A functional (Monadic) second-order theory of infinite trees

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A FUNCTIONAL (MONADIC) SECOND-ORDER THEORY OF INFINITE TREES

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ABSTRACT. This paper presents a complete axiomatization of Monadic Second-Order Logic (MSO) over infinite trees. MSO on infinite trees is a rich system, and its decidability ("Rabin's Tree Theorem") is one of the most powerful known results concerning the decidability of logics.

By a complete axiomatization we mean a complete deduction system with a polynomial-time recognizable set of axioms. By naive enumeration of formal derivations, this formally gives a proof of Rabin's Tree Theorem. The deduction system consists of the usual rules for second-order logic seen as two-sorted first-order logic, together with the natural adaptation to infinite trees of the axioms of MSO on ω-words. In addition, it contains an axiom scheme expressing the (positional) determinacy of certain parity games.

The main difficulty resides in the limited expressive power of the language of MSO. We actually devise an extension of MSO, called *Functional (Monadic) Second-Order Logic* (FSO), which allows us to uniformly manipulate (hereditarily) finite sets and corresponding labeled trees, and whose language allows for higher abstraction than that of MSO.

1. INTRODUCTION

This paper presents a complete axiomatization of Monadic Second-Order Logic (MSO) over infinite trees. MSO on infinite trees is a rich system which contains non trivial mathematical theories (see e.g. [Rab69, BGG97]) and subsumes many logics, including modal logics (see e.g. [BdRV02]) and logics for verification (see e.g. [VW08]). Rabin's Tree Theorem [Rab69], the decidability of MSO on infinite trees, is one of the most powerful known results concerning the decidability of logics (see e.g. [BGG97]).

The original decidability proof of [Rab69] relied on an effective translation of formulae to finite state automata running on infinite trees. Since then, there has been considerable work on Rabin's Tree Theorem, culminating in streamlined decidability proofs, as presented e.g. in [Tho97, GTW02, PP04]. Most current approaches to MSO on infinite trees (with the notable exception of [Blu13]) are based on translations of formulae to automata.

By a 'complete axiomatization' we mean a complete deduction system with a *polynomial-time* recognizable set of axioms and rules. This condition on axiom/rule recognizability is typical in proof theory, where it is known as the *Cook-Reckhow criterion* [CR79]. The point is that proofs should be 'easily checkable', which rules out axiomatizations based on enumerations of all true formulae. In

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this way, a complete axiomatization not only constitutes an alternative demonstration of Rabin's Tree theorem itself, by naive enumeration of formal derivations, but also yields a meaningful notion of 'proof certificate' for theorems.

Our deduction system consists of the usual rules for second-order logic seen as two-sorted first-order logic (see e.g. [Rib12]), together with the natural adaptation to infinite trees of the axioms of MSO on ω-words [Sie70]. In addition, it contains an axiom scheme expressing the (positional) determinacy of certain parity games.

We continue a line of work begun by Büchi and Siefkes, who gave axiomatizations of MSO on various classes of linear orders (see e.g. [Sie70, BS73]), as well as an axiomatization of *Weak* MSO (WMSO) over infinite trees [Sie78] (WMSO is MSO with set quantifications restricted to *finite* sets). These works essentially rely on formalizations of automata in the logic. A major result in the axiomatic treatment of logics over infinite structures is Walukiewicz's proof of completeness of Kozen's axiomatization of the modal μ-calculus [Wal00] (see also [AL17] for an alternative recent proof of this result). Another trend relies on model-theoretic techniques. For instance [tCF10, GtC12] give complete axiomatizations of MSO and the modal μ-calculus over finite trees; a reworking of the completeness of MSO on ω-words [Sie70] is proposed in [Rib12]; and [SV10] gives a model-theoretic completeness proof for a fragment of the modal μ-calculus. An attractive feature of model-theoretic completeness proofs for the aforementioned logics is that they allow elegant reformulations of algebraic approaches to these logics. Unfortunately, in the case of MSO over infinite trees, the only known algebraic approach [Blu13] seems too complex to be easily formalized. We therefore directly formalize a translation of formulae to automata in the axiomatic theory.

Mirroring usual automata based decidability proofs (see e.g. [Tho97, GTW02, PP04]), our method for proving completeness proceeds in two steps. We first formalize a translation of MSO-formulae to tree automata (using the positional determinacy of parity games to prove the complementation lemma), so that each closed formula is provably equivalent to an automaton over the singleton alphabet. The second (and much shorter) step is a variant of the Büchi-Landweber Theorem [BL69] which states that MSO decides winning for (definable) games of finite graphs, and which is obtained thanks to the completeness of MSO over ω-words.

The main expositional difficulty resides in the limited expressive power of the language of MSO. To ameliorate this we actually devise an extension of MSO, called *Functional (Monadic) Second-Order Logic* (FSO), allowing uniform manipulation of (hereditarily) finite sets and corresponding labeled infinite trees. We intuitively see FSO as providing a language for higher abstraction than that of MSO, allowing a uniform formalization of automata and games which would have been difficult to write down in MSO. However, since FSO is interpretable in MSO (as we show), its language has the same intrinsic limitations as the language of MSO. In particular it suffers from the inexpressibility of choice over tree positions [GS83, CL07], and so predicates such as length comparison of tree positions are not expressible in FSO. This implies that only positional strategies (w.r.t. our specific notion of acceptance games), are expressible in FSO and moreover that usually unproblematic reasoning on infinite plays can become cumbersome in this setting.

There are several ways to translate MSO to tree automata. We choose to translate formulae to alternating parity automata, following [Wal02]. The two non-trivial steps in the translation are negation and (existential) quantification. Negation requires the complementation of automata, relying on the determinacy of acceptance games, while existential quantifiers require us to simulate an alternating automaton by an equivalent non-deterministic one (this is the *Simulation Theorem* [EJ91, MS95]), thence obtaining an automaton computing the appropriate projection.

As usual with translations of MSO to tree automata, we rely on McNaughton's Theorem [McN66] (see also e.g. [Tho90, PP04]), stating that non-deterministic Büchi automata on ω-words are effectively equivalent to deterministic *parity* (or Muller, Rabin, Streett) automata on ω-words. In translations of MSO to alternating tree automata, McNaughton's Theorem is usually invoked for the Simulation Theorem.[1] In our context, the relevant instances of McNaughton's Theorem are imported into FSO via the completeness of MSO on ω-words [Sie70].

It is well-known that the MSO theory of k-ary trees can be embedded in that of the binary tree [Rab69]. However, it does not seem that such an embedding yields an axiomatization of k-ary trees from an axiomatization of the binary tree. Therefore, in this work, we axiomatize the MSO theory of the full infinite $\mathscr{D}$-ary tree for an arbitrary non-empty finite set $\mathscr{D}$.

This paper is a corrected version of [DR15], which contains a flaw in the positional determinacy argument (Thm. VI.15). In the present paper, we augment the systems FSO and MSO with an axiom expressing the positional determinacy of parity games, thereby obtaining complete axiomatizations. We do not know yet whether the theory MSO of [DR15] is complete, but let us mention that the axiomatization of WMSO over infinite trees given in [Sie78] augments the natural analogue for trees of Peano's arithmetic with an axiom of induction over finite trees.

Outline. The paper is organized as follows. We present the basic formal theory for MSO in §2 Our theory FSO is then presented in §3 and we sketch its mutual interpretability with MSO. §4 and §5 discuss a formalization of two-players infinite games in FSO, and, in particular, we give a formulation of the axiom $(PosDet)$ of positional determinacy of parity games. This provides us with the required tools to formalize in §6 (alternating) tree automata, acceptance games and basic operations on them (including complementation in $\mathsf{FSO} + (PosDet)$). §7 is an interlude discussing a complete theory of MSO over ω-words within the infinite paths of FSO. Building on §6 and §7, we then give our completeness argument for $\mathsf{FSO} + (PosDet)$ and $\mathsf{MSO} + (PosDet)$ in §8. Finally, §9 contains a proof of the Simulation Theorem in FSO, and the mutual interpretations of FSO and MSO are proved correct in Appendix A.

2. Preliminaries: MSO on Infinite Trees as a Second-Order Logic

We present here a basic formal theory of *Monadic Second-Order Logic* (MSO) over infinite trees. This theory can be seen as an analogue for trees of Peano's axioms for second order arithmetic. In order to obtain a complete theory, MSO will be augmented with an axiom of positional determinacy of parity games (see §3.6, §5.6 and §8).

We are going to define the theory $\mathsf{MSO}_{\mathscr{D}}$ of the infinite full $\mathscr{D}$-ary tree $\mathscr{D}^*$, for $\mathscr{D}$ a finite non-empty set. Both the language and the axioms of $\mathsf{MSO}_{\mathscr{D}}$ will depend on $\mathscr{D}$. The language of $\mathsf{MSO}_{\mathscr{D}}$ is the usual language of two-sorted first-order logic, with one sort for *Individuals* and one sort for *(Monadic) Predicates*. The axioms of $\mathsf{MSO}_{\mathscr{D}}$ are the expected axioms on the relational structure of the full $\mathscr{D}$-ary tree, together with induction and comprehension. The theory $\mathsf{MSO}_{\mathscr{D}}$ is essentially that of [Sie78], but with second-order quantifications intended to range over arbitrary subsets of $\mathscr{D}^*$ (instead of just finite ones), and without the axiom of induction over finite trees.

We fix for the rest of this Section a finite non-empty set $\mathscr{D}$ of *tree directions*.

[1] The approach of [MS95] to the Simulation Theorem actually contains a proof of McNaughton Theorem, but we do not see how to easily formalize it in our context.

$$\frac{}{\Phi \vdash \varphi \vee \neg\varphi} \qquad \frac{}{\Phi, \varphi \vdash \varphi} \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \neg\varphi}{\Phi \vdash \psi}$$

$$\frac{\Phi \vdash \varphi}{\Phi \vdash \varphi \vee \psi} \qquad \frac{\Phi \vdash \psi}{\Phi \vdash \varphi \vee \psi} \qquad \frac{\Phi \vdash \varphi \vee \psi \qquad \Phi, \varphi \vdash \vartheta \qquad \Phi, \psi \vdash \vartheta}{\Phi \vdash \vartheta}$$

Figure 1: Deduction Rules for Propositional Logic.

