# The ( $\leq 5$ ) -hypomorphy of digraphs up to complementation 

Aymen Ben Amira, Bechir Chaari, Jamel Dammak, Hamza Si Kaddour

## To cite this version:

Aymen Ben Amira, Bechir Chaari, Jamel Dammak, Hamza Si Kaddour. The ( $\leq 5$ ) -hypomorphy of digraphs up to complementation. Arab Journal of Mathematical Sciences, 2019, 25 (1), pp.1-16. 10.1016/j.ajmsc.2018.06.002 . hal-02067995

## HAL Id: hal-02067995

## https://hal.science/hal-02067995

Submitted on 21 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

@(®)

# The ( $\leq 5$ )-hypomorphy of digraphs up to complementation 

AYMEN BEN AMIRA ${ }^{a}$, BECHIR CHAARI ${ }^{a}$, JAMEL DAMMAK ${ }^{a}$, HAMZA SI KADDOUR ${ }^{b, *}$<br>${ }^{a}$ Department of Mathematics, Faculty of Sciences of Sfax, B.P. 1171, 3000 Sfax, Tunisia<br>${ }^{b}$ Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France


#### Abstract

Two digraphs $G=(V, E)$ and $G^{\prime}=\left(V, E^{\prime}\right)$ are isomorphic up to complementation if $G^{\prime}$ is isomorphic to $G$ or to the complement $\bar{G}:=\left(V,\left\{(x, y) \in V^{2}: x \neq y,(x, y) \notin E\right\}\right)$ of $G$. The boolean sum $G \dot{+} G^{\prime}$ is the symmetric digraph $U=(V, E(U))$ defined by $\{x, y\} \in E(U)$ if and only if $(x, y) \in E$ and $(x, y) \notin E^{\prime}$, or $(x, y) \notin E$ and $(x, y) \in E^{\prime}$. Let $k$ be a nonnegative integer. The digraphs $G$ and $G^{\prime}$ are ( $\leq k$ )-hypomorphic up to complementation if for every $t$-element subset $X$ of $V$, with $t \leq k$, the induced subdigraphs $G_{\lceil X}$ and $G_{\lceil X}^{\prime}$ are isomorphic up to complementation. The digraphs $G$ and $G^{\prime}$ are hereditarily isomorphic (resp. hereditarily isomorphic up to complementation) if for each subset $X$ of $V$, the induced subdigraphs $G_{\lceil X}$ and $G_{\lceil X}^{\prime}$ are isomorphic (resp. isomorphic up to complementation). Here, we give the form of the pair $\left\{G, G^{\prime}\right\}$ whenever $G$ and $G^{\prime}$ are two digraphs, ( $\leq 5$ )-hypomorphic up to complementation and such that the boolean sum $U:=G \dot{+} G^{\prime}$ and the complement $\bar{U}$ are both connected, and thus we deduce that $G$ and $G^{\prime}$ are hereditarily isomorphic up to complementation; we prove also that the value 5 is optimal. The case $U$ or $\bar{U}$ is not connected is studied in a forth come paper.


Keywords: Digraph, isomorphism, $k$-hypomorphy up to complementation, hereditary isomorphy up to complementation, boolean sum, symmetric digraph, tournament, indecomposability

2010 Mathematics Subject Classification: 05C50; 05C60

## 1 Introduction

In this paper, we study the reconstruction of digraphs up to complementation (definitions and notations are given in section 2). Ulam's reconstruction conjecture on digraphs [22], still unsolved for graphs, is well-known (see [2, 3]). Fraïssé made a related conjecture about relational structures. The case of binary relations was solved by Lopez [14, 15, 16], he showed that all binary relations are ( $\leq 6$ )-reconstructible. The case of ternary relations

[^0]was solved negatively by Pouzet [17]. On the other hand, Stockmeyer [20] showed that the tournaments are not, in general, ( -1 )-reconstructible, invalidating so the conjecture of Ulam for digraphs. In 1993, Hagendorf raised the $(\leq k)$-half-reconstruction problem for digraphs and solved it with Lopez [12, 13], they showed that the finite digraphs are ( $\leq 12$ )-half-reconstructible. In 1995, Boudabbous and Lopez [6] showed that the finite tournaments are $(\leq 7)$-half-reconstructible. This motivated, in 2013, M.Alzohairi, M.Bouaziz and Y.Boudabbous to introduce the concept of $(\leq k)$-hereditary reconstructibility of posets [1]. In 2015, Y.Boudabbous proposed the problem of $(\leq k)$-hereditarily reconstruction of digraphs. He solved this problem for tournaments with A.Boussaïri, A. Chaïchaâ and N. El Amri [5].

We say that a symmetric digraph $G$ is connected if for any distinct vertices $a$ and $b$ of $G$, there are vertices $a=x_{0}, x_{1}, \cdots, x_{m}=b$ of $G$, such that $x_{i-{ }_{G}} x_{i+1}$ for each $i \in\{0, \cdots, m-1\}$. Otherwise $G$ is said disconnected. A component of $G$ is a maximal connected subdigraph of $G$. Let $G=(V, E)$ and $G^{\prime}=\left(V, E^{\prime}\right)$ be two digraphs, 2-hypomorphic up to complementation. The boolean sum $G \dot{+} G^{\prime}$ of $G$ and $G^{\prime}$ is the symmetric digraph $U=(V, E(U))$ defined by $\{x, y\} \in E(U)$ if and only if $(x, y) \in E$ and $(x, y) \notin E^{\prime}$, or $(x, y) \notin E$ and $(x, y) \in E^{\prime}$. Clearly $\bar{U}=\bar{G} \dot{+} G^{\prime}$. Denote $\mathfrak{D}_{G, G^{\prime}}$ the binary relation on $V$ such that: for $x \in V, x \mathfrak{D}_{G, G^{\prime}} x$; and for $x \neq y \in V, x \mathfrak{D}_{G, G^{\prime}} y$ if there exists a sequence $x=x_{0}, x_{1}, \ldots, x_{m}=y$ of elements of $V$ satisfying $\left(x_{i}, x_{i+1}\right) \in E$ if and only if $\left(x_{i}, x_{i+1}\right) \notin E^{\prime}$, for each $i, 0 \leq i \leq m-1$. The relation $\mathfrak{D}_{G, G^{\prime}}$ is an equivalence relation called the difference relation, its classes are called difference classes, this relation was introduced by Lopez [14]. Then clearly $C$ is a connected component of $U:=G \dot{+} G^{\prime}$ if and only if $C$ is an equivalence class of $\mathfrak{D}_{G, G^{\prime}}$, and thus $\mathfrak{D}_{G, G^{\prime}}$ and $\mathfrak{D}_{\bar{G}, G^{\prime}}$ have only one class if and only if $U$ and $\bar{U}$ are connected. In 2003, Dammak [8] proved the following result.

Proposition 1.1 ([8]) Let $T$ and $T^{\prime}$ be two finite tournaments, ( $\leq 5$ )-hypomorphic up to complementation, and $U:=T \dot{+} T^{\prime}$. If $U$ and $\bar{U}$ are connected, then $T$ and $T^{\prime}$ are total orders.

In 1999, Ille raised the problem of the $(\leq k)$-reconstruction up to complementation of digraphs. The case of symmetric digraphs was solved by Dammak, Lopez, Pouzet and Si Kaddour [9, 10], they proved that, the symmetric digraphs on $v$ vertices are $t$ reconstructible up to complementation for every $4 \leq t \leq v-3$. In fact, the case $t=v-3$ was solved in [10] using the following result established by Pouzet, Si Kaddour and Trotignon [18].

Theorem 1.2 [18] If $G$ and $G^{\prime}$ are two symmetric digraphs, 3-hypomorphic up to complementation and $|V(G)| \geq 10$, then the connected components of $U:=G \dot{+} G^{\prime}$, or of its complement $\bar{U}$, are cycles of even length or paths.

We define the symmetric digraph $P_{n}$ in the following manner, $V\left(P_{n}\right)=\{0,1, \cdots, n-1\}$, and for $i \neq j \in\{0,1, \ldots, n-1\},\{i, j\}$ is an edge of $P_{n}$ when $|i-j|=1$. Thus $P_{n}:=0 \_\_1 \_\ldots \_n-2 \_n-1$. A path is a symmetric digraph isomorphic to $P_{n}$. A cycle is a symmetric digraph isomorphic to $C_{n}:=\left(V\left(P_{n}\right), E\left(P_{n}\right) \cup\{\{0, n-1\}\}\right)$ for some integer $n \geq 3$.


Figure 1: $C_{n}$
We define the digraph $\overrightarrow{P_{n}}$ by, for $i \neq j \in\{0,1, \ldots, n-1\}, i \longrightarrow_{\overrightarrow{P_{n}}} j$ when $j=i+1$. Thus $\overrightarrow{P_{n}}:=0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n-2 \longrightarrow n-1$. We call directed path or oriented path a digraph isomorphic to $\overrightarrow{P_{n}}$, and directed cycle or oriented cycle a digraph isomorphic to $\overrightarrow{C_{n}}:=\left(V\left(\overrightarrow{P_{n}}\right), E\left(\overrightarrow{P_{n}}\right) \cup\{(n-1,0)\}\right)$ for some integer $n \geq 3$.


Figure 2: $\overrightarrow{C_{n}}$
We define $\overrightarrow{P_{n}^{f}}$ (resp. $\overrightarrow{C_{n}^{f}}$ ) obtained from $\overrightarrow{P_{n}}$ (resp. $\overrightarrow{C_{n}}$ ) by switching the void pairs by the full pairs. Thus $\overrightarrow{P_{n}^{f}}=\overrightarrow{\left(\overrightarrow{P_{n}}\right)^{*}}$ and $\overrightarrow{C_{n}^{f}}=\overrightarrow{\left(\overrightarrow{C_{n}}\right)^{*}}$.

A total order is a tournament $T$ such that for $x, y, z \in V(T)$, if $x \longrightarrow_{T} y$ and $y \longrightarrow_{T} z$ then $x \longrightarrow_{T} z$. Given a total order $O=(V, E)$, for $x, y \in V, x<y$ means $x \longrightarrow_{o} y$. Thus, a total order on $n$ vertices can be denoted by $v_{0}<v_{1}<\cdots<v_{n-1}$.

Our main result is the following.
Theorem 1.3 Let $G$ and $G^{\prime}$ be two digraphs on the same set $V$ of $n \geq 4$ vertices such that $G$ and $G^{\prime}$ are $(\leq 5)$-hypomorphic up to complementation. Let $U:=G \dot{+} G^{\prime}$. If $U$ and $\bar{U}$ are connected, then $G^{\prime}$ and $G$ are hereditarily isomorphic up to complementation; more precisely one of the following holds:

1) $G$ and $G^{\prime}$ are two total orders.
2) $G \simeq \overrightarrow{P_{n}}$ or $G \simeq \overrightarrow{C_{n}}$, and $G^{\prime}=G^{*}$.
3) $G \simeq \overrightarrow{P_{n}}$ or $G \simeq \overrightarrow{C_{n}}$, and $G^{\prime}=\overrightarrow{G^{*}}$.
4) $G \simeq \overrightarrow{P_{n}^{f}}$ or $G \simeq \overrightarrow{C_{n}^{f}}$, and $G^{\prime}=G^{*}$.
5) $G \simeq \overrightarrow{P_{n}^{f}}$ or $G \simeq \overrightarrow{C_{n}^{f}}$, and $G^{\prime}=\overline{G^{*}}$.

In Proposition 3.5, we prove that the value 5 is optimal by giving two digraphs $G, G^{\prime}$, on the same vertex set $V$ with $|V| \geq 5$, which are ( $\leq 4$ )-hypomorphic up to complementation and not ( $\leq 5$ )-hypomorphic up to complementation, $U:=G \dot{+} G^{\prime}$ and $\bar{U}$ are connected but $G$ and $G^{\prime}$ are not isomorphic up to complementation, and thus not hereditarily isomorphic up to complementation.

From Theorem 1.3, we deduce trivially the following result for digraphs which is similar to Theorem 1.2.

Corollary 1.4 Let $G$ and $G^{\prime}$ be two digraphs on the same set $V$ of $n \geq 4$ vertices such that $G$ and $G^{\prime}$ are ( $\leq 5$ )-hypomorphic up to complementation and $U:=G \dot{+} G^{\prime}$. If $U$ and $\bar{U}$ are connected and $G$ is not a total order, then $U$ or $\bar{U}$ is a cycle or a path.

