



**HAL**  
open science

# Efficient parametric estimation for a signal-plus-noise Gaussian model from discrete time observations

Dominique Dehay, Khalil El Waled, Vincent Monsan

► **To cite this version:**

Dominique Dehay, Khalil El Waled, Vincent Monsan. Efficient parametric estimation for a signal-plus-noise Gaussian model from discrete time observations. *Statistical Inference for Stochastic Processes*, 2021, 24 (1), pp.17-33. 10.1007/s11203-020-09225-1 . hal-02067554

**HAL Id: hal-02067554**

**<https://hal.science/hal-02067554v1>**

Submitted on 14 Mar 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Parametric estimation for a signal-plus-noise model from discrete time observations

Dominique Dehay<sup>1</sup>, Khalil El Waled<sup>2,3</sup>, Vincent Monsan<sup>4</sup>

<sup>1</sup>Univ Rennes, CNRS, IRMAR – UMR 6625,  
F–35000 Rennes, France.

<sup>2</sup> University of Nouakchott Al Asriya, Mauritania,

<sup>3</sup> Qassim University, Saudi Arabia.

<sup>4</sup> Université Félix Houphouët-Boigny, Abidjan, Côte d’Ivoire.

dominique.dehay@univ-rennes2.fr, khalil.elwaled@gmail.com, vmonsan@yahoo.fr

February 28, 2019

**Abstract:** This paper deals with the parametric inference for integrated signals embedded in an additive Gaussian noise and observed at deterministic discrete instants which are not necessarily equidistant. The unknown parameter is multidimensional and compounded of a signal-of-interest parameter and a variance parameter of the noise. We state the consistency and the minimax efficiency of the maximum likelihood estimator and of the Bayesian estimator when the time of observation tends to  $\infty$  and the delays between two consecutive observations tend to 0 or are only bounded. The class of signals in consideration contains among others, almost periodic signals and also non-continuous periodic signals. However the problem of frequency estimation is not considered here.

**Keywords:** Maximum likelihood estimation; Bayesian estimation; high frequency sampling; low frequency sampling; minimax efficiency; asymptotic properties of estimators.

# 1 Introduction

Consider the following integrated signal-plus-noise model

$$dX_t = f(\boldsymbol{\alpha}, t) dt + \sigma(\boldsymbol{\beta}, t) dW_t, \quad t \geq 0 \quad (1)$$

where the functions  $f : \mathbf{A} \times \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $\sigma : \mathbf{B} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are measurable,  $f(\boldsymbol{\alpha}, t)$ , respectively  $\sigma(\boldsymbol{\beta}, t)$  is continuous in the first component  $\boldsymbol{\alpha} \in \mathbf{A}$ , respectively in  $\boldsymbol{\beta} \in \mathbf{B}$ ;  $\mathbf{A}$  is a bounded open convex subset of  $\mathbb{R}^p$ ,  $\mathbf{B}$  is a bounded open convex subset of  $\mathbb{R}^q$ ,  $p, q \geq 0$ ,  $p + q > 0$ , and  $\{W_t\}$  is a Wiener process defined over a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We assume that the initial random variable  $X_0$  is independent on Wiener process  $\{W_t\}$  and does not depend on the unknown parameter  $\boldsymbol{\theta} := (\boldsymbol{\alpha}, \boldsymbol{\beta})$ .

Since very long time, this model has received a considerable amount of investigation. The statistical analysis of such signals has attracted much interest, its applications ranging from telecommunications, mechanics, to econometrics and financial studies. For the continuous time observation framework, we cite the well-known work of Ibragimov and Has'minskii (1981) as well as the contributions of Kutoyants (1984) who studied the consistency and the minimax efficiency of the maximum likelihood estimator and the Bayesian estimator.

However, in practice it is difficult to record numerically a continuous time process and generally the observations take place at discrete moments (Mishra and Prakasa-Rao 2001). Most of the publications on discrete time observation concern regular sampling, that is the discrete time observations are usually equally spaced. Nevertheless, many applications make use of non equidistant sampling. The sampled points can be associated with quantiles of some distribution (see e.g. in another context Blanke and Vial 2014; see also Sacks and Ylvisaker 1968) or can be perturbed by some jitter effect (see e.g. Dehay, Dudek and El Badaoui 2017).

The aim of the paper is the study of the maximum likelihood estimator and the Bayesian estimator of the unknown parameter  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$  from a discrete time observation  $\{X_{t_0}, \dots, X_{t_n}\}$ ,  $0 = t_0 < t_1 < \dots < t_n = T_n$  of the process  $\{X_t\}$  as  $n$  and  $T_n \rightarrow \infty$ , and the delays between two consecutive observations tend to 0 or are only bounded. The non uniform sampling scheme is scarcely taken in consideration in the usual literature on the inference of such a model (1) of integrated signal-plus-noise. We obtain that for this scheme of observation, the rate of convergence of the maximum likelihood estimator and the Bayesian estimator for the parameter  $\boldsymbol{\alpha}$  of the signal-of-interest is  $\sqrt{T_n}$  while the rate of convergence for the parameter  $\boldsymbol{\beta}$  of the noise variance

is  $\sqrt{n}$ , without any condition on the speed of convergence to 0 of the delay between two observations as  $n \rightarrow \infty$ , in contrary to the model of an ergodic diffusion (Stoyanov 1984, Florens-Zmirou 1989, Genon-Catalot 1992, Mishra and Prakara Rao 2001, Uchida and Yoshida 2012). This fact is due to the non-randomness of the signal-of-interest  $f(\boldsymbol{\alpha}, t)$  and of the variance  $\sigma^2(\boldsymbol{\beta}, t)$ . Notice that model (1) is not ergodic, and the signal-of-interest is not necessarily continuous or periodic in time. The problem of frequency estimation is not tackled in this work.

The paper is organized as follows. In Section 2, we introduce the framework and the assumptions on the model and the scheme of observation. We also state that these assumptions are fulfilled for almost periodic models. Model (1) being Gaussian, the exact log-likelihood of the increments of the observations is given by relation (3) and in Section 3 we deduce the local asymptotic normality property of the model of observation when the delays between two consecutive observations tend to 0 or are only bounded, in any case the total time of observation  $T_n$  goes to infinity. Then in Sections 4.1 and 4.2 we prove that the maximum likelihood estimator and the Bayesian estimator are consistent, asymptotically normal and asymptotically optimal, following the theory of minimax efficiency from Chapter III in (Ibragimov and Has'minskii 1981). Then examples of linear models are provided in Section 5. Some technical results are gathered in Appendix A. We complete this work by stating in Appendix B some expressions of the Fisher information matrices and of the identifiability functions in the cases of almost periodic and periodic functions.

## 2 Framework

From now on we concentrate on the consistency and the efficiency of the maximum likelihood estimator and the Bayesian estimator of the parameter  $\boldsymbol{\theta}$  for the integrated signal-plus-noise model (1). For that purpose we assume that the observations occur at instants  $0 = t_0 < t_1 < \dots < t_n = T_n$  of the interval  $[0, T_n]$ , where  $0 < t_i - t_{i-1} \leq h_n := \max_i \{t_i - t_{i-1}\}$ . We assume that  $T_n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $\{h_n\}$  is bounded. Notice that the observation of the sequence  $X_{t_i}$ ,  $i \in \{0, \dots, n\}$  corresponds to the observation of  $Y_0 := X_0$  and of the increments defined by  $Y_i := X_{t_i} - X_{t_{i-1}}$ ,  $i \in$

$\{1, \dots, n\}$ . Denote

$$F_i(\boldsymbol{\alpha}) := \int_{t_{i-1}}^{t_i} f(\boldsymbol{\alpha}, t) dt \quad \text{and} \quad G_i(\boldsymbol{\beta}) := \left( \int_{t_{i-1}}^{t_i} \sigma^2(\boldsymbol{\beta}, t) dt \right)^{1/2}.$$

When the true value of the parameter is  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ , the increment  $Y_i$ ,  $i \geq 1$ , is equal to

$$Y_i^{(\boldsymbol{\theta})} := F_i(\boldsymbol{\alpha}) + \int_{t_{i-1}}^{t_i} \sigma(\boldsymbol{\beta}, t) dW_t. \quad (2)$$

Thus the random variable  $Y_i$ ,  $i \geq 1$ , is Gaussian with mean  $F_i(\boldsymbol{\alpha})$  and variance  $G_i^2(\boldsymbol{\beta})$ :  $\mathcal{L}_{\boldsymbol{\theta}}[Y_i] = \mathcal{N}(F_i(\boldsymbol{\alpha}), G_i^2(\boldsymbol{\beta}))$ . Moreover the random variables  $Y_i$ ,  $i = 0, \dots, n$ , are independent. Therefore we can compute the log-likelihood of the increments  $\{Y_i : i = 1 \dots, n\}$  which is equal to

$$\Lambda_n(\boldsymbol{\theta}) = \frac{-n \ln(2\pi)}{2} - \sum_{i=1}^n \ln G_i(\boldsymbol{\beta}) - \sum_{i=1}^n \frac{(Y_i - F_i(\boldsymbol{\alpha}))^2}{2G_i^2(\boldsymbol{\beta})}. \quad (3)$$

Henceforth we assume that the following conditions are fulfilled.

**Assumption A1** The functions  $f : \mathbf{A} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ , and  $\sigma : \mathbf{B} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , are measurable. The function  $\boldsymbol{\alpha} \mapsto f(\boldsymbol{\alpha}, t)$  is differentiable and the gradient function  $\boldsymbol{\alpha} \mapsto \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t)$  is uniformly continuous in  $\boldsymbol{\alpha} \in \mathbf{A}$  uniformly with respect to the time  $t$  varying in  $\mathbb{R}^+$ . The function  $\boldsymbol{\beta} \mapsto \sigma^2(\boldsymbol{\beta}, t)$  is two-times differentiable and the functions  $\boldsymbol{\beta} \mapsto \nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t)$  and  $\boldsymbol{\beta} \mapsto \nabla_{\boldsymbol{\beta}}^2 \sigma^2(\boldsymbol{\beta}, t)$ , are uniformly continuous in  $\boldsymbol{\beta} \in \mathbf{B}$  uniformly with respect to the time  $t$  varying in  $\mathbb{R}^+$ . Hence, for every  $\gamma > 0$  there exists  $\eta > 0$  such that for  $|\boldsymbol{\alpha} - \boldsymbol{\alpha}'| \leq \eta$  and  $|\boldsymbol{\beta} - \boldsymbol{\beta}'| \leq \eta$  we have

$$\sup_t |\nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t) - \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}', t)| \leq \gamma$$

and

$$\sup_t \left( |\nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t) - \nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}', t)| + |\nabla_{\boldsymbol{\beta}}^2 \sigma^2(\boldsymbol{\beta}, t) - \nabla_{\boldsymbol{\beta}}^2 \sigma^2(\boldsymbol{\beta}', t)| \right) \leq \gamma.$$

Here the  $p$ -dimensional vector  $\nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t)$  is the gradient (derivative) function of  $f(\boldsymbol{\alpha}, t)$  with respect to  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)$ :  $\nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t) := (\partial_{\alpha_1} f(\boldsymbol{\alpha}, t), \dots, \partial_{\alpha_p} f(\boldsymbol{\alpha}, t))$ ; the  $q$ -dimensional vector  $\nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t)$  is the gradient function of  $\sigma^2(\boldsymbol{\beta}, t)$  with respect to  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_q)$ ; the  $q \times q$ -matrix  $\nabla_{\boldsymbol{\beta}}^2 \sigma^2(\boldsymbol{\beta}, t)$  is the second order derivative of  $\sigma^2(\boldsymbol{\beta}, t)$  with respect to  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_q)$ :  $\nabla_{\boldsymbol{\beta}}^2 \sigma^2(\boldsymbol{\beta}, t) := \left( \partial_{\beta_j} \partial_{\beta_k} \sigma^2(\boldsymbol{\beta}, t) \right)_{1 \leq j, k \leq q}$ .

**Assumptions A2** The function  $t \mapsto f(\boldsymbol{\alpha}, t)$  is locally integrable in  $\mathbb{R}$  for any  $\boldsymbol{\alpha} \in \mathbf{A}$ ; moreover

$$0 < \inf_{\boldsymbol{\beta}, t} \sigma^2(\boldsymbol{\beta}, t) \leq \sup_{\boldsymbol{\beta}, t} \sigma^2(\boldsymbol{\beta}, t) < \infty;$$

$$\sup_{\boldsymbol{\alpha}, t} |\nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t)| < \infty \quad \text{and} \quad \sup_{\boldsymbol{\beta}, t} |\nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t)| < \infty.$$

**Assumptions A3** There exist two positive definite matrices  $\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$  and  $\mathbf{J}_q^{(\boldsymbol{\beta})}$  such that

$$\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})} = \lim_{n \rightarrow \infty} \frac{1}{T_n} \sum_{i=1}^n \frac{\nabla_{\boldsymbol{\alpha}}^* F_i(\boldsymbol{\alpha}) \nabla_{\boldsymbol{\alpha}} F_i(\boldsymbol{\alpha})}{G_i^2(\boldsymbol{\beta})}$$

$$\mathbf{J}_q^{(\boldsymbol{\beta})} = \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{i=1}^n \nabla_{\boldsymbol{\beta}}^* \ln G_i^2(\boldsymbol{\beta}) \nabla_{\boldsymbol{\beta}} \ln G_i^2(\boldsymbol{\beta})$$

the convergences being uniform with respect to  $\boldsymbol{\theta}$  varying in  $\Theta = \mathbf{A} \times \mathbf{B}$ . Here and henceforth the superscript  $*$  designates the transpose operator for vectors and matrices.

**Assumptions A4** For every  $\nu > 0$  there exists  $\mu_\nu > 0$  and  $n_\nu > 0$  such that

$$\frac{1}{T_n} \sum_{i=1}^n \frac{(F_i(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha} + \boldsymbol{\delta}))^2}{t_i - t_{i-1}} \geq \mu_\nu \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n \frac{(G_i^2(\boldsymbol{\beta}) - G_i^2(\boldsymbol{\beta}'))^2}{(t_i - t_{i-1})^2} \geq \mu_\nu$$

for any  $n \geq n_\nu$  and all  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ ,  $\boldsymbol{\theta}' = (\boldsymbol{\alpha}', \boldsymbol{\beta}')$  in  $\Theta$  with  $|\boldsymbol{\alpha} - \boldsymbol{\alpha}'| \geq \nu$  and  $|\boldsymbol{\beta} - \boldsymbol{\beta}'| \geq \nu$ .