2.1. **The Language of** $\mathsf{MSO}_{\mathscr{D}}$**.** The language of $\mathsf{MSO}_{\mathscr{D}}$ has two sorts:

- The sort of *Individuals*, intended to range over tree positions $p \in \mathscr{D}^*$. We have infinitely many Individual variables x, y, z etc. We also have one constant symbol $\dot{\varepsilon}$ (for the root of $\mathscr{D}^*$), and one unary function symbol S_d for each $d \in \mathscr{D}$ (for the successor function $p \mapsto p.d$). Individual terms, written $t, u,$ etc. are given by:

$$t \quad ::= \quad x \quad | \quad \dot{\varepsilon} \quad | \quad \mathsf{S}_d(t) \qquad \text{(for } d \in \mathscr{D}\text{)}$$

- The sort of *(Monadic) Predicates*, with variables $X, Y, Z,$ etc, intended to range over sets of tree positions $A \in \mathcal{P}(\mathscr{D}^*)$. There are no other term formers for this sort.

Formulae of $\mathsf{MSO}_{\mathscr{D}}$ are given by the following grammar:

$$\varphi, \psi \in \mathbf{\Lambda}_{\mathscr{D}} \quad ::= \quad X(t) \quad | \quad t \doteq u \quad | \quad t \mathrel{\dot{<}} u \quad | \quad (\varphi \vee \psi) \quad | \quad \neg\varphi \quad | \quad (\exists x)\varphi \quad | \quad (\exists X)\varphi$$

where t and u are Individual terms. We use the usual derived formulae:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
(\forall x)\varphi & := & \neg(\exists x)(\neg\varphi) & \varphi \wedge \psi & := & \neg(\neg\varphi \vee \neg\psi) \\
(\forall X)\varphi & := & \neg(\exists X)(\neg\varphi) & \varphi \Rightarrow \psi & := & \neg\varphi \vee \psi \\
\top & := & (\forall x)(x \doteq x) & \bot & := & \neg\top \\
(t \mathrel{\dot{\leq}} u) & := & (t \mathrel{\dot{<}} u) \vee (t \doteq u) & & &
\end{array}$$

We employ usual writing conventions for formulae, for instance omitting internal and external brackets when appropriate.

2.2. **The Deduction System of** $\mathsf{MSO}_{\mathscr{D}}$**.** Deduction for $\mathsf{MSO}_{\mathscr{D}}$ is defined by the system presented in Figure 1 and Figure 2 (where Φ stands for a multiset of formulae), together with the following axioms.

- *Equality on Individuals:*

$$(\forall x)(x \doteq x) \qquad \text{and} \qquad (\forall x)(\forall y)\big(x \doteq y \implies \varphi[x/z] \implies \varphi[y/z]\big) \qquad \text{(for each } \varphi\text{)}$$

- The *Tree Axioms* of Figure 3.
- *Comprehension Scheme:*

$$(\exists X)(\forall y)\big[X(y) \iff \varphi\big] \qquad \text{(for each } \varphi\text{, with } X \text{ not free in } \varphi\text{)}$$

- *Induction Axiom:*

$$(\forall X)\left(X(\dot{\varepsilon}) \implies \bigwedge_{d \in \mathscr{D}} (\forall y)\big[X(y) \implies X(\mathsf{S}_d(y))\big] \implies (\forall y)X(y)\right)$$

Remark 2.1. As usual, one can derive $\vdash (\varphi \Rightarrow \psi \Rightarrow \vartheta) \Leftrightarrow ((\varphi \wedge \psi) \Rightarrow \vartheta)$ and we have the *Deduction Theorem*:

$$\Phi, \varphi \vdash \psi \qquad \text{iff} \qquad \Phi \vdash \varphi \Rightarrow \psi$$

$$\frac{\Phi \vdash \varphi[t/x]}{\Phi \vdash (\exists x)\varphi} \qquad\qquad \frac{\Phi \vdash (\exists x)\varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (x \text{ not free in } \Phi, \psi)$$

$$\frac{\Phi \vdash \varphi[Y/X]}{\Phi \vdash (\exists X)\varphi} \qquad\qquad \frac{\Phi \vdash (\exists X)\varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (X \text{ not free in } \Phi, \psi)$$

Figure 2: Deduction Rules for Predicate Logic.

$$\neg(\exists x) \bigvee_{d \neq d'} \big(\mathsf{S}_d(x) \doteq \mathsf{S}_{d'}(x)\big) \qquad\qquad (\forall x)(\forall y) \bigwedge_{d \in \mathscr{D}} \big(\mathsf{S}_d(x) \doteq \mathsf{S}_d(y) \quad \Rightarrow \quad x \doteq y\big)$$

$$\neg(\exists x)\big(x \mathrel{\dot{<}} x\big) \qquad\qquad (\forall x)(\forall y)(\forall z)\big(x \mathrel{\dot{<}} y \quad \Rightarrow \quad y \mathrel{\dot{<}} z \quad \Rightarrow \quad x \mathrel{\dot{<}} z\big)$$

$$(\forall x)\big(\dot{\varepsilon} \mathrel{\dot{\leq}} x\big) \qquad\qquad (\forall x)(\forall y)\Big(\big(\bigvee_{d \in \mathscr{D}} x \mathrel{\dot{<}} \mathsf{S}_d(y)\big) \quad \Leftrightarrow \quad x \mathrel{\dot{\leq}} y\Big)$$

Figure 3: Tree Axioms of $\mathsf{MSO}_{\mathscr{D}}$ and $\mathsf{FSO}_{\mathscr{D}}$ (where $(x \mathrel{\dot{\leq}} y)$ stands for $(x \mathrel{\dot{<}} y \vee x \doteq y)$).

Indeed, if $\Phi, \varphi \vdash \psi$, then one gets $\Phi \vdash \neg\varphi \vee \psi$ by $\vee$-Elimination on the Excluded Middle $\Phi \vdash \varphi \vee \neg\varphi$. Conversely, if $\Phi \vdash \neg\varphi \vee \psi$, then one gets $\Phi, \varphi \vdash \psi$ by $\vee$-Elimination. One similarly obtains the *Modus Ponens* as a derived rule

$$\frac{\Phi \vdash \psi \Rightarrow \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi}$$

Notation 2.2. Henceforth, we write MSO instead of $\mathsf{MSO}_{\mathscr{D}}$ when the set of directions $\mathscr{D}$ is clear from the context.

3. A Functional Extension of MSO on Infinite Trees

In this Section, we present (bounded) *Functional (Monadic) Second-Order Logic* over the full $\mathscr{D}$-ary tree ($\mathsf{FSO}_{\mathscr{D}}$), an extension of $\mathsf{MSO}_{\mathscr{D}}$ with (hereditarily) *finite sets* and bounded quantification over them. As with $\mathsf{MSO}_{\mathscr{D}}$ in §2, we will simply write FSO for $\mathsf{FSO}_{\mathscr{D}}$ when $\mathscr{D}$ is irrelevant or clear from the context.

$\mathsf{FSO}_{\mathscr{D}}$ is equipped with a basic axiomatization which will allow us, in §4-§6, to formalize a basic theory of games and automata, and in particular to state an axiom scheme $(PosDet)$ expressing the positional determinacy of (suitably represented) parity games (§5.6). We will then show in §8 that $\mathsf{FSO}_{\mathscr{D}} + (PosDet)$ is complete.

3.1. Motivations and Overview.

Motivations and Overview. Let us first discuss the motivations and guiding principles in the design of $\mathsf{FSO}_{\mathscr{D}}$. As usual, within the language of $\mathsf{MSO}_{\mathscr{D}}$ presented in §2, we can simulate a labeling of $\mathscr{D}^*$ over a finite non-empty set Σ

$$T : \mathscr{D}^* \longrightarrow \Sigma$$

There are different ways to achieve this. A possibility is, for say $\Sigma = \{\mathtt{a}_1, \dots, \mathtt{a}_n\}$, to code $T : \mathscr{D}^* \to \Sigma$ using a tuple of Monadic variables $X_1, \dots, X_n$ such that

$$x \in X_i \quad \text{iff} \quad T(x) = \mathtt{a}_i \qquad\qquad (\text{for } i = 1, \dots, n)$$

A more succinct coding could be obtained using $\lceil \log n \rceil$ monadic variables to encode the letter index i of $\mathtt{a}_i$ in binary. However, directly working with such codings would make it cumbersome to formalize games and automata as presented in this paper. We will therefore rather work in the system $\mathsf{FSO}_{\mathscr{D}}$, which is built around the following principles:

(1) $\mathsf{FSO}_{\mathscr{D}}$ has no primitive notion of *Monadic variables*. Instead, $\mathsf{FSO}_{\mathscr{D}}$ has a primitive notion of *Function variables*, of the form
$$F : \mathscr{D}^* \longrightarrow \Sigma \qquad (\Sigma \text{ a finite set})$$
(2) In addition, $\mathsf{FSO}_{\mathscr{D}}$ allows us to work *uniformly* with arbitrary finite sets. In particular, we have an explicit sort for them, including terms, variables and quantifications.
(3) $\mathsf{FSO}_{\mathscr{D}}$ is faithfully interpretable in $\mathsf{MSO}_{\mathscr{D}}$. To this end, all quantifications over finite sets in $\mathsf{FSO}_{\mathscr{D}}$-formulae are required to be bounded.

In particular, there is a syntactic translation $\langle - \rangle$ of $\mathsf{FSO}_{\mathscr{D}}$-formulae to $\mathsf{MSO}_{\mathscr{D}}$-formulae. The basic idea of this translation is to interpret finite sets using propositional logic, and to interpret Functions $F : \mathscr{D}^* \to \{\mathtt{a}_1, \dots, \mathtt{a}_n\}$ as partitions $X_1, \dots, X_n$ of $\mathscr{D}^*$. But while $\mathsf{FSO}_{\mathscr{D}}$ handles free variables over finite sets in a uniform way, the translation $\langle - \rangle$ only applies to $\mathsf{FSO}_{\mathscr{D}}$-formulae without free variables over finite sets. This means that for an $\mathsf{FSO}_{\mathscr{D}}$-formula $\varphi(k)$ with k a variable over finite sets, for each finite set κ we will have a specific $\mathsf{MSO}_{\mathscr{D}}$-formula $\langle \varphi(\kappa) \rangle$.