## 2 Definitions and notations

A directed graph or simply digraph $G$ consists of a finite and nonempty set $V$ of vertices together with a prescribed collection $E$ of ordered pairs of distinct vertices, called the set of the edges of $G$. Such a digraph is denoted by $(V(G), E(G))$ or simply ( $V, E)$. Given a digraph $G=(V, E)$, to each nonempty subset $X$ of $V$ associate the subdigraph $(X, E \cap(X \times X))$ of $G$ induced by $X$ denoted by $G_{\mid X}$. Given a proper subset $X$ of $V$, $G_{\mid V \backslash X}$ is also denoted by $G-X$, and by $G-v$ whenever $X=\{v\}$. With each digraph $G=(V, E)$ associate its dual $G^{*}=\left(V, E^{*}\right)$ and its complement $\bar{G}=(V, \bar{E})$ defined as follows. Given $x \neq y \in V,(x, y) \in E^{*}$ if $(y, x) \in E$, and $(x, y) \in \bar{E}$ if $(x, y) \notin E$.

Let $G=(V, E)$ be a digraph, for $x \neq y \in V, x \longrightarrow{ }_{G} y$ or $y \longleftarrow_{G} x$ (or simply $x \longrightarrow y$ if there is no confusion) means $(x, y) \in E$ and $(y, x) \notin E ; x_{-{ }_{G}} y$ (or simply $\left.x \__{-} y\right)$ means $(x, y) \in E$ and $(y, x) \in E ; x \ldots{ }_{G} y$ (or $x \ldots y$ or $\left.x{ }_{G} y\right)$ means $(x, y) \notin E$ and $(y, x) \notin E$. For $X, Y \subseteq V, X-{ }_{G} Y$ and $X \ldots{ }_{G} Y$ (or $X{ }_{G} Y$ ) are defined in the same way. If $X=\{x\}$ or $Y=\{y\}$, we can replace $X$ by $x$ and $Y$ by $y$.

Given a digraph $G=(V, E)$, two distinct vertices $x$ and $y$ of $G$ form a directed pair or oriented pair if either $x \longrightarrow_{G} y$ or $x \longleftarrow_{G} y$. Otherwise, $\{x, y\}$ is a neutral pair; it is full if $x ـ_{G} y$, and void if $x \ldots{ }_{G} y$. Two interesting types of digraphs are symmetric digraphs and tournaments. A digraph $G=(V, E)$ is a symmetric digraph or graph (resp. tournament) whenever for $x \neq y \in V, x-_{G} y$ or $x \ldots_{G} y$ (resp. $x \longrightarrow_{G} y$ or $y \longrightarrow_{G} x$ ). If $G=(V, E)$ is a graph, each edge $(x, y)$ of $G$ is identified with the pair $\{x, y\}$ and is called an edge of $G$. For instance, given a set $V,(V, \emptyset)$ is the empty graph on $V$ whereas $\left(V,[V]^{2}\right)$ is the complete graph on $V$, where $[V]^{2}$ is the set of pairs $\{x, y\}$ of distinct elements of $V$.

Given two digraphs $G=(V, E)$ and $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$, a bijection $f$ from $V$ onto $V^{\prime}$ is an isomorphism from $G$ onto $G^{\prime}$ provided that for any $x, y \in V,(x, y) \in E$ if and only if $(f(x), f(y)) \in E^{\prime}$. The digraphs $G$ and $G^{\prime}$ are isomorphic, which is denoted by $G \simeq G^{\prime}$, if there exists an isomorphism from one onto the other, otherwise $G \not \approx G^{\prime}$. A digraph $H$ embeds into $G$, or $H$ is embeddable in $G$, if $H$ is isomorphic to an induced subdigraph of $G$.

Given two digraphs $G$ and $G^{\prime}$ on the same vertex set $V$. They are equal up to complementation if $G^{\prime}=G$ or $G^{\prime}=\bar{G}$. Let $k$ be an integer with $0<k<|V|$, the digraphs $G$ and $G^{\prime}$ are $k$-hypomorphic (resp. ( $-k$ )-hypomorphic) if for every $k$-element (resp. ( $|V|-k$ )element) subset $X$ of $V$, the induced subdigraphs $G_{\lceil X}$ and $G_{\mid X}^{\prime}$ are isomorphic. The digraphs $G$ and $G^{\prime}$ are $(\leq k)$-hypomorphic if they are $t$-hypomorphic for each integer $t \leq k$. A digraph $G$ is $k$-reconstructible (resp. ( $-k$ )-reconstructible) if any digraph $k$-hypomorphic (resp. ( $-k$ )-hypomorphic) to $G$ is isomorphic to $G$. A digraph $G$ is $(\leq k)$-reconstructible if any digraph $(\leq k)$-hypomorphic to $G$ is isomorphic to $G$. The digraphs $G$ and $G^{\prime}$ are isomorphic up to complementation (resp. hemimorphic) if $G^{\prime}$ is isomorphic to $G$ or $\bar{G}$ (resp. to $G$ or $G^{*}$ ). The digraphs $G^{\prime}$ and $G$ are hereditarily isomorphic [19] if for each nonempty subset $X$ of $V$, the digraphs $G_{\mid X}$ and $G^{\prime}{ }_{\mid X}$ are isomorphic. They are hered-
itarily isomorphic up to complementation [4] if they are hereditarily isomorphic, or $G^{\prime}$ and $\bar{G}$ are hereditarily isomorphic. Let $k$ be a positive integer, the digraphs $G$ and $G^{\prime}$ are $k$-hypomorphic up to complementation (resp. $k$-hemimorphic) if for every $k$-element subset $X$ of $V$, the induced subdigraphs $G_{\mid X}$ and $G_{{ }_{X X}}$ are isomorphic up to complementation (resp. hemimorphic). The digraphs $G$ and $G^{\prime}$ are ( $\leq k$ )-hypomorphic up to complementation (resp. $(\leq k)$-hemimorphic) if they are $t$-hypomorphic up to complementation (resp. $t$-hemimorphic) for each integer $t \leq k$. A digraph $G$ is $k$-reconstructible up to complementation (resp. $k$-half-reconstructible) if any digraph $k$-hypomorphic up to complementation (resp. $k$-hemimorphic) to $G$ is isomorphic up to complementation (resp. hemimorphic) to $G$. A digraph $G$ is $(\leq k)$-reconstructible up to complementation (resp. ( $\leq k$ )-half-reconstructible) if any digraph $(\leq k)$-hypomorphic up to complementation (resp. ( $\leq k$ )-hemimorphic) to $G$ is isomorphic up to complementation (resp. hemimorphic) to $G$.
A 3-cycle is a tournament isomorphic to $\overrightarrow{C_{3}}:=(\{0,1,2\},\{(0,1),(1,2),(2,0)\})$. A flag is a digraph hemimorphic to $(\{0,1,2\},\{(0,1),(1,2),(2,1)\})$. A peak is a digraph hemimorphic to $(\{0,1,2\},\{(0,1),(0,2),(1,2),(2,1)\})$ or to $(\{0,1,2\},\{(0,1),(0,2)\})$. Let $G$ be a digraph, the positive degree (resp. negative degree) of a vertex $x$ of $G$, denoted $d_{G}^{+}(x)$ (resp. $d_{G}^{-}(x)$ ), is the number of $y \in V(G)$ such that $x \longrightarrow_{G} y$ (resp. $y \longrightarrow_{G} x$ ). Notice that, here, $d_{G}^{+}(x)$ (resp. $\left.d_{G}^{-}(x)\right)$ is not the outdegree (resp. indegree) of the vertex $x$. The type of $G$ is $\left(e, e^{\prime}\right)$ where $e$ and $e^{\prime}$ are respectively the number of full pairs of $G$ and $\bar{G}$. Let $G=(V, E)$ and $G^{\prime}=\left(V, E^{\prime}\right)$ be two digraphs and $a, b \in V$. We say that $\{a, b\}$ have the same character in $G$ and $G^{\prime}$ if and only if $G_{\{\{a, b\}} \simeq G_{\uparrow\{a, b\}}^{\prime}$.

Let $G=(V, E)$ be a graph, the degree of a vertex $x$ of $G$, denoted $d_{G}(x)$, is the number of $y \in V(G)$ such that $x-_{G} y$.

## 3 The Gallai decomposition Theorem

Given a digraph $G=(V, E)$, a subset $I$ of $V$ is an interval of $G$ if for every $x \in V \backslash I$ either $x \longrightarrow_{G} I$ or $x \longleftarrow_{G} I$ or $x-_{G} I$ or $x \ldots_{G} I$. For instance, $\emptyset, V$ and $\{x\}$ (where $x \in V)$ are intervals of $G$, called trivial intervals. A digraph is indecomposable if all its intervals are trivial, otherwise it is decomposable.

The graph $Q_{n}$ (see Figure 3) is defined in the following manner. For $i \neq j \in$ $\{0,1, \ldots, n-1\},\{i, j\}$ is an edge of $Q_{n}$ whenever either $i, j \in\{0,1, \ldots, n-3\}$ and $|i-j|=1$ or $\{i, j\}=\{n-2, \ell\}$, where $\ell \in\{0,1, \ldots, n-4\} \cup\{n-1\}$.


Figure 3: $Q_{n}$.
Theorem 3.1 [7] Let $S=(V, E)$ be an indecomposable graph with $|V| \geq 4$. Let $W$ denotes the set of $x \in V$ such that there is a subset $X$ of $V$ satisfying $S_{\mid X}$ is isomorphic
to $P_{4}$ and $x \in X$. We have: $|V \backslash W| \leq 1$. Furthermore, if $V \backslash W=\{x\}$, then there are $a$ subset $X$ of $V$ containing $x$ and an isomorphism $f$ from $S_{\mid X}$ onto $Q_{5}$ such that $f(x)=v_{0}$.

Theorem 3.2 [7] Let $S=(V, E)$ be an indecomposable graph with $|V| \geq 5$. For $a \neq b \in$ $V$, there is a subset $X$ of $V$ satisfying: $a, b \in X$ and there is an isomorphism $f$ from $S_{\mid X}$ or $\bar{S}_{\mid X}$ onto $P_{k}$ or $Q_{k}$, where $k \geq 5$, such that $f(\{a, b\})=\{0, k-1\}$.

We begin with a well-known property of the intervals. Given a digraph $G=(V, E)$, if X and Y are disjoint intervals of G , then $X \longrightarrow_{G} Y$, or $X \longleftarrow_{G} Y$, or $X-_{G} Y$, or $X \ldots{ }_{G} Y$. This property leads to consider interval partitions of $G$, that is, partitions of $V$, all the elements of which are intervals of $G$. The elements of such a partition $P$ become the vertices of the quotient $G / P=(P, E / P)$ of $G$ by $P$ defined as follows: given $X \neq Y \in P$, $(X, Y) \in E / P$ if $(x, y) \in E$ for $x \in X$ and $y \in Y$. Given a digraph $G=(V, E)$, a subset $X$ of $V$ is a strong interval [11] of $G$ provided that $X$ is an interval of $G$ and for each interval $Y$ of $G$, we have: if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. For $|V| \geq 2$, the family of the maximal strong intervals under inclusion which are distinct from $V$ is denoted by $P(G)$. The family $P(G)$ constitues an interval partition of $V$. Now we state the Gallai decomposition theorem.

Theorem 3.3 [11] Given a digraph $G=(V, E)$, with $|V| \geq 2$. The corresponding quotient $G / P(G)$ is a complete digraph or an empty digraph or a total order or an indecomposable digraph with at least 3 vertices.

The inverse operation of the quotient is the lexicographic sum defined as follows: let $m$ be an integer, $m \geq 1, S=(\{0,1, \ldots, m-1\}, E)$ be a digraph and for $i<m, G_{i}=\left(V_{i}, E_{i}\right)$ be a digraph such that the $V_{i}$ 's are nonempty and pairwise disjoint. The lexicographic sum over $S$ of the $G_{i}$ 's or simply the S -sum of the $G_{i}$ 's, is the digraph denoted by $S\left(G_{0}, G_{1}, \ldots, G_{m-1}\right)$ and defined on the union of the $V_{i}$ 's as follows: given $x \in V_{i}$ and $y \in V_{j}$, where $i, j \in\{0,1, \ldots, m-1\},(x, y)$ is an edge of $S\left(G_{0}, G_{1}, \ldots, G_{m-1}\right)$ if either $i=j$ and $(x, y) \in E_{i}$ or $i \neq j$ and $(i, j) \in E$ : this digraph replaces each vertex $i$ of $S$ by $G_{i}$. We say that the vertex $i$ of $S$ is dilated by $G_{i}$.

