_1, T_1 + \check{T})$ such that $x(\cdot)$ crosses the line $\{x_1 = x_1^r\}$ at time $\bar{T}_1$ with $x_2(\bar{T}_1) < \check{x}_{2,0}$ (see step 2 of the proof of Proposition 3.1). Therefore, the state $x(\bar{T}_1)$ is below the point $\check{P}^r$. As $x(\cdot)$ cannot cross the orbit of $\check{P}^r$ (denoted $\check{\gamma}$) on $(\bar{T}_1, T_2)$, $x(\cdot)$ crosses the line $\{x_1 = x_1^r\}$ at some time T_2 such that

$$x_2(T_2) > \underline{x}_2. \tag{4.2}$$

Hence, T_2 can be defined as the first time $t > T_1 + \check{T}$ such that $x(\cdot)$ reaches the set

$$\mathcal{T}_2 := \{x \in \mathcal{S}\ ;\ x_1 \leq x_1^r \text{ and } x_2 \geq \underline{x}_2\},$$

at time t. On the positive set $\mathcal{S}^+ := \{x \in \mathcal{S}\ ;\ x_1 > 0\ ;\ x_2 > 0\}$, that is invariant by (2.3), we are in a position to introduce the first entry time functions:

$$R_1(x_0) := \inf\{t \geq 0\ ;\ z(t, x_0, D_M) \in \mathcal{T}_1\},\ R_2(x_0) := \inf\{t \geq 0\ ;\ z(t, x_0, D_m) \in \mathcal{T}_2\},$$

with $x_0 \in \mathcal{S}^+$. Then, for $x_{2,0} \in (0, s_{in} - x_1^r]$, Proposition 3.1 allows to write the composition

$$\mathcal{O}(x_{2,0}) = z_2(T_2 - T_1 - \check{T}, x(T_1 + \check{T}), D_m), \tag{4.3}$$

with

$$\begin{cases} T_1 & := & R_1(x_1^r, x_{2,0}), \\ x(T_1) & := & z(T_1, (x_1^r, x_{2,0}), D_M), \\ x(T_1 + \check{T}) & := & z(\check{T}, x(T_1), D_m), \\ T_2 & := & T_1 + \check{T} + R_2(x(T_1 + \check{T})), \end{cases}$$

thanks to the definitions of T_1, T_2, and $\check{T}$. From the continuous dependency of an ODE w.r.t. initial conditions, (see, *e.g.*, [21]), the maps $x_0 \mapsto z(t, x_0, D)$ (for a fixed t) and $t \mapsto z(t, x_0, D)$ (for a fixed x_0) are Lipschitz continuous, given a constant $D \in [D_m, D_M]$. Therefore, proving the Lipschitz continuity of T_1 and T_2 w.r.t $x_{2,0}$ essentially requires to prove the Lipschitz continuity of R_1 and R_2 over the set $\mathcal{S}^+$. Notice first that for constant $D = D_M$, resp. $D = D_m$, any solution in $\mathcal{S}^+$ converges asymptotically to a steady state that belongs to the interior of $\mathcal{T}_1$, resp. $\mathcal{T}_2$. Therefore, the targets $\mathcal{T}_1$ and $\mathcal{T}_2$ can be reached in a finite horizon from any initial condition in $\mathcal{S}^+$, and thus, R_1, R_2 are well defined with finite values in $\mathcal{S}^+$. To prove their Lipschitz continuity, we shall use Theorem 4.1 recalled in Appendix 2, showing that the inward pointing condition (4.8) is fulfilled on the boundary of $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathcal{S}^+$.