## Remarks

1) Assumptions A1 and A2 are technical conditions. We readily see that assumptions A1 and A2 are satisfied when the parameter set  $\Theta = \mathbf{A} \times \mathbf{B}$  is compact and the functions  $(\boldsymbol{\alpha}, t) \mapsto (f(\boldsymbol{\alpha}, t), \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t))$  and  $(\boldsymbol{\beta}, t) \mapsto (\sigma^2(\boldsymbol{\beta}, t), \nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t), \nabla_{\boldsymbol{\beta}}^2 \sigma^2(\boldsymbol{\beta}, t))$  are continuous and periodic in  $t$ . More generally these assumptions A1 and A2 are also satisfied when we replace the periodicity by the almost periodicity in  $t$  uniformly with respect to  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Theta$  (see Appendix B).

2) Assumption A1 is generally not satisfied when we consider the problem of frequency estimation. For example, the signal-of-interest  $f(\alpha, t) = \sin(\alpha t)$  with  $\alpha \in \mathbf{A}$ ,  $\mathbf{A} \subset \mathbb{R}$  does not satisfied assumption A1 since  $\sup_t |\cos(\alpha t) - \cos(\alpha' t)| = 2$  when  $\alpha \neq \alpha'$ .

3) With assumption A3 we can define the asymptotic Fisher information  $d \times d$ -matrix  $\mathbf{J}^{(\boldsymbol{\theta})}$  of the model,  $d := p + q$ , by

$$\mathbf{J}^{(\boldsymbol{\theta})} := \text{diag} \left[ \mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}, \mathbf{J}_q^{(\boldsymbol{\beta})} \right] = \begin{bmatrix} \mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})} & 0_{p \times q} \\ 0_{q \times p} & \mathbf{J}_q^{(\boldsymbol{\beta})} \end{bmatrix}.$$

Under conditions A1, A2 and A3, the function  $\boldsymbol{\theta} \mapsto \mathbf{J}^{(\boldsymbol{\theta})}$  is continuous on  $\Theta = \mathbf{A} \times \mathbf{B}$ . Furthermore as  $\mathbf{J}^{(\boldsymbol{\theta})}$  is a positive definite matrix, its square root  $(\mathbf{J}^{(\boldsymbol{\theta})})^{-1/2} = \text{diag}[(\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})})^{-1/2}, (\mathbf{J}_q^{(\boldsymbol{\beta})})^{-1/2}]$  is well defined and is continuous on  $\boldsymbol{\theta} \in \Theta$ .

Besides the limits  $\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$  and  $\mathbf{J}_q^{(\boldsymbol{\beta})}$  exist when the functions  $\nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t)$ ,  $\sigma^2(\boldsymbol{\beta}, t)$  and  $\nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t)$  are almost periodic in time  $t$  and  $h_n \rightarrow 0$  or when these functions are periodic and the delay between two observations is constant  $h = P/\nu$ ,  $\nu \in \mathbb{N}$  being fixed (see Appendix B).

4) Assumption A4 is an identifiability condition. Assume that the following limits exist

$$\mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}') := \liminf_{n \rightarrow \infty} \frac{1}{T_n} \sum_{i=1}^n \frac{(F_i(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha}'))^2}{t_i - t_{i-1}} \quad (4)$$

$$\mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}') := \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{(G_i^2(\boldsymbol{\beta}) - G_i^2(\boldsymbol{\beta}'))^2}{(t_i - t_{i-1})^2}, \quad (5)$$

the convergences being uniform with respect to  $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathbf{A}$ ,  $\boldsymbol{\beta}, \boldsymbol{\beta}' \in \mathbf{B}$  with  $|\boldsymbol{\alpha} - \boldsymbol{\alpha}'| \geq \nu$  and  $|\boldsymbol{\beta} - \boldsymbol{\beta}'| \geq \nu$ , and that

$$\mu_\nu := \frac{1}{2} \min \left\{ \inf_{|\boldsymbol{\alpha} - \boldsymbol{\alpha}'| \geq \nu} \mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}'), \inf_{|\boldsymbol{\beta} - \boldsymbol{\beta}'| \geq \nu} \mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}') \right\} > 0$$

for any  $\nu > 0$ , then Assumption A4 is fulfilled.

When the functions  $f(\boldsymbol{\alpha}, t)$  and  $\sigma^2(\boldsymbol{\beta}, t)$  are almost periodic in time  $t$ , then  $\mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$  and  $\mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}')$  exist and if in addition for  $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$  there exists  $t$  such  $f(\boldsymbol{\alpha}, t) \neq f(\boldsymbol{\alpha}', t)$ , and for  $\boldsymbol{\beta} \neq \boldsymbol{\beta}'$  there exists  $t$  such that  $\sigma^2(\boldsymbol{\beta}, t) \neq \sigma^2(\boldsymbol{\beta}', t)$ , then  $\mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$  and  $\mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}')$  are positive. See also Appendix B.

5) Expressions for  $\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$ ,  $\mathbf{J}_q^{(\boldsymbol{\beta})}$ ,  $\mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$  and  $\mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}')$  are given in Appendix B when the functions  $f(\boldsymbol{\theta}, t)$  and  $\sigma^2(\boldsymbol{\beta}, t)$  are periodic in time  $t$  as well as when the delays between two observations tend to 0 than when the delays are constant.

### 3 LAN property of the model

To establish the asymptotic normality and the asymptotic efficiency of the maximum likelihood estimator and of the Bayesian estimator, we will apply the method from (Ibragimov and Has'minskii 1981) on minimax efficiency. Thus, we study the asymptotic behaviour of the likelihood of the observation in the neighbourhood of the true value of the parameter. For this purpose we define the log-likelihood ratio

$$\Lambda_n^{(\boldsymbol{\theta}, \boldsymbol{w})} := \ln \left( \frac{d\mathbb{P}_n^{\boldsymbol{\theta} + \boldsymbol{w} \Phi_n^{(\boldsymbol{\theta})}}}{d\mathbb{P}_n^{\boldsymbol{\theta}}} ((Y_0, \dots, Y_n)) \right)$$

for  $\mathbf{w} \in \mathcal{W}_{\boldsymbol{\theta},n} := \{\mathbf{w} \in \mathbb{R}^d : \boldsymbol{\theta} + \mathbf{w}\Phi_n^{(\boldsymbol{\theta})} \in \Theta\}$ . Here the invertible  $d \times d$ -matrix (local normalizing matrix)  $\Phi_n^{(\boldsymbol{\theta})}$  is equal to  $\Phi_n^{(\boldsymbol{\theta})} := \text{diag}[\boldsymbol{\varphi}_n^{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \boldsymbol{\psi}_n^{(\boldsymbol{\beta})}]$  where  $\boldsymbol{\varphi}_n^{(\boldsymbol{\theta})} := (T_n \mathbf{J}_p^{(\boldsymbol{\alpha},\boldsymbol{\beta})})^{-1/2}$ ,  $\boldsymbol{\psi}_n^{(\boldsymbol{\beta})} := (n \mathbf{J}_q^{(\boldsymbol{\beta})})^{-1/2}$ . Furthermore  $P_{\boldsymbol{\theta} + \mathbf{w}\Phi_n^{(\boldsymbol{\theta})}}^n$  is the distribution of  $(Y_0, \dots, Y_n)$  when the value of the parameter is  $\boldsymbol{\theta} + \mathbf{w}\Phi_n^{(\boldsymbol{\theta})}$ , and  $P_{\boldsymbol{\theta}}^n$  is the distribution of  $(Y_0, \dots, Y_n)$  when the value of the parameter is  $\boldsymbol{\theta}$ . Now we state that the family of distribution densities  $\{dP_{\boldsymbol{\theta} + \mathbf{w}\Phi_n^{(\boldsymbol{\theta})}}^n / dP_{\boldsymbol{\theta}}^n\}$  is asymptotically normal as  $n \rightarrow \infty$ . More precisely

**Proposition 1** *Assume that  $\Theta = \mathbf{A} \times \mathbf{B}$  is open and convex, and conditions A1, A2 and A3 are fulfilled. Then the family  $\{P_{\boldsymbol{\theta}}^{(n)} : \boldsymbol{\theta} \in \Theta\}$  is uniformly locally asymptotically normal (uniformly LAN) in any compact subset  $\mathcal{K}$  of  $\Theta$ . That is for any compact subset  $\mathcal{K}$  of  $\Theta$ , for arbitrary sequences  $\{\boldsymbol{\theta}_n\} \subset \mathcal{K}$  and  $\{\mathbf{w}_n\} \subset \mathbb{R}^d$  such that  $\boldsymbol{\theta}_n + \mathbf{w}_n \Phi_n^{(\boldsymbol{\theta}_n)} \in \mathcal{K}$  and  $\mathbf{w}_n \rightarrow \mathbf{w} \in \mathbb{R}^d$  as  $n \rightarrow \infty$ , the log-likelihood ratio  $\Lambda_n^{(\boldsymbol{\theta}_n, \mathbf{w}_n)}$  can be decomposed as*

$$\Lambda_n^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} = \Delta_n^{(\boldsymbol{\theta}_n)} \mathbf{w}^* - \frac{1}{2} |\mathbf{w}|^2 + r_n(\boldsymbol{\theta}_n, \mathbf{w}_n)$$

where the random vector  $\Delta_n^{(\boldsymbol{\theta}_n)}$  converges in law to the standard normal distribution:

$$\lim_{n \rightarrow \infty} \mathcal{L}_{\boldsymbol{\theta}_n} [\Delta_n^{(\boldsymbol{\theta}_n)}] = \mathcal{N}_d(0_d, \mathbb{I}_{d \times d}),$$

$d = p + q$ , and the random variable  $r_n(\boldsymbol{\theta}_n, \mathbf{w}_n)$  converges in  $P_{\boldsymbol{\theta}_n}$ -probability to 0.

**Proof** Since the random variables  $Y_i$ ,  $i = 0, \dots, n$  are independent, the distribution of  $Y_0$  does not depend on  $\boldsymbol{\theta}$  and the random variable  $Y_i$ ,  $i \geq 1$ , is Gaussian with  $\mathcal{L}_{\boldsymbol{\theta}}[Y_i] = \mathcal{N}(F_i(\boldsymbol{\alpha}), G_i^2(\boldsymbol{\beta}))$ , the log-likelihood  $\Lambda_n^{(\boldsymbol{\theta}, \mathbf{w})}$  is equal to

$$\Lambda_n^{(\boldsymbol{\theta}, \mathbf{w})} = - \sum_{i=1}^n \ln \left( \frac{G_i(\boldsymbol{\beta} + \mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})})}{G_i(\boldsymbol{\beta})} \right) - \sum_{i=1}^n \left( \frac{(Y_i - F_i(\boldsymbol{\alpha} + \mathbf{u}\boldsymbol{\varphi}_n^{(\boldsymbol{\alpha},\boldsymbol{\beta})}))^2}{2G_i^2(\boldsymbol{\beta} + \mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})})} - \frac{(Y_i - F_i(\boldsymbol{\alpha}))^2}{2G_i^2(\boldsymbol{\beta})} \right)$$

where  $\mathbf{w} := (\mathbf{u}, \mathbf{v})$ . Then, plugging in the right hand side of the previous equality the expression (2) of  $Y_i$  when the true value of the parameter is  $\boldsymbol{\theta}$ , we can write

$$\Lambda_n^{(\boldsymbol{\theta}, \mathbf{w})} = \sum_{i=1}^n \left( M_{n,i}^{(\boldsymbol{\theta}, \mathbf{w})} + R_{n,i}^{(\boldsymbol{\theta}, \mathbf{w})} \right)$$

where

$$M_{n,i}^{(\boldsymbol{\theta}, \mathbf{w})} := \frac{(F_i(\boldsymbol{\alpha} + \mathbf{u}\boldsymbol{\varphi}_n^{(\boldsymbol{\alpha},\boldsymbol{\beta})}) - F_i(\boldsymbol{\alpha})) G_i(\boldsymbol{\beta})}{G_i^2(\boldsymbol{\beta} + \mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})})} W_i^{(\boldsymbol{\beta})} + \frac{1}{2} \left( 1 - \frac{G_i^2(\boldsymbol{\beta})}{G_i^2(\boldsymbol{\beta} + \mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})})} \right) ((W_i^{(\boldsymbol{\beta})})^2 - 1)$$



and

$$W_i^{(\beta)} := \frac{1}{G_i(\beta)} \int_{t_{i-1}}^{t_i} \sigma(\beta, t) dW_t.$$

Thus

$$R_{n,i}^{(\theta, \mathbf{w})} = \frac{-(F_i(\boldsymbol{\alpha} + \mathbf{u}\boldsymbol{\varphi}_n^{(\alpha, \beta)}) - F_i(\boldsymbol{\alpha}))^2}{2G_i^2(\beta + \mathbf{v}\boldsymbol{\psi}_n^{(\beta)})} + \frac{1}{2} \left( 1 - \frac{G_i^2(\beta)}{G_i^2(\beta + \mathbf{v}\boldsymbol{\psi}_n^{(\beta)})} + \ln \left( \frac{G_i^2(\beta)}{G_i^2(\beta + \mathbf{v}\boldsymbol{\psi}_n^{(\beta)})} \right) \right).$$

Finally to approximate  $M_{n,i}^{(\theta, \mathbf{w})}$  we define the  $d$ -dimensional random vector  $\Delta_n^{(\theta)} := \sum_{i=1}^n \Delta_{n,i}^{(\theta)}$  by

$$\Delta_{n,i}^{(\theta)} := \left( \frac{\nabla_{\boldsymbol{\alpha}} F_i(\boldsymbol{\alpha}) \boldsymbol{\varphi}_n^{(\alpha, \beta)}}{G_i(\beta)} W_i^{(\beta)}, \frac{(\nabla_{\beta} \ln G_i^2(\beta)) \boldsymbol{\psi}_n^{(\beta)}}{2} ((W_i^{(\beta)})^2 - 1) \right).$$

The random vectors  $\Delta_{n,i}^{(\theta)}$ ,  $i = 1, \dots, n$  are independent with mean zero and variance  $d \times d$ -matrix given by

$$\text{Var}_{\boldsymbol{\theta}} [\Delta_{n,i}^{(\theta)}] = \text{diag} \left[ \frac{\boldsymbol{\varphi}_n^{(\alpha, \beta)} \nabla_{\boldsymbol{\alpha}}^* F_i(\boldsymbol{\alpha}) \nabla_{\boldsymbol{\alpha}} F_i(\boldsymbol{\alpha}) \boldsymbol{\varphi}_n^{(\alpha, \beta)}}{G_i^2(\beta)}, \frac{\boldsymbol{\psi}_n^{(\beta)} \nabla_{\beta}^* \ln G_i^2(\beta) \nabla_{\beta} \ln G_i^2(\beta) \boldsymbol{\psi}_n^{(\beta)}}{2} \right].$$