From Theorem 3.3, we have immediately this result.
Corollary 3.4 Given a graph $G=(V, E)$. If $G$ and $\bar{G}$ are connected, if and only if $G=S\left(G_{0}, G_{1}, \ldots, G_{m-1}\right)$, where $S$ is an indecomposable graph with at least 4 vertices and $G_{i}$ is a graph for each $i \in\{0,1, \ldots, m-1\}$.

The following result shows the optimality of the value 5 in Theorem 1.3.
Proposition 3.5 Let $A_{3}:=\left\{\left\{a_{0}, b_{0}, c_{0}\right\},\left\{\left(a_{0}, b_{0}\right),\left(b_{0}, c_{0}\right),\left(c_{0}, a_{0}\right)\right\}\right\}$. Let $G$ (resp. $\left.G^{\prime}\right)$ be the digraph obtained from $\overrightarrow{P_{n}}\left(\right.$ resp. $\left.\left(\overrightarrow{P_{n}}\right)^{*}\right)$ by dilating the vertex 0 by $A_{3}$. Let $U:=G \dot{+} G^{\prime}$. Then $G$ and $G^{\prime}$ are $(\leq 4)$-hypomorphic up to complementation, not $(\leq 5)$ hypomorphic up to complementation, $U$ and $\bar{U}$ are connected, but $G$ and $G^{\prime}$ are not isomorphic up to complementation, and thus not hereditarily isomorphic up to complementation.

Proof. Note that $A_{3}$ is an oriented cycle isomorphic to $C_{3}$. The graph $U$ is obtained from $P_{n}$ by dilating the vertex 0 by the empty graph with vertex set $\left\{a_{0}, b_{0}, c_{0}\right\}$. By Corollary 3.4, $U$ and $\bar{U}$ are connected. Clearly $G$ and $G^{\prime}$ are ( $\leq 4$ )-hypomorphic up to complementation. The subdigraphs $G_{\mid\left\{a_{0}, b_{0}, c_{0}, 1,2\right\}}$ and $G_{\mid\left\{a_{0}, b_{0}, c_{0}, 1,2\right\}}^{\prime}$ are not isomorphic because $d_{G_{\left\{\left\{a_{0}, b_{0}, c_{0}, 1,2\right\}\right.}^{\prime}}^{+}(1)=3$ but $d_{G_{\left\{\left\{a_{0}, b_{0}, c_{0}, 1,2\right\}\right.}}^{+}(x) \leq 2$ for all vertex $x$. The subdigraphs $\bar{G}_{\left\{\left\{a_{0}, b_{0}, c_{0}, 1,2\right\}\right.}$ and $G_{\left\{\left\{a_{0}, b_{0}, c_{0}, 1,2\right\}\right.}^{\prime}$ are not isomorphic because there are full edges in $\bar{G}_{\backslash\left\{a_{0}, b_{0}, c_{0}, 1,2\right\}}$ whereas there are none in $G_{\left\lceil\left\{a_{0}, b_{0}, c_{0}, 1,2\right\}\right.}^{\prime}$. Thus $G$ and $G^{\prime}$ are not $(\leq 5)$ hypomorphic up to complementation. As $d_{G^{\prime}}^{+}(1)=3$ and there is no vertex $x$ in $G$ of degree 3 , and there are full edges in $\bar{G}$ whereas there are none in $G^{\prime}$, then $G$ and $G^{\prime}$ are not isomorphic up to complementation. Thus $G$ and $G^{\prime}$ are not hereditarily isomorphic up to complementation.

## 4 Preliminary results

Theorem 4.1 [21] Let $G$ be a graph. If $G$ and $\bar{G}$ are connected then $G$ embeds a $P_{4}$.
Remark 4.2 Let $G$ and $G^{\prime}$ be two digraphs on the same set $V$ such that $G$ and $G^{\prime}$ are ( $\leq 3$ )-hypomorphic up to complementation. Let $U:=G \dot{+} G^{\prime}$ and $a, b, c \in V$. If $G_{\mid\{a, b, c\}}$ is a peak or a flag, then $U_{\{\{a, b, c\}}$ is a complete or an empty graph.

Lemma 4.3 Let $G$ and $G^{\prime}$ be two digraphs on the same set $V$ such that $G$ and $G^{\prime}$ are ( $\leq 3$ )-hypomorphic up to complementation. Let $U:=G \dot{+} G^{\prime}$ and $a, b, c \in V$.

1) If $E\left(U_{\mid\{a, b, c\}}\right)$ or $E\left(\bar{U}_{\mid\{a, b, c\}}\right)$ is the set $\{\{a, b\},\{b, c\}\}$, then $\{a, b\}$ is an oriented pair in $G$ if and only if $\{b, c\}$ is an oriented pair in $G$.
2) If $E\left(U_{\mid\{a, b, c\}}\right)$ or $E\left(\bar{U}_{\{\{a, b, c\}}\right)$ is the set $\{\{a, b\}\}$ and $\{a, b\}$ is an oriented pair in $G$, then $\{a, b\}$ is an interval of $G_{\mid\{a, b, c\}}$ and $G_{\mid\{a, b, c\}}^{\prime}$.
3) If $E\left(U_{\uparrow\{a, b, c\}}\right)$ or $E\left(\bar{U}_{\mid\{a, b, c\}}\right)$ is the set $\{\{a, b\}\}$ and $\{a, b\}$ is a neutral pair in $G$, then $\{a, b\}$ is not an interval of $G_{\{\{a, b, c\}}$, and $\{b, c\}$ is an oriented pair in $G$ if and only if $\{a, c\}$ is an oriented pair in $G$. Moreover if $c \longrightarrow_{G} a\left(\right.$ resp. $c-_{G} a$ ) then $b \longrightarrow_{G} c$ (resp. $c \ldots{ }_{G} b$ ).

Proof. 1) By contradiction. Without loss of generality (W.l.o.g.), we assume that $a \longrightarrow_{G} b$ and $b-_{G} c$, then $a \longleftarrow_{G^{\prime}} b$ and $b \ldots{ }_{G^{\prime}} c$. If $\{a, c\}$ is an oriented pair in $G$ not reversed in $G^{\prime}$, then $G_{\lceil\{a, b, c\}}^{\prime} \not 千 G_{\mid\{a, b, c\}}$ and $G_{\mid\{a, b, c\}}^{\prime} \nsim \bar{G}_{\mid\{a, b, c\}}$ because exactly one of $G_{\mid\{a, b, c\}}$ and $G_{\{\{a, b, c\}}^{\prime}$ is a peak, which contradicts the 3-hypomorphy up to complementation. If $\{a, c\}$ is a neutral pair in $G$ not reversed in $G^{\prime}$, then $G_{\mid\{a, b, c\}}^{\prime} \nsim G_{\{\{a, b, c\}}$ and $G_{\lceil\{a, b, c\}}^{\prime} \nsim \bar{G}_{\{\{a, b, c\}}$ because exactly one of $G_{\mid\{a, b, c\}}$ and $G_{\lceil\{a, b, c\}}^{\prime}$ is a flag, which contradicts the 3-hypomorphy up to complementation.
2) W.l.o.g, we assume that $E\left(U_{\mid\{a, b, c\}}\right)=\{\{a, b\}\}$ Then $E\left(\bar{U}_{\mid\{a, b, c\}}\right)=\{\{a, c\},\{b, c\}\}$ and $\bar{U}=G \dot{+} \overline{G^{\prime}}$. We can assume that $a \longrightarrow_{G} b$, then $a \longrightarrow_{\overline{G^{\prime}}} b$.

- Case 1. $\{b, c\}$ is an oriented pair in $G$.
W.l.o.g we assume $b \longrightarrow_{G} c$, thus $b \longleftarrow_{\overline{G^{\prime}}} c$. Since $a \longrightarrow_{G} b, b \longrightarrow_{G} c$ and $\{a, c\} \longrightarrow_{\bar{G}^{\prime}} b$, from the 3-hypomorphic up to complementation we have $a \longrightarrow_{G} c$ and the conclusion follows.
- Case 2. $\{b, c\}$ is not an oriented pair in $G$.
W.l.o.g we can assume $b ـ_{G} c$, thus $b \cdots_{\bar{G}^{\prime}} c$. From 1) of this lemma, $\{a, c\}$ is a neutral pair in $G$. Since $G$ and $\overline{G^{\prime}}$ are 3-hypomorphic up to complementation, $a-{ }_{G} c$ and the conclusion follows.

3) We have $E\left(U_{\lceil\{a, b, c\}}\right)$ or $E\left(\bar{U}_{\mid\{a, b, c\}}\right)=\{\{a, b\}\}$ and $\{a, b\}$ is a neutral pair in $\bar{G}$.
W.l.o.g, we can assume that $E\left(U_{\{\{a, b, c\}}\right)=\{\{a, b\}\}$ and $a-_{G} b$, so $a \ldots_{G^{\prime}} b$.

- Case 1. $\{a, c\}$ is an oriented pair in $G$ not reversed in $G^{\prime}$.
W.l.o.g, we assume that $a \longrightarrow_{G} c$, so $a \longrightarrow{ }_{G^{\prime}} c$. We have $U_{\{\{a, b, c\}}$ is neither a complete graph nor an empty graph, so from Remark 4.2, each of $G_{\mid\{a, b, c\}}$ and $G_{\mid\{a, b, c\}}^{\prime}$ is neither a peak nor a flag, so $b \longleftarrow_{G} c$ and $b \longleftarrow_{G^{\prime}} c$.
- Case 2. $\{a, c\}$ is a neutral pair in $G$ not reversed in $G^{\prime}$.
W.l.o.g, we assume that $a-_{G} c$, so $a-_{G^{\prime}} c$. As $a-_{G}\{b, c\}$ and $a-_{G^{\prime}} c$ and $a \ldots_{G^{\prime}}$, then the 3-hypomorphy up to complementation applied to $G_{\{\{a, b, c\}}$ gives $b \ldots{ }_{G} c$, so $b \ldots{ }_{G^{\prime}} c$.
In the two cases we have $\{a, b\}$ is not an interval of $G_{\backslash\{a, b, c\}}$.
Lemma 4.4 Let $G$ and $G^{\prime}$ be two digraphs on the same vertex set $V$ such that $G$ and $G^{\prime}$ are $(\leq 3)$-hypomorphic up to complementation, and $U:=G \dot{+} G^{\prime}$. Let $n \geq 3$ be an integer, $X:=\left\{v_{0}, v_{1}, \ldots, v_{n-1}\right\} \subset V$ and $x \in V \backslash X$.
We assume that $U_{\mid X \cup\{x\}}=x \_v_{0-\_} v_{1}-\cdots-v_{n-1}$.

1) If $G_{\mid X}=v_{0} \longrightarrow v_{1} \longrightarrow \cdots \longrightarrow v_{n-1}$, then $G_{\mid X \cup\{x\}}=x \longrightarrow v_{0} \longrightarrow v_{1} \longrightarrow \cdots \longrightarrow v_{n-1}$ and $G_{\lceil X \cup\{x\}}^{\prime}=G_{\lceil X \cup\{x\}}^{*}$.
2) If $G_{\mid X}=\overrightarrow{P_{n}^{f}}$, then $G_{\mid X \cup\{x\}}$ is isomorphic to $\overrightarrow{P_{n+1}^{f}}$ by an isomorphism $f$ such that $f\left(v_{i}\right)=i+1$ for each $i \in\{0, \ldots, n-1\}$ and $f(x)=0$, and $G_{\lceil X \cup\{x\}}^{\prime}=G_{\upharpoonright X \cup\{x\}}^{*}$.