Technically, the finite sets of $\mathsf{FSO}_{\mathscr{D}}$ will be the usual *hereditarily* finite sets.

Definition 3.1. Let $V_0 := \varnothing$, and $V_{n+1} := \mathcal{P}(V_n)$ for each $n \in \mathbb{N}$. The set V_ω of *hereditarily finite* sets (HF-sets) is defined as
$$V_\omega \quad := \quad \bigcup_{n \in \mathbb{N}} V_n$$

Remark 3.2. In the context of this paper, it is useful to note that, as is well-known (see e.g. [Jec06, Exercise 12.9]), V_ω is a model of ZFC^- (*i.e.* of ZFC without the infinity axiom).

Convention 3.3. We will always assume the finite non-empty set $\mathscr{D}$ of *tree directions* to be an HF-set.

The language of $\mathsf{FSO}_{\mathscr{D}}$ will have the same sort of Individuals as $\mathsf{MSO}_{\mathscr{D}}$ and a sort for HF-sets, and its Function variables will be of the form $F : \mathscr{D}^* \to K$ for K a term over HF-sets (HF-term). The design of $\mathsf{FSO}_{\mathscr{D}}$ is obtained as a compromise between the following two conflicting desiderata:

(1) To be as flexible as possible to allow an easy formalization of games and automata.
(2) To be as simple as possible to allow an easy translation to $\mathsf{MSO}_{\mathscr{D}}$.

This leads us to two peculiar design choices.

(1) We have, in addition to the above mentioned sorts, a distinct sort of *Functions over* HF-*sets*. This sort contains only constants (so these functions cannot be quantified over), whose purpose is to provide Skolem functions for those $\forall\exists$ (bounded) statements over HF-sets which are provable in ZFC^-.
(2) In order to facilitate the translation of $\mathsf{FSO}_{\mathscr{D}}$ to $\mathsf{MSO}_{\mathscr{D}}$, Function variables, written $(F : K)$ ("F has codomain K"), cannot occur in HF-terms. Formally, Functions $(F : K)$ are only allowed in atomic formulae of the form
$$F(t) \doteq L \qquad (\text{for } L \text{ an HF-term})$$

The axioms of $\mathsf{FSO}_{\mathscr{D}}$ will contain the obvious adaptation of the Tree Axioms and the Induction Axiom of $\mathsf{MSO}_{\mathscr{D}}$. We also have axioms defining the aforementioned Skolem functions. In addition, the Comprehension Scheme of $\mathsf{MSO}_{\mathscr{D}}$ will be replaced by *Functional Choice Axioms* allowing us to define Functions $F : \mathscr{D}^* \to K$ from $\forall\exists$-statements:

$$(\forall x)(\exists k \in K)\varphi(x,k) \implies (\exists F : \mathscr{D}^* \to K)(\forall x)\varphi(x, F(x))$$

Remark 3.4. *Functional Choice Axioms* as above actually amount to *Comprehension* in MSO (§2.2). Such axioms do not create choice predicates for Individuals, which are known to be undefinable in MSO [GS83, CL07], and moreover to break decidability when added to the language of MSO [BG00, CL07].

The rest of this Section is organized as follows. The system $\mathsf{FSO}_{\mathscr{D}}$ is defined in §3.2-3.4, and its (expected) interpretation in the standard model of $\mathscr{D}$-ary trees is given in §3.5. Then in §3.6 we discuss the interpretation of $\mathsf{FSO}_{\mathscr{D}}$ in $\mathsf{MSO}_{\mathscr{D}}$ and a straightforward embedding of $\mathsf{MSO}_{\mathscr{D}}$ in $\mathsf{FSO}_{\mathscr{D}}$. Finally, §3.7 presents notation whose purpose is to allow some flexibility in the manipulation of functions. The language and axioms of $\mathsf{FSO}_{\mathscr{D}}$ are summarized in Figure 5, with references to the relevant parts of the text.

3.2. **The Language of $\mathsf{FSO}_{\mathscr{D}}$.** We now formally define the language of $\mathsf{FSO}_{\mathscr{D}}$, for $\mathscr{D}$ an HF-set. It consists of the the following sorts:

- The sort of *Hereditarily finite (*HF*) sets*, with infinitely many HF-variables k, ℓ etc., and with one constant symbol $\dot{\kappa}$ for each $\kappa \in V_\omega$ (we often simply write κ for $\dot{\kappa}$ in formulae, omitting the overset dot).
- The same sort of *Individuals* as $\mathsf{MSO}_{\mathscr{D}}$ (see §2.1).
- The sort of *Functions*, with infinitely many variables F, G, H, etc.
- The sort of HF-*Functions*, with no variable. For each pair $(n,m) \in \mathbb{N} \times \mathbb{N}$, we assume given a constant symbol $\dot{\mathsf{g}}_{n,m}$ of arity n. The interpretation of these constant symbols is discussed in §3.4.4.

The language of $\mathsf{FSO}_{\mathscr{D}}$ has two kinds of terms. The *Individual terms* are the same as those of $\mathsf{MSO}_{\mathscr{D}}$. In addition, $\mathsf{FSO}_{\mathscr{D}}$ also has HF-*terms*, which are given by

$$K, L \quad ::= \quad k \quad | \quad \dot{\kappa} \quad | \quad \dot{\mathsf{g}}_{n,m}(L_1, \dots, L_n)$$

The *formulae* of $\mathsf{FSO}_{\mathscr{D}}$ are built as follows:

$$\begin{array}{rcl} \varphi, \psi & ::= & t \doteq u \quad | \quad t \mathrel{\dot{<}} u \\ & | & K \doteq L \quad | \quad K \mathrel{\dot{\in}} L \quad | \quad K \mathrel{\dot{\subseteq}} L \quad | \quad F(t) \doteq K \\ & | & \varphi \vee \psi \quad | \quad \neg\varphi \\ & | & (\exists x)\varphi \quad | \quad (\exists F : K)\varphi \quad | \quad (\exists k \mathrel{\dot{\in}} L)\varphi \quad | \quad (\exists k \mathrel{\dot{\subseteq}} L)\varphi \end{array}$$

An $\mathsf{FSO}_{\mathscr{D}}$-formula φ is HF-*closed* if it contains no free HF-variable.

Notation 3.5.

(1) Usual derived formulae are defined similarly as with MSO (where $\star$ is either $\dot{\in}$ or $\dot{\subseteq}$):

$$\begin{array}{rclcrcl} (\forall x)\varphi & := & \neg(\exists x)(\neg\varphi) & & \varphi \wedge \psi & := & \neg(\neg\varphi \vee \neg\psi) \\ (\forall F : L)\varphi & := & \neg(\exists F : L)(\neg\varphi) & & \varphi \Rightarrow \psi & := & \neg\varphi \vee \psi \\ (\forall k \star L)\varphi & := & \neg(\exists k \star L)(\neg\varphi) & & (t \mathrel{\dot{\leq}} u) & := & (t \mathrel{\dot{<}} u) \vee (t \doteq u) \\ \top & := & (\forall x)(x \doteq x) & & \bot & := & \neg\top \end{array}$$

(2) In addition to bounded quantification $(\exists F : K)$, we use the notation $(F : K)$ within formulae as the defined formula:

$$(F : K) \quad := \quad (\forall x)(\exists k \mathrel{\dot{\in}} K)(F(x) \doteq k)$$

$$\frac{\Phi \vdash \varphi[t/x]}{\Phi \vdash (\exists x)\varphi} \qquad \frac{\Phi \vdash (\exists x)\varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi} \ (x \text{ not free in } \Phi, \psi)$$

$$\frac{\Phi \vdash \varphi[G/F] \qquad \Phi \vdash (G : K)}{\Phi \vdash (\exists F : K)\varphi} \qquad \frac{\Phi \vdash (\exists F : K)\varphi \qquad \Phi,\ (F : K),\ \varphi \vdash \psi}{\Phi \vdash \psi} \ (F \text{ not free in } \Phi, \psi)$$

$$\frac{\Phi \vdash \varphi[K/k] \qquad \Phi \vdash K \star L}{\Phi \vdash (\exists k \star L)\varphi} \ (\text{for } \star \text{ either } \dot{\in} \text{ or } \dot{\subseteq})$$

$$\frac{\Phi \vdash (\exists k \star K)\varphi \qquad \Phi,\ k \star K,\ \varphi \vdash \psi}{\Phi \vdash \psi} \ (\text{for } \star \text{ either } \dot{\in} \text{ or } \dot{\subseteq}, \text{ and } k \text{ not free in } \Phi, \psi)$$

$$\frac{\Phi \vdash \varphi}{\Phi[F(t)/k] \vdash \varphi[F(t)/k]} \ (\Phi[F(t)/k], \varphi[F(t)/k] \ \mathsf{FSO}\text{-formulae})$$

Figure 4: Deduction Rules for $\mathsf{FSO}_{\mathscr{D}}$.

(3) For variables $\boldsymbol{K} = K_1, \dots, K_n$ and $\boldsymbol{L} = L_1, \dots, L_n$, and $\star$ either $\doteq$, $\dot{\in}$ or $\dot{\subseteq}$, we let

$$\boldsymbol{K} \star \boldsymbol{L} \quad = \quad (K_1, \dots, K_n) \star (L_1, \dots, L_n) \quad := \quad \bigwedge_{1 \le i \le n} K_i \star L_i$$

Remark 3.6. The (hereditarily) finite set $\mathscr{D}$ of tree directions is considered both as a parameter in the definition of $\mathsf{FSO}_{\mathscr{D}}$, via the successor term constructors S_d (for $d \in \mathscr{D}$) and the corresponding axioms (see §3.4), and as a (constant) HF set, which can occur as such in $\mathsf{FSO}_{\mathscr{D}}$ formulae. Strictly speaking, we should write $\dot{\mathscr{D}}$ rather than $\mathscr{D}$ in the latter case, but we usually simply omit the overset dot, as with other HF-sets.