Now let  $\mathcal{K}$  be a compact subset of  $\Theta$ . Let  $\{\boldsymbol{\theta}_n = (\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n)\} \subset \mathcal{K}$  and  $\{\mathbf{w}_n = (\mathbf{u}_n, \mathbf{v}_n)\} \subset \mathbb{R}^d$  such that  $\boldsymbol{\theta}_n + \mathbf{w}_n \Phi_n^{(\boldsymbol{\theta}_n)} \in \mathcal{K}$  and  $\mathbf{w}_n \rightarrow \mathbf{w} = (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^d$  as  $n \rightarrow \infty$ ,

In Lemma 1 in Appendix A we prove that the random vector  $\Delta_n^{(\boldsymbol{\theta}_n)}$  converges in distribution to the  $d$ -dimension standard Gaussian distribution, that is

$$\lim_{n \rightarrow \infty} \mathcal{L}_{\boldsymbol{\theta}_n} [\Delta_n^{(\boldsymbol{\theta}_n)}] = \mathcal{N}_d(0_d, \mathbb{I}_{d \times d}).$$

Now we show that  $M_n^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} - \Delta_n^{(\boldsymbol{\theta}_n)} \mathbf{w}_n^*$  converges to 0 in quadratic mean. Indeed, from the independence of the Gaussian variables  $W_i^{(\beta)}$ ,  $i = 1, \dots, n$

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\theta}_n} \left[ (M_n^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} - \Delta_n^{(\boldsymbol{\theta}_n)} \mathbf{w}_n^*)^2 \right] &= \sum_{i=1}^n \mathbb{E}_{\boldsymbol{\theta}_n} \left[ (M_{n,i}^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} - \Delta_{n,i}^{(\boldsymbol{\theta}_n)} \mathbf{w}_n^*)^2 \right] \\ &= \sum_{i=1}^n \left( \frac{(F_i(\boldsymbol{\alpha}_n + \mathbf{u}_n \boldsymbol{\varphi}_n^{(\alpha_n, \beta_n)}) - F_i(\boldsymbol{\alpha}_n)) G_i(\beta_n)}{G_i^2(\beta_n + \mathbf{v}_n \boldsymbol{\psi}_n^{(\beta_n)})} - \frac{\mathbf{u}_n \boldsymbol{\varphi}_n^{(\alpha_n, \beta_n)} \nabla_{\boldsymbol{\alpha}}^* F_i(\boldsymbol{\alpha}_n)}{G_i(\beta_n)} \right)^2 \\ &\quad + \frac{1}{2} \sum_{i=1}^n \left( 1 - \frac{G_i^2(\beta_n)}{G_i^2(\beta_n + \mathbf{v}_n \boldsymbol{\psi}_n^{(\beta_n)})} - \mathbf{v}_n \boldsymbol{\psi}_n^{(\beta_n)} \nabla_{\beta}^* \ln G_i^2(\beta_n) \right)^2. \end{aligned}$$

As  $\boldsymbol{\varphi}_n^{(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n)} = (T_n \mathbf{J}_p^{(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n)})^{-1/2}$  and  $\boldsymbol{\psi}_n^{(\boldsymbol{\beta}_n)} = (n \mathbf{J}_q^{(\boldsymbol{\beta}_n)})^{-1/2}$ , thanks to conditions A1 and A2 Taylor expansion expansion formula with integral remainder gives us

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\theta}_n} \left[ \left( M_n^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} - \Delta_n^{(\boldsymbol{\theta}_n)} \mathbf{w}_n^* \right)^2 \right] &= \sum_{i=1}^n \mathbb{E}_{\boldsymbol{\theta}_n} \left[ \left( M_{n,i}^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} - \Delta_n^{(\boldsymbol{\theta}_n)} \mathbf{w}_n^* \right)^2 \right] \\ &\leq c \left( |\mathbf{u}_n|^2 |(\mathbf{J}_p^{(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n)})^{-1/2}|^2 + |\mathbf{v}_n|^2 |(\mathbf{J}_q^{(\boldsymbol{\beta}_n)})^{-1/2}|^2 \right) \times \\ &\quad \times \sup_t \left( \sup_{|\boldsymbol{\alpha}' - \boldsymbol{\alpha}_n| \leq |\mathbf{u}_n \boldsymbol{\varphi}_n^{(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n)}|} \left| \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}', t) - \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}_n, t) \right|^2 \right. \\ &\quad \left. + \sup_{|\boldsymbol{\beta}' - \boldsymbol{\beta}_n| \leq |\mathbf{v}_n \boldsymbol{\psi}_n^{(\boldsymbol{\beta}_n)}|} \left( \left| \sigma^2(\boldsymbol{\beta}', t) - \sigma^2(\boldsymbol{\beta}_n, t) \right|^2 + \left| \nabla_{\boldsymbol{\alpha}} \sigma^2(\boldsymbol{\beta}', t) - \nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}_n, t) \right|^2 \right) \right) \end{aligned}$$

where  $c$  is some positive constant which can depend on  $\mathcal{K}$ . Under condition A1 and A3, the positive functions  $\boldsymbol{\theta} \mapsto |(\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})})^{-1/2}|$  and  $\boldsymbol{\theta} \mapsto |(\mathbf{J}_q^{(\boldsymbol{\beta})})^{-1/2}|$  are continuous in the compact subset  $\mathcal{K}$ , so they are bounded in  $\mathcal{K}$ . Since  $|\mathbf{u}_n \boldsymbol{\varphi}_n^{(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n)}|$  and  $|\mathbf{v}_n \boldsymbol{\psi}_n^{(\boldsymbol{\beta}_n)}|$  converge to 0 as  $n \rightarrow \infty$  and  $\{\boldsymbol{\theta}_n\} \subset \mathcal{K}$ , condition A1 implies that  $M_n^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} - \Delta_n^{(\boldsymbol{\theta}_n)} \mathbf{w}_n^*$  converges in quadratic mean to 0. Next, thanks again to Taylor expansion formula we have

$$\begin{aligned} \left| R_n^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} + \frac{|\mathbf{w}_n^*|^2}{2} \right| &\leq c \left( |\mathbf{u}_n|^2 |(\mathbf{J}_p^{(\boldsymbol{\theta}_n)})^{-1/2}|^2 + |\mathbf{v}_n|^2 |(\mathbf{J}_q^{(\boldsymbol{\beta}_n)})^{-1/2}|^2 \right) \times \\ &\quad \times \sup_t \left( \sup_{|\boldsymbol{\alpha}' - \boldsymbol{\alpha}| \leq |\mathbf{u}_n \boldsymbol{\varphi}_n^{(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n)}|} \left| \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}', t) - \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t) \right|^2 \right. \\ &\quad \left. + \sup_{|\boldsymbol{\beta}' - \boldsymbol{\beta}| \leq |\mathbf{v}_n \boldsymbol{\psi}_n^{(\boldsymbol{\beta}_n)}|} \left( \left| \sigma^2(\boldsymbol{\beta}', t) - \sigma^2(\boldsymbol{\beta}, t) \right|^2 \right. \right. \\ &\quad \left. \left. + \left| \nabla_{\boldsymbol{\alpha}} \sigma^2(\boldsymbol{\beta}', t) - \nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t) \right|^2 + \left| \nabla_{\boldsymbol{\alpha}}^2 \sigma^2(\boldsymbol{\beta}', t) - \nabla_{\boldsymbol{\beta}}^2 \sigma^2(\boldsymbol{\beta}, t) \right|^2 \right) \right). \end{aligned}$$

Then we readily deduce that the random variable

$$r_n(\boldsymbol{\theta}_n, \mathbf{w}_n) := M_n^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} + R_n^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} - \Delta_n^{(\boldsymbol{\theta}_n, \mathbf{w}_n)} \mathbf{w}_n^* + \frac{|\mathbf{w}_n^*|^2}{2}$$

converges in  $\mathbb{P}_{\boldsymbol{\theta}_n}$ -probability to 0 as  $n \rightarrow \infty$ . This achieves the proof of the LAN property of the model. ■

## 4 Efficient estimation

Cramér-Rao lower bound of the mean square risk is not entirely satisfactory to define the asymptotic efficiency of a sequence of estimators. See e.g. Section I.9 in (Ibragimov

Khasminskii 1981), see also Section 1.3 in (Kutoyants 2009). Then we consider here the asymptotic optimality in the sense of local asymptotic minimax lower bound of the risk of the sequence  $\{\bar{\boldsymbol{\theta}}_n\} := \{\bar{\boldsymbol{\theta}}_n, n > 0\}$  for the estimation of  $\boldsymbol{\theta}$ , that is

$$\mathcal{R}_{\boldsymbol{\theta}}(\{\bar{\boldsymbol{\theta}}_n\}) := \lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \sup_{|\boldsymbol{\theta}' - \boldsymbol{\theta}| \leq \epsilon} \mathbb{E}_{\boldsymbol{\theta}'} \left[ L \left( \sqrt{T_n}(\bar{\boldsymbol{\alpha}}_n - \boldsymbol{\alpha}'), \sqrt{n}(\bar{\boldsymbol{\beta}}_n - \boldsymbol{\beta}') \right) \right]$$

where  $\bar{\boldsymbol{\theta}}_n$  is any statistic function of the observation  $\{X_{t_i}, i = 0, \dots, n\}$  or, which is equivalent, of  $\{Y_i, i = 0, \dots, n\}$ . The loss function  $L(\cdot)$  belongs to the set  $\mathcal{L}$  of non-negative Borel functions on  $\mathbb{R}^d$  which are continuous at 0 with  $L(0_d) = 0$ ,  $L(-x) = L(x)$ , the set  $\{x : L(x) < c\}$  is a convex set for any  $c > 0$ , and we also assume that the function  $L(\cdot) \in \mathcal{L}$  admits a polynomial majorant. Clearly all functions  $L(\boldsymbol{\theta}) = |\boldsymbol{\theta}|^a$ ,  $a > 0$ , as well as  $L(\boldsymbol{\theta}) = 1_{\{|\boldsymbol{\theta}| > a\}}$ ,  $a > 0$ , belong to  $\mathcal{L}$ . (Here  $1_{\{x > a\}}$  denotes the indicator function of  $(a, \infty)$ .)

Since the model of observation is locally asymptotically normal then the local asymptotic minimax risk  $\mathcal{R}_{\boldsymbol{\theta}}(\{\bar{\boldsymbol{\theta}}_n\})$  for any sequence  $\{\bar{\boldsymbol{\theta}}_n = (\bar{\boldsymbol{\alpha}}_n, \bar{\boldsymbol{\beta}}_n)\}$  of estimators of  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$  admits a lower bound for any loss function  $L \in \mathcal{L}$ . More precisely

$$\mathcal{R}_{\boldsymbol{\theta}}(\{\bar{\boldsymbol{\theta}}_n\}) \geq \mathbb{E}[L(\boldsymbol{\xi}^{(\boldsymbol{\theta})})] \quad (6)$$

where  $\boldsymbol{\xi}^{(\boldsymbol{\theta})}$  is a random  $d$ -dimensional vector whose distribution is centered Gaussian with  $d \times d$ -matrix variance equal to  $(\mathbf{J}^{(\boldsymbol{\theta})})^{-1}$  (see Le Cam 1969 and Hajek 1972; see also Ibragimov Has'minskii 1981).

## 4.1 Maximum Likelihood estimator

The maximum likelihood estimator  $\hat{\boldsymbol{\theta}}_n$  is any statistics defined from the observation such that

$$\hat{\boldsymbol{\theta}}_n \in \arg \sup_{\boldsymbol{\theta} \in \Theta} \Lambda_n(\boldsymbol{\theta}).$$

In the next theorem we establish that  $\hat{\boldsymbol{\theta}}_n$  is an efficient estimator of  $\boldsymbol{\theta}$  in the sense that its asymptotic minimax risk  $\mathcal{R}_{\boldsymbol{\theta}}(\{\hat{\boldsymbol{\theta}}_n\})$  is equal to the lower bound  $\mathbb{E}[L(\boldsymbol{\xi}^{(\boldsymbol{\theta})})]$ .

**Theorem 1** *Let  $\Theta = \mathbf{A} \times \mathbf{B}$  be open, convex and bounded. Assume that conditions A1–A4 are fulfilled. Then the maximum likelihood estimator  $\hat{\boldsymbol{\theta}}_n = (\hat{\boldsymbol{\alpha}}_n, \hat{\boldsymbol{\beta}}_n)$  of  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$  is consistent. It is asymptotically normal uniformly with respect to  $\boldsymbol{\theta}$  varying in any compact subset  $\mathcal{K}$  of  $\Theta = \mathbf{A} \times \mathbf{B}$ :*

$$\lim_{n \rightarrow \infty} \mathcal{L}_{\boldsymbol{\theta}} \left[ \left( \sqrt{T_n}(\hat{\boldsymbol{\alpha}}_n - \boldsymbol{\alpha}), \sqrt{n}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \right) \right] = \mathcal{N}_d(0_d, (\mathbf{J}^{(\boldsymbol{\theta})})^{-1})$$

where  $\mathbf{J}^{(\theta)} = \text{diag}[\mathbf{J}_p^{(\alpha, \beta)}, \mathbf{J}_q^{(\beta)}]$ . Moreover it is locally asymptotically minimax at any  $\theta \in \Theta$  for any loss function  $L(\cdot) \in \mathcal{L}$ , in the sense that inequality (6) becomes an equality for  $\bar{\theta}_n = \hat{\theta}_n$ .