Proof. 1) We have $E\left(U_{\mid\left\{x, v_{0}, v_{1}\right\}}\right)=\left\{\left\{x, v_{0}\right\},\left\{v_{0}, v_{1}\right\}\right\}$ and $v_{0} \longrightarrow_{G} v_{1}$, then 1) of Lemma 4.3 gives $\left\{x, v_{0}\right\}$ is an oriented pair in $G$, reversed in $G^{\prime}$, let $j \in\{2,3 \ldots n-1\}$, we have $E\left(U_{\mid\left\{x, v_{0}, v_{j}\right\}}\right)=\left\{\left\{x, v_{0}\right\}\right\}$, then 2) of Lemma 4.3 applied to $\left\{x, v_{0}, v_{j}\right\}$ gives that $\left\{x, v_{0}\right\}$ is an interval of $G_{\backslash\left\{x, v_{0}, v_{j}\right\}}$. As $v_{0} \ldots_{G} v_{j}$, thus $x \ldots{ }_{G} v_{j}$ and $x \ldots_{G^{\prime}} v_{j}$. We have $U_{\mid\left\{x, v_{1}, v_{2}\right\}}=x \ldots v_{1} \ldots v_{2}, x \ldots{ }_{G} v_{2}$ and $v_{1} \longrightarrow_{G} v_{2}$, then 2) of Lemma 4.3 gives $x \ldots{ }_{G} v_{1}$ and $x \ldots_{G^{\prime}} v_{1}$. As $v_{0-{ }_{U}} v_{1}$ and $x \ldots_{U} v_{1}$ then, from Remark 4.2, $G_{\left\lceil\left\{x, v_{0}, v_{1}\right\}\right.}$ is not a peak, thus $x \longrightarrow{ }_{G} v_{0}$ and $x \longleftarrow{ }_{G^{\prime}} v_{0}$. Then, $G_{\{X \cup\{x\}}=x \longrightarrow v_{0} \longrightarrow v_{1} \longrightarrow \cdots \longrightarrow v_{n-1}$ and $G_{\mid X \cup\{x\}}^{\prime}=G_{\lceil X \cup\{x\}}^{*}$.
2) The proof is similar to that of first assertion.

Lemma 4.5 Let $G$ and $G^{\prime}$ be two digraphs on the same vertex set $V$ such that $G$ and $G^{\prime}$ are $(\leq 4)$-hypomorphic up to complementation, and $U:=G \dot{+} G^{\prime}$. Let $X:=\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\} \subset V$. If $U_{\mid X}=v_{0}-v_{1} \_v_{2} \_v_{3}$, we have :

1) If $\left\{v_{0}, v_{1}\right\}$ is a neutral pair in $G$, then

$$
\left\{G_{\lceil X}, G_{\lceil X}^{\prime}\right\}=\left\{H, \overline{H^{*}}\right\} \text { or }\left\{G_{\lceil X}, G_{\lceil X}^{\prime}\right\}=\left\{H^{*}, \bar{H}\right\},
$$

where $H:=v_{1} \longrightarrow v_{3} \longrightarrow v_{0} \longrightarrow v_{2}$.
2) If $\left(v_{0} \longrightarrow_{G} v_{1}\right.$ and $\left.v_{1} \longrightarrow{ }_{G} v_{2}\right)$ or $\left(v_{0} \longleftarrow{ }_{G} v_{1}\right.$ and $\left.v_{1} \longleftarrow{ }_{G} v_{2}\right)$, then

$$
\left\{G_{\lceil X}, G_{\lceil X}^{\prime}\right\}=\left\{\overrightarrow{P_{4}},\left(\overrightarrow{P_{4}}\right)^{*}\right\} \text { or }\left\{G_{\lceil X}, G_{\lceil X}^{\prime}\right\}=\left\{\overrightarrow{P_{4}^{f}},\left(\overrightarrow{P_{4}^{f}}\right)^{*}\right\}
$$

3) If $\left(v_{0} \longrightarrow{ }_{G} v_{1}, v_{1} \longleftarrow{ }_{G} v_{2}\right)$ or $\left(v_{0} \longleftarrow{ }_{G} v_{1}\right.$ and $\left.v_{1} \longrightarrow_{G} v_{2}\right)$, then

$$
\begin{gathered}
\left\{G_{\lceil X}, G_{\lceil X}^{\prime}\right\}=\left\{v_{0}<v_{2}<v_{1}<v_{3}, v_{1}<v_{0}<v_{3}<v_{2}\right\} \\
\text { or }\left\{G_{\lceil X}, G_{\lceil X}^{\prime}\right\}=\left\{v_{2}<v_{3}<v_{0}<v_{1}, v_{3}<v_{1}<v_{2}<v_{0}\right\} .
\end{gathered}
$$

Proof. 1) As $\left\{v_{0}, v_{1}\right\}$ is a neutral pair in $G$, w.l.o.g, we assume that $v_{0}-{ }_{G} v_{1}$. Then $v_{0} \ldots{ }_{G^{\prime}} v_{1}$. We have $E\left(U_{\left\{\left\{v_{0}, v_{1}, v_{2}\right\}\right.}\right)=\left\{\left\{v_{0}, v_{1}\right\},\left\{v_{1}, v_{2}\right\}\right\}$ and $v_{0-{ }_{G}} v_{1}$, so 1$)$ of Lemma 4.3 applied to $\left\{v_{0}, v_{1}, v_{2}\right\}$ gives $\left\{v_{1}, v_{2}\right\}$ is a neutral pair in $G$ reversed in $G^{\prime}$. We have $E\left(U_{\left\{\left\{v_{1}, v_{2}, v_{3}\right\}\right.}\right)=\left\{\left\{v_{1}, v_{2}\right\},\left\{v_{2}, v_{3}\right\}\right\}$, so 1$)$ of Lemma 4.3 applied to $\left\{v_{1}, v_{2}, v_{3}\right\}$ gives $\left\{v_{2}, v_{3}\right\}$ is a neutral pair in $G$ reversed in $G^{\prime}$. According to the nature of the pair $\left\{v_{1}, v_{2}\right\}$, we have the following cases:

- Case 1. $v_{1-{ }_{G}} v_{2}$.

Then $v_{1} \ldots_{G^{\prime}} v_{2}$. We have $v_{0} \ldots_{U} v_{2}$, then the 3 -hypomorphy up to complementation applied to $\left\{v_{0}, v_{1}, v_{2}\right\}$ gives $\left\{v_{0}, v_{2}\right\}$ is an oriented pair in $G$ not reversed in $G^{\prime}$. We assume that $v_{0} \longrightarrow_{G} v_{2}$ and $v_{0} \longrightarrow{ }_{G^{\prime}} v_{2}$ (resp. $v_{0} \longleftarrow_{G} v_{2}$ and $v_{0} \longleftarrow_{G^{\prime}} v_{2}$ ). As $U_{\mid\left\{v_{0}, v_{2}, v_{3}\right\}}=$ $v_{0} \ldots v_{2} \_v_{3}$, then 3) of Lemma 4.3 gives $v_{0} \longleftarrow{ }_{G} v_{3}$ and $v_{0} \longleftarrow{ }_{G^{\prime}} v_{3}$ (resp. $v_{0} \longrightarrow_{G} v_{3}$ and $\left.v_{0} \longrightarrow{ }_{G^{\prime}} v_{3}\right)$. As $U_{\left\{\left\{v_{0}, v_{1}, v_{3}\right\}\right.}=v_{3} \ldots v_{0}-v_{1}$, then 3) of Lemma 4.3 gives $v_{1} \longrightarrow_{G} v_{3}$ and $v_{1} \longrightarrow{ }_{G^{\prime}} v_{3}$ (resp. $v_{1} \longleftarrow_{G} v_{3}$ and $v_{1} \longleftarrow_{G^{\prime}} v_{3}$ ). Since $v_{1-{ }_{U}} v_{2}$ and $v_{1} \ldots_{U} v_{3}$, from Remark 4.2, $G_{\left\lceil\left\{v_{1}, v_{2}, v_{3}\right\}\right.}$ is not a flag, so $v_{2-{ }_{G}} v_{3}$ and $v_{2} \ldots_{G^{\prime}} v_{3}$. Then $G_{\left\{\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}^{\prime}=H$ and $G_{\mid\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}=\overline{H^{*}}\left(\right.$ resp. $G_{\left\lceil\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}=\bar{H}$ and $\left.G^{\prime}{ }_{\left\{\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}=H^{*}\right)$.

- Case 2. $v_{1} \cdots{ }_{G} v_{2}$.

Then $v_{1-{ }_{G}} v_{2}$. As $v_{0} \ldots_{U} v_{2}$ and $v_{1-—_{U}} v_{2}$ then, from Remark 4.2, $G_{\backslash\left\{v_{0}, v_{1}, v_{2}\right\}}$ is not a flag, so $\left\{v_{0}, v_{2}\right\}$ is a neutral pair in $G$ not reversed in $G^{\prime}$. W.l.o.g. we can assume that $v_{0}-{ }_{G} v_{2}$, so $v_{0-{ }_{G}{ }^{\prime}} v_{2}$. Since $E\left(U_{\mid\left\{v_{0}, v_{2}, v_{3}\right\}}\right)=\left\{\left\{v_{2}, v_{3}\right\}\right\}$ and $\left\{v_{2}, v_{3}\right\}$ is a neutral pair in $G$, then 3$)$ of Lemma 4.3 gives $v_{0} \ldots_{G} v_{3}$ and $v_{0} \cdots_{G^{\prime}} v_{3}$. We have $v_{0} \ldots_{G^{\prime}}\left\{v_{1}, v_{3}\right\}, v_{0} \cdots_{G} v_{3}$ and $v_{0-{ }_{G}} v_{1}$, so the 3 -hypomorphy up to complementation applied to $\left\{v_{0}, v_{1}, v_{3}\right\}$ gives $v_{1-{ }_{G}} v_{3}$, so $v_{1-{ }_{G}} v_{3}$. We have $v_{1-{ }_{G^{\prime}}}\left\{v_{2}, v_{3}\right\}, v_{1} \cdots{ }_{G} v_{2}$ and $v_{1-{ }_{G}} v_{3}$, then the 3-hypomorphy up to complementation applied to $\left\{v_{1}, v_{2}, v_{3}\right\}$ gives $v_{2} \ldots{ }_{G^{\prime}} v_{3}$, so $v_{2}{ }_{G_{G}} v_{3}$. Then $G_{\upharpoonright\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}$ and $G_{\mid\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}^{\prime}$ have respectively the types (4,2) and (3,3), so $G_{\left\lceil\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}^{\prime} \nsim G_{\left\lceil\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}$ and $G_{\left\lceil\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}^{\prime} \not 千 \bar{G}_{\left\{\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}$, that contradict the 4 -hypomorphy up to complementation.
2) • Case 1. $v_{0} \longrightarrow_{G} v_{1}$ and $v_{1} \longrightarrow_{G} v_{2}$.

Then $v_{1} \longrightarrow_{G^{\prime}} v_{0}$ and $v_{2} \longrightarrow{ }_{G^{\prime}} v_{1}$. We have $v_{0} \ldots_{U} v_{2}$, if $\left\{v_{0}, v_{2}\right\}$ is an oriented pair in $G$, then one of the subdigraphs $G_{\mid\left\{v_{0}, v_{1}, v_{2}\right\}}$ and $G_{\left\{\left\{v_{0}, v_{1}, v_{2}\right\}\right.}^{\prime}$ is a 3 -cycle and the other is a total order of order 3, that contradict the 3-hypomorphy up to complementation, so $\left\{v_{0}, v_{2}\right\}$ is a neutral pair in $G$ not reversed in $G^{\prime}$, thus $G_{\left\{\left\{v_{0}, v_{1}, v_{2}\right\}\right.}=\overrightarrow{P_{3}}$ or $\overrightarrow{P_{3}^{f}}$, and $G_{\left\{\left\{v_{0}, v_{1}, v_{2}\right\}\right.}^{\prime}=G_{\left\lceil\left\{v_{0}, v_{1}, v_{2}\right\}\right.}^{*}$. As $U_{\left\lceil\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}$ is a $P_{4}$ then, from Lemma 4.4, $G_{\left\{\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}=\vec{P}_{4}$ or $\overrightarrow{P_{4}^{f}}$, and $G_{\uparrow\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}^{\prime}=G_{\uparrow\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}^{*}$.

- Case 2. $v_{0} \longleftarrow{ }_{G} v_{1}$ and $v_{1} \longleftarrow{ }_{G} v_{2}$.