Lipschitz continuity of R_1. Observe first that $\mathcal{T}_1$ is convex, hence the (convex) normal cone to $\mathcal{T}_1$ at some point $x \in \mathcal{S}^+$ of its boundary is given by the expression

$$N_{\mathcal{T}_1}(x) = \begin{cases} \mathbb{R}_+ \times \{0\}, & x_1 = \bar{x}_1,\ x_2 < \varepsilon, \\ \{0\} \times \mathbb{R}_+, & x_1 < \bar{x}_1,\ x_2 = \varepsilon, \\ \mathbb{R}_+ \times \mathbb{R}_+, & x_1 = \bar{x}_1,\ x_2 = \varepsilon. \end{cases}$$

Then, we can easily check that

$$\{x_1 = \bar{x}_1, x_2 \leq \varepsilon\} \ \Rightarrow\ f_1(x, D_M) < \phi_1 := (\mu_1(s_{in}) - D_M)\bar{x}_1 < 0,$$
$$\{x_1 \leq \bar{x}_1, x_2 = \varepsilon\} \ \Rightarrow\ f_2(x, D_M) < \phi_2 := (\mu_2(s_{in} - \varepsilon) - D_M)\varepsilon < 0.$$

From the preceding inequalities, we deduce that for any point x on the boundary of $\mathcal{T}_1$ in $\mathcal{S}^+$, one has

$$f(x, D_M) \cdot \nu < \min(\phi_1, \phi_2)\|\nu\|, \quad \nu \in N_{\mathcal{T}_1}(x) \setminus \{0\}.$$

This allows us to conclude that R_1 is Lipschitz continuous over $\mathcal{S}^+$, thanks to Theorem 4.1.

Lipschitz continuity of R_2. Observe that $\mathcal{T}_2$ is also convex, hence the (convex) normal cone to $\mathcal{T}_2$ at some point $x \in \mathcal{S}^+$ of its boundary is given by the expression

$$N_{\mathcal{T}_2}(x) = \begin{cases} \mathbb{R}_+ \times \{0\}, & x_1 = x_1^r, x_2 > \underline{x}_2, \\ \{0\} \times \mathbb{R}_-, & x_1 < x_1^r, x_2 = \underline{x}_2, \\ \mathbb{R}_+ \times \mathbb{R}_-, & x_1 = x_1^r, x_2 = \underline{x}_2. \end{cases}$$

One can easily that the following properties are fulfilled:

$$\begin{aligned} \{x_1 = x_1^r, x_2 > \underline{x}_2\} &\Rightarrow f_1(x, D_m) < \psi_1 := f_1((x_1^r, \underline{x}_2), D_m), \\ \{x_1 < x_1^r, x_2 = \underline{x}_2\} &\Rightarrow f_2(x, D_m) > \psi_2 := f_2((x_1^r, \underline{x}_2), D_m), \\ \{x_1 = x_1^r, x_2 = \underline{x}_2\} &\Rightarrow f(x, D_m) \cdot \nu = \psi_1 \nu_1 + \psi_2 \nu_2. \end{aligned}$$

As a consequence of (3.9), (4.1) and (4.2), note that one has $x_1^m - x_1^r < \underline{x}_2 < x_2^m - x_1^r$. This allows us to conclude that $\psi_1 < 0$ and that $\psi_2 > 0$ yielding the inequality

$$f(x, D_m) \cdot \nu \leq \min(\psi_1, -\psi_2) \|\nu\|, \quad \nu \in N_{\mathcal{T}_2}(x) \setminus \{0\},$$

for any x on the boundary of the set $\mathcal{T}_2$ in $\mathcal{S}^+$. This proves the Lipschitz continuity of R_2 on $\mathcal{S}^+$ using again Theorem 4.1. We conclude that both T_1 and T_2 are Lipschitz continuous w.r.t $x_{2,0}$.

Let us prove now (ii). Recall that $\mathcal{O}$ can be written as function of T_1 and T_2 (see (4.3)). As T_1 and T_2 are Lipschitz continuous w.r.t $x_{2,0}$ then $\mathcal{O}$ as well. Combining (3.9) and (4.2) gives that $\mathcal{O}(x_{2,0}) \in J$ as was to be proved. □

Remark 4.1. *Since T_1 and T_2 are continuous then times T_i, $i \geq 3$, introduced in Proposition 3.1 are also continuous functions of $x_{2,0}$.*

Let us now study the monotonicity of the operator $\mathcal{O}$.