3.3. **The Deduction System of** $\mathsf{FSO}_{\mathscr{D}}$**.** Deduction for $\mathsf{FSO}_{\mathscr{D}}$ is defined by the system presented on Figure 1 (with $\mathsf{FSO}_{\mathscr{D}}$ formulae instead of $\mathsf{MSO}_{\mathscr{D}}$ formulae) and Figure 4, together with all the axioms of §3.4. The language and axioms of $\mathsf{FSO}_{\mathscr{D}}$ are summarized in Figure 5.

3.4. **Basic Axiomatization.** We now present the axioms of $\mathsf{FSO}_{\mathscr{D}}$. The first group (Equality, Tree Axioms and Induction, §3.4.1-§3.4.2) corresponds to its counterpart in $\mathsf{MSO}_{\mathscr{D}}$. We then present our specific axioms for HF-Sets in §3.4.4 and our Functional Choice Axioms in §3.4.5.

3.4.1. *Equality.* The theory $\mathsf{FSO}_{\mathscr{D}}$ has usual equality axioms for individuals and HF-Sets.

- *Equality on Individuals.*

$$(\forall x)(x \doteq x) \qquad \text{and} \qquad (\forall x)(\forall y)\big(x \doteq y \implies \varphi[x/z] \implies \varphi[y/z]\big) \qquad (\text{for all formula } \varphi)$$

- *Equality on* HF-*Sets* (for all formula φ, all HF-terms K, L and all HF-variable m):

$$K \doteq K \qquad \text{and} \qquad (K \doteq L \implies \varphi[K/m] \implies \varphi[L/m])$$

Language

Individual Terms	$t \quad ::= \quad x \quad \mid \quad \dot\varepsilon \quad \mid \quad \mathsf{S}_d(t)$	$(d \in \mathscr{D})$	(§2.1)
Functions	F, G, H, etc	(only variables)	(§3.2)
HF-Terms	$K, L \quad ::= \quad k$	(k HF-variable)	(§3.2)
	$\mid \quad \dot\kappa$	$(\kappa \in V_\omega)$	
	$\mid \quad \dot{\mathsf{g}}_{n,m}(L_1, \ldots, L_n)$	($\dot{\mathsf{g}}_{n,m}$ HF-Function)	
Formulae	$\varphi, \psi \quad ::=$		(§3.2)
	$\mid \quad t \doteq u \quad \mid \quad t \mathrel{\dot<} u \quad \mid \quad F(t) \doteq K$		
	$\mid \quad K \doteq L \quad \mid \quad K \mathrel{\dot\in} L \quad \mid \quad K \mathrel{\dot\subseteq} L$		
	$\mid \quad \varphi \vee \psi \quad \mid \quad \neg\varphi$		
	$\mid \quad (\exists x)\varphi \quad \mid \quad (\exists F : K)\varphi \quad \mid \quad (\exists k \mathrel{\dot\in} L)\varphi \quad \mid \quad (\exists k \mathrel{\dot\subseteq} L)\varphi$		

Axioms

Equality: (§3.4.1)

$$(\forall x)(x \doteq x) \qquad (\forall x)(\forall y)\big(x \doteq y \implies \varphi[x/z] \implies \varphi[y/z]\big)$$

$$K \doteq K \qquad (K \doteq L \implies \varphi[K/m] \implies \varphi[L/m])$$

Induction: (§3.4.2)

$$\varphi(\dot\varepsilon) \implies \textstyle\bigwedge_{d\in\mathscr{D}}(\forall x)\big[\varphi(x) \implies \varphi(\mathsf{S}_d(x))\big] \implies (\forall x)\varphi(x)$$

Tree Axioms: (§3.4.3)

$$\neg(\exists x) \textstyle\bigvee_{d\neq d'} \big(\mathsf{S}_d(x) \doteq \mathsf{S}_{d'}(x)\big) \qquad (\forall x)(\forall y) \textstyle\bigwedge_{d\in\mathscr{D}} \big(\mathsf{S}_d(x) \doteq \mathsf{S}_d(y) \;\Rightarrow\; x \doteq y\big)$$

$$\neg(\exists x)\big(x \mathrel{\dot<} x\big) \qquad (\forall x)(\forall y)(\forall z)\big(x \mathrel{\dot<} y \;\Rightarrow\; y \mathrel{\dot<} z \;\Rightarrow\; x \mathrel{\dot<} z\big)$$

$$(\forall x)\big(\dot\varepsilon \mathrel{\dot\leq} x\big) \qquad (\forall x)(\forall y)\Big(\big(\textstyle\bigvee_{d\in\mathscr{D}} x \mathrel{\dot<} \mathsf{S}_d(y)\big) \;\Leftrightarrow\; x \mathrel{\dot\leq} y\Big)$$

Axioms on HF-*Sets:* (§3.4.4)

$$\varphi_{n,m}[\boldsymbol{K}/\boldsymbol{k}][\dot{\mathsf{g}}_{n,m}(\boldsymbol{K})/\ell]$$

(provided $\mathbf{Sk}(\mathsf{ZFC}^-) \vdash (\forall k_1, \ldots, k_n)(\exists! \ell)\varphi_{n,m}$)

Functional Choice Axioms: (§3.4.5)

$$(\forall k \mathrel{\dot\in} K)(\exists \ell \mathrel{\dot\in} L)\varphi(k, \ell) \implies \big(\exists f \mathrel{\dot\in} L^K\big)(\forall k \mathrel{\dot\in} K)\varphi(k, f(k))$$

$$(\forall x)(\exists k \mathrel{\dot\in} K)\varphi(x, k) \implies (\exists F : K)(\forall x)(\exists k \mathrel{\dot\in} K)\,(F(x) \doteq k \;\wedge\; \varphi(x, k))$$

$$(\forall k \mathrel{\dot\in} K)(\exists F : L)\varphi(k, F) \implies \big(\exists G : L^K\big)(\forall k \mathrel{\dot\in} K)\varphi(k, F)[G(k) \mathbin{/\!\!/} F]$$

Figure 5: Summary of $\mathsf{FSO}_{\mathscr{D}}$.

Remark 3.7. Note that $\mathsf{FSO}_{\mathscr{D}}$ is equipped with an explicit *Substitution Rule*

$$\frac{\Phi \vdash \varphi}{\Phi[F(t)/k] \vdash \varphi[F(t)/k]} \qquad (\Phi[F(t)/k], \varphi[F(t)/k] \text{ FSO-formulae})$$

Substitution entails the following (where $\varphi(F(t))$ is an FSO-formula):

$$(F(t) \doteq K) \implies \varphi(K) \implies \varphi(F(t))$$

as well as the derived rule

$$\frac{\Phi \vdash \varphi(F(t)) \qquad \Phi \vdash (F : K)}{\Phi \vdash (\exists k \mathrel{\dot\in} K)\varphi(k)}$$

The former is a direct consequence of the Substitution rule together with elimination of equality on HF-Sets. For the latter, first note that Remark 2.1 also holds for $\mathsf{FSO}_{\mathscr{D}}$. In particular, one can derive

$$(k \mathrel{\dot\in} K) \implies \varphi(k) \implies (\exists k \mathrel{\dot\in} K)\varphi(k)$$

On the other hand, we have

$$(\exists \ell \mathrel{\dot\in} K)(k \doteq \ell) \implies (k \mathrel{\dot\in} K)$$

We therefore get

$$(\exists \ell \mathrel{\dot\in} K)(k \doteq \ell) \implies \varphi(k) \implies (\exists k \mathrel{\dot\in} K)\varphi(k)$$

and the Substitution rule gives

$$(\exists \ell \mathrel{\dot\in} K)(F(t) \doteq \ell) \implies \varphi(F(t)) \implies (\exists k \mathrel{\dot\in} K)\varphi(k)$$

□

3.4.2. *Induction.* We have the following *Induction Scheme*:

$$\varphi(\dot\varepsilon) \implies \bigwedge_{d\in\mathscr{D}} (\forall x)\big[\varphi(x) \implies \varphi(\mathsf{S}_d(x))\big] \implies (\forall x)\varphi(x) \qquad (\text{for each formula } \varphi)$$

3.4.3. *Tree Axioms.* For the tree structure of $\mathscr{D}^*$, we have the same Tree Axioms as $\mathsf{MSO}_{\mathscr{D}}$, displayed in Figure 3 (recall that $\mathsf{FSO}_{\mathscr{D}}$ has the same Individuals as $\mathsf{MSO}_{\mathscr{D}}$).

We now state expected results on the axioms so far introduced. To this end, let $\mathsf{FSO}^0_{\mathscr{D}}$ be the system consisting of the deduction rules of Figures 1 and 4, together with the Equality Axioms (§3.4.1) the Induction Scheme (§3.4.2) and the Tree Axioms (Figure 3).

Proposition 3.8. $\mathsf{FSO}^0_{\mathscr{D}}$ *proves the following.*

(1) $(\forall x)(\forall y)(x \mathrel{\dot\leq} y \mathrel{\dot\leq} x \implies x \doteq y)$
(2) $(\forall x)(x \mathrel{\dot\leq} \dot\varepsilon \implies x \doteq \dot\varepsilon)$
(3) $\neg(\exists x)(x \mathrel{\dot<} \dot\varepsilon)$
(4) $\neg(\exists x)(\mathsf{S}_d(x) \doteq \dot\varepsilon)$
(5) $(\forall x)(x \doteq \dot\varepsilon \;\vee\; (\exists y) \vee_{d\in\mathscr{D}}\, x \doteq \mathsf{S}_d(y))$
(6) $(\forall x)(\forall y)(x \mathrel{\dot<} y \implies \vee_{d\in\mathscr{D}}\mathsf{S}_d(x) \mathrel{\dot\leq} y)$

Proof.