Then $v_{0} \longrightarrow{ }_{G^{\prime}} v_{1}$ and $v_{1} \longrightarrow{ }_{G^{\prime}} v_{2}$. From Case 1, by exchanging the roles of $G$ and $G^{\prime}$, we have $G_{\left\lceil\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}^{\prime}=\overrightarrow{P_{4}}$ or $\overrightarrow{P_{4}^{f}}$, and $G_{\mid\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}=\left(G^{\prime}\right)_{\mid\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}^{*}$.
3) • Case 1. $v_{0} \longrightarrow_{G} v_{1}$ and $v_{1} \longleftarrow{ }_{G} v_{2}$.

Then $v_{0} \longleftarrow{ }_{G^{\prime}} v_{1}$ and $v_{1} \longrightarrow{ }_{G^{\prime}} v_{2}$. As $v_{1-\coprod_{U}} v_{2}$ and $v_{0} \ldots_{U} v_{2}$ then, from Remark 4.2, $G_{\left\{\left\{v_{0}, v_{1}, v_{2}\right\}\right.}$ is not a peak, so $\left\{v_{0}, v_{2}\right\}$ is an oriented pair in $G$ not reversed in $G^{\prime}$. We assume that $v_{0} \longrightarrow_{G} v_{2}$ and $v_{0} \longrightarrow_{G^{\prime}} v_{2}$ (resp. $v_{0} \longleftarrow_{G} v_{2}$ and $v_{0} \longleftarrow_{G^{\prime}} v_{2}$ ). We have $E\left(U_{\left\{\left\{v_{1}, v_{2}, v_{3}\right\}\right.}\right)=\left\{\left\{v_{1}, v_{2}\right\},\left\{v_{2}, v_{3}\right\}\right\}$ and $v_{1} \longleftarrow{ }_{G} v_{2}$, then 1) of Lemma 4.3 gives $\left\{v_{2}, v_{3}\right\}$ is an oriented pair in $G$ reversed in $G^{\prime}$, we have $E\left(U_{\mid\left\{v_{0}, v_{2}, v_{3}\right\}}\right)=\left\{\left\{v_{2}, v_{3}\right\}\right\}$, then 2) of Lemma 4.3 applied to $\left\{v_{0}, v_{2}, v_{3}\right\}$ gives $\left\{v_{2}, v_{3}\right\}$ is an interval of $G_{\mid\left\{v_{0}, v_{2}, v_{3}\right\}}$, so $v_{0} \longrightarrow{ }_{G} v_{3}$ and $v_{0} \longrightarrow{ }_{G^{\prime}} v_{3}$ (resp. $v_{0} \longleftarrow{ }_{G} v_{3}$ and $\left.v_{0} \longleftarrow{ }_{G^{\prime}} v_{3}\right)$. We have $E\left(U_{\left\{\left\{v_{0}, v_{1}, v_{3}\right\}\right.}\right)=\left\{\left\{v_{0}, v_{1}\right\}\right\}$ and $v_{0} \longrightarrow_{G} v_{1}$, then 2) of Lemma 4.3 applied to $\left\{v_{0}, v_{1}, v_{3}\right\}$ gives $\left\{v_{0}, v_{1}\right\}$ is an interval of $G_{\left\lceil\left\{v_{0}, v_{1}, v_{3}\right\}\right.}$, so $v_{1} \longrightarrow_{G} v_{3}$ and $v_{1} \longrightarrow{ }_{G^{\prime}} v_{3}$ (resp. $v_{1} \longleftarrow_{G} v_{3}$ and $v_{1} \longleftarrow_{G^{\prime}} v_{3}$ ). We have $v_{1} \longleftarrow_{G} v_{2}, v_{1} \longrightarrow{ }_{G} v_{3}$ and $v_{1} \longrightarrow_{G^{\prime}}\left\{v_{2}, v_{3}\right\}$ (resp. $v_{1} \longrightarrow_{G^{\prime}} v_{2}, v_{1} \longleftarrow_{G^{\prime}} v_{3}$ and $\left.v_{1} \longleftarrow{ }_{G}\left\{v_{2}, v_{3}\right\}\right)$, then the 3 -hypomorphy up to complementation applied to $\left\{v_{1}, v_{2}, v_{3}\right\}$ gives $v_{2} \longrightarrow_{G} v_{3}$, so $v_{2} \longleftarrow{ }_{G^{\prime}} v_{3}$ (resp. $v_{2} \longleftarrow{ }_{G^{\prime}} v_{3}$, so $v_{2} \longrightarrow_{G} v_{3}$ ), thus $G_{\lceil X}=v_{0}<$ $v_{2}<v_{1}<v_{3}$ and $G_{\lceil X}^{\prime}=v_{1}<v_{0}<v_{3}<v_{2}$ (resp. $G_{\lceil X}=v_{2}<v_{3}<v_{0}<v_{1}$ and $\left.G_{\lceil X}^{\prime}=v_{3}<v_{1}<v_{2}<v_{0}\right)$.
$\bullet$ Case 2. $v_{0} \longleftarrow{ }_{G} v_{1}$ and $v_{1} \longrightarrow_{G} v_{2}$.
Then $v_{0} \longrightarrow{ }_{G^{\prime}} v_{1}$ and $v_{1} \longleftarrow{ }_{G^{\prime}} v_{2}$. From Case 1, by exchanging the roles of $G$ and $G^{\prime}$, we have $G_{\lceil X}^{\prime}=v_{0}<v_{2}<v_{1}<v_{3}$ and $G_{\lceil X}=v_{1}<v_{0}<v_{3}<v_{2}$ or $G_{\lceil X}^{\prime}=v_{2}<v_{3}<v_{0}<v_{1}$ and $G_{\lceil X}=v_{3}<v_{1}<v_{2}<v_{0}$.

Proposition 4.6 Let $G$ and $G^{\prime}$ be two digraphs on the same vertex set $V$, ( $\leq 5$ )hypomorphic up to complementation. Let $U:=G \dot{+} G^{\prime}$. If $U$ and $\bar{U}$ are connected and $G$ is not a tournament, then there exists $X \subset V$, such that $G_{\mid X} \simeq \overrightarrow{P_{4}}$ or $\overrightarrow{P_{4}^{f}}$, and $G_{\lceil X}^{\prime}=G_{\lceil X}^{*}$ or $G_{\mid X}^{\prime}=\overline{G^{*}}{ }_{\mid X}$.

Proof. From Theorem 4.1, there exists $X:=\left\{u_{0}, u_{1}, u_{2}, u_{3}\right\} \subset V$ such that $u_{0} \_u_{1} \_u_{2} \_u_{3}$ is an induced $P_{4}$ of $U$. The hypotheses of Lemma 4.5 are satisfied. If we have 1 ) or 2 ) of Lemma 4.5, then we conclude.

Now we consider that only the situation 3) of Lemma 4.5 holds.
That is if $X:=\left\{u_{0}, u_{1}, u_{2}, u_{3}\right\} \subset V$ such that $u_{0} \_u_{1} \_u_{2} \_u_{3}$ is an induced $P_{4}$ of $U$, then $\left\{G_{\lceil X}, G_{\lceil X}^{\prime}\right\}=\left\{u_{0}<u_{2}<u_{1}<u_{3}, u_{1}<u_{0}<u_{3}<u_{2}\right\}$ or $\left\{u_{2}<u_{3}<u_{0}<u_{1}, u_{3}<\right.$ $\left.u_{1}<u_{2}<u_{0}\right\}$. From this, if $u_{i} \longrightarrow_{G} u_{i+1}$ then $u_{i+1} \longleftarrow{ }_{G} u_{i+2}$ for each $i \in\{0,1\}$.

We will show that the situation $(\star)$ is impossible, which completes our proof. As $G$ is not a tournament, there exist $a, b \in V(G)$ such that $\{a, b\}$ is a neutral pair in $G$. From Corollary 3.4, $U=S\left(U_{0}, U_{1}, \ldots, U_{m-1}\right)$, where $S$ is an indecomposable graph with at least 4 vertices and the $U_{i}$ 's are graphs, for each $i \in\{0,1, \ldots, m-1\}$.

Claim $4.7\{a, b\} \nsubseteq V\left(U_{i}\right)$, for each $i \in\{0,1, \ldots, m-1\}$.
Proof. We assume by contradiction, that there exists $i \in\{0,1 \ldots, m-1\}$ such that $a, b \in V\left(U_{i}\right)$. Then from Theorem 3.1, there exist $v_{0}, v_{1}, v_{2}, v_{3} \in V(U)$ such that one of the following cases holds.

- Case 1. In $U$, we have $v_{0}-v_{1} \_v_{2} \_\{a, b\}$.

Let $x \in\{a, b\}$. We have $U_{\mid\left\{v_{0}, v_{1}, v_{2}, x\right\}}=v_{0}-v_{1 \_\_} v_{2 \_\_} x$, so from $(\star), G_{\mid\left\{v_{0}, v_{1}, v_{2}, x\right\}}$ and $G_{\left\{\left\{v_{0}, v_{1}, v_{2}, x\right\}\right.}^{\prime}$ are two total orders of order 4, w.l.o.g., we can assume that $G_{\left\{\left\{v_{0}, v_{1}, v_{2}, x\right\}\right.}=$ $v_{0}<v_{2}<v_{1}<x$ and $G_{\mid\left\{v_{0}, v_{1}, v_{2}, x\right\}}^{\prime}=v_{1}<v_{0}<x<v_{2}$. Then $G_{\mid\left\{v_{0}, v_{1}, v_{2}, a, b\right\}}=v_{0}<v_{2}<$ $v_{1}<\{a, b\}$ and $G_{\mid\left\{v_{0}, v_{1}, v_{2}, a, b\right\}}^{\prime}=v_{1}<v_{0}<\{a, b\}<v_{2}$. Clearly, since $G_{\mid\left\{v_{2}, a, b\right\}}$ is a peak and $v_{2-}\{a, b\}$, from Remark 4.2, $a_{-_{U}} b$. Since $G_{\mid\left\{v_{1}, a, b\right\}}$ is a peak and $v_{1} \ldots_{U}\{a, b\}$, from Remark 4.2, $a \ldots_{U} b$. A contradiction.

- Case 2. In $U$, we have $v_{0}-\{a, b\}-v_{2}-v_{3}$.

The proof is similar to that of Case 1 .

- Case 3. In $U$, we have


As $U_{\left\{\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}=v_{0} \_v_{1 —} v_{2}-v_{3}$, from $(\star), G_{\mid\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}$ and $G_{\mid\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}^{\prime}$ are two total orders of order 4. W.l.o.g., we assume that $G_{\mid\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}=v_{0}<v_{2}<v_{1}<v_{3}$ and $G_{\left\{\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}^{\prime}=v_{1}<v_{0}<v_{3}<v_{2}$. Let $x \in\{a, b\}$. We have $E\left(U_{\left\{\left\{x, v_{2}, v_{3}\right\}\right.}\right)=$ $\left\{\left\{x, v_{2}\right\},\left\{v_{2}, v_{3}\right\}\right\}$ (resp. $\left.E\left(U_{\mid\left\{x, v_{0}, v_{1}\right\}}\right)=\left\{\left\{v_{0}, v_{1}\right\},\left\{x, v_{1}\right\}\right\}\right)$ and $\left\{v_{2}, v_{3}\right\}$ (resp. $\left.\left\{v_{0}, v_{1}\right\}\right)$ is an oriented pair in $G$, then 1) of Lemma 4.3 applied to $\left\{x, v_{2}, v_{3}\right\}$ (resp. $\left\{x, v_{0}, v_{1}\right\}$ ) gives $\left\{x, v_{2}\right\}$ (resp. $\left\{x, v_{1}\right\}$ ) is an oriented pair in $G$ reversed in $G^{\prime}$. Since $E\left(U_{\left\lceil\left\{x, v_{1}, v_{3}\right\}\right.}\right)=$ $\left.\left\{\left\{x, v_{1}\right\}\right\}, 2\right)$ of Lemma 4.3 applied to $\left\{x, v_{1}, v_{3}\right\}$ gives $\left\{x, v_{1}\right\}$ is an interval of $G_{\left\{\left\{x, v_{1}, v_{3}\right\}\right.}$, we have $v_{1} \longrightarrow_{G} v_{3}$, so $x \longrightarrow_{G} v_{3}$ and $x \longrightarrow_{G^{\prime}} v_{3}$. We have $\left\{x, v_{2}\right\} \longrightarrow_{G} v_{3}, x \longrightarrow_{G^{\prime}} v_{3}$, $v_{3} \longrightarrow{ }_{G^{\prime}} v_{2}$, then the 3-hypomorphy up to complementation applied to $\left\{x, v_{2}, x_{3}\right\}$ gives $x \longrightarrow{ }_{G^{\prime}} v_{2}$, so $x \longleftarrow_{G} v_{2}$. We have $G_{\left\{\left\{v_{2}, a, b\right\}\right.}$ is a peak and $v_{2-_{U}}\{a, b\}$, then $a_{-_{U}} b$. We have $G_{\left\{\left\{v_{3}, a, b\right\}\right.}$ is a peak and $v_{3} \ldots_{U}\{a, b\}$, then $a \ldots_{U} b$. A contradiction.