Proposition 4.2. *The operator $\mathcal{O}$ is decreasing.*

Proof. Take two points $x_{2,0}^{\pm} \in (0, s_{in} - x_1^r]$ such that $x_{2,0}^- < x_{2,0}^+$, and let us show that $\mathcal{O}(x_{2,0}^-) > \mathcal{O}(x_{2,0}^+)$. Denote by $x^+(\cdot)$, $x^-(\cdot)$ the solutions generated by the time-varying $D(\cdot)$ given by Proposition 3.1 and for the initial conditions $x^+(0) = (x_1^r, x_{2,0}^+)$ and $x^-(0) = (x_1^r, x_{2,0}^-)$ respectively. One can then write $x^+(0) \preccurlyeq x^-(0)$. For convenience, we denote by T_1^+, T_2^+ and T_1^-, T_2^- the times T_1, T_2 (as in Proposition 3.1) associated with $x^+(\cdot)$ and $x^-(\cdot)$ respectively, and let us set

$$\bar{T}_1^+ := \inf\{t > T_1^+ \ ; \ x_1^+(t) > x_1^r\} \quad \text{and} \quad \bar{T}_1^- := \inf\{t > T_1^- \ ; \ x_1^-(t) > x_1^r\}.$$

First, let us note that the time-varying removal rate given by Proposition 3.1 satisfies $D(t) = D_M$ for both trajectories in a neighborhood of $t = 0$. Using the order property (3.4), one can then write

$$z(t, x^+(0), D_M) \preccurlyeq z(t, x^-(0), D_M), \quad t \geq 0. \tag{4.4}$$

Thanks to this property, we must have $T_1^+ \geq T_1^-$ (otherwise, x^+ reaches $\mathcal{E}$ at some time $T_1^+ < T_1^-$ implying a contradiction with (4.4)). Therefore one gets (recall that trajectories with $D = D_M$ decrease), we obtain the inequality

$$x_1^+(T_1^+) = z_1(T_1^+, x^+(0), D_M) \leq z_1(T_1^-, x^+(0), D_M) \leq z_1(T_1^-, x^-(0), D_M) = x_1^-(T_1^-).$$

Since the orbits of (2.3) with $D = D_M$ do not intersect, we also obtain that $x_2^+(T_1^+) \geq x_2^-(T_1^-)$ which implies that $x^+(T_1^+) \preccurlyeq x^-(T_1^-)$. Because $x_1^+(T_1^+) < x_1^-(T_1^-)$, the time needed by $x^+(\cdot)$ to reach the liner $\{x_1 = x_1^r\}$ from $x_1^+(T_1^+)$ is greater than the time of $x^-(\cdot)$ to reach the line $\{x_1 = x_1^r\}$ from $x_1^-(T_1^-)$. This gives $\bar{T}_1^+ - T_1^+ \geq \bar{T}_1^- - T_1^-$.

We now consider $x^+(T_1^+)$ and $x^-(T_1^-)$ as initial conditions for (2.3) with $D = D_m$. Then one gets

$$z(t, x^+(T_1^+), D_m) \preccurlyeq z(t, x^-(T_1^-), D_m), \quad t \geq 0.$$

In the same way as previously, we deduce the inequality:

$$x_2^+(\bar{T}_1^+) = z_2(\bar{T}_1^+ - T_1^+, x^+(T_1^+), D_m) \geq z_2(\bar{T}_1^- - T_1^-, x^+(T_1^+), D_m) \geq z_2(\bar{T}_1^- - T_1^-, x^-(T_1^-), D_m) = x_2^-(\bar{T}_1^-).$$