**Proof** To prove this theorem, we state that in our framework the following conditions B1–B4 from Theorem 1.1 and Corollary 1.1 of Chapter III in (Ibragimov and Has'minskii 1981) are fulfilled. Denote by  $Z_n^{(\theta, \mathbf{w})}$  the likelihood ratio

$$Z_n^{(\theta, \mathbf{w})} := e^{\Lambda_n^{(\theta, \mathbf{w})}} = \frac{d\mathbf{P}^{(n)}_{\theta + \mathbf{w}\Phi_n^{(\theta)}}}{d\mathbf{P}^{(n)}_{\theta}}((Y_0, \dots, Y_n)).$$

Then we are going to establish the following properties

- B1 The family  $\{\mathbf{P}_{\theta}^{(n)}, \theta \in \Theta\}$  is uniformly LAN in any compact subset of  $\Theta$ .
- B2 For every  $\theta \in \Theta$ , the  $d \times d$ -matrix  $\Phi_n^{(\theta)}$  is positive definite,  $d = p + q$ , and there exists a continuous  $d \times d$ -matrix valued function  $(\theta, \theta') \mapsto B(\theta, \theta')$  such that for every compact subset  $\mathcal{K}$  of  $\Theta$

$$\lim_{n \rightarrow \infty} \sup_{\theta \in \mathcal{K}} |\Phi_n^{(\theta)}| = 0$$

and

$$\lim_{n \rightarrow \infty} (\Phi_n^{(\theta)})^{-1} \Phi_n^{(\theta')} = B(\theta, \theta')$$

where the lastest convergence is uniform with respect to  $\theta$  and  $\theta'$  varying in  $\mathcal{K}$ .

- B3 For every compact subset  $\mathcal{K}$  of  $\Theta$ , there exist  $b > p + q$ ,  $m > 0$ ,  $B = B(\mathcal{K}) > 0$ ,  $a = a(\mathcal{K}) \in \mathbb{R}$ , such that

$$\sup_{\theta \in \mathcal{K}} \sup_{\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}_{\theta, r, n}} |\mathbf{w}_1 - \mathbf{w}_2|^{-b} \mathbf{E}_{\theta} \left[ \left( (Z_n^{(\theta, \mathbf{w}_1)})^{1/2m} - (Z_n^{(\theta, \mathbf{w}_2)})^{1/2m} \right)^{2m} \right] < B(1 + r^a)$$

for any  $r > 0$ . Here  $\mathcal{W}_{\theta, r, n} := \{\mathbf{w} \in \mathbb{R}^d : |\mathbf{w}| < r \text{ and } \theta + \mathbf{w}\Phi_n^{(\theta)} \in \Theta\}$ .

- B4 For any compact subset  $\mathcal{K}$  of  $\Theta$ , and for every  $N > 0$ , there exists  $n_1 = n_1(N, \mathcal{K}) > 0$  such that

$$\sup_{\theta \in \mathcal{K}} \sup_{n > n_1} \sup_{\mathbf{w} \in \mathcal{W}_{\theta, n}} |\mathbf{w}|^N \mathbf{E}_{\theta} \left[ (Z_n^{(\theta, \mathbf{w})})^{1/2} \right] < \infty.$$

Recall that  $\mathcal{W}_{\theta, n} := \{\mathbf{w} \in \mathbb{R}^d : \theta + \mathbf{w}\Phi_n^{(\theta)} \in \Theta\}$ .

In Proposition 1 we have stated that the family  $\{\mathbb{P}_\theta^{(n)}, \theta \in \Theta\}$  is uniformly LAN in any compact subset of  $\Theta$  (condition B1). In addition, as  $\Phi_n^{(\theta)} = \text{diag}[\varphi_n^{(\alpha, \beta)}, \psi_n^{(\beta)}]$ ,  $\varphi_n^{(\alpha, \beta)} := (T_n \mathbf{J}_p^{(\alpha, \beta)})^{-1/2}$  and  $\psi_n^{(\beta)} := (n \mathbf{J}_q^{(\beta)})^{-1/2}$ , from the continuity of  $\theta \mapsto \mathbf{J}_p^{(\alpha, \beta)}$  and  $\beta \mapsto \mathbf{J}_q^{(\beta)}$  we deduce that condition B2 is fulfilled with

$$B(\theta, \theta') = \text{diag} \left[ (\mathbf{J}_p^{(\alpha, \beta)})^{1/2} (\mathbf{J}_p^{(\alpha', \beta')})^{-1/2}, (\mathbf{J}_q^{(\beta)})^{1/2} (\mathbf{J}_q^{(\beta')})^{-1/2} \right].$$

Now we check condition B3. Let the compact subset  $\mathcal{K} \subset \Theta$  the integer  $m > 0$  and  $r > 0$  be fixed. As  $\mathbb{E}_\theta [Z_n^{(\theta, \mathbf{w}_1)}] = \mathbb{E}_\theta [Z_n^{(\theta, \mathbf{w}_2)}] = 1$ , we have

$$\begin{aligned} & \mathbb{E}_\theta \left[ \left| (Z_n^{(\theta, \mathbf{w}_1)})^{1/2m} - (Z_n^{(\theta, \mathbf{w}_2)})^{1/2m} \right|^{2m} \right] \\ &= \sum_{k=0}^{2m} (-1)^{2m-k} \binom{2m}{k} \mathbb{E}_\theta \left[ (Z_n^{(\theta, \mathbf{w}_1)})^{k/2m} (Z_n^{(\theta, \mathbf{w}_2)})^{(2m-k)/2m} \right] \\ &\leq \sum_{k=0}^{2m} \binom{2m}{k} \mathbb{E}_\theta [Z_n^{(\theta, \mathbf{w}_1)}]^{k/2m} \mathbb{E}_\theta [Z_n^{(\theta, \mathbf{w}_2)}]^{(2m-k)/2m} = 2^{2m}. \end{aligned} \quad (7)$$

Thus we deduce that

$$|\mathbf{w}_1 - \mathbf{w}_2|^{-b} \mathbb{E}_\theta \left[ \left| (Z_n^{(\theta, \mathbf{w}_1)})^{1/2m} - (Z_n^{(\theta, \mathbf{w}_2)})^{1/2m} \right|^{2m} \right] \leq 2^{2m} |\mathbf{w}_1 - \mathbf{w}_2|^{-b} \leq 2^{2m} R^{-b} \quad (8)$$

for any  $b > 0$ , any  $R > 0$  and for any  $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}_{\theta, n}$  such that  $|\mathbf{w}_2 - \mathbf{w}_1| \geq R$ . Henceforth we choose  $R = 1 > 0$  and we consider that  $|\mathbf{w}_2 - \mathbf{w}_1| < 1$ . Assumption A1 entails that

$$\begin{aligned} & \mathbb{E}_\theta \left[ \left| (Z_n^{(\theta, \mathbf{w}_1)})^{1/2m} - (Z_n^{(\theta, \mathbf{w}_2)})^{1/2m} \right|^{2m} \right] \\ &= \mathbb{E}_\theta \left[ \left( \int_0^1 \partial_s \exp \left\{ \frac{1}{2m} \Lambda_n(\theta + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1)) \Phi_n^{(\theta)}) \right\} ds \right)^{2m} \right] \\ &= (2m)^{-2m} \int_0^1 \mathbb{E}_{\theta + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1)) \Phi_n^{(\theta)}} \left[ (\partial_s \Lambda_n(\theta + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1)) \Phi_n^{(\theta)}))^{2m} \right] ds. \end{aligned}$$

Now let

$$\begin{aligned} W_i^{(\beta, \mathbf{v}_1, \mathbf{v}_2, s)} &:= \frac{1}{G(\beta + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1)) \psi_n^{(\beta)})} \int_{t_{i-1}}^{t_i} \sigma(\beta + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1)) \psi_n^{(\beta)}, t) dW_t \\ U_n^{(\theta, \mathbf{w}_1, \mathbf{w}_2, s)} &:= \varphi_n^{(\alpha, \beta)} \sum_{i=1}^n \frac{\nabla_\alpha^* F_i(\alpha + (\mathbf{u}_1 + s(\mathbf{u}_2 - \mathbf{u}_1)) \varphi_n^{(\alpha, \beta)})}{G_i(\beta + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1)) \psi_n^{(\beta)})} W_i^{(\beta, \mathbf{v}_1, \mathbf{v}_2, s)} \end{aligned}$$

and

$$V_n^{(\beta, \mathbf{w}_1, \mathbf{w}_2, s)} := \boldsymbol{\psi}_n^{(\beta)} \sum_{i=1}^n \frac{\nabla_{\beta}^* G_i^2(\boldsymbol{\beta} + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1))\boldsymbol{\psi}_n^{(\beta)})}{2G_i^2(\boldsymbol{\beta} + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1))\boldsymbol{\psi}_n^{(\beta)})} \left( (W_i^{(\beta, \mathbf{v}_1, \mathbf{v}_2, s)})^2 - 1 \right).$$

When

$$\begin{aligned} Y_i &= F_i(\boldsymbol{\alpha} + (\mathbf{u}_1 + s(\mathbf{u}_2 - \mathbf{u}_1))\boldsymbol{\varphi}_n^{(\alpha, \beta)}) + \int_{t_{i-1}}^{t_i} \sigma(\boldsymbol{\beta} + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1))\boldsymbol{\psi}_n^{(\beta)}, t) dW_t \\ &= F_i(\boldsymbol{\alpha} + (\mathbf{u}_1 + s(\mathbf{u}_2 - \mathbf{u}_1))\boldsymbol{\varphi}_n^{(\alpha, \beta)}) + G_i(\boldsymbol{\beta} + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1))\boldsymbol{\psi}_n^{(\beta)}) W_i^{(\beta, \mathbf{v}_1, \mathbf{v}_2, s)}, \end{aligned}$$

expression (3) of the log-likelihood implies that

$$\partial_s \Lambda_n(\boldsymbol{\theta} + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1))\Phi_n^{(\boldsymbol{\theta})}) = (\mathbf{u}_2 - \mathbf{u}_1)U_n^{(\boldsymbol{\theta}, \mathbf{w}_1, \mathbf{w}_2, s)} + (\mathbf{v}_2 - \mathbf{v}_1)V_n^{(\boldsymbol{\theta}, \mathbf{w}_1, \mathbf{w}_2, s)}.$$

Thus

$$\begin{aligned} & \left( \mathbb{E}_{\boldsymbol{\theta} + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1))\Phi_n^{(\boldsymbol{\theta})}} \left[ \left( \partial_s \Lambda_n(\boldsymbol{\theta} + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1))\Phi_n^{(\boldsymbol{\theta})}) \right)^{2m} \right] \right)^{1/2m} \\ & \leq \left( \mathbb{E}_{\boldsymbol{\theta} + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1))\Phi_n^{(\boldsymbol{\theta})}} \left[ \left( (\mathbf{u}_2 - \mathbf{u}_1)U_n^{(\boldsymbol{\theta}, \mathbf{v}_1, \mathbf{v}_2, s)} \right)^{2m} \right] \right)^{1/2m} \\ & \quad + \left( \mathbb{E}_{\boldsymbol{\theta} + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1))\Phi_n^{(\boldsymbol{\theta})}} \left[ \left( (\mathbf{v}_2 - \mathbf{v}_1)V_n^{(\boldsymbol{\theta}, \mathbf{v}_1, \mathbf{v}_2, s)} \right)^{2m} \right] \right)^{1/2m} \end{aligned}$$

Since the random variables  $W_i^{(\beta, \mathbf{v}_1, \mathbf{v}_2, s)}$ ,  $i = 1, \dots, n$  are independent with the same standard Gaussian distribution  $\mathcal{N}(0, 1)$ , the random variable  $(\mathbf{u}_2 - \mathbf{u}_1)U_n^{(\boldsymbol{\theta}, \mathbf{v}_1, \mathbf{v}_2, s)}$  is Gaussian with variance

$$\begin{aligned} & \mathbb{E}_{\boldsymbol{\theta} + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1))\Phi_n^{(\boldsymbol{\theta})}} \left[ \left( (\mathbf{u}_2 - \mathbf{u}_1)U_n^{(\boldsymbol{\theta}, \mathbf{v}_1, \mathbf{v}_2, s)} \right)^2 \right] \\ & = \sum_{i=1}^n \left( (\mathbf{u}_2 - \mathbf{u}_1)\boldsymbol{\varphi}_n^{(\alpha, \beta)} \frac{\nabla_{\alpha}^* F_i(\boldsymbol{\alpha} + (\mathbf{u}_1 + s(\mathbf{u}_2 - \mathbf{u}_1))\boldsymbol{\varphi}_n^{(\alpha, \beta)})}{G_i(\boldsymbol{\beta} + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1))\boldsymbol{\psi}_n^{(\beta)})} \right)^2. \end{aligned}$$

Recall that  $\boldsymbol{\varphi}_n^{(\alpha, \beta)} = (T_n \mathbf{J}_p^{(\alpha, \beta)})^{-1/2}$ . Then the moment of order  $2m$  of the random variable  $(\mathbf{u}_2 - \mathbf{u}_1)U_n^{(\boldsymbol{\theta}, \mathbf{v}_1, \mathbf{v}_2, s)}$  is equal to

$$\begin{aligned} & \frac{(2m)!}{2^m m!} \left( \sum_{i=1}^n \left( (\mathbf{u}_2 - \mathbf{u}_1)\boldsymbol{\varphi}_n^{(\alpha, \beta)} \frac{\nabla_{\alpha}^* F_i(\boldsymbol{\alpha} + (\mathbf{u}_1 + s(\mathbf{u}_2 - \mathbf{u}_1))\boldsymbol{\varphi}_n^{(\alpha, \beta)})}{G_i(\boldsymbol{\beta} + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1))\boldsymbol{\psi}_n^{(\beta)})} \right)^2 \right)^m \\ & \leq \frac{(2m)!}{2^m m!} |\mathbf{u}_2 - \mathbf{u}_1|^{2m} |(J_p^{(\boldsymbol{\theta})})^{-1/2}|^{2m} \sup_{\alpha, \beta} \left| \frac{1}{T_n} \sum_{i=1}^n \frac{\nabla_{\alpha}^* F_i(\alpha) \nabla_{\alpha} F_i(\alpha)}{G_i^2(\beta)} \right|^m \\ & \leq \frac{(2m)!}{2^m m!} |\mathbf{u}_2 - \mathbf{u}_1|^{2m} |(J_p^{(\boldsymbol{\theta})})^{-1/2}|^{2m} \frac{\sup_{\alpha', t} |\nabla_{\alpha} f(\alpha', t)|^{2m}}{\inf_{\beta', t} \sigma^{2m}(\beta', t)}. \end{aligned}$$

To estimate the moment of order  $2m$  of the random variable  $(\mathbf{v}_2 - \mathbf{v}_1)V_n^{(\beta, \mathbf{v}_1, \mathbf{v}_2, s)}$  we can compute the Laplace function

$$L_n^{(V)}(z) = \mathbb{E}_{\boldsymbol{\theta} + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1))\Phi_n^{(\boldsymbol{\theta})}} \left[ \exp \left\{ z(\mathbf{v}_2 - \mathbf{v}_1)V_n^{(\beta, \mathbf{v}_1, \mathbf{v}_2, s)} \right\} \right]$$

of this random variable and we apply the well-known relationship between the moment of order  $2m$  and the  $2m$ -th derivative of the Laplace function at 0 :

$$\partial_z^{2m} L_n^{(V)}(0) = \mathbb{E}_{\boldsymbol{\theta} + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1))\Phi_n^{(\boldsymbol{\theta})}} \left[ \left( (\mathbf{v}_2 - \mathbf{v}_1)V_n^{(\beta, \mathbf{v}_1, \mathbf{v}_2, s)} \right)^{2m} \right].$$

This is done in Appendix A, and Lemma 3 ensures that there exists  $n_0 > 0$  such for every integers  $n > n_0$  and  $m \geq 1$

$$\begin{aligned} |\partial_z^{2m} L_n^{(V)}(0)| &= \mathbb{E}_{\boldsymbol{\theta} + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1))\Phi_n^{(\boldsymbol{\theta})}} \left[ \left( (\mathbf{v}_2 - \mathbf{v}_1)V_n^{(\beta, \mathbf{v}_1, \mathbf{v}_2, s)} \right)^{2m} \right] \\ &\leq c_{2m} |\mathbf{v}_2 - \mathbf{v}_1|^{2m} \times \left| (\mathbf{J}_q^{(\beta)})^{-1/2} \right|^{2m} \times \frac{\sup_{\boldsymbol{\beta}', t} |\nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}', t)|^{2m}}}{\inf_{\boldsymbol{\beta}', t} \sigma^{4m}(\boldsymbol{\beta}', t)} \end{aligned}$$

where  $c_{2m}$  is some constant value depending only on  $m$ .