(1) If $x \mathrel{\dot<} y \mathrel{\dot<} x$ then by transitivity of $\dot<$ we have $x \mathrel{\dot<} x$, contradicting the irreflexivity of $\dot<$.
(2) If $x \mathrel{\dot\leq} \dot\varepsilon$, we have $x \mathrel{\dot\leq} \dot\varepsilon \mathrel{\dot\leq} x$, so that $x \doteq \dot\varepsilon$.
(3) If $x \mathrel{\dot\leq} \dot\varepsilon$ then $x \doteq \dot\varepsilon$ so that we cannot have $(x \mathrel{\dot\leq} \dot\varepsilon) \wedge \neg(X \doteq \dot\varepsilon)$.
(4) If $\mathsf{S}_d(x) \doteq \dot\varepsilon$, then since $x \mathrel{\dot<} \mathsf{S}_d(x)$ we have $x \mathrel{\dot<} \dot\varepsilon$, a contradiction.
(5) A direct application of the Induction Scheme.

(6) Assuming given x, we apply the Induction Scheme on the formula $\varphi(y) := (x \mathrel{\dot{<}} y \Rightarrow \bigvee_{d \in \mathscr{D}} \mathsf{S}_d(x) \mathrel{\dot{\leq}} y)$.

We trivially get $\varphi(\dot{\varepsilon})$ since $\neg(x \mathrel{\dot{<}} \dot{\varepsilon})$. Assuming now $\varphi(y)$ we show $\varphi(\mathsf{S}_{d'}(y))$. So assume $x \mathrel{\dot{<}} \mathsf{S}_{d'}(y)$. Then the Tree Axioms give $x \mathrel{\dot{\leq}} y$. If $x \mathrel{\dot{<}} y$, then we are done thanks to $\varphi(y)$. Otherwise, we have $x \doteq y$, so that $\mathsf{S}_{d'}(x) \doteq \mathsf{S}_{d'}(y)$ and we are done. □

A consequence of Proposition 3.8 is that the Induction Scheme of $\mathsf{FSO}_{\mathscr{D}}$ (§3.4.2) implies the usual scheme of *Well-Founded Induction* w.r.t. the strict prefix order $\dot{<}$.

Theorem 3.9 (Well-Founded Induction)**.** $\mathsf{FSO}^0_{\mathscr{D}}$ *proves the following form of well-founded induction:*

$$(\forall x)\big[(\forall y \mathrel{\dot{<}} x)(\varphi(y)) \implies \varphi(x)\big] \implies (\forall x)\varphi(x)$$

Proof. Assume

$$\forall x \, [\forall y \, (y \mathrel{\dot{<}} x \implies \varphi(y)) \implies \varphi(x)]$$

We apply induction on the formula

$$\psi(x) \quad := \quad (\forall y \mathrel{\dot{\leq}} x)\varphi(y)$$

We have $\psi(\dot{\varepsilon})$ since $\forall y \neg(y \mathrel{\dot{<}} \dot{\varepsilon})$. Assuming $\psi(x)$, we get $\psi(\mathsf{S}_d(x))$ as follows. If $y \mathrel{\dot{<}} \mathsf{S}_d(x)$, we have $y \mathrel{\dot{\leq}} x$, hence $\varphi(y)$ since we assumed $\psi(x)$. Moreover, $\varphi(\mathsf{S}_d(x))$ follows from the fact that $y \mathrel{\dot{<}} \mathsf{S}_d(x)$ implies $y \mathrel{\dot{\leq}} x$, hence $\varphi(y)$ since we assumed $\psi(x)$. □

Remark 3.10. Both Proposition 3.8 and Theorem 3.9 also hold for $\mathsf{MSO}_{\mathscr{D}}$.

3.4.4. HF-*Sets.* We now present our axioms on HF-Sets. Their purpose is to ease formalization in $\mathsf{FSO}_{\mathscr{D}}$. Recall that HF-sets range over V_ω (Definition 3.1). The idea of these axioms is to incorporate in $\mathsf{FSO}_{\mathscr{D}}$ as much of the theory of V_ω as possible, while keeping $\mathsf{FSO}_{\mathscr{D}}$ interpretable in $\mathsf{MSO}_{\mathscr{D}}$ and with a semi-recursive notion of provability. The interpretation of $\mathsf{FSO}_{\mathscr{D}}$ in $\mathsf{MSO}_{\mathscr{D}}$ relies on the fact that in $\mathsf{FSO}_{\mathscr{D}}$-formulae, all quantifications over HF-Sets are bounded (either by $\dot{\in}$ or $\dot{\subseteq}$), so that in a closed $\mathsf{FSO}_{\mathscr{D}}$-formula, quantifications over HF-Sets can be interpreted using usual propositional logic.

We will have, as particular cases of our axioms on HF-Sets, all bounded formulae over HF-Sets which are true in V_ω. Moreover, w.r.t. the interpretation of $\mathsf{FSO}_{\mathscr{D}}$ in $\mathsf{MSO}_{\mathscr{D}}$ (§3.6) and in particular w.r.t. its application to $\mathsf{MSO}_{\mathscr{D}}$ over ω-words (§7, §8 and §9), it is important to have sufficiently many functions over V_ω available within *closed* HF-terms. This is the main purpose of our axioms on HF-Sets. They state that the HF-Functions $\dot{\mathsf{g}}_{n,m}$ are Skolem functions for $\forall\exists!$-statements over HF-Sets. These axioms are further commented in §8.5.

Definition 3.11 (HF-Formula)**.** An HF-*formula* is an $\mathsf{FSO}_{\mathscr{D}}$-formula with atoms of the form $K \doteq L$, $K \mathrel{\dot{\in}} L$ or $K \mathrel{\dot{\subseteq}} L$ where K and L are HF-terms.

Fix a distinguished HF-variable ℓ, and an enumeration $k_1, k_2, \dots$ of distinct HF-variables all different from ℓ. Furthermore, fix an enumeration $(\varphi_{n,m})_{n,m \in \mathbb{N}}$ of HF-formulae satisfying the following conditions:

(1) Each formula $\varphi_{n,m}$ has free variables among $k_1, \dots, k_n, \ell$.
(2) All HF-Functions occurring in $\varphi_{n,m}$ have the form $\dot{\mathsf{g}}_{n',m'}$ with $m' < m$.
(3) Each HF-formula φ satisfying (1) and (2) occurs infinitely often in $(\varphi_{n,m})_{n,m \in \mathbb{N}}$, in the following sense. If φ has free variables among $k_1, \dots, k_n, \ell$, then there are infinitely many $m \in \mathbb{N}$ such that φ is $\varphi_{n,m}$.

Recall from Remark 3.2 that V_ω is a model of ZFC^-. The idea of our Axioms on HF-Sets is that

$$\mathsf{FSO}_{\mathscr{D}} \vdash \varphi_{n,m}[\dot{\mathsf{g}}_{n,m}(k_1,\dots,k_n)/\ell] \qquad \text{whenever} \qquad \mathsf{ZFC}^- \vdash (\forall k_1,\dots,k_n)(\exists!\ell)\varphi_{n,m} \tag{3.1}$$

(where $(k \dot{\subseteq} k')$ is interpreted as $(\forall m \dot{\in} k)(m \dot{\in} k')$). However, recall that $\varphi_{n,m}$ in (3.1) may contain HF-Functions $\dot{\mathsf{g}}_{n',m'}$ with $m' < m$. The premise of (3.1) can thus not be formulated in ZFC^-, but requires a suitable extension of it. We let $\mathbf{Sk}(\mathsf{ZFC}^-)$ consist of ZFC^- augmented with the axioms

$$(\forall k_1,\dots,k_n)(\exists!\ell)(\varphi_{n,m}) \implies (\forall k_1,\dots,k_n)\varphi_{n,m}[\dot{\mathsf{g}}_{n,m}(k_1,\dots,k_n)/\ell] \quad (\text{for each } n,m \in \mathbb{N})$$

It is well-known that $\mathbf{Sk}(\mathsf{ZFC}^-)$ is thus a conservative extension of ZFC^- (see e.g. [vD04, §3.4]). $\mathsf{FSO}_{\mathscr{D}}$ has the following axiom scheme for HF-Sets, which simply consists of (3.1) formulated for $\mathbf{Sk}(\mathsf{ZFC}^-)$ rather than ZFC^-:

- *Axioms on* HF-*Sets.* For each $n,m \in \mathbb{N}$ such that

$$\mathbf{Sk}(\mathsf{ZFC}^-) \vdash \quad (\forall k_1,\dots,k_n)(\exists!\ell)\varphi_{n,m} \tag{3.2}$$

  and for all HF-terms $\boldsymbol{K} = K_1,\dots,K_n$, we have the axiom

$$\varphi_{n,m}[\boldsymbol{K}/\boldsymbol{k}][\dot{\mathsf{g}}_{n,m}(\boldsymbol{K})/\ell]$$

Remark 3.12. Note that this axiom scheme makes the axiom set of $\mathsf{FSO}_{\mathscr{D}}$ not recursive. But as expected for a proof system, *provability* in $\mathsf{FSO}_{\mathscr{D}}$ remains semi-recursive.

We fix here once and for all an interpretation of the HF-Function symbols $\dot{\mathsf{g}}_{n,m}$ as functions over V_ω. The idea is that if (3.2) holds, then $\dot{\mathsf{g}}_{n,m}$ is interpreted as a computable function $\mathsf{g}_{n,m} : V_\omega^n \to V_\omega$ such that

$$V_\omega \models \quad (\forall k_1,\dots,k_n)\varphi_{n,m}[\mathsf{g}_{n,m}(k_1,\dots,k_n)/\ell] \tag{3.3}$$

But again recall that $\varphi_{n,m}$ may contain HF-Functions $\dot{\mathsf{g}}_{n',m'}$ with $m' < m$. We therefore proceed inductively, as follows.