Since the orbits of (2.3) with $D = D_m$ do not intersect, one necessarily has $x_2^+(\bar{T}_1^+) > x_2^-(\bar{T}_1^-)$ (and also $x_1^+(\bar{T}_1^+) = x_1^-(\bar{T}_1^-)$). Finally, D is constant equal to D_m for both trajectories until the first instant at which one of the two trajectory leaves the set $\mathcal{K}(x_1^r)$. Since $x^+(\bar{T}_1^+)$ is above $x^-(\bar{T}_1^-)$, the point $x^-(T_2^-)$ is necessarily above $x^+(T_2^+)$, that is

$$x_2^-(T_2^-) > x_2^+(T_2^+),$$

which implies the desired inequality $\mathcal{O}(x_{2,0}^+) < \mathcal{O}(x_{2,0}^-)$. □

4.2 Existence and attractivity of periodic solutions

In this section, we study how for any initial condition, the time-varying removal rate $D(\cdot)$ given in Proposition 3.1 allow system (2.3) to synchronize with a periodic solution (*i.e.* any solution of (2.3) associated with $D(\cdot)$ converges asymptotically to a periodic solution).

4.2.1 Existence of periodic solutions

The existence of a weakly resilient periodic trajectory follows from the previous results about the operator $\mathcal{O}$.

Corollary 4.1. *There exists a unique positive periodic solution $x^\star(\cdot)$ associated with the time-varying $D(\cdot)$ given by Proposition 3.1 such that $x^\star(0) = (x_1^r, x_{2,0}^\star)$ with $x_{2,0}^\star$ satisfying $\mathcal{O}(x_{2,0}^\star) = x_{2,0}^\star$.*

Proof. Consider the function $\varphi : (0, s_{in} - x_{1,r}] \to \mathbb{R}$ defined as

$$\varphi(x_2) := \mathcal{O}(x_2) - x_2, \quad x_2 \in (0, s_{in} - x_{1,r}].$$

From Propositions 4.1 and 4.2, $\varphi(\cdot)$ is continuous and decreasing. Moreover, it verifies $\varphi(x_2) > 0$ for $x_2 < \underline{x}_2$ and $\varphi(x_2) < 0$ for $x_2 > \bar{x}_2$ (because $\mathcal{O}$ is with values in J). We can then conclude that $\varphi(\cdot)$ possesses a unique zero in the interval $(0, s_{in} - x_{1,r}]$, or equivalently that there exists a unique fixed point $x_{2,0}^\star$ of $\mathcal{O}$. The solution $x^\star(\cdot)$ for the initial condition $(x_1^r, x_{2,0}^\star)$ verifies $x^\star(T_2^\star) = x^\star(0)$, where $T - 2^\star$ is equal to the time T_2 generated by the time-varying $D(\cdot)$ given in Proposition 3.1, and is thus $T_2^\star$-periodic. We conclude that $x^\star(\cdot)$ is the unique periodic solution such that $x_1^\star(0) = x_1^r$ and $\mathcal{O}(x_2^\star(0)) = x_2^\star(0)$. □

4.2.2 Attractivity of the periodic solution

Due to the particular structure of the non-autonomous dynamics (the times T_i are not known explicitly), it appears that determining explicitly a bound on the Lipschitz rank of $\mathcal{O}$ is quite difficult. However, in all the simulations we performed, the operator $\mathcal{O}$ appears to be contractive, providing ε to be sufficiently small. We thus posit the following conjecture.