Then for any  $n \geq n_0$

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\theta}} \left[ \left( \left( Z_n^{(\boldsymbol{\theta}, \mathbf{w}_1)} \right)^{1/2m} - \left( Z_n^{(\boldsymbol{\theta}, \mathbf{w}_2)} \right)^{1/2m} \right)^{2m} \right]^{1/2m} \\ \leq c_{m,1} |\mathbf{w}_2 - \mathbf{w}_1| \times \left| (\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})})^{-1/2} \right| \times \frac{\sup_{\boldsymbol{\alpha}', t} |\nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}', t)|}{\inf_{\boldsymbol{\beta}', t} \sigma^2(\boldsymbol{\beta}', t)} \\ + c_{m,2} |\mathbf{v}_2 - \mathbf{v}_1| \times \left| (\mathbf{J}_q^{(\beta)})^{-1/2} \right| \times \frac{\sup_{\boldsymbol{\beta}', t} |\nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}', t)|}{\inf_{\boldsymbol{\beta}', t} \sigma^2(\boldsymbol{\beta}', t)} \end{aligned}$$

where  $c_{1,m} \geq 0$  and  $c_{2,m} \geq 0$  are two constant values depending only on  $m$ . Since  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \mapsto \left| (\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})})^{-1/2} \right|$  and  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \mapsto \left| (\mathbf{J}_q^{(\beta)})^{-1/2} \right|$  are positive continuous functions on the compact set  $\mathcal{K}$  we deduce that

$$|\mathbf{w}_2 - \mathbf{w}_1|^{-2m} \mathbb{E}_{\boldsymbol{\theta}} \left[ \left( \left( Z_n^{(\boldsymbol{\theta}, \mathbf{w}_1)} \right)^{1/2m} - \left( Z_n^{(\boldsymbol{\theta}, \mathbf{w}_2)} \right)^{1/2m} \right)^{2m} \right] \leq c_m \quad (9)$$

for any  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{K}$ , for any  $w_1, w_2 \in \mathcal{W}_{\boldsymbol{\theta}, n}$  such that  $|\mathbf{w}_2 - \mathbf{w}_1| \leq 1$  and for any  $n \geq n_0$ . Here  $c_m$  is some constant which depends on  $m$  and  $\mathcal{K}$ .

From inequalities (8) and (9) we readily deduce that condition B3 is satisfied with  $b = 2m > p + q$ ,  $a = 0$ , and at least for any  $n \geq n_0$ .

Finally, we establish that condition B4 is fulfilled. To do that we study the term  $\mathbb{E}_{\boldsymbol{\theta}} \left[ \left( Z_n^{(\boldsymbol{\theta}, \mathbf{w})} \right)^{1/2} \right]$ , first in the case  $|\mathbf{w}\Phi_n^{(\boldsymbol{\theta})}|$  is "small" for which we use Taylor expansion

formula (assumptions A1, A2) and then in the case  $|\mathbf{w}\Phi_n^{(\theta)}|$  is "large" for which we use the identifiability condition A4. Thanks to equality (12),

$$\begin{aligned} & \ln \mathbb{E}_{\theta} \left[ e^{\frac{1}{2}(\Lambda_n(\theta+\mu) - \Lambda_n(\theta))} \right] \\ & \leq - \sum_{i=1}^n \frac{(F_i(\alpha + \delta) - F_i(\alpha))^2}{8 \sup_{\beta,t} \sigma^2(\beta, t)(t_i - t_{i-1})} - \sum_{i=1}^n \frac{(G_i^2(\beta + \gamma) - G_i^2(\beta))^2}{16 \sup_{\beta,t} \sigma^4(\beta, t)(t_i - t_{i-1})^2} \\ & \leq - \frac{\inf_{\beta,t} \sigma^2(\beta, t)}{8 \sup_{\beta,t} \sigma^2(\beta, t)} \sum_{i=1}^n \frac{(F_i(\alpha + \delta) - F_i(\alpha))^2}{G_i^2(\beta)} \\ & \quad - \frac{\inf_{\beta,t} \sigma^4(\beta, t)}{16 \sup_{\beta,t} \sigma^4(\beta, t)} \sum_{i=1}^n \left( 1 - \frac{G_i^2(\beta)}{G_i^2(\beta + \gamma)} \right)^2 \end{aligned}$$

for any  $\theta$  and  $\theta + \mu \in \Theta$  where  $\theta = (\alpha, \beta)$  and  $\mu = (\delta, \gamma)$ .

(i) From assumptions A1, A2 and A3 with Taylor expansion formula, there exist  $\nu > 0$  and  $n_1 > 0$  such that for every  $n > n_1$ ,  $\theta = (\alpha, \beta) \in \mathcal{K}$  and  $\mathbf{w} \in \mathcal{W}_{\theta,n}$  such that  $|\mathbf{w}\Phi_n^{(\theta)}| < \nu$ , we have

$$\sum_{i=1}^n \frac{(F_i(\alpha + \mathbf{w}\varphi_n^{(\alpha,\beta)}) - F_i(\alpha))^2}{G_i^2(\beta)} \geq \frac{|\mathbf{w}|^2}{2}$$

and

$$\sum_{i=1}^n \left( 1 - \frac{G_i^2(\beta)}{G_i^2(\beta + \mathbf{v}\psi_n^{(\beta)})} \right)^2 \geq \frac{|\mathbf{v}|^2}{2}.$$

Thus

$$\ln \mathbb{E}_{\theta} \left[ \exp \left\{ \frac{1}{2} (\Lambda_n(\theta + \mathbf{w}\Phi_n^{(\theta)}) - \Lambda_n(\theta)) \right\} \right] \leq -c_1(\nu) |\mathbf{w}|^2$$

where

$$c_1(\nu) := \min \left\{ \frac{\inf_{\beta,t} \sigma^2(\beta, t)}{16 \sup_{\beta,t} \sigma^2(\beta, t)}, \frac{\inf_{\beta,t} \sigma^4(\beta, t)}{32 \sup_{\beta,t} \sigma^4(\beta, t)} \right\} > 0.$$

(ii) Besides from the identifiability condition A4, for every  $\nu > 0$ , there exist  $\mu_\nu > 0$  and  $n_\nu > 0$  such that for  $n > n_\nu$ ,  $\theta = (\alpha, \beta) \in \mathcal{K}$  and  $\mu = (\delta, \gamma)$  with  $\theta + \mu \in \Theta$  and  $|\mu| \geq \nu$  we have

$$\frac{1}{T_n} \sum_{i=1}^n \frac{(F_i(\alpha + \delta) - F_i(\alpha))^2}{t_i - t_{i-1}} \geq \mu_\nu.$$



Let  $\mathbf{w} = (\mathbf{u}, \mathbf{v}) \in \mathcal{W}_{\boldsymbol{\theta}, n}$  such that  $|\mathbf{w}\Phi_n^{(\boldsymbol{\theta})}| \geq \nu$ . As  $|\mathbf{u}\boldsymbol{\varphi}_n^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}| \leq \text{diam}(\mathbf{A})$ , we deduce that

$$\sum_{i=1}^n \frac{(F_i(\boldsymbol{\alpha} + \mathbf{u}\boldsymbol{\varphi}_n^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}) - F_i(\boldsymbol{\alpha}))^2}{t_i - t_{i-1}} \geq \frac{T_n \mu_\nu |\mathbf{u}\boldsymbol{\varphi}_n^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}|^2}{\text{diam}(\mathbf{A})^2} \geq \frac{\mu_\nu |\mathbf{u}|^2}{\text{diam}(\mathbf{A})^2 |(\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})})^{1/2}|^2} > 0.$$

Notice that we have used the relation  $|\mathbf{u}(\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})})^{-1/2}|^2 \geq |\mathbf{u}|^2 |(\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})})^{1/2}|^{-2}$ . Let

$$c_2(\nu) := \frac{\mu_\nu}{8 \text{diam}(\mathbf{A})^2 \sup_{\boldsymbol{\theta} \in \mathcal{K}} |(\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})})^{1/2}|^2 \sup_{\boldsymbol{\beta}, t} \sigma^2(\boldsymbol{\beta}, t)} > 0.$$

Then for every  $n \geq \eta := \max\{n_1, n_\nu\}$

$$\sum_{i=1}^n \frac{(F_i(\boldsymbol{\alpha} + \mathbf{u}\boldsymbol{\varphi}_n^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}) - F_i(\boldsymbol{\alpha}))^2}{4(G_i^2(\boldsymbol{\beta} + \mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})}) + G_i^2(\boldsymbol{\beta}))} \geq \min\{c_1(\nu), c_2(\nu)\} |\mathbf{u}|^2. \quad (10)$$

(iii) From the identifiability condition A4, for  $n \geq n_\nu$  and  $|\boldsymbol{\mu}| \geq \nu$  we have

$$\frac{1}{n} \sum_{i=1}^n \frac{(G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma}) - G_i^2(\boldsymbol{\beta}))^2}{(t_i - t_{i-1})^2} \geq \mu_\nu.$$

Let  $\mathbf{w} = (\mathbf{u}, \mathbf{v}) \in \mathcal{W}_{\boldsymbol{\theta}, n}$  such that  $|\mathbf{w}\Phi_n^{(\boldsymbol{\theta})}| \geq \nu$ . Since  $|\mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})}| \leq \text{diam}(\mathbf{B})$  we have

$$\sum_{i=1}^n \frac{(G_i^2(\boldsymbol{\beta}) - G_i^2(\boldsymbol{\beta} + \mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})}))^2}{(t_i - t_{i-1})^2} \geq \frac{n \mu_\nu |\mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})}|^2}{\text{diam}(\mathbf{B})^2} \geq \frac{\mu_\nu |\mathbf{v}|^2}{\text{diam}(\mathbf{B})^2 |(\mathbf{J}_q^{(\boldsymbol{\beta})})^{1/2}|^2}.$$

Let

$$c_3(\nu) := \frac{\mu_\nu}{16 \text{diam}(\mathbf{B})^2 \sup_{\boldsymbol{\theta} \in \mathcal{K}} |(\mathbf{J}_q^{(\boldsymbol{\beta})})^{1/2}|^2 \sup_{\boldsymbol{\beta}, t} \sigma^4(\boldsymbol{\beta}, t)} > 0.$$

Hence for every  $n > \eta$

$$\sum_{i=1}^n \int_{G_i^2(\boldsymbol{\beta})}^{G_i^2(\boldsymbol{\beta} + \mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})})} \frac{G_i^2(\boldsymbol{\beta} + \mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})}) - x}{4x(x + G_i^2(\boldsymbol{\beta} + \mathbf{v}\boldsymbol{\psi}_n^{(\boldsymbol{\beta})}))} dx \geq \min\{c_1(\nu), c_3(\nu)\} |\mathbf{v}|^2. \quad (11)$$

(iv) Denote

$$c := \min\{c_1(\nu), c_2(\nu), c_3(\nu)\} > 0.$$

So thanks to inequalities (10) and (11), for  $n > \eta$ ,  $\boldsymbol{\theta} \in \mathcal{K}$  and  $\mathbf{w} = (\mathbf{u}, \mathbf{v}) \in \mathcal{W}_{\boldsymbol{\theta}, n}$  such that  $|\mathbf{w}\Phi_n^{(\boldsymbol{\theta})}| \geq \nu$  we obtain that

$$\mathbb{E}_{\boldsymbol{\theta}} \left[ (Z_n^{(\boldsymbol{\theta}, \mathbf{w})})^{1/2} \right] \leq e^{-c|\mathbf{w}|^2}.$$

As

$$\lim_{|\mathbf{w}| \rightarrow \infty} \mathbf{w}^N e^{-c|\mathbf{w}|^2} = 0,$$

we deduce that for all  $N > 0$

$$\sup_{\boldsymbol{\theta} \in \mathcal{K}} \sup_{n > \eta} \sup_{\mathbf{w} \in \mathcal{W}_{\boldsymbol{\theta}, n}} |\mathbf{w}|^N \mathbb{E}_{\boldsymbol{\theta}} \left[ (Z_n^{(\boldsymbol{\theta}, \mathbf{w})})^{1/2} \right] < \infty.$$

Thus condition B4 is satisfied. This achieves the proof of the theorem. ■

## 4.2 Bayesian estimator

Here the unknown parameter  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$  is supposed to be a random vector with known prior density distribution  $\pi(\cdot)$  on the parameter set  $\Theta = \mathbf{A} \times \mathbf{B}$ . We are going to study the property of the Bayesian estimator  $\tilde{\boldsymbol{\theta}}_n$  that minimizes the mean Bayesian risk defined as

$$R_n(\bar{\boldsymbol{\theta}}_n) := \int_{\Theta} \mathbb{E}_{\boldsymbol{\theta}} \left[ l((\bar{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \boldsymbol{\delta}_n) \right] \pi(\boldsymbol{\theta}) d\boldsymbol{\theta},$$

where for simplicity of presentation the loss function  $l(\cdot)$  is equal to  $l(\boldsymbol{\theta}) = |\boldsymbol{\theta}|^a$  for some  $a > 0$  (see e.g. Ibragimov and Has'minskii 1981). Here  $\boldsymbol{\delta}_n = \text{diag}[\sqrt{T_n} \mathbb{I}_{p \times p}, \sqrt{n} \mathbb{I}_{q \times q}]$ . From Fubini theorem we can write

$$R_n(\bar{\boldsymbol{\theta}}_n) = \mathbb{E}_{\boldsymbol{\theta}_o} \left[ \int_{\Theta} l((\bar{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \boldsymbol{\delta}_n) L_n^{(\boldsymbol{\theta}_o, \boldsymbol{\theta})} \pi(\boldsymbol{\theta}) d\boldsymbol{\theta} \right]$$

for any fixed value  $\boldsymbol{\theta}_o$  of  $\Theta$ , where  $L_n^{(\boldsymbol{\theta}_o, \boldsymbol{\theta})}$  is the likelihood ratio.