Convention 3.13. By induction on $m \in \mathbb{N}$, we interpret the HF-Function symbols $\dot{\mathsf{g}}_{n,m}$ as computable functions

$$\mathsf{g}_{n,m} \ : \ V_\omega^n \ \longrightarrow \ V_\omega$$

Consider the formula $\varphi_{n,m}$, and assume that all HF-Functions $\dot{\mathsf{g}}_{n',m'}$ with $m' < m$ are already interpreted. If (3.2) holds then by the Countable Axiom of Choice we interpret $\dot{\mathsf{g}}_{n,m}$ as the unique function $\mathsf{g}_{n,m} : V_\omega^n \to V_\omega$ such that (3.3) holds. Note that such functions are computable. Otherwise, we interpret $\dot{\mathsf{g}}_{n,m}$ as the function with constant value $\varnothing$.

Remark 3.14. Note each HF-Function symbol is interpreted by a recursive function in Convention 3.13. However, since (3.2) is undecidable, there is no algorithm taking $(n,m) \in \mathbb{N}^2$ to the interpretation of $\dot{\mathsf{g}}_{n,m}$. This point is further discussed in §8.5, where a natural workaround is proposed, as well as some explanations for our present choice of Axioms on HF-Sets.

We now discuss some consequences of these axioms. First note that if φ is a *closed* HF-formula, then it is provable in $\mathbf{Sk}(\mathsf{ZFC}^-)$ if and only if it holds in V_ω. We state this fact as a Remark for the record, and also to reiterate how much deductive power underlies the axioms on HF-sets.

Remark 3.15. Given a closed HF-formula φ,

$$\mathsf{FSO}_{\mathscr{D}} \vdash \varphi \qquad \text{whenever} \qquad V_\omega \models \varphi$$

Moreover, we have all instances of the following:

(a) *Extensionality*.

$$(\forall m \mathrel{\dot\in} k)(m \mathrel{\dot\in} \ell) \;\Rightarrow\; (\forall m \mathrel{\dot\in} \ell)(m \mathrel{\dot\in} k) \;\Rightarrow\; k \doteq \ell$$

(b) *Finite sets*. For each $n \in \mathbb{N}$ we have an n-ary HF-Function symbol $\{-,\dots,-\}$ such that

$$\bigwedge_{1\le i\le n} (k_i \mathrel{\dot\in} \{k_1,\dots,k_n\}) \quad\wedge\quad (\forall m \mathrel{\dot\in} \{k_1,\dots,k_n\}) \bigvee_{1\le i\le n} (m \doteq k_i)$$

We have in particular singletons $\{-\}$ and unordered pairs $\{-,-\}$. Using Extensionality, $\mathsf{FSO}_{\mathscr{D}}$ proves that

$$\{\{k\},\{k,\ell\}\} \doteq \{\{k'\},\{k',\ell'\}\} \quad\Longleftrightarrow\quad k \doteq k' \wedge \ell \doteq \ell'$$

We use the following shorthand:

$$(k,\ell) \quad:=\quad \{\{k\},\{k,\ell\}\}$$

(c) *Union*. We have an HF-Function symbol $\cup(-)$ such that

$$(\forall \ell \mathrel{\dot\in} \cup(k))(\exists m \mathrel{\dot\in} k)(\ell \mathrel{\dot\in} m) \quad\wedge\quad (\forall \ell \mathrel{\dot\in} k)(\forall m \mathrel{\dot\in} \ell)(m \mathrel{\dot\in} \cup(k))$$

(d) *Powerset.* We have an HF-Function symbol $\mathcal{P}(-)$ such that

$$(\forall \ell \mathrel{\dot\in} \mathcal{P}(k))(\forall m \mathrel{\dot\in} \ell)(m \mathrel{\dot\in} k) \quad\wedge\quad (\forall \ell \mathrel{\dot\subseteq} k)(\ell \mathrel{\dot\in} \mathcal{P}(k))$$

The powerset is the reason for our introduction of inclusion ($\dot\subseteq$) as an atomic formula: It is well-known that the powerset cannot be defined by a $\Delta_0(\dot\in)$-formula. A possible formula defining it is:

$$(\forall \ell \mathrel{\dot\in} \mathcal{P}(k))(\forall m \mathrel{\dot\in} \ell)(m \mathrel{\dot\in} k) \quad\wedge\quad (\forall \ell)\big[(\forall m \mathrel{\dot\in} \ell)(m \mathrel{\dot\in} k) \implies \ell \mathrel{\dot\in} \mathcal{P}(k)\big]$$

The quantification $\forall\ell$ in the right conjunct is not $\dot\in$-bounded, and cannot be so. In addition, we also have an HF-Function symbol $\mathcal{P}_*(-)$ for the *non-empty powerset*, that is such that

$$k \mathrel{\dot\in} \mathcal{P}_*(\ell) \quad\Longleftrightarrow\quad \big(k \mathrel{\dot\in} \mathcal{P}(\ell) \;\wedge\; (\exists m \mathrel{\dot\in} k)\big)$$

(e) *Comprehension.* Given an HF-formula φ with free variables among $k_1,\dots,k_n,k$, we have an HF-Function $\{k \mathrel{\dot\in} (-) \mid \varphi[-,\dots,-,k]\}$ such that

$$m \mathrel{\dot\in} \{k \mathrel{\dot\in} k_0 \mid \varphi[k_1,\dots,k_n,k]\} \quad\Longleftrightarrow\quad m \mathrel{\dot\in} k_0 \wedge \varphi[k_1,\dots,k_n,m]$$

(f) *Products.* We have a binary HF-Function $(-)\times(-)$ such that

$$k \times \ell \;:=\; \big\{m \mathrel{\dot\in} \mathcal{P}(\mathcal{P}(k \cup \ell)) \mid (\exists k_0 \mathrel{\dot\in} k)(\exists \ell_0 \mathrel{\dot\in} \ell)\big[m \doteq (k_0,\ell_0)\big]\big\}$$

Moreover, we have binary projections given by HF-Functions $\pi_1^{-,-}(-)$ and $\pi_2^{-,-}(-)$ such that

$$\pi_1^{k,\ell}(m) \;=\; \big\{n \in \ell \mid m \mathrel{\dot\subseteq} k \times \ell \;\wedge\; (\exists n' \in k)\big[(n,n') \mathrel{\dot\in} m\big]\big\}$$

and similarly for $\pi_2^{-,-}(-)$. Whenever possible, we write $\pi_i(-)$ instead of $\pi_i^{k,\ell}(-)$. Note that by composing binary projections π_1 and π_2 we obtain projections

$$\pi_i^n \;:\; k_1 \times \cdots \times k_n \;\longrightarrow\; k_i$$

for any $k_1,\dots,k_n$ and $i \in \{1,\dots,n\}$

(g) *Function Spaces and Application.* We have an exponent HF-Function $(-)^{(-)}$ such that

$$\ell^k \quad := \quad \{m \mathrel{\dot\in} \mathcal{P}(k \times \ell) \mid (\forall k_0 \mathrel{\dot\in} k)(\exists! \ell_0 \mathrel{\dot\in} \ell)\big[(k_0, \ell_0) \mathrel{\dot\in} m\big]\}$$

Moreover, function application is given by the HF-Function $@_{-,-}(-,-)$ with

$$@_{k,\ell}(f,a) \quad := \quad \{m \mathrel{\dot\in} \cup(\ell) \mid (\exists \ell_0 \mathrel{\dot\in} \ell)\big[m \mathrel{\dot\in} \ell_0 \wedge (a, \ell_0) \mathrel{\dot\in} f\big]\}$$

(here f and a are HF-variables). Whenever possible, we omit the subscripts k, ℓ of $@_{k,\ell}(f,a)$ and write simply $f(a)$ for $@_{k,\ell}(f,a)$.

(h) *Disjoint Unions.* We have a binary HF-Function $(-) + (-)$ with

$$k + \ell \quad \doteq \quad (\{0\} \times k) \cup (\{1\} \times \ell)$$

(see Convention 5.18 for a further discussion on finite ordinals in our context). We moreover have HF-Functions $\mathsf{in}^{k,\ell}_k(-)$, $\mathsf{in}^{k,\ell}_\ell(-)$, and $[-,-]^\imath_{k,\ell}$ such that (dropping subscripts and superscripts)

$$\mathsf{in}(m) \doteq (0,m) \qquad \mathsf{in}(n) \doteq (1,n) \qquad [f,g](0,m) \doteq f(m) \qquad [f,g](1,n) \doteq g(n)$$

for $m \mathrel{\dot\in} k$, $\ell \mathrel{\dot\in} n$ and $f \mathrel{\dot\in} \imath^k$, $g \mathrel{\dot\in} \imath^\ell$.

Convention 3.16. Regarding function spaces as in (g) above, $\mathsf{FSO}_{\mathscr{D}}$ proves

$$(k^\ell)^m \quad \cong \quad k^{\ell \times m}$$

In the following, we reason modulo that bijection, and simply identify $(k^\ell)^m$ with $k^{\ell \times m}$.

Remark 3.17. An HF-relation $\preceq \subseteq K \times K$ is a *partial order* on an HF-term K, if the formula $\mathrm{PO}(\preceq, K)$ holds in V_ω, where $\mathrm{PO}(\preceq, K)$ is the HF-formula:

$$(\forall k, \ell, m \mathrel{\dot\in} K)\Big[\big(k \preceq \ell \;\Rightarrow\; \ell \preceq k \;\Rightarrow\; k \doteq \ell\big) \;\wedge\; \big(k \preceq \ell \;\Rightarrow\; \ell \preceq m \;\Rightarrow\; k \preceq m\big)\Big]$$

A partial order $\preceq \subseteq K \times K$ is a *well-order* if every subset of K has a $\preceq$-least element, that is, if the following formula $\mathrm{WO}(\preceq, K)$ holds in V_ω:

$$(\forall \ell \mathrel{\dot\subseteq} K)\big[\ell \neq \varnothing \;\Rightarrow\; (\exists m \mathrel{\dot\in} \ell)(\forall n \mathrel{\dot\in} \ell)(m \preceq n)\big]$$

Since every HF-set is finite and can be well-ordered, we have

$$V_\omega \quad \models \quad (\forall k)\,\underbrace{(\exists \preceq \mathrel{\dot\subseteq} k \times k)\big[\mathrm{PO}(\preceq, k) \;\wedge\; \mathrm{WO}(\preceq, k) \;\wedge\; \mathrm{WO}(\succeq, k)\big]}_{\varphi(k)}$$

Since $\varphi(k)$ is an HF-formula $\varphi(k)$, it follows that $\mathsf{FSO}_{\mathscr{D}}$ proves $\varphi(k)$, hence in particular that every HF-set is well-ordered. □

3.4.5. *Functional Choice Axioms.* We have the following functional choice axiom schemes.