Conjecture 4.1. *For ε sufficiently small, the operator $\mathcal{O}$ is contractive on J.*

Proposition 4.3. *Under the conjecture, for ε sufficiently small an any initial condition in $\mathcal{K}(x_1^r)$, the solution $x(\cdot)$ generated by the time-varying $D(\cdot)$ given in Proposition 3.1 converges asymptotically to the periodic solution $x^\star(\cdot)$ up to a time shift $\bar{\sigma}$:*

$$\lim_{t \to +\infty} x(t + \bar{\sigma}) - x^\star(t) = 0.$$

Proof. Fix an initial condition in $\mathcal{K}(x_1^r)$. From Proposition 3.1 we know that the solution $x_1(\cdot)$ reaches x_1^r in finite time. Let t_0 be the first time t for which $x_1(t_0) = x_1^r$ and let $x_{2,0} = x_2(t_0)$. We can then consider, without any loss of generality, $(x_1^r, x_{2,0})$ as initial condition. Let T_i be the sequence of times given by Proposition 3.1. The trajectory $x(\cdot)$ is then solution of the non-autonomous dynamics $\dot{x} = F(t, x)$ with

$$F(t, x) = \begin{cases} f(x, D_M), \ t \in [T_{2k}, T_{2k+1}) \\ f(x, D_m), \ t \in [T_{2k+1}, T_{2k+2}) \end{cases} \qquad (k \in \mathbb{N})$$

As $\mathcal{O}$ is contractive, ona has the following limit

$$\lim_{k \to +\infty} x_2(T_{2k}) = \lim_{k \to +\infty} \mathcal{O}^k(x_{2,0}) = x_{2,0}^\star. \tag{4.5}$$

Let $T_i^\star$ be the sequence of times given by Proposition 3.1 for the initial condition $x_0^\star = (x_1^\tau, x_{2,0}^\star)$ and posit

$$F^\star(t,x) = \begin{cases} f(x, D_M), \; t \in [T_{2k}^\star, T_{2k+1}^\star) \\ f(x, D_m), \; t \in [T_{2k+1}^\star, T_{2k+2}^\star) \end{cases} \qquad (k \in \mathbb{N})$$

Clearly, on has $T_{i+2}^\star = T_i^\star + T_2^\star$ i.e. $F^\star$ is a $T_2^\star$-periodic dynamics.

We define now the following functions

$$\tilde{F}(\tau, x) = \frac{T_{i+1} - T_i}{T_{i+1}^\star - T_i^\star} F(\tau, x), \quad \tau \in [T_i^\star, T_{i+1}^\star)$$

$$g(t) = T_i^\star + \frac{T_{i+1}^\star - T_i^\star}{T_{i+1} - T_i}(t - T_i), \quad t \in [t_i, T_{i+1})$$

Clearly, the solution $x(\cdot)$ of $\dot{x} = F(t,x)$ satisfies $x(t) = \tilde{x}(g(t))$ for any $t \geq 0$, where $\tilde{x}(\cdot)$ is solution of

$$\frac{d\tilde{x}}{d\tau}(\tau) = \tilde{F}(\tau, \tilde{x}(\tau)), \quad \tilde{x}(0) = x(0).$$

Thanks to the continuity property of T_i (see Proposition 4.1 and Remark 4.1) and (4.5), one concludes that

$$\lim_{k \to +\infty} T_{2k+i} - T_{2k} = \lim_{k \to +\infty} T_i(x_2(T_{2k})) = T_i(x_{2,0}^\star), \; i = 1, 2, \tag{4.6}$$

which gives

$$\lim_{k \to +\infty} T_{2(k+1)} - T_{2k} = T_2(x_{2,0}^\star) = T_2^\star, \tag{4.7}$$

and

$$\lim_{k \to +\infty} T_{2k+i+1} - T_{2k+i} = T_{i+1}(x_{2,0}^\star) - T_i(x_{2,0}^\star), \; i = 0, 1.$$