$$L_n^{(\boldsymbol{\theta}_o, \boldsymbol{\theta})} := \frac{d\mathbb{P}_{\boldsymbol{\theta}}^{(n)}}{d\mathbb{P}_{\boldsymbol{\theta}_o}^{(n)}}(Y_0, Y_1, \dots, Y_n).$$

If there exists an estimator  $\tilde{\boldsymbol{\theta}}_n$  which minimizes

$$\int_{\Theta} l((\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \boldsymbol{\delta}_n) L_n^{(\boldsymbol{\theta}_o, \boldsymbol{\theta})} \pi(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

then it will be Bayesian. For a quadratic loss function ( $a = 2$ ) this minimization gives the expression of the Bayesian estimator through a conditional expectation

$$\tilde{\boldsymbol{\theta}}_n = \int_{\Theta} \boldsymbol{\theta} \pi(\boldsymbol{\theta} | Y_0, \dots, Y_n) d\boldsymbol{\theta}$$

where

$$\pi(\boldsymbol{\theta} | Y_0, \dots, Y_n) := \frac{L_n^{(\boldsymbol{\theta}_o, \boldsymbol{\theta})} \pi(\boldsymbol{\theta})}{\int_{\Theta} L_n^{(\boldsymbol{\theta}_o, \boldsymbol{\theta})} \pi(\boldsymbol{\theta}) d\boldsymbol{\theta}}.$$

Then, from Theorem 2.1 in Chapter III of (Ibragimov and Has'minskii 1981) we state that

**Theorem 2** *Let  $\Theta = \mathbf{A} \times \mathbf{B}$  be open convex and bounded. Assume that the conditions of Theorem 1 are fulfilled. Assume that the prior density  $\pi(\boldsymbol{\theta})$  is continuous and positive on  $\Theta$  and that the loss function  $l(\boldsymbol{\theta}) = |\boldsymbol{\theta}|^a$  for some  $a > 0$ . Then, uniformly with respect to  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$  varying in any compact subset  $\mathcal{K}$  of  $\Theta$ , the corresponding Bayesian estimator  $\tilde{\boldsymbol{\theta}}_n = (\tilde{\boldsymbol{\alpha}}_n, \tilde{\boldsymbol{\beta}}_n)$  converges in probability and is asymptotically normal:*

$$\lim_{n \rightarrow \infty} \mathcal{L}_{\boldsymbol{\theta}} \left[ \left( \sqrt{T_n}(\tilde{\boldsymbol{\alpha}}_n - \boldsymbol{\alpha}), \sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \right) \right] = \mathcal{N}_d(0_d, (\mathbf{J}^{(\boldsymbol{\theta})})^{-1}).$$

Moreover, the Bayesian estimator  $\tilde{\boldsymbol{\theta}}_n$  is locally asymptotically minimax at any  $\boldsymbol{\theta} \in \Theta$  for any loss function  $L(\cdot) \in \mathcal{L}$ , in the sense that inequality (6) becomes an equality for  $\bar{\boldsymbol{\theta}}_n = \tilde{\boldsymbol{\theta}}_n$ .

**Proof** This is a direct consequence of Theorem 2.1 in Chapter III of (Ibragimov and Has'minskii 1981) and the proof of Theorem 1. ■

## 5 Linear parameter models

### 5.1 Non-parametrized variance

Here we consider the specific case where  $f(\boldsymbol{\alpha}, t) = \boldsymbol{\alpha} \mathbf{f}(t)^* = \alpha_1 f_1(t) + \dots + \alpha_p f_p(t)$ ,  $\boldsymbol{\theta} = \boldsymbol{\alpha}$  and  $\Theta = \mathbf{A} \subset \mathbb{R}^p$ :

$$dX_t = \boldsymbol{\alpha} \mathbf{f}(t)^* dt + \sigma(t) dW_t.$$

The functions  $f_1(\cdot), \dots, f_p(\cdot)$  are such that there exists a positive definite  $p \times p$ -matrix  $\mathbf{J}$  which fulfils

$$\mathbf{J} = \lim_{n \rightarrow \infty} \frac{1}{T_n} \sum_{i=1}^n \frac{\mathbf{F}_i^* \mathbf{F}_i}{G_i^2}.$$

Here  $\mathbf{f}(t) := (f_1(t), \dots, f_p(t))$ ,  $\mathbf{F}_i := \int_{t_{i-1}}^{t_i} \mathbf{f}(t) dt$  and  $G_i^2 := \int_{t_{i-1}}^{t_i} \sigma^2(t) dt$ .

Then  $F_i(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \mathbf{F}_i$ ,  $\nabla_{\boldsymbol{\alpha}} F_i(\boldsymbol{\alpha}) = \mathbf{F}_i$ ,  $\mathbf{J}^{(\boldsymbol{\alpha})} = \mathbf{J}_p^{(\boldsymbol{\alpha})} = \mathbf{J}$  and

$$\mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = \liminf_{n \rightarrow \infty} (\boldsymbol{\alpha} - \boldsymbol{\alpha}') \left( \frac{1}{T_n} \sum_{i=1}^n \frac{\mathbf{F}_i^* \mathbf{F}_i}{t_i - t_{i-1}} \right) (\boldsymbol{\alpha} - \boldsymbol{\alpha}')^*.$$

Furthermore the maximum likelihood estimator  $\widehat{\boldsymbol{\alpha}}_n$  has an explicit expression obtained as a zero of the gradient of  $\Lambda_n(\boldsymbol{\alpha})$  so

$$\widehat{\boldsymbol{\alpha}}_n = \left( \sum_{i=1}^n \frac{\mathbf{F}_i^* \mathbf{F}_i}{G_i^2} \right)^{-1} \left( \sum_{i=1}^n \frac{Y_i \mathbf{F}_i}{G_i^2} \right).$$

Since

$$Y_i = Y_i^{(\boldsymbol{\alpha})} := \boldsymbol{\alpha} \mathbf{F}_i^* + \int_{t_{i-1}}^{t_i} \sigma(t) dW_t.$$

we obtain that

$$\widehat{\boldsymbol{\alpha}}_n = \boldsymbol{\alpha} + \left( \sum_{i=1}^n \frac{\mathbf{F}_i^* \mathbf{F}_i}{G_i^2} \right)^{-1} \left( \sum_{i=1}^n \frac{\mathbf{F}_i}{G_i^2} \int_{t_{i-1}}^{t_i} \sigma(t) dW_t \right).$$

Then the maximum likelihood estimator  $\widehat{\boldsymbol{\alpha}}_n$  is a Gaussian variable which converges in norm  $L^r$  to  $\boldsymbol{\alpha}$  as  $n \rightarrow \infty$  for any  $r \geq 1$ . The variance of  $\widehat{\boldsymbol{\alpha}}_n$  is equal to

$$\text{Var}_{\boldsymbol{\alpha}}[\widehat{\boldsymbol{\alpha}}_n] = \left( \sum_{i=1}^n \frac{\mathbf{F}_i^* \mathbf{F}_i}{G_i^2} \right)^{-1}.$$

The asymptotic variance matrix of  $\sqrt{T_n}(\widehat{\boldsymbol{\alpha}}_n - \boldsymbol{\alpha})$  is equal to  $\mathbf{J}^{-1}$

## 5.2 Linear parametrized noise

Here we consider the case where  $f(\boldsymbol{\alpha}, t) = \boldsymbol{\alpha} \mathbf{f}(t)^* = \alpha_1 f_1(t) + \dots + \alpha_p f_p(t)$ , and  $\sigma^2(\beta, t) = \beta \sigma^2(t)$ .  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \beta)$  and  $\Theta = \mathbf{A} \times \mathbf{B} \subset \mathbb{R}^p \times \mathbb{R}^+$ :

$$dX_t = \boldsymbol{\alpha} \mathbf{f}(t)^* dt + \sqrt{\beta} \sigma(t) dW_t.$$

In this case  $p \geq 1$ ,  $q = 1$ . Moreover

$$\mathbf{J}_p^{(\boldsymbol{\alpha}, \beta)} = \lim_{n \rightarrow \infty} \frac{1}{T_n \beta^2} \sum_{i=1}^n \frac{\mathbf{F}_i^* \mathbf{F}_i}{G_i^2}, \quad \mathbf{J}_q^{(\beta)} = \frac{1}{\beta^2}$$

$$\mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = \liminf_{n \rightarrow \infty} (\boldsymbol{\alpha} - \boldsymbol{\alpha}') \left( \frac{1}{T_n} \sum_{i=1}^n \frac{\mathbf{F}_i^* \mathbf{F}_i}{t_i - t_{i-1}} \right) (\boldsymbol{\alpha} - \boldsymbol{\alpha}')^*$$

and

$$\mu_q(\beta, \beta') = \left( \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{G_i^4}{(t_i - t_{i-1})^2} \right) (\beta - \beta')^2.$$

Furthermore with the same notation than in subsection 5.1 we have

$$\begin{aligned}\widehat{\boldsymbol{\alpha}}_n &= \left( \sum_{i=1}^n \frac{\mathbf{F}_i^* \mathbf{F}_i}{G_i^2} \right)^{-1} \left( \sum_{i=1}^n \frac{Y_i \mathbf{F}_i}{G_i^2} \right) \\ &= \boldsymbol{\alpha} + \left( \sum_{i=1}^n \frac{\mathbf{F}_i \mathbf{F}_i^*}{G_i^2} \right)^{-1} \left( \sum_{i=1}^n \frac{\mathbf{F}_i}{G_i^2} \int_{t_{i-1}}^{t_i} \sigma(t) dW_t \right)\end{aligned}$$

and

$$\widehat{\beta}_n = \frac{1}{n} \sum_{i=1}^n \frac{(Y_i - \widehat{\boldsymbol{\alpha}}_n \mathbf{F}_i^*)^2}{G_i^2}.$$

We readily obtain that the maximum likelihood estimator  $(\widehat{\boldsymbol{\alpha}}_n, \widehat{\beta}_n)$  converges in norm  $L^r$  to  $(\boldsymbol{\alpha}, \beta)$  as  $n \rightarrow \infty$  for any  $r \geq 1$ .

## 6 Appendix A

Next, we establish some technical results.

**Lemma 1** *The random vector  $\Delta_n^{(\boldsymbol{\theta}_n)}$  converges in distribution to the  $d$ -dimension standard Gaussian distribution:*

$$\lim_{n \rightarrow \infty} \mathcal{L}_{\boldsymbol{\theta}_n} [\Delta_n^{(\boldsymbol{\theta}_n)}] = \mathcal{N}_d(0_d, \mathbb{I}_{d \times d}).$$

**Proof** We know that if  $W$  is a standard Gaussian random variable, that is  $\mathcal{L}(W) = \mathcal{N}(0, 1)$ , then for every  $a, b, c \in \mathbb{R}$  with  $a < 1/2$  we have

$$\mathbb{E} \left[ e^{aW^2 + bW + c} \right] = \exp \left[ \frac{b^2}{2(1-2a)} + c - \frac{1}{2} \ln(1-2a) \right].$$

First let  $n_1$

$$n_1 \geq \frac{4 \sup_{\boldsymbol{\beta}} |(\mathbf{J}_q^{(\boldsymbol{\beta})})^{-1/2}|^2 \sup_{\boldsymbol{\beta}, t} |\nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t)|}{\inf_{\boldsymbol{\beta}, t} \sigma^4(\boldsymbol{\beta}, t)}$$

thus, as  $\boldsymbol{\psi}_n^{(\boldsymbol{\beta})} = (n \mathbf{J}_q^{(\boldsymbol{\beta})})^{-1/2}$ , for every  $n > n_1$  we have

$$|(\nabla_{\boldsymbol{\beta}} \ln G_i^2(\boldsymbol{\beta})) \boldsymbol{\psi}_n^{(\boldsymbol{\beta})}|^2 \leq \frac{|(\mathbf{J}_q^{(\boldsymbol{\beta})})^{-1/2}|^2 \sup_{\boldsymbol{\beta}, t} |\nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t)|}{n \inf_{\boldsymbol{\beta}, t} \sigma^4(\boldsymbol{\beta}, t)} \leq \frac{1}{2}.$$

Hence for  $n > n_1$  and  $\mathbf{w} = (\mathbf{u}, v) \in \mathbb{R}^d$  with  $|\mathbf{w}| < 1$ , we have

$$\mathbb{E}_{\boldsymbol{\theta}_n} \left[ e^{\Delta_{n,i}^{(\boldsymbol{\theta}_n)} \mathbf{w}^*} \right] = \exp \left[ \frac{b_{n,i}^2}{2(1-2a_{n,i})} + c_{n,i} - \frac{1}{2} \ln(1-2a_{n,i}) \right]$$

where

$$a_{n,i} = -c_{n,i} = \frac{(\nabla_{\beta} \ln G_i^2(\beta_n), \psi_n^{(\beta_n)} \mathbf{v}^*)}{2}$$

$$b_{n,i} = \frac{(\nabla_{\alpha} F_i(\alpha_n)) \phi_n^{(\theta_n)} \mathbf{u}^*}{G_i(\beta_n)}$$

The independence of the Gaussian variables  $W_i$ ,  $i = 1, \dots, n$ , implies that

$$\ln \mathbb{E}_{\theta_n} \left[ e^{\Delta_n^{(\theta_n)} \mathbf{w}^*} \right] = \sum_{i=1}^n \frac{b_{n,i}^2}{2(1-2a_{n,i})} + \sum_{i=1}^n \left( c_{n,i} - \frac{\ln(1-2a_{n,i})}{2} \right).$$