From Claim 4.7, there are $i, j \in\{0,1 \ldots, m-1\}, i \neq j$, such that $a \in V\left(U_{i}\right)$ and $b \in V\left(U_{j}\right)$. For each $X:=\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\} \subset V$, if $v_{0} \_v_{1} \_v_{2} \_v_{3}$ is an induced $P_{4}$ of $U$, then $\operatorname{from}(\star), G_{\mid\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}}$ and $G_{\left\lceil\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\}\right.}^{\prime}$ are total orders, so $\{a, b\}$ is not a subset of $X$ and $m \geq 5$. From Theorem 3.2, there is a subset $Y:=\left\{v_{0}, v_{1}, \ldots, v_{m-1}\right\}$ of $V(S)$ satisfying: $a, b \in Y$ and there is an isomorphism $f$ from $U_{\mid Y}$ or $\bar{U}_{\mid Y}$ onto $P_{m}$ or $Q_{m}$, such that $f(\{a, b\})=\left\{v_{0}, v_{m-1}\right\}$.

- Case 1. $U_{\mid\left\{v_{0}, v_{1}, \ldots, v_{m-1}\right\}} \simeq P_{m}$.
W.l.o.g, we can assume that $a=v_{0}, b=v_{m-1}$ and $U_{\left\lceil\left\{v_{0}, v_{1}, \ldots, v_{m-1}\right\}\right.}=P_{m}$. We have for each $i \in\{0,1 \ldots, m-4\}, U_{\mid\left\{v_{i}, v_{i+1}, v_{i+2}, v_{i+3}\right\}} \simeq P_{4}$ then, from $(\star), G_{\mid\left\{v_{i}, v_{i+1}, v_{i+2}, v_{i+3}\right\}}$ and $G_{\left\{\left\{v_{i}, v_{i+1}, v_{i+2}, v_{i+3}\right\}\right.}^{\prime}$ are total orders, thus $\left\{v_{j}, v_{j+1}\right\}$ is an oriented pair in $G$ reversed in $G^{\prime}$ for each $j \in\{0,1, \ldots, m-2\}$. For $i \in\{0,1 \ldots, m-4\}$, we have $U_{\mid\left\{v_{m-1}, v_{i}, v_{i+1}\right\}}=$ $v_{m-1} \ldots v_{i-} v_{i+1}$ and $\left\{v_{i}, v_{i+1}\right\}$ is an oriented pair in $G$, so 2 ) of Lemma 4.3 applied to $\left\{v_{m-1}, v_{i}, v_{i+1}\right\}$ gives $\left\{v_{i}, v_{i+1}\right\}$ is an interval of $G_{\mid\left\{v_{m-1}, v_{i}, v_{i+1}\right\}}$, then $\left\{v_{0}, v_{1}, \ldots, v_{m-3}\right\}$ is an interval of $G_{\mid\left\{v_{m-1}, v_{0}, v_{1}, \ldots, v_{m-3}\right\}}$. As $G_{\mid\left\{v_{m-4}, v_{m-3}, v_{m-2}, v_{m-1}\right\}}$ is a total order, $\left\{v_{m-3}, v_{m-1}\right\}$ is an oriented pair in $G$, then $\left\{v_{0}, v_{m-1}\right\}=\{a, b\}$ is oriented in $G$. A contradiction.
- Case 2. $U_{\mid\left\{v_{0}, v_{1}, \ldots, v_{m-1}\right\}} \simeq Q_{m}$.
W.l.o.g, we can assume that $a=v_{0}, b=v_{m-1}$ and $U_{\left\{\left\{v_{0}, v_{1}, \ldots, v_{m-1}\right\}\right.}=Q_{m}$.

Case 2.1. $m=5$.
Then $U_{\left\{\left\{v_{0}, v_{1}, v_{2}, v_{3}, v_{4}\right\}\right.}=Q_{5}=$


As $U_{\left\{\left\{v_{2}, v_{1}, v_{3}, v_{4}\right\}\right.}=v_{2}-v_{1 —} v_{3}-v_{4}$, thus from $(\star), G_{\left\{\left\{v_{2}, v_{1}, v_{3}, v_{4}\right\}\right.}$ and $G_{\left\{\left\{v_{2}, v_{1}, v_{3}, v_{4}\right\}\right.}^{\prime}$ are total orders. We have $E\left(U_{\mid\left\{v_{0}, v_{1}, v_{2}\right\}}\right)=\left\{\left\{v_{0}, v_{1}\right\},\left\{v_{1}, v_{2}\right\}\right\}$ and $\left\{v_{1}, v_{2}\right\}$ is an oriented pair in $G$, so 1 ) of Lemma 4.3 applied to $\left\{v_{0}, v_{1}, v_{2}\right\}$ gives $\left\{v_{0}, v_{1}\right\}$ is an oriented pair in $G$ reversed in $G^{\prime}$, we have $E\left(U_{\left\{\left\{v_{0}, v_{1}, v_{4}\right\}\right.}\right)=\left\{\left\{v_{0}, v_{1}\right\}\right\}$, thus 2) of Lemma 4.3 applied to $\left\{v_{0}, v_{1}, v_{4}\right\}$ gives $\left\{v_{0}, v_{1}\right\}$ is an interval of $G_{\mid\left\{v_{0}, v_{1}, v_{4}\right\}}$. As $\left\{v_{1}, v_{4}\right\}$ is an oriented pair in $G$, then $\left\{v_{0}, v_{4}\right\}=\{a, b\}$ is an oriented pair in $G$. A contradiction.

Case 2.2. $m>5$.
We have $U_{\mid\left\{v_{0}, v_{m-1}, v_{m-2}, v_{m-3}, v_{m-4}\right\}}=\left\{v_{0}, v_{m-1}\right\} \_v_{m-2 \_\_} v_{m-4 \_\_} v_{m-3}$, where $\left\{v_{0}, v_{m-1}\right\}$ is an interval of $U_{\left\{\left\{v_{0}, v_{m-1}, v_{m-2}, v_{m-3}, v_{m-4}\right\}\right.}$. A contradiction is obtained by making a proof similar to that of Case 1 of the proof of Claim 4.7.

## 5 Reconstruction up to complementation

Lemma 5.1 Let $G$ and $G^{\prime}$ be two digraphs on the same vertex set $V$ such that $G$ and $G^{\prime}$ are $(\leq 5)$-hypomorphic up to complementation. Let $U:=G \dot{+} G^{\prime}, X:=\left\{v_{0}, v_{1}, v_{2}, v_{3}\right\} \subset V$ and $x \in V \backslash X$. If $G_{\mid X}=\vec{P}_{4}$ or $\overrightarrow{P_{4}^{f}}$, and $G_{\lceil X}^{\prime}=G_{\uparrow X}^{*}$ then,

1) $x \ldots U\left\{v_{1}, v_{2}\right\}$.
2) Up to isomorphism, $U_{\lceil X \cup\{x\}}$ is one of the following graphs:


Proof. W.l.o.g., we can assume that $G_{\lceil X}=v_{0} \longrightarrow v_{1} \longrightarrow v_{2} \longrightarrow v_{3}$, so $G_{\lceil X}^{\prime}=G_{\lceil X}^{*}=$ $v_{3} \longrightarrow v_{2} \longrightarrow v_{1} \longrightarrow v_{0}$ and $U_{\Gamma X}=v_{0}-v_{1} \_v_{2} \_v_{3}$. It suffices to prove 1) because 2) is a consequence of 1 ). By contradiction $x-{ }_{U} v_{1}$ or $x-_{U} v_{2}$.

- Case 1. $x_{-}{ }_{U} v_{1}$ and $x \ldots_{U} v_{2}$, or $x-_{U} v_{2}$ and $x \ldots_{U} v_{1}$.
W.l.o.g., we can assume that $x-_{U} v_{1}$ and $x \ldots{ }_{U} v_{2}$.

Case 1.1. $x \ldots{ }_{U} v_{3}$.
Since $U_{\left\lceil\left\{x, v_{1}, v_{2}, v_{3}\right\}\right.}=x-v_{1}-v_{2} \_v_{3}$ and $G_{\left\{\left\{v_{1}, v_{2}, v_{3}\right\}\right.}=v_{1} \longrightarrow v_{2} \longrightarrow v_{3}$, by Lemma 4.4, $G_{\mid\left\{x, v_{1}, v_{2}, v_{3}\right\}}=x \longrightarrow v_{1} \longrightarrow v_{2} \longrightarrow v_{3}$ and $G_{\mid\left\{x, v_{1}, v_{2}, v_{3}\right\}}^{\prime}=G_{\left\lceil\left\{x, v_{1}, v_{2}, v_{3}\right\}\right.}^{*}$. Thus in $G$ we have $\left\{v_{0}, x\right\} \longrightarrow v_{1} \longrightarrow v_{2} \longrightarrow v_{3}$, and in $G^{\prime}$ we have $\left\{v_{0}, x\right\} \longleftarrow v_{1} \longleftarrow v_{2} \longleftarrow v_{3}$. So $G_{\{X \cup\{x\}}^{\prime} \nsim G_{\mid X \cup\{x\}}$. On the other hand, $G_{\mid X \cup\{x\}}^{\prime}$ and $\bar{G}_{\{X \cup\{x\}}$ are not isomorphic because their types are distinct. Then $G_{\mid X \cup\{x\}}^{\prime}$ and $G_{\{X \cup\{x\}}$ are not 5-hypomorphic up to complementation, a contradiction.

Case 1.2. $x x_{U} v_{3}$ and $x \ldots_{U} v_{0}$.
As $E\left(\bar{U}_{\left\{\left\{x, v_{0}, v_{3}\right\}\right.}\right)=\left\{\left\{x, v_{0}\right\},\left\{v_{0}, v_{3}\right\}\right\}$ and $\left.v_{0} \ldots{ }_{G} v_{3}, 1\right)$ of Lemma 4.3 gives $\left\{x, v_{0}\right\}$ is a neutral pair in $G$. We have $E\left(\bar{U}_{\left\{\left\{x, v_{0}, v_{1}\right\}\right.}\right)=\left\{\left\{x, v_{0}\right\}\right\}$ and $v_{0} \longrightarrow_{G} v_{1}$, then 3) of Lemma 4.3 gives $v_{1} \longrightarrow_{G} x$ and $v_{1} \longleftarrow{ }_{G^{\prime}} x$. As $E\left(\bar{U}_{\mid\left\{x, v_{1}, v_{3}\right\}}\right)=\left\{\left\{v_{1}, v_{3}\right\}\right\}$ and $v_{1} \ldots{ }_{G} v_{3}$, 3) of Lemma 4.3 gives $x \longrightarrow_{G} v_{3}$ and $x \longleftarrow_{G^{\prime}} v_{3}$. Since $E\left(U_{\mid\left\{x, v_{0}, v_{3}\right\}}\right)=\left\{\left\{x, v_{3}\right\}\right\}$, thus 2) of Lemma 4.3 applied to $\left\{v_{0}, x, v_{3}\right\}$ gives $\left\{x, v_{3}\right\}$ is an interval for $G_{\mid\left\{x, v_{0}, v_{3}\right\}}$, since $v_{0} \ldots_{G} v_{3}$,
so $x \ldots{ }_{G} v_{0}$ and $x \ldots{ }_{G^{\prime}} v_{0}$.
In $G$ we have $v_{0} \longrightarrow v_{1} \longrightarrow\left\{x, v_{2}\right\} \longrightarrow v_{3}$ and in $G^{\prime}$ we have $v_{0} \longleftarrow v_{1} \longleftarrow\left\{x, v_{2}\right\} \longleftarrow v_{3}$, so $G_{\{X \cup\{x\}}^{\prime} \nsim G_{\mid X \cup\{x\}}$. By types, $G_{\mid X \cup\{x\}}^{\prime} \nsim \bar{G}_{\mid X \cup\{x\}}$. We get a contradiction with the 5 -hypomorphy up to complementation.