- HF-*Bounded Choice for* HF-*Sets*.

$$(\forall k \mathrel{\dot\in} K)(\exists \ell \mathrel{\dot\in} L)\varphi(k,\ell) \implies \big(\exists f \mathrel{\dot\in} L^K\big)(\forall k \mathrel{\dot\in} K)\varphi(k, f(k))$$

- HF-*Bounded Choice for Functions*.

$$(\forall x)(\exists k \mathrel{\dot\in} K)\varphi(x,k) \implies (\exists F : K)(\forall x)(\exists k \mathrel{\dot\in} K)\,(F(x) \doteq k \;\wedge\; \varphi(x,k))$$

- *Iterated* HF-*Bounded Choice*.

$$(\forall k \mathrel{\dot\in} K)(\exists F : L)\varphi(k,F) \implies \big(\exists G : L^K\big)(\forall k \mathrel{\dot\in} K)\varphi(k,F)[G(k) /\!\!/ F]$$

where the substitution $[G(k) /\!\!/ F]$ is defined as the usual substitution operation but with

$$(F(t) \doteq M)[G(k) /\!\!/ F] \quad := \quad \big(\exists f \mathrel{\dot\in} L^K\big)\,(G(t) \doteq f \;\wedge\; f(k) \doteq M)$$

We insist that none of these axioms create choice functions for the *individuals* of $\mathsf{FSO}_{\mathscr{D}}$ (cf Remark 3.4). Despite their common shape, these three axiom schemes are actually of different nature. First, the axiom of HF-*Bounded Choice for Functions*

$$(\forall x)(\exists k \mathrel{\dot\in} K)\varphi(x,k) \implies (\exists F : K)(\forall x)(\exists k \mathrel{\dot\in} K)\,(F(x) \doteq k \;\wedge\; \varphi(x,k)) \tag{3.4}$$

is a counterpart in $\mathsf{FSO}_{\mathscr{D}}$ of the Comprehension Scheme of $\mathsf{MSO}_{\mathscr{D}}$. Recalling the informal discussion in §3.1 and anticipating §3.5 and §3.6, let us assume a translation $\langle - \rangle$ from (HF-closed) $\mathsf{FSO}_{\mathscr{D}}$-formulae to $\mathsf{MSO}_{\mathscr{D}}$-formulae, and let us assume that K is a closed HF-term representing the HF-set $\{\kappa_1,\dots,\kappa_n\}$. Then the premise of (3.4) can be read as

$$(\forall x) \bigvee_{1\le i\le n} \langle \varphi(x,\kappa_i)\rangle$$

The conclusion easily follows from the fact that using Comprehension, one can define in $\mathsf{MSO}_{\mathscr{D}}$ a partition $X_1,\dots,X_n$ of $\mathscr{D}^*$ such that

$$(\forall x) \bigvee_{1\le i\le n} \Big(X_i(x) \;\;\wedge\;\; \langle \varphi(x,\kappa_i)\rangle \Big)$$

Second, HF-*Bounded Choice for* HF-*Sets*

$$(\forall k \mathrel{\dot\in} K)(\exists \ell \mathrel{\dot\in} L)\varphi(k,\ell) \implies \big(\exists f \mathrel{\dot\in} L^K\big)(\forall k \mathrel{\dot\in} K)\varphi(k,f(k)) \tag{3.5}$$

may look similar to the Axioms on HF-Sets of §3.4.4. The differences are that the formula φ here is an *arbitrary* formula of $\mathsf{FSO}_{\mathscr{D}}$, not necessarily an HF-formula in the sense of Definition 3.11, and moreover that this axiom only involves $\mathsf{FSO}_{\mathscr{D}}$ (*i.e. bounded*) quantifications, contrary to (3.2). Note that for HF-formulae φ, this axiom is indeed an instance of the axioms of §3.4.4. In the general case, this axiom can be seen as following from the fact that quantifications over HF-Sets in $\mathsf{FSO}_{\mathscr{D}}$ are ultimately interpreted in propositional logic. Assume that the HF-terms K and L are closed, and correspond to the HF-sets κ and λ respectively. Then the premise of (3.5) can be read as

$$\bigwedge_{\kappa_0\in\kappa}\bigvee_{\lambda_0\in\lambda} \langle \varphi(\kappa_0,\lambda_0)\rangle$$

which is equivalent in propositional logic to the interpretation of the conclusion

$$\bigvee_{f\in\lambda^\kappa}\bigwedge_{\kappa_0\in\kappa} \langle \varphi(\kappa_0,f(\kappa_0))\rangle$$

Similarly, *Iterated* HF-*Bounded Choice* reduces to an equivalence of the form

$$\bigwedge_{1\le i\le n} (\exists \boldsymbol{X})\varphi(\kappa_i,\boldsymbol{X}) \iff (\exists \boldsymbol{X}_1)\dots(\exists \boldsymbol{X}_n) \bigwedge_{1\le i\le n} \varphi(\kappa_i,\boldsymbol{X}_i)$$

and follows from Comprehension.

The definition of $\mathsf{FSO}_{\mathscr{D}}$ is now complete.

Notation 3.18. Similarly as with $\mathsf{MSO}_{\mathscr{D}}$, we shall write FSO for $\mathsf{FSO}_{\mathscr{D}}$ when the set of tree directions $\mathscr{D}$ is clear from the context.

3.5. **The Standard Model of FSO.** The *standard model* $\mathfrak{T}$ of $\mathsf{FSO}_{\mathscr{D}}$ is the full $\mathscr{D}$-ary tree $\mathscr{D}^*$ equipped with suitable domains for each sort:

- HF-*Sets* range over V_ω, and each constant $\dot{\kappa}$ is interpreted by the corresponding HF-set $\kappa \in V_\omega$.
- *Individuals* range over $\mathscr{D}^*$, the constant $\dot{\varepsilon}$ is interpreted by the empty sequence $\varepsilon \in \mathscr{D}^*$ and S_d as the map taking $p \in \mathscr{D}^*$ to $p.d \in \mathscr{D}^*$. Moreover, we write $<$ for the strict prefix order on $\mathscr{D}^*$.
- *Functions* range over
$$\bigcup_{\kappa \in V_\omega} (\mathscr{D}^* \quad \longrightarrow \quad \kappa)$$
- For each $n, m \in \mathbb{N}$, the HF-Function $\dot{\mathsf{g}}_{n,m}$ (of arity n) is interpreted as the function
$$\mathsf{g}_{n,m} \quad : \quad V_\omega^n \quad \longrightarrow \quad V_\omega$$
fixed in Convention 3.13.

Remark 3.19. Note that $\mathfrak{T}$ has the same individuals as the standard model of $\mathsf{MSO}_{\mathscr{D}}$. Moreover we write $\mathfrak{T}$ for both the standard model of $\mathsf{FSO}_{\mathscr{D}}$ and that of $\mathsf{MSO}_{\mathscr{D}}$, as an abuse of notation.

We have the usual interpretation $[\![t]\!] \in \mathscr{D}^*$ for each closed individual term t with parameters in $\mathfrak{T}$, and an interpretation $[\![K]\!] \in V_\omega$ for each closed HF-term K with parameters in $\mathfrak{T}$. The relation $\mathfrak{T} \models \varphi$, for a closed FSO-formula φ with parameters in $\mathfrak{T}$, is defined by induction on φ as follows:

$$\begin{array}{llll}
\mathfrak{T} \models K \mathrel{\dot{\star}} L & \text{iff} & [\![K]\!] \star [\![L]\!] & \text{(for } \star \text{ either } =, \in, \text{ or } \subseteq) \\
\mathfrak{T} \models t \mathrel{\dot{\star}} u & \text{iff} & [\![t]\!] \star [\![u]\!] & \text{(for } \star \text{ either } = \text{ or } <) \\
\mathfrak{T} \models \varphi \vee \psi & \text{iff} & (\mathfrak{T} \models \varphi) \text{ or } (\mathfrak{T} \models \psi) & \\
\mathfrak{T} \models \neg\varphi & \text{iff} & \mathfrak{T} \not\models \varphi & \\
\mathfrak{T} \models (\exists x)\varphi & \text{iff} & \mathfrak{T} \models \varphi[p/x] \ \text{ for some } p \in \mathscr{D}^* & \\
\mathfrak{T} \models (\exists k \mathrel{\dot{\star}} L)\varphi & \text{iff} & \mathfrak{T} \models \varphi[\kappa/k] \ \text{ for some } \kappa \star [\![L]\!] & \text{(for } \star \text{ either } \in \text{ or } \subseteq) \\
\mathfrak{T} \models (\exists F : L)\varphi & \text{iff} & \mathfrak{T} \models \varphi[\mathtt{F}/F] \ \text{ for some } \mathtt{F} \in [\![L]\!]^{\mathscr{D}^*} &
\end{array}$$

By a routine induction argument, we can show the soundness of $\mathsf{FSO}_{\mathscr{D}}$ w.r.t. $\mathfrak{T}$:

Proposition 3.20. *Given* FSO*-formulae* $\psi_1, \dots, \psi_n, \varphi$ *with free* HF*-variables among* $\boldsymbol{k}$*, free Individual variables among* $\boldsymbol{x}$*, and free Function variables among* $\boldsymbol{F}$*, if*

$$\psi_1, \dots, \psi_n \vdash_{\mathsf{FSO}} \varphi$$

then for all HF*-Sets* $\boldsymbol{\kappa}$*, all* $\boldsymbol{p} \in \mathscr{D}^*$ *and all* $\mathtt{F} \in \bigcup_{\kappa \in V_\omega} (\mathscr{D}^* \to \kappa)$*, we have*

$$\mathfrak{T} \models \varphi[\boldsymbol{\kappa}/\boldsymbol{k}, \boldsymbol{p}/\boldsymbol{x}, \mathtt{F}/\boldsymbol{F}] \qquad \textit{whenever} \qquad \mathfrak{T} \models \bigwedge_{1 \le i \le n} \psi_i[\boldsymbol{\kappa}/\boldsymbol{k}, \boldsymbol{p}/\boldsymbol{x}, \mathtt{F}/\boldsymbol{F}]$$

Remark 3.21. It follows from Remark 3.14 that the map $[\![-]\!]$ is not computable on HF-terms. We refer to §8.5 for a discussion and a workaround.