Since  $n > n_1$  and  $|\mathbf{v}_1| < 1$  we have  $|a_{n,i}| \leq 1/4$  and

$$\left| \frac{b_{n,i}^2}{1-2a_{n,i}} - b_{n,i}^2 \right| \leq \int_0^1 \frac{2b_{n,i}^2 |a_{n,i}|}{(1-2a_{n,i}x)^2} dx \leq \frac{2b_{n,i}^2 |a_{n,i}|}{(1-2|a_{n,i}|)^2} \leq 8b_{n,i}^2 |a_{n,i}|$$

as well as

$$\left| c_{n,i} - \frac{\ln(1-2a_{n,i})}{2} - a_{n,i}^2 \right| = \left| \frac{\ln(1-2a_{n,i})}{2} + a_{n,i} + a_{n,i}^2 \right| \leq \int_0^1 \frac{4|a_{n,i}|^3 x^2}{|1-2a_{n,i}x|} dx \leq 8|a_{n,i}|^3.$$

Thus

$$\left| \ln \mathbb{E} \left[ e^{\mathbf{w}^* \Delta_n^{(\theta)}} \right] - \frac{1}{2} \sum_{i=1}^n (b_{n,i}^2 + 2a_{n,i}^2) \right| \leq 4 \sum_{i=1}^n (b_{n,i}^2 + 2a_{n,i}^2) |a_{n,i}|.$$

We know that  $\psi_n^{(\beta_n)} = (n\mathbf{J}_q^{(\beta_n)})^{-1/2}$ , and  $\varphi_n^{(\alpha_n, \beta_n)} = (T_n \mathbf{J}_p^{(\theta_n)})^{-1/2}$ . We deduce that

$$|a_{n,i}|^2 \leq |\mathbf{v} \psi_n^{(\beta_n)}|^2 \frac{\sup_{\beta, t} |\nabla_{\beta} \sigma^2(\beta, t)|^2}{\inf_{\beta, t} |\sigma^2(\beta, t)|^2} \leq \frac{|\mathbf{v}|^2 |\mathbf{J}_q(\beta_n)^{-1/2}|^2}{n} \times \frac{\sup_{\beta, t} |\nabla_{\beta} \sigma^2(\beta, t)|^2}{\inf_{\beta, t} |\sigma^2(\beta, t)|^2}.$$

and assumption A3 entails that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (b_{n,i}^2 + 2a_{n,i}^2) = |\mathbf{u}|^2 + |\mathbf{v}|^2.$$

We deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\theta_n} \left[ e^{\Delta_n^{(\theta_n)} \mathbf{w}^*} \right] = e^{\frac{|\mathbf{w}|^2}{2}}$$

for any  $\mathbf{w}$  such  $|\mathbf{w}| < 1$ . This is sufficient to conclude that the random vector  $\Delta_n^{(\theta_n)}$  converges in distribution to the  $d$ -dimension standard Gaussian distribution. ■

**Lemma 2** Let  $0 < z < 1$  be fixed. For  $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$  and  $\boldsymbol{\mu} = (\boldsymbol{\delta}, \boldsymbol{\gamma})$  such  $\boldsymbol{\theta}, \boldsymbol{\theta} + \boldsymbol{\mu} \in \Theta$  convex, we have

$$\begin{aligned} \ln \mathbb{E}_{\boldsymbol{\theta}} \left[ e^{z(\Lambda_n(\boldsymbol{\theta} + \boldsymbol{\mu}) - \Lambda_n(\boldsymbol{\theta}))} \right] &= - \sum_{i=1}^n \frac{(F_i(\boldsymbol{\alpha} + \boldsymbol{\delta}) - F_i(\boldsymbol{\alpha}))^2}{2((1-z)^{-1}G_i^2(\boldsymbol{\beta}) + z^{-1}G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma}))} \\ &\quad - \sum_{i=1}^n \int_{G_i^2(\boldsymbol{\beta})}^{G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})} \frac{G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma}) - x}{2x((1-z)^{-1}x + z^{-1}G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma}))} dx. \end{aligned} \quad (12)$$

**Proof** When  $Y_i = F_i(\boldsymbol{\alpha}) + \int_{t_{i-1}}^{t_i} \sigma(\boldsymbol{\beta}, t) dW_t$  we can write

$$\Lambda_n(\boldsymbol{\theta} + \boldsymbol{\mu}) - \Lambda_n(\boldsymbol{\theta}) = \sum_{i=1}^n (a_{n,i}W_i^2 + b_{n,i}W_i + c_{n,i})$$

where

$$\begin{aligned} a_{n,i} &= \frac{1}{2} \left( 1 - \frac{G_i^2(\boldsymbol{\beta})}{G_i^2(\boldsymbol{\beta} + \boldsymbol{\mu})} \right), \\ b_{n,i} &= \frac{(F_i(\boldsymbol{\alpha} + \boldsymbol{\delta}) - F_i(\boldsymbol{\alpha}))G_i(\boldsymbol{\beta})}{G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})}, \\ c_{n,i} &= \frac{-(F_i(\boldsymbol{\alpha} + \boldsymbol{\delta}) - F_i(\boldsymbol{\alpha}))^2}{2G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})} + \ln \left( \frac{G_i(\boldsymbol{\beta})}{G_i(\boldsymbol{\beta} + \boldsymbol{\gamma})} \right) \end{aligned}$$

and

$$W_i = \frac{1}{G_i(\boldsymbol{\beta})} \int_{t_{i-1}}^{t_i} \sigma(\boldsymbol{\beta}, t) dW_t.$$

Let  $0 < z < 1$  be fixed. Since the random variables  $W_i$ ,  $i = 1, \dots, n$  are independent with the standard Gaussian distribution  $\mathcal{N}(0, 1)$ , and since  $za_{n,i} < 1/2$ , we obtain

$$\ln \mathbb{E}_{\boldsymbol{\theta}} \left[ e^{z(\Lambda_n(\boldsymbol{\theta} + \boldsymbol{\mu}) - \Lambda_n(\boldsymbol{\theta}))} \right] = \sum_{i=1}^n \left( \frac{z^2 b_{n,i}^2}{2(1 - 2za_{n,i})} + zc_{n,i} - \frac{1}{2} \ln(1 - 2za_{n,i}) \right).$$

Moreover

$$1 - 2a_{n,i} = 1 - z + \frac{zG_i^2(\boldsymbol{\beta})}{G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})},$$

$$\ln(1 - 2a_{n,i}) = \int_{G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})}^{G_i^2(\boldsymbol{\beta})} \frac{zdx}{zx + (1-z)G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})}$$

and

$$\ln \left( \frac{G_i^2(\boldsymbol{\beta})}{G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})} \right) = - \int_{G_i^2(\boldsymbol{\beta})}^{G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})} \frac{dx}{x}.$$

Hence

$$\frac{z^2 b_{n,i}^2}{1 - 2za_{n,i}} - \frac{z(F_i(\boldsymbol{\alpha} + \boldsymbol{\delta}) - F_i(\boldsymbol{\alpha}))^2}{G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})} = \frac{-z(1-z)(F_i(\boldsymbol{\alpha} + \boldsymbol{\delta}) - F_i(\boldsymbol{\alpha}))^2}{zG_i^2(\boldsymbol{\beta}) + (1-z)G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})}$$

and

$$-\ln(1 - 2za_{n,i}) + z \ln \left( \frac{G_i(\boldsymbol{\beta})}{G_i(\boldsymbol{\beta} + \boldsymbol{\gamma})} \right) = \int_{G_i^2(\boldsymbol{\beta})}^{G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma})} \frac{z(1-z)(x - G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma}))}{x(zx + (1-z)G_i^2(\boldsymbol{\beta} + \boldsymbol{\gamma}))} dx.$$

Then the lemma is proved ■

**Lemma 3** *There exists  $n_0 > 0$  such that for every  $n \geq n_0$ ,  $\boldsymbol{\theta} \in \Theta$ ,  $s \in [0, 1]$  and  $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}_{\boldsymbol{\theta}, n}$  with  $|\mathbf{w}_1 - \mathbf{w}_2| \leq 1$ , the Laplace function  $z \mapsto L_n^{(V)}(z)$  is well defined, differentiable with respect to  $z$  in the interval  $(-1, 1)$ , and we have*

$$|\partial_z^r L_n^{(V)}(0)| \leq c_r |\mathbf{v}_2 - \mathbf{v}_1|^r \times |(\mathbf{J}_q^{(\boldsymbol{\beta})})^{-1/2}|^r \times \frac{\sup_{\boldsymbol{\beta}', t} |\nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}', t)|^r}{\inf_{\boldsymbol{\beta}', t} \sigma^{2r}(\boldsymbol{\beta}', t)} \quad (13)$$

for every integer  $r \geq 2$ , where  $c_r$  is some constant value depending only on  $r$ .

**Proof** For this purpose, let

$$\mathbf{a}_i := \boldsymbol{\psi}_n^{(\boldsymbol{\beta})} \frac{\nabla_{\boldsymbol{\beta}}^* G_i^2(\boldsymbol{\beta} + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1))\boldsymbol{\psi}_n^{(\boldsymbol{\beta})})}{2G_i^2(\boldsymbol{\beta} + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1))\boldsymbol{\psi}_n^{(\boldsymbol{\beta})})}$$

Then

$$V_n^{(\boldsymbol{\beta}, \mathbf{v}_1, \mathbf{v}_2, s)} = (\mathbf{v}_2 - \mathbf{v}_1) \sum_{i=1}^n \mathbf{a}_i ((W_i^{(\boldsymbol{\beta}, \mathbf{v}_1, \mathbf{v}_2, s)})^2 - 1) = \sum_{i=1}^n V_{n,i}$$

where  $V_{n,i} = (\mathbf{v}_2 - \mathbf{v}_1) \mathbf{a}_i ((W_i^{(\boldsymbol{\beta}, \mathbf{v}_1, \mathbf{v}_2, s)})^2 - 1)$ . As  $\boldsymbol{\psi}_n^{(\boldsymbol{\beta})} = (n\mathbf{J}_q^{(\boldsymbol{\beta})})^{-1/2}$ , assumptions A1 and A2 imply that

$$|\mathbf{a}_i| \leq \frac{1}{\sqrt{n}} \times \frac{\sup_{\boldsymbol{\beta}} |(\mathbf{J}_q^{(\boldsymbol{\beta})})^{-1/2}| \sup_{\boldsymbol{\beta}, t} \sigma^2(\boldsymbol{\beta}, t)}{2 \inf_{\boldsymbol{\beta}, t} \sigma^2(\boldsymbol{\beta}, t)},$$

thus there exist  $n_1$  such for  $n > n_1$ ,  $|a_i| \leq 1/2$ . Moreover as  $|\mathbf{v}_1 - \mathbf{v}_2| \leq |\mathbf{w}_1 - \mathbf{w}_2| \leq 1$  and the random variables  $W_i^{(\boldsymbol{\beta}, \mathbf{v}_1, \mathbf{v}_2, s)}$ ,  $i = 1, \dots, n$ , have the same standard Gaussian distribution  $\mathcal{N}(0, 1)$ , for  $n > n_1$  the Laplace function  $z \mapsto L_{n,i}^{(V)}(z)$  of  $V_{n,i}$  is well defined on  $(-1, 1)$  and

$$\begin{aligned} L_{n,i}^{(V)}(z) &:= \mathbf{E}_{\boldsymbol{\theta} + (\mathbf{w}_1 + s(\mathbf{w}_2 - \mathbf{w}_1))\Phi_n^{(\boldsymbol{\theta})}} \left[ \exp \left[ z(\mathbf{v}_2 - \mathbf{v}_1) \mathbf{a}_i ((W_i^{(\boldsymbol{\beta}, \mathbf{v}_1, \mathbf{v}_2, s)})^2 - 1) \right] \right] \\ &= \exp \left[ -z(\mathbf{v}_2 - \mathbf{v}_1) \mathbf{a}_i - \frac{1}{2} \ln(1 - 2z(\mathbf{v}_2 - \mathbf{v}_1) \mathbf{a}_i) \right]. \end{aligned}$$



Recall that the random variables  $W_i^{(\boldsymbol{\beta}, \mathbf{v}_1, \mathbf{v}_2, s)}$ ,  $i = 1, \dots, n$ , are independent, hence the Laplace function of  $(\mathbf{v}_2 - \mathbf{v}_1)\boldsymbol{\psi}_n^{(\boldsymbol{\beta})}V_n^{(\boldsymbol{\beta}, \mathbf{v}_1, \mathbf{v}_2, s)}$  is equal to

$$L_n^{(V)}(z) = \prod_{i=1}^n L_{n,i}^{(V)}(z).$$

Consequently

$$\ln L_n^{(V)}(z) = z \sum_{i=1}^n (\mathbf{v}_1 - \mathbf{v}_2)\mathbf{a}_i - \frac{1}{2} \sum_{i=1}^n \ln(1 + 2z(\mathbf{v}_1 - \mathbf{v}_2)\mathbf{a}_i)$$

for any  $z \in (-1, 1)$ .

Notice that in one hand for every integer  $r \geq 1$ , the  $r$ th derivative of  $L_n^{(V)}(z)$  at  $z \in (-1, 1)$  exists and is equal to

$$\partial_z^r L_n^{(V)}(z) = \sum_{k=0}^{r-1} \binom{r-1}{k} \partial_z^k L_n^{(V)}(z) \partial_z^{r-k} \ln L_n^{(V)}(z). \quad (14)$$

In the other hand

$$\partial_z \ln L_n^{(V)}(z) = \sum_{i=1}^n (\mathbf{v}_1 - \mathbf{v}_2)\mathbf{a}_i - \sum_{i=1}^n \frac{(\mathbf{v}_1 - \mathbf{v}_2)\mathbf{a}_i}{1 + 2z(\mathbf{v}_1 - \mathbf{v}_2)\mathbf{a}_i}$$

and for every integer  $r \geq 2$

$$\partial_z^r \ln L_n^{(V)}(z) = (-1)^r \times 2^{r-1} \sum_{i=1}^n \frac{((\mathbf{v}_1 - \mathbf{v}_2)\mathbf{a}_i)^r}{(1 + 2z(\mathbf{v}_1 - \mathbf{v}_2)\mathbf{a}_i)^r}.$$

Then  $\partial_z \ln L_n^{(V)}(0) = 0$  and for every integer  $r \geq 2$

$$\begin{aligned} |\partial_z^r \ln L_n^{(V)}(0)| &\leq 2^{r-1} \sum_{i=1}^n |(\mathbf{v}_1 - \mathbf{v}_2)\mathbf{a}_i|^r \\ &\leq 2^{r-1} \sum_{i=1}^n \left| (\mathbf{v}_1 - \mathbf{v}_2)\boldsymbol{\psi}_n^{(\boldsymbol{\beta})} \frac{\nabla_{\boldsymbol{\beta}}^* G_i^2(\boldsymbol{\beta} + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1))\boldsymbol{\psi}_n^{(\boldsymbol{\beta})})}{2G_i^2(\boldsymbol{\beta} + (\mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1))\boldsymbol{\psi}_n^{(\boldsymbol{\beta})})} \right|^r \\ &\leq 2^{r-2} |\mathbf{v}_2 - \mathbf{v}_1|^r |(\mathbf{J}_q^{(\boldsymbol{\theta})})^{-1/2}|^r n^{1-r/2} \frac{\sup_{\boldsymbol{\beta}', t} |\nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}', t)|^r}{\inf_{\boldsymbol{\beta}', t} \sigma^{2r}(\boldsymbol{\beta}', t)} \end{aligned}$$

as  $\boldsymbol{\psi}_n^{(\boldsymbol{\beta})} = (n\mathbf{J}_q^{(\boldsymbol{\beta})})^{-1/2}$ . Thanks to equality (14), by induction we obtain inequality 13).