Case 1.3. $x_{-{ }_{U}}\left\{v_{0}, v_{3}\right\}$.
As $E\left(\bar{U}_{\left\{\left\{x, v_{0}, v_{2}\right\}\right.}\right)=\left\{\left\{v_{0}, v_{2}\right\},\left\{v_{2}, x\right\}\right\}$ and $\left.v_{0} \cdots_{G} v_{2}, 1\right)$ of Lemma 4.3 gives $\left\{x, v_{2}\right\}$ is a neutral pair in $G$. Since $E\left(\bar{U}_{\left\{\left\{x, v_{2}, v_{3}\right\}\right.}\right)=\left\{\left\{x, v_{2}\right\}\right\}$ (resp. $E\left(\bar{U}_{\mid\left\{x, v_{1}, v_{2}\right\}}\right)=\left\{\left\{x, v_{2}\right\}\right\}$ and $v_{1} \longrightarrow_{G} v_{2}$ ), 3) of Lemma 4.3 applied to $\left\{x, v_{3}, v_{2}\right\}$ (resp. $\left\{x, v_{1}, v_{2}\right\}$ ) gives $x \longleftarrow_{G} v_{3}$ and $x \longrightarrow{ }_{G^{\prime}} v_{3}$ (resp. $x \longrightarrow_{G} v_{1}$ and $\left.x \longleftarrow_{G^{\prime}} v_{1}\right)$. We have $E\left(\bar{U}_{\left\{\left\{x, v_{0}, v_{3}\right\}\right.}\right)=\left\{\left\{v_{0}, v_{3}\right\}\right\}$, thus 3) of Lemma 4.3 applied to $\left\{x, v_{0}, v_{3}\right\}$ gives $x \longrightarrow_{G} v_{0}$ and $x \longleftarrow_{G^{\prime}} v_{0}$. As $E\left(U_{\left\{\left\{x, v_{0}, v_{2}\right\}\right.}\right)=$ $\left.\left\{\left\{x, v_{0}\right\}\right\}, 2\right)$ of Lemma 4.3 applied to $\left\{x, v_{0}, v_{2}\right\}$ gives $\left\{x, v_{0}\right\}$ is an interval of $G_{\mid\left\{x, v_{0}, v_{2}\right\}}$, then $x \ldots{ }_{G} v_{2}$ and $x \ldots{ }_{G^{\prime}} v_{2}$. The induced digraphs $G_{\left\lceil\left\{v_{0}, v_{1}, v_{2}, x\right\}\right.}$ and $G_{\left\lceil\left\{v_{0}, v_{1}, v_{2}, x\right\}\right.}^{\prime}$. $a$ not isomorphic up to complementation, which gives a contradiction with the hypothesis $G$ and $G^{\prime}$ are ( $\leq 5$ )-hypomorphic up to complementation.

- Case 2. $x_{-}\left\{v_{1}, v_{2}\right\}$.

Case 2.1. $x \ldots{ }_{U}\left\{v_{0}, v_{3}\right\}$.
We have $E\left(\bar{U}_{\left\{\left\{x, v_{1}, v_{3}\right\}\right.}\right)=\left\{\left\{x, v_{3}\right\},\left\{v_{1}, v_{3}\right\}\right\}$ and $v_{1} \ldots{ }_{G} v_{3}$ (resp. $E\left(\bar{U}_{\left\{\left\{x, v_{0}, v_{2}\right\}\right.}\right)=$ $\left\{\left\{x, v_{0}\right\},\left\{v_{0}, v_{2}\right\}\right\}$ and $v_{0} \ldots_{G} v_{2}$ ), then 1) of Lemma 4.3 gives $\left\{x, v_{3}\right\}$ (resp. $\left\{x, v_{0}\right\}$ ) is a neutral pair in $G$. We have $E\left(\bar{U}_{\left\{\left\{x, v_{2}, v_{3}\right\}\right.}\right)=\left\{\left\{x, v_{3}\right\}\right\}$ and $v_{2} \longrightarrow_{G} v_{3}$ (resp. $E\left(\bar{U}_{\mid\left\{x, v_{0}, v_{1}\right\}}\right)=\left\{\left\{x, v_{0}\right\},\left\{x, v_{1}\right\}\right\}$ and $\left.v_{0} \longrightarrow_{G} v_{1}\right)$, then 3) of Lemma 4.3 applied to $\left\{x, v_{2}, v_{3}\right\}$ (resp. $\left\{v_{0}, v_{1}, x\right\}$ ) gives $v_{2} \longleftarrow_{G} x$ and $v_{2} \longrightarrow_{G^{\prime}} x$ (resp. $v_{1} \longrightarrow_{G} x$ and $\left.v_{1} \longleftarrow{ }_{G^{\prime}} x\right)$. We have $E\left(U_{\left\{\left\{x, v_{1}, v_{3}\right\}\right.}\right)=\left\{\left\{x, v_{1}\right\}\right\}$ (resp. $E\left(U_{\mid\left\{x, v_{0}, v_{2}\right\}}\right)=\left\{\left\{x, v_{2}\right\}\right\}$ ), then 2) of Lemma 4.3 applied to $\left\{x, v_{1}, v_{3}\right\}$ (resp. $\left\{v_{0}, v_{2}, x\right\}$ ) gives $\left\{x, v_{1}\right\}$ (resp. $\left\{x, v_{2}\right\}$ ) is an interval of $G_{\backslash\left\{x, v_{1}, v_{3}\right\}}$ (resp. $G_{\left\lceil\left\{v_{0}, v_{2}, x\right\}\right.}$ ) then $x \ldots{ }_{G} v_{3}$ and $x \ldots_{G^{\prime}} v_{3}$ (resp. $x \ldots_{G} v_{0}$ and $\left.x \ldots{ }_{G^{\prime}} v_{0}\right)$. By types, $G_{\mid\left\{v_{0}, v_{1}, v_{2}, x\right\}}^{\prime} \not \subset \bar{G}_{\left\{\left\{v_{0}, v_{1}, v_{2}, x\right\}\right.}$. If $\sigma$ is an isomorphism from $G_{\mid\left\{v_{0}, v_{1}, v_{2}, x\right\}}$ onto $G_{\left\lceil\left\{v_{0}, v_{1}, v_{2}, x\right\}\right.}^{\prime}$, then $\sigma\left(v_{1}\right)=v_{1}$ because $v_{1}$ is the only vertex in $\left\{v_{0}, v_{1}, v_{2}, x\right\}$, whose not adjacent to any neutral pair in $G_{\mid\left\{v_{0}, v_{1}, v_{2}, x\right\}}$; now since $d_{G_{\left\{\left\{v_{0}, v_{1}, v_{2}, x\right\}\right.}^{+}}\left(v_{1}\right)=2$ and $d_{G_{\left\{\left\{v_{0}, v_{1}, v_{2}, x\right\}\right.}^{\prime},}^{+}\left(v_{1}\right)=1$, we deduce that $G_{\left\{\left\{v_{0}, v_{1}, v_{2}, x\right\}\right.}^{\prime} \nsim G_{\left\{\left\{v_{0}, v_{1}, v_{2}, x\right\}\right.}$. A contradiction.

Case 2.2. $x_{-ـ_{U}} v_{3}$ and $x \ldots_{U} v_{0}$ or $x_{-{ }_{U}} v_{0}$ and $x \ldots_{U} v_{3}$.
W.l.o.g., we can assume that $x_{-_{U}} v_{3}$ and $x \ldots{ }_{U} v_{0}$.

We do the same proof as Case 1.2. In $G$ we have $v_{0} \longrightarrow v_{1} \longrightarrow\left\{x, v_{2}\right\} \longrightarrow v_{3}$ and in $G^{\prime}$ we have $v_{0} \longleftarrow v_{1} \longleftarrow\left\{x, v_{2}\right\} \longleftarrow v_{3}$, so $G_{\lceil X \cup\{x\}}^{\prime}$ an $G_{\lceil X \cup\{x\}}$ are not 5 -hypomorphic up to complementation, a contradiction.

Case 2.3. $x_{-}\left\{v_{0}, v_{3}\right\}$.
According to the nature of $\left\{x, v_{2}\right\}$ in $G$, we can distinguish the following subcases.
Case 2.3.1. $x \longrightarrow_{G} v_{2}$ or $x \longleftarrow{ }_{G} v_{2}$.
W.l.o.g. we can suppose $x \longrightarrow_{G} v_{2}$. As $x \varlimsup_{U} v_{2}$ then $x \longleftarrow_{G^{\prime}} v_{2}$. We have $E\left(\bar{U}_{\left\{\left\{x, v_{0}, v_{2}\right\}\right.}\right)=$ $\left\{\left\{v_{0}, v_{2}\right\}\right\}, v_{0} \ldots{ }_{G} v_{2}$ and $x \longrightarrow_{G} v_{2}$. So, 3) of Lemma 4.3 applied to $\left\{x, v_{0}, v_{2}\right\}$ gives $x \longleftarrow{ }_{G} v_{0}$ and $x \longrightarrow{ }_{G^{\prime}} v_{0}$. We have $E\left(\bar{U}_{\mid\left\{x, v_{0}, v_{3}\right\}}\right)=\left\{\left\{v_{0}, v_{3}\right\}\right\}$ and $v_{0} \ldots_{G} v_{3}$, then 3$)$ of Lemma 4.3 applied to $\left\{x, v_{0}, v_{3}\right\}$ gives $x \longrightarrow_{G} v_{3}$ and $x \longleftarrow_{G^{\prime}} v_{3}$. We have $E\left(\bar{U}_{\mid\left\{x, v_{1}, v_{3}\right\}}\right)=$ $\left\{\left\{v_{1}, v_{3}\right\}\right\}$ and $v_{1} \ldots{ }_{G} v_{3}$. So 3 ) of Lemma 4.3 applied to $\left\{x, v_{1}, v_{3}\right\}$ gives $x \longleftarrow_{G} v_{1}$ and $x \longrightarrow{ }_{G^{\prime}} v_{1}$. We have that $x$ is the only vertex in $\left\{v_{0}, v_{2}, v_{3}, x\right\}$, which is not adjacent to any neutral pair in $G_{\left\{\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}$. As $d_{G_{\left\{\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}^{+}}(x) \neq d_{G_{\left\{\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}^{\prime}}^{+}(x)$, then $G_{\left\{\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}^{\prime} \neq$
$G_{\left\{\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}$. Moreover $G_{\left\lceil\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}^{\prime} \nsim \bar{G}_{\left\{\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}$ because their types are distinct. We get a contradiction with the 4 -hypomorphy up to complementation.

Case 2.3.2. $x \ldots{ }_{G} v_{2}$.
 As $v_{0} \ldots{ }_{G^{\prime}}\left\{x, v_{3}\right\}, v_{0} \ldots_{G} v_{3}$ and $x-_{G} v_{0}$, then $x-_{G^{\prime}} v_{3}$, so $x \ldots_{G} v_{3}$. As $v_{3} \ldots{ }_{G}\left\{x, v_{1}\right\}$, $v_{1} \ldots_{G^{\prime}} v_{3}$ and $x$ - $_{G^{\prime}} v_{3}$, then $x$ - $_{G} v_{1}$, so $x \ldots_{G^{\prime}} v_{1}$. Since $G_{\left\{\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}$ and $G_{\left\{\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}^{\prime}$ have respectively the types $(1,4)$ and $(2,3), G_{\left\{\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}^{\prime}$ and $G_{\left\{\left\{v_{0}, v_{2}, v_{3}, x\right\}\right.}$ are not isomorphic up to complementation, a contradiction.