Then, one has

$$\lim_{i \to +\infty} \frac{T_{i+1} - T_i}{T_{i+1}^\star - T_i^\star} = 1$$

and deduce that $\tilde{F}$ is an asymptotically periodic dynamics with $F^\star$ as limit (see Definition 4.2 in Appendix 3). Moreover, one has

$$\lim_{k \to +\infty} \tilde{x}(kT_2^\star) = x(T_{2k}) = x_0^\star.$$

Therefore, we can apply the Theorem 4.3 (from [27], recalled in Appendix 3) which gives

$$\lim_{t \to +\infty} \tilde{x}(t) - x^\star(t) = 0$$

and we have obtained thus the following limit

$$\lim_{t \to +\infty} x(t) - x^\star(g(t)) = 0.$$

We show now that the time shift $\sigma(t) := t - g(t)$ admits a limit $\bar{\sigma}$. Notice that one has $\delta(T_i) = T_i - T_i^\star$ for any i. Therefore, it is enough to prove that the sequence $u_k = T_{2k} - T_{2k}^\star$ converges.

As $\mathcal{O}$ is contractive on J, there exists $\alpha \in (0,1)$ such that

$$|x_2(T_{2(k-1)}) - x_{2,0}^\star| = |\mathcal{O}(x_2(T_{2(k-2)}) - x_{2,0}^\star| \leq \alpha^{k-1}|x_{2,0} - x_{2,0}^\star|, \quad k > 1$$

As the map $x_{2,0} \mapsto T_2$ is Lipschitz continuous (cf Proposition 4.1), say of rank L, one obtains

$$|u_k - u_{k-1}| = |T_2(x_2(T_{2(k-1)}) - T_2^\star| \leq L\alpha^{k-1}|x_{2,0} - x_{2,0}^\star|$$

and let us finally show that u_k is a Cauchy sequence. For any $n > 1$ and $k > 1$, one has

$$|u_{n+k} - u_n| \leq L \sum_{i=0}^{k-1} \alpha^{n+i}|x_{2,0} - x_{2,0}^\star| = L\alpha^n \frac{1 - \alpha^k|}{1 - \alpha}|x_{2,0} - x_{2,0}^\star|$$

As $\alpha < 1$, on obtains that $|u_{n+k} - u_n|$ tends to 0 when n tends to ∞ uniformly in k. We conclude that the Cauchy sequence u_k admits a limit, which gives the asymptotic shift $\bar{\sigma}$. □

Appendix 1: list of notations

We remind in the next table all the parameters used in Section 3 and 4.

	Notation	Definition
$D = D_M$	$\hat{P}^r$	$(x_1^r, s_{in} - x_1^r)$
	$\hat{x}(\cdot)$	$z(\cdot, \hat{P}^r, D_M)$
	$\hat{\gamma}^+$	$\{\hat{x}(t)\ ;\ t \geq 0\}$
	$\hat{t}$	$\inf\{t > 0\ ;\ \hat{x}_1(t) < \bar{x}_1\}$
	δ	$\hat{x}_2(\hat{t})$
$D = D_m$	$\check{x}_{2,0}$	$\check{x}_2(0)$ with $\check{x}_2(0) < \min(x_1^m - x_1^r, \delta)$
	$\check{P}^r$	$(x_1^r, \check{x}_{2,0})$
	$\check{x}(\cdot)$	$z(\cdot, \check{P}^r, D_m)$
	$\check{\gamma}^-$	$\{\check{x}(t)\,;\, t \leq 0\}$
	$\check{\gamma}$	$\{\check{x}(t)\,;\, \forall t\}$
	$\bar{x}_2$	$W^u(E_1^m) \cap \{x_1 = x_1^r\}$
	η	$x_1^m - x_1^r - \check{x}_{2,0}$
	P_ε	$(\underline{x}_1, \varepsilon)$ intersection between $\hat{\gamma}^+$ and $\check{\gamma}^-$
	$\hat{T}$	such that $\hat{x}(\hat{T}) = P_\varepsilon$
	$\check{T}$	such that $\check{x}(-\check{T}) = P_\varepsilon$
	$\mathcal{E}$	$(0, \bar{x}_1] \times (0, \varepsilon]$
	$\check{Q}^r$	$(x_1^r, \underline{x}_2)$ intersection, different from $\check{P}^r$, between $\check{\gamma}$ and $\{x_1 = x_1^r\} \cap \mathcal{S}$