This achieves the proof of the lemma. ■

## 7 Appendix B

To illustrate assumptions A3 and A4 we state the following results the proofs of which are left to the reader. (See also the particular case of the linear parameters models in Section 5).

First, when we assume the continuity with respect to  $t$  and that the delays between observations go to 0, the sums in the definitions of  $\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$ ,  $\mathbf{J}_q^{(\boldsymbol{\beta})}$ ,  $\mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$  and  $\mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}')$  can be replaced by integrals.

**Lemma 4** *Assume that the function  $t \mapsto (f(\boldsymbol{\alpha}, t), \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t))$  uniformly continuous in  $\mathbb{R}^+$  uniformly with respect to  $\boldsymbol{\alpha}$  varying in  $\mathbf{A}$  and that the function  $t \mapsto (\sigma^2(\boldsymbol{\beta}, t), \nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t))$  is uniformly continuous in  $\mathbb{R}^+$  uniformly with respect to  $\boldsymbol{\beta}$  varying in  $\mathbf{B}$ . Assume also that  $\inf_{\boldsymbol{\beta}, t} \sigma^2(\boldsymbol{\beta}, t) > 0$  and  $h_n \rightarrow 0$ . Then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{T_n} \sup_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \left| \sum_{i=1}^n \frac{\nabla_{\boldsymbol{\alpha}}^* F_i(\boldsymbol{\alpha}) \nabla_{\boldsymbol{\alpha}} F_i(\boldsymbol{\alpha})}{G_i^2(\boldsymbol{\beta})} - \int_0^{T_n} \frac{\nabla_{\boldsymbol{\alpha}}^* f(\boldsymbol{\alpha}, t) \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t)}{\sigma^2(\boldsymbol{\beta}, t)} dt \right| &= 0, \\ \lim_{n \rightarrow \infty} \frac{1}{T_n} \sup_{\boldsymbol{\alpha}, \boldsymbol{\alpha}'} \left| \sum_{i=1}^n \frac{(F_i(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha}'))^2}{t_i - t_{i-1}} - \int_0^{T_n} (f(\boldsymbol{\alpha}, t) - f(\boldsymbol{\alpha}', t))^2 dt \right| &= 0.. \end{aligned}$$

When, in addition  $t_i - t_{i-1} = h_n \rightarrow 0$ , then

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{\boldsymbol{\beta}} \left| \frac{1}{n} \sum_{i=1}^n \nabla_{\boldsymbol{\beta}}^* \ln G_i^2(\boldsymbol{\beta}) \nabla_{\boldsymbol{\beta}} \ln G_i^2(\boldsymbol{\beta}) - \frac{1}{T_n} \int_0^{T_n} \nabla_{\boldsymbol{\beta}}^* \ln \sigma^2(\boldsymbol{\beta}, t) \nabla_{\boldsymbol{\beta}} \ln \sigma^2(\boldsymbol{\beta}, t) dt \right| &= 0, \\ \lim_{n \rightarrow \infty} \sup_{\boldsymbol{\beta}, \boldsymbol{\beta}'} \left| \frac{1}{n} \sum_{i=1}^n \frac{(G_i^2(\boldsymbol{\beta}) - G_i^2(\boldsymbol{\beta}'))^2}{(t_i - t_{i-1})^2} - \frac{1}{T_n} \int_0^{T_n} (\sigma^2(\boldsymbol{\beta}, t) - \sigma^2(\boldsymbol{\beta}', t))^2 dt \right| &= 0. \end{aligned}$$

**Almost periodic functions** First recall that a function  $t \mapsto \phi(\boldsymbol{\chi}, t)$  is almost periodic in  $\mathbb{R}$  uniformly with respect to  $\boldsymbol{\chi}$  varying in a set  $X$  when for every  $\epsilon > 0$ , there exists  $l_\epsilon > 0$  such that for any  $a \in \mathbb{R}$  there is  $\rho \in [a, a + l_\epsilon]$  for which

$$\sup_{\boldsymbol{\chi}, t} |\phi(\boldsymbol{\chi}, t + \rho) - \phi(\boldsymbol{\chi}, t)| \leq \epsilon.$$

See e.g chapters II and IV in (Corduneanu 1968). As an example, let  $k$  be a positive number and let  $\lambda_1, \dots, \lambda_k$  be  $k$  distinct real numbers. Then the function  $\phi(\boldsymbol{\chi}, t) = \chi_1 \cos(\lambda_1 t) + \dots + \chi_k \cos(\lambda_k t)$  is almost periodic in  $t$  uniformly with respect to  $\boldsymbol{\chi} = (\chi_1, \dots, \chi_k)$  varying in any compact subset  $X$  of  $\mathbb{R}^k$ .

Now Lemma 4 can be applied when the function  $t \mapsto (f(\boldsymbol{\alpha}, t), \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t))$  is almost periodic in  $\mathbb{R}$  uniformly with respect to  $\boldsymbol{\alpha}$  varying in  $\mathbf{A}$ , the function  $t \mapsto (\sigma^2(\boldsymbol{\beta}, t), \nabla_{\boldsymbol{\beta}} \sigma^2(\boldsymbol{\beta}, t))$  is almost periodic in  $\mathbb{R}$  uniformly with respect to  $\boldsymbol{\beta}$  varying in  $\mathbf{B}$ ,  $\inf_{\boldsymbol{\beta}, t} \sigma^2(\boldsymbol{\beta}, t) > 0$  and  $h_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Furthermore  $\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$  and  $\mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$  exists and

$$\begin{aligned}\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\nabla_{\boldsymbol{\alpha}}^* f(\boldsymbol{\alpha}, t) \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t)}{\sigma^2(\boldsymbol{\beta}, t)} dt, \\ \mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}') &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (f(\boldsymbol{\alpha}, t) - f(\boldsymbol{\alpha}', t))^2 dt.\end{aligned}$$

the convergences being uniform with respect to  $\boldsymbol{\alpha}$  varying in  $\mathbf{A}$  and with respect to  $\boldsymbol{\beta}$  varying in  $\mathbf{B}$ . If in addition,  $t_i - t_{i-1} = h_n$  then  $\mathbf{J}_q^{(\boldsymbol{\beta})}$  and  $\mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}')$  exist and

$$\begin{aligned}\mathbf{J}_q^{(\boldsymbol{\beta})} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_0^T \nabla_{\boldsymbol{\beta}}^* \ln \sigma^2(\boldsymbol{\beta}, t) \nabla_{\boldsymbol{\beta}} \ln \sigma^2(\boldsymbol{\beta}, t) dt, \\ \mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}') &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\sigma^2(\boldsymbol{\beta}, t) - \sigma^2(\boldsymbol{\beta}', t))^2 dt.\end{aligned}$$

the convergences being uniform with respect to  $\boldsymbol{\beta}$  varying in  $\mathbf{B}$ .

**Periodic functions** When the functions  $f(\boldsymbol{\alpha}, t)$  and  $\sigma^2(\boldsymbol{\beta}, t)$  are periodic in  $t$  with the same period  $P > 0$ , we obtain expressions for  $\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$ ,  $\mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$ ,  $\mathbf{J}_q^{(\boldsymbol{\beta})}$  and  $\mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}')$ . For continuous functions when  $h_n \rightarrow 0$  Lemma 4 entails that

$$\begin{aligned}\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})} &= \frac{1}{P} \int_0^P \frac{\nabla_{\boldsymbol{\alpha}}^* f(\boldsymbol{\alpha}, t) \nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, t)}{\sigma^2(\boldsymbol{\beta}, t)} dt, \\ \mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}') &= \frac{1}{P} \int_0^P (f(\boldsymbol{\alpha}, t) - f(\boldsymbol{\alpha}', t))^2 dt.\end{aligned}$$

If in addition,  $t_i - t_{i-1} = h_n$  then

$$\begin{aligned}\mathbf{J}_q^{(\boldsymbol{\beta})} &= \frac{1}{2P} \int_0^P \nabla_{\boldsymbol{\beta}}^* \ln \sigma^2(\boldsymbol{\beta}, t) \nabla_{\boldsymbol{\beta}} \ln \sigma^2(\boldsymbol{\beta}, t) dt, \\ \mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}') &= \frac{1}{P} \int_0^P (\sigma^2(\boldsymbol{\beta}, t) - \sigma^2(\boldsymbol{\beta}', t))^2 dt.\end{aligned}$$

By now we no longer assume that the delays between two observations tend to 0, but we assume that the sampling scheme has some periodic feature, that is  $t_{j\nu+k} = t_k + jP$  and  $t_{j\nu} = jP$ , for some  $\nu \in \mathbb{N}$ , and for any  $j, k \in \mathbb{N}$ . Then we have

$$\mathbf{J}_p^{(\boldsymbol{\alpha}, \boldsymbol{\beta})} = \frac{1}{P} \sum_{k=1}^{\nu} \frac{\nabla_{\boldsymbol{\alpha}}^* F_k(\boldsymbol{\alpha}) \nabla_{\boldsymbol{\alpha}} F_k(\boldsymbol{\alpha})}{G_k^2(\boldsymbol{\beta})} = \frac{1}{T_n} \sum_{i=1}^n \frac{\nabla_{\boldsymbol{\alpha}}^* F_i(\boldsymbol{\alpha}) \nabla_{\boldsymbol{\alpha}} F_i(\boldsymbol{\alpha})}{G_i^2(\boldsymbol{\beta})}$$

and

$$\mu_p(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = \frac{1}{P} \sum_{i=1}^{\nu} \frac{(F_i(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha}'))^2}{t_i - t_{i-1}} = \frac{1}{T_n} \sum_{i=1}^n \frac{(F_i(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha}'))^2}{t_i - t_{i-1}}$$

for any  $T_n = nP/\nu$  with  $n/\nu \in \mathbb{N}$ . Furthermore

$$\mathbf{J}_q^{(\boldsymbol{\beta})} = \frac{1}{2\nu} \sum_{i=1}^{\nu} \nabla_{\boldsymbol{\beta}}^* \ln G_i^2(\boldsymbol{\alpha}) \nabla_{\boldsymbol{\beta}} \ln G_i^2(\boldsymbol{\beta}) = \frac{1}{2n} \sum_{i=1}^n \nabla_{\boldsymbol{\beta}}^* \ln G_i^2(\boldsymbol{\alpha}) \nabla_{\boldsymbol{\beta}} \ln G_i^2(\boldsymbol{\beta})$$

and

$$\mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}') = \frac{1}{\nu} \sum_{i=1}^{\nu} \frac{(G_i^2(\boldsymbol{\beta}) - G_i^2(\boldsymbol{\beta}'))^2}{(t_i - t_{i-1})^2} = \frac{1}{n} \sum_{i=1}^n \frac{(G_i^2(\boldsymbol{\beta}) - G_i^2(\boldsymbol{\beta}'))^2}{(t_i - t_{i-1})^2}.$$

Furthermore when we assume that  $t_i - t_{i-1} = h > 0$  fixed and  $P = \nu h$ ,  $\nu \in \mathbb{N}$ , we obtain that

$$\mu_q(\boldsymbol{\beta}, \boldsymbol{\beta}') = \frac{1}{Ph} \sum_{i=1}^{\nu} (G_i^2(\boldsymbol{\beta}) - G_i^2(\boldsymbol{\beta}'))^2.$$

## References

- [1] Corduneanu, C. (1968). *Almost Periodic Functions*, Interscience Publishers (John Wiley & Sons)
- [2] Blanke, D. and Vial, C. (2014). Global smoothness estimation of a Gaussian process from general sequence designs, *Electronic Journal of Statistics* 8 (2014) 1152–1187.
- [3] Dehay, D., Dudek, A.E. and El Badaoui, M. (2017). Bootstrap for almost cyclostationary processes with jitter effect, *Digital Signal Processing* 73 (2017) 93–105.
- [4] Florens-Zmirou, D. (1989). Approximate discrete-time schemes for statistics of diffusion processes, *Statistics* 20 (4) (1989) 547–557.
- [5] Le Cam, L. (1969). *Théorie Asymptotique de la Décision Statistique*, Presses de l'Université de Montréal, Montréal
- [6] Dacunha-Castelle, D. and Florens-Zmirou, D. (1986). Estimation of the coefficient of a diffusion from discrete observations, *Stochastics* 19, 263–284.
- [7] Höpfner, R. and Kutoyants, Yu. (2010). On LAN for parametrized continuous periodic signals in a time inhomogeneous diffusion, *Statistics & Decisions* 27, 309–326.
- [8] Genon-Catalot, V. (1990). Maximum contrast estimation for diffusion process from discrete observations, *Statistics* 21, 99–116.
- [9] Ibragimov, I.A. and Has'minskii, R.Z. (1981). *Statistical Estimation, Asymptotic Theory*, Springer-Verlag, New York.
- [10] Kutoyants, Yu. (1984). *Parameter estimation for stochastic processes*, In Research and Exposition in Math. 6, Heldermann-Verlag, Berlin.
- [11] Mishra, M.N. and Prakasa Rao, B.L.S. (2001). Asymptotic minimax estimation in nonlinear stochastic differential equations from discrete observations, *Statistics & Decisions* 19, 121–136.
- [12] Sacks, J. and Ylvisaker, D. (1968). Designs for regression problems with correlated errors: many parameters. *Ann. Math. Statist.* 39, 49–69.
- [13] Uchida, M. and Yoshida, N. (2012). Adaptive estimation of an ergodic diffusion process based on sampled data. *Stochastic Process. Appl.* 122, 2885–2924.