Case 2.3.3. $x-{ }_{G} v_{2}$.
We do the same proof as Case 2.3.2.
From Lemma 5.1, we obtain the following result.
Corollary 5.2 Let $G$ and $G^{\prime}$ be two digraphs on the same vertex set $V$ such that $G$ and $G^{\prime}$ are ( $\leq 5$ )-hypomorphic up to complementation. Let $U:=G \dot{+} G^{\prime}, X:=\left\{v_{0}, v_{1}, \cdots, v_{k-1}\right\} \subset$ $V$ and $x \in V \backslash X$. If $G_{\lceil X}=\overrightarrow{P_{k}}$ or $\overrightarrow{P_{k}^{f}}$, and $G_{\lceil X}^{\prime}=G_{\lceil X}^{*}$ then,

1) $x \cdots u\left\{v_{1}, \cdots, v_{k-2}\right\}$.
2) Up to isomorphism, $U_{\lceil X \cup\{x\}}$ is one of the following graphs:


Proposition 5.3 Let $G$ and $G^{\prime}$ be two digraphs on the same set $V$ of $n \geq 4$ vertices, such that $G$ and $G^{\prime}$ are $(\leq 5)$-hypomorphic up to complementation and $U:=G \dot{+} G^{\prime}$ is connected. Let $X \subset V$.

1) If $G_{\mid X} \simeq \overrightarrow{P_{4}}$ and $G_{\lceil X}^{\prime}=G_{\lceil X}^{*}$, then $G \simeq \overrightarrow{P_{n}}$ or $G \simeq \overrightarrow{C_{n}}$, and $G^{\prime}=G^{*}$.
2) If $G_{\mid X} \simeq \overrightarrow{P_{4}}$ and $G_{\mid X}^{\prime}=\overline{G^{*}}{ }_{X X}$, then $G \simeq \overrightarrow{P_{n}}$ or $G \simeq \overrightarrow{C_{n}}$, and $G^{\prime}=\overline{G^{*}}$.
3) If $G_{\lceil X} \simeq \overrightarrow{P_{4}^{f}}$ and $G_{\lceil X}^{\prime}=G_{\lceil X}^{*}$, then $G \simeq \overrightarrow{P_{n}^{f}}$ or $G \simeq \overrightarrow{C_{n}^{f}}$, and $G^{\prime}=G^{*}$.
4) If $G_{\mid X} \simeq \overrightarrow{P_{4}^{f}}$ and $G_{\mid X}^{\prime}=\overline{G^{*}}{ }_{\mid X}$, then $G \simeq \overrightarrow{P_{n}^{f}}$ or $G \simeq \overrightarrow{C_{n}^{f}}$, and $G^{\prime}=\overline{G^{*}}$.

Proof. It suffices to prove 1) because 2), 3) and 4) are consequences of 1). As $G_{\mid X} \simeq \overrightarrow{P_{4}}$, let $\vec{P}_{\ell}$ be a largest induced oriented path in $G$ reversed in $G^{\prime}$. Clearly, $\ell \geq 4$. W.l.o.g. we can assume $\vec{P}_{\ell}=v_{0} \longrightarrow v_{1} \longrightarrow \cdots \longrightarrow v_{\ell-1}$ and $G_{\mid V\left(\vec{P}_{\ell}\right)}^{\prime}=v_{0} \longleftarrow v_{1} \longleftarrow \cdots \longleftarrow v_{\ell-1}$.

So $U_{\mid V\left(\vec{P}_{\ell}\right)}=v_{0 \_-} v_{1} \ldots \cdots \_v_{\ell-2 \_-v_{\ell-1}}$. If $V\left(\vec{P}_{\ell}\right)=V$, then $G=\vec{P}_{\ell}$ and $G^{\prime}=G^{*}$. In the rest of this proof, we assume that $V \backslash V\left(\vec{P}_{\ell}\right) \neq \emptyset$. As $U$ is connected, there exists $v_{\ell} \in V \backslash V\left(\overrightarrow{P_{\ell}}\right)$, such that $U_{\mid V\left(\overrightarrow{P_{\ell}}\right) \cup\left\{v_{\ell}\right\}}$ is connected. From 2) of Corollary 5.2, up to isomorphism, $U_{\mid V\left(\overrightarrow{P_{\ell}}\right) \cup\left\{v_{\ell}\right\}}$ is one of the following graphs:

$$
v_{\ell}-v_{0}-v_{1}-\cdots-v_{\ell-2}-v_{\ell-1}
$$



If $U_{\mid V\left(\overrightarrow{P_{\ell}}\right) \cup\left\{v_{\ell}\right\}}$ is the graph $v_{\ell \_v_{0}} v_{0} v_{1} \ldots \cdots-v_{\ell-2 \_v_{\ell-1}}$ then, from Lemma 4.4, we have $G_{\mid V\left(\overrightarrow{P_{\ell}}\right) \cup\left\{v_{\ell}\right\}}=v_{\ell} \longrightarrow v_{0} \longrightarrow v_{1} \longrightarrow \cdots \longrightarrow v_{\ell-1}$ and $G_{\mid V\left(\overrightarrow{P_{\ell}}\right) \cup\left\{v_{\ell}\right\}}^{\prime}=G_{\mid V\left(\overrightarrow{P_{\ell}}\right) \cup\left\{v_{\ell}\right\}}^{*}$, that contradict the fact that $\vec{P}_{\ell}$ is the largest induced oriented path in $G$ reversed in $G^{\prime}$. Then $U_{\mid V\left(\vec{P}_{\ell}\right) \cup\left\{v_{\ell}\right\}}$ is the second graph.
If there is $x$ in $V \backslash\left(V\left(\vec{P}_{\ell}\right) \cup\left\{v_{\ell}\right\}\right)$, we have $v_{i-1 \_} v_{i \_\_} v_{i+1} \_v_{i+2}$ for each $i \in\{1, \ldots, \ell-2\}$, then from 1) of Lemma 5.1, $x \ldots v_{i}$ for each $i \in\{1, \ldots, \ell-1\}$, also we have
 As $U$ is connected, $V=V\left(\vec{P}_{\ell}\right) \cup\left\{v_{\ell}\right\}$.
We have $G_{\left\lceil\left\{v_{0}, v_{1}, \ldots, v_{\ell-2}\right\}\right.}=v_{0} \longrightarrow v_{1} \longrightarrow \cdots \longrightarrow v_{\ell-2}$ and $U_{\mid\left\{v_{\ell}, v_{0}, v_{1}, \ldots, v_{\ell-2}\right\}}=$ $v_{\ell \_-} v_{0}-v_{1} \_\cdots-v_{\ell-3 \_-} v_{\ell-2}$, then from Lemma 4.4, $G_{\left\{\left\{v_{\ell}, v_{0}, v_{1}, \ldots, v_{\ell-2}\right\}\right.}=v_{\ell} \longrightarrow$ $v_{0} \longrightarrow v_{1} \longrightarrow \cdots \longrightarrow v_{\ell-2}$ and $G_{\left\{\left\{v_{\ell}, v_{0}, v_{1}, \ldots, v_{\ell-2}\right\}\right.}^{\prime}=G_{\left\{\left\{v_{\ell}, v_{0}, v_{1}, \ldots, v_{\ell-2}\right\}\right.}^{*}$.
We have $\bar{U}_{\left\{\left\{v_{\ell-2}, v_{\ell-1}, v_{\ell}\right\}\right.}=v_{\ell-2}-v_{\ell} \ldots v_{\ell-1}, v_{\ell-2} \longrightarrow_{G} v_{\ell-1}$ and $v_{\ell-2} \ldots{ }_{G} v_{\ell}$, then 3) Lemma 4.3 applied to $\left\{v_{\ell-2}, v_{\ell-1}, v_{\ell}\right\}$ gives $v_{\ell-1} \longrightarrow_{G} v_{\ell}$ and $v_{\ell-1} \longleftarrow_{G^{\prime}} v_{\ell}$. Then $G=\overrightarrow{C_{\ell+1}}$ and $G^{\prime}=G^{*}$.

Proof of Theorem 1.3. If $G$ is a tournament then from Proposition 1.1, $G$ and $G^{\prime}$ are total orders. If $G$ is not a tournament, then using Proposition 4.6, there exists a subset $X$ of $V(G)$ such that, $G_{\mid X} \simeq \overrightarrow{P_{4}}$ or $\overrightarrow{P_{4}^{f}}$, and $G_{\lceil X}^{\prime}=G_{\lceil X}^{*}$ or $\overline{G^{*}}{ }_{\mid X}$; then we conclude using Proposition 5.3.

## Acknowledgements

We warmly thank the two referees for their helpfull comments.

## References

[1] M. Alzohairi, M. Bouaziz and Y. Boudabbous, Orders and (4)-hemimorphy. J. Mult.Valued Logic Soft Comput. 21 (2013), no. 3-4, 355-371.
[2] J. A. Bondy, A graph reconstructor's manual. Surveys in combinatorics, 1991 (Guildford, 1991), 221-252, London Math. Soc. Lecture Note Ser., 166, Cambridge Univ. Press, Cambridge, 1991.
[3] J. A. Bondy and R. L. Hemminger, Graph reconstruction-a survey. J. Graph Theory 1 (1977), no. 3, 227-268.
[4] M. Bouaziz, Y. Boudabbous and N. El Amri, Hereditary hemimorphy of $\{-k\}-$ hemimorphic tournaments for $k \geq 5$. J. Korean Math. Soc. 48 (2011), no. 3, 599-626.
[5] Y. Boudabbous, A. Boussaïri, A. Chaïchaâ and N. El Amri, Self-duality of modules and reconstruction of tournaments up to duality. J. Mult.-Valued Logic Soft Comput. 27 (2016), no. 5-6, 501-530.
[6] Y. Boudabbous and G. Lopez, La relation différence et l'anti-isomorphie. Math. Logic Quart. 41 (1995), no. 2, 268-280.
[7] A. Cournier and P. Ille, Minimal indecomposable graphs. Discrete Math. 183 (1998), 61-80.
[8] J. Dammak, La (-5)-demi-reconstructibilité des relations binaires connexes finies. Proyecciones 22 (2003), no. 3, 181-199.
[9] J. Dammak, G. Lopez, M. Pouzet and H. Si Kaddour, Hypomorphy of graphs up to complementation. JCTB, Series B 99 (2009), no. 1, 84-96.
[10] J. Dammak, G. Lopez, M. Pouzet and H. Si Kaddour, Boolean sum of graphs and reconstruction up to complementation. Adv. Pure Appl. Math. 4 (2013), no. 3, 315349.
[11] T. Gallai, Transitiv orientierbare Graphen, Acta Math. Acad. Sci. Hungar 18 (1967), 25-66.
[12] J. G. Hagendorf and G. Lopez, La demi-reconstructibilité des relations binaires d'au moins 13 éléments. C. R. Acad. Sci. Paris Sér. I Math. 317 (1993), 7-12.
[13] J. G. Hagendorf and G. Lopez, Un théorème de demi-reconstruction des relations binaires de cardinal > 12 (2017), hal-01501643v1.
[14] G. Lopez, Deux résultats concernant la détermination d'une relation par les types d'isomorphie de ses restrictions. C. R. Acad. Sci. Paris Sér. A 274 (1972), 15251528.
[15] G. Lopez, Sur la détermination d'une relation par les types d'isomorphie de ses restrictions. C. R. Acad. Sci. Paris Sér. A 275 (1972), 951-953.
[16] G. Lopez, L'indéformabilité des relations et multirelations binaires. Z. Math. Logik Grundlag. Math. 24 (1978), no. 4, 303-317.
[17] M. Pouzet, Relations non reconstructibles par leurs restrictions. J. Combin. Theory Ser. B 26 (1979), no. 1, 22-34.
[18] M. Pouzet, H. Si Kaddour and N. Trotignon, Claw freeness, 3-homogeneous subsets of a graph and a reconstruction problem. Contrib. Discrete Math. 6 (2011), 92-103.
[19] K. B. Reid and C. Thomassen, Strongly self-complementary and hereditarily isomorphic tournaments. Monatsh. Math. 81 (1976), no. 4, 291-304.
[20] P. K. Stockmeyer, The falsity of the reconstruction conjecture for tournaments. J. Graph Theory 1 (1977), 19-25.
[21] D. Sumner, Graphs indecomposable with respect to the $X$-join. Discrete Math. 6 (1973), 281-298.
[22] S. M. Ulam, A collection of mathematical problems. Interscience Tracts in Pure and Applied Mathematics, no. 8 Interscience Publishers, New York-London 1960 xiii+150 pp.


[^0]:    * Corresponding author.

    E-mail addresses: ayman.benamira@yahoo.fr (A. Ben Amira), chaari.bechir@ymail.com (B. Chaari), jdammak@yahoo.fr (J. Dammak), sikaddour@univ-lyon1.fr (H. Si Kaddour).