Appendix 2: Petrov's condition

We recall here a result about the continuity of the first entry time function (see Theorem 8.25 in [11]), that is stated here for a non-controlled dynamics. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping of class C^1 with linear growth, and denote by $y(\cdot, y_0)$ the unique solution of the Cauchy problem:

$$\left\{ \begin{array}{ll} \dot{y} & = g(y), \\ y(0) & = y_0, \end{array} \right.$$

defined over $\mathbb{R}_+$. Hereafter, we are given a non-empty compact subset $\mathcal{T}$ of $\mathbb{R}^n$ and for $y \in \mathcal{T}$, the set $N_{\mathcal{T}}^p(y)$ stands for the *proximal normal cone* to the set $\mathcal{T}$ at the point y (see [10]). The standard inner product is written $a \cdot b$ for $a, b \in \mathbb{R}^n$, and $\|a\|$ denotes the euclidean norm of the vector a.

Theorem 4.1. *Suppose that the Petrov condition*

$$\exists \gamma < 0,\ \forall y \in \partial\mathcal{T},\ \forall \nu \in N_{\mathcal{T}}^p(y) \setminus \{0\}, \quad g(y) \cdot \nu < \gamma \|\nu\|, \tag{4.8}$$

is fulfilled. Then, the first entry time function

$$R(y_0) := \inf \{t \geq 0\ ;\ y(t, y_0) \in \mathcal{T}\}, \quad y_0 \in \mathbb{R}^n,$$

is Lipschitz continuous in its open domain $\{y_0 \in \mathbb{R}^n\ ;\ R(y_0) < +\infty\}$.

If $\mathcal{T}$ is convex, the set $N_{\mathcal{T}}^p(y)$ coincides with the *convex normal cone* (see, *e.g.*, [10]) defined for $y \in \mathcal{T}$ as:

$$N_{\mathcal{T}}(y) := \{q \in \mathbb{R}^n\ ;\ q \cdot (z - y) \leq 0, \quad \forall z \in \mathcal{T}\}.$$

Appendix 3: Asymptotically periodic systems

We recall first the definition of asymptotically periodic semi-flows in $\mathbb{R}^n$.

Definition 4.2. *A non-autonomous semiflow* Φ*:* $\{(t,s);\ 0 \le s \le t < \infty\} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ *is asymptotically periodic with limit* ω*-periodic semi-flow* $T(t)$*:* $\mathbb{R}^n \mapsto \mathbb{R}^n$*,* $t \ge 0$ *if*

$$\Phi(t_j + n_j\omega, n_j\omega, x_j) \to T(t)x,\ j \to \infty$$

for any sequences $t_j \to t$*,* $n_j \to \infty$*,* $x_j \to x$ *when* $j \to \infty$*, with* x*,* x_j *in* $\mathbb{R}^n$*.*

The following result can be found in [27] (Theorem 3.1).

Theorem 4.3. *Let* Φ*:* $\{(t,s);\ 0 \le s \le t < \infty\} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ *be an asymptotically periodic semi-flow with limit* ω*-periodic semi-flow* $T(t)$*:* $\mathbb{R}^n \mapsto \mathbb{R}^n$*,* $t \ge 0$*. Denote* $T_n(x) = \phi(n\omega, 0, x)$ *and* $S(x) = T(\omega)x$*,* $n \ge 0$*,* $x \in \mathbb{R}^n$*. If* A_0 *is a compact subset of* $\mathbb{R}^n$ *invariant by the semi-flow* S *and* $y \in \mathbb{R}^n$ *is such that* $d(T_n(t), A_0) \to 0$ *when* $n \to \infty$ *then*

$$\lim_{t \to +\infty} d(\Phi(t, 0, y), T(t)A_0) = 0.$$

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