3.6. **Mutual Interpretability of FSO and MSO.** While FSO seems more expressive than MSO (and, indeed, is easier to work with), the two theories can mutually interpret each other via two formula-level translations:

$$\langle - \rangle \quad : \quad \mathsf{FSO} \quad \longrightarrow \quad \mathsf{MSO} \qquad \text{and} \qquad (-)^\circ \quad : \quad \mathsf{MSO} \quad \longrightarrow \quad \mathsf{FSO}$$

As we shall see, both translations preserve and reflect provability:

$$\begin{array}{lll}
\mathsf{FSO} \vdash \varphi \quad \text{if and only if} \quad \mathsf{MSO} \vdash \langle \varphi \rangle & & (\varphi \text{ closed } \mathsf{FSO}\text{-formula}) \\
\mathsf{MSO} \vdash \varphi \quad \text{if and only if} \quad \mathsf{FSO} \vdash \varphi^\circ & & (\varphi \text{ closed } \mathsf{MSO}\text{-formula})
\end{array}$$

The interpretation $(-)^\circ$ of MSO in FSO simply amounts to simulate the (Monadic) Predicate variables of MSO by FSO-Function variables $\mathscr{D}^* \to 2$. We therefore see $(-)^\circ$ as an embedding, and see FSO as a conservative extension of MSO which is faithfully interpretable in MSO. This property is not only a sanity check: we actually rely on it in our completeness argument (see Rem. 3.28). We discuss the translations $\langle - \rangle$ and $(-)^\circ$ separately in §3.6.1 and §3.6.2 below. In both cases, detailed proofs are deferred to Appendix A.

3.6.1. *From* FSO *to* MSO. The translation $\langle - \rangle : \mathsf{FSO} \to \mathsf{MSO}$ interprets the HF-part of FSO using propositional logic. It is essentially straightforward, except for the case of Functions, which require some care. We will work with the following convention:

Convention 3.22. We assume that each HF-set κ comes with a fixed enumeration $\boldsymbol{\kappa} = \kappa_1, \dots, \kappa_n$ of its elements.

The translation $\langle - \rangle$ will map an HF-closed FSO-formula φ without free Function variables to an MSO-formula $\langle \varphi \rangle$. As stated earlier, quantifications over HF-Sets will be interpreted using propositional logic. For instance we have,

$$\langle (\exists k \mathrel{\dot\in} K)\varphi \rangle \quad = \quad \bigvee_{\kappa \in [\![K]\!]} \langle \varphi[\kappa/k] \rangle$$

where $[\![K]\!] \in V_\omega$ is the standard interpretation of the closed HF-term K defined in §3.5. As a consequence, the translation $\langle - \rangle$ is *non-uniform* w.r.t. HF-Sets. In particular, for an FSO-formula φ with free HF-variables among $\boldsymbol{k} = k_1, \dots, k_p$, each tuple of HF-sets $\boldsymbol{\kappa} = \kappa_1, \dots, \kappa_p$ will induce a specific MSO-formula $\langle \varphi[\boldsymbol{\kappa}/\boldsymbol{k}] \rangle$.

The interpretation of Function variables is more complex. Consider a closed HF-term K and assume $[\![K]\!] = \{\kappa_1, \dots, \kappa_c\}$. Then a Function $(F : K)$ can be seen as a function

$$F \;\;:\;\; \mathscr{D}^* \;\;\longrightarrow\;\; \{\kappa_1, \dots, \kappa_c\}$$

As indicated in §3.1, we interpret F as a tuple $X_1, \dots, X_c$ of Monadic variables such that

$$x \in X_i \quad \text{iff} \quad F(x) = \kappa_i \qquad\qquad (\text{for } i = 1, \dots, c)$$

In other words, $F : \mathscr{D}^* \to \{\kappa_1, \dots, \kappa_c\}$ is seen as a partition $X_1, \dots, X_c$ of $\mathscr{D}^*$. To handle the interpretation of Functions in the inductive definition of $\langle - \rangle$, it is actually convenient to temporarily work in an extension of FSO with the following atomic formulae:

- $|X_1 \dots X_n(t) \doteq_{\boldsymbol{\kappa}} L|$ where $\boldsymbol{\kappa} = \kappa_1, \dots, \kappa_n$ enumerates an HF-set and $X_1, \dots, X_n$ are monadic variables of MSO.

Extended FSO-formulae are built just like FSO-formulae, but possibly using the atomic formulae above. Extended atomic formulae are useful for dealing with HF-bounded quantifications over Functions, say $(\exists F : K)\varphi$. The point is that F occurs in subformulae of φ of the form $F(t) \doteq L$, where the HF-term L may contain free HF-variables. Hence the value of L is not known when the translation of $(\exists F : K)$ has to be computed. Extended atomic formulae allow us to delay the interpretation of $F(t) \doteq L$ until $[\![L]\!]$ is known.

The interpretation of an extended HF-closed FSO-formula φ *without free Function variables* is the MSO-formula $\langle\varphi\rangle$ defined by induction on φ as follows:

$$
\begin{array}{rcll}
\langle |X_1 \dots X_n(t) \doteq_{\boldsymbol{\kappa}} L| \rangle & := & \bigvee_{1 \le i \le n \ \& \ \kappa_i = [\![L]\!]} X_i(t) & \\
\langle K \mathrel{\dot\star} L \rangle & := & \begin{cases} \top & \text{if } [\![K]\!] \star [\![L]\!] \\ \bot & \text{otherwise} \end{cases} & (\text{where } \star \in \{=, \in, \subseteq\}) \\
\langle t \star u \rangle & := & t \star u & (\text{where } \star \in \{\doteq, \dot<\}) \\
\langle \neg\varphi \rangle & := & \neg\langle\varphi\rangle & \\
\langle \varphi \vee \psi \rangle & := & \langle\varphi\rangle \vee \langle\psi\rangle & \\
\langle (\exists k \mathrel{\dot\star} K)\varphi \rangle & := & \bigvee_{\kappa \star [\![K]\!]} \langle \varphi[\kappa/k] \rangle & (\text{where } \star \in \{\in, \subseteq\}) \\
\langle (\exists x)\varphi \rangle & := & (\exists x)\langle\varphi\rangle & \\
\langle (\exists F : K)\varphi \rangle & := & (\exists X_1) \dots (\exists X_c) & \\
 & & \quad \begin{cases} & \Upsilon_c(X_1, \dots, X_c) \\ \wedge & \langle \varphi[|X_1 \dots X_c(t) \doteq_{\boldsymbol{\kappa}} L| \ / \ (F(t) \doteq L)] \rangle \end{cases} &
\end{array}
$$

where in the last clause, $[\![K]\!]$ is enumerated by $\boldsymbol{\kappa} = \kappa_1, \dots, \kappa_c$, and $\Upsilon_c(X_1, \dots, X_c)$ is the following MSO-formula, expressing that $X_1, \dots, X_c$ form a partition of $\mathscr{D}^*$:

$$
\Upsilon_c(X_1, \dots, X_c) \quad := \quad (\forall x) \left[\bigvee_{1 \le i \le c} \left(X_i(x) \quad \wedge \quad \bigwedge_{j \ne i} \neg X_j(x) \right) \right]
$$

Note that in the definition of $\langle\varphi\rangle$ above, since φ is assumed to be HF-closed, the displayed HF-terms K and L are closed, so that their $\mathfrak{T}$-interpretation $[\![K]\!], [\![L]\!] \in V_\omega$ is defined (see §3.5).

Remark 3.23. Since it involves the standard interpretation map $[\![-]\!]$ on HF-terms, it follows from Remark 3.21 (§3.5) that the interpretation $\langle - \rangle$ is not recursive. We refer to §5.6.1 and §8.5 for discussions and workarounds.

Theorem 3.24. *For every closed* FSO*-formula* φ*, we have*

$$
\mathsf{MSO} \vdash \langle\varphi\rangle \qquad \textit{whenever} \qquad \mathsf{FSO} \vdash \varphi
$$

The proof of Theorem 3.24 is deferred to Appendix A. The logical rules of FSO are handled routinely. The interpretations of most of the axioms of FSO are almost trivially provable from the corresponding axioms of MSO. The Functional Choice Axioms are dealt-with essentially as explained in §3.4.5.

3.6.2. *From* MSO *to* FSO. The translation $(-)^\circ : \mathsf{MSO} \to \mathsf{FSO}$ is much simpler than $\langle - \rangle$. Assume given a FSO-Function variable F_X for each monadic MSO-variable X. The map $(-)^\circ$ is inductively defined as follows:

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
(X(t))^\circ & := & F_X(t) \doteq 1 & (\varphi \vee \psi)^\circ & := & \varphi^\circ \vee \psi^\circ \\
(t \doteq u)^\circ & := & t \doteq u & (\neg\varphi)^\circ & := & \neg(\varphi^\circ) \\
(t \mathrel{\dot\le} u)^\circ & := & t \mathrel{\dot\le} u & ((\exists x)\varphi)^\circ & := & (\exists x)\varphi^\circ \\
 & & & ((\exists X)\varphi)^\circ & := & (\exists F_X : 2)\varphi^\circ
\end{array}
$$

It is easy to see that $(-)^\circ$ is truth preserving (and reflecting) w.r.t. the standard model $\mathfrak{T}$, by a direct induction on formulae relying on the bijection $\mathcal{P}(\mathscr{D}^*) \cong 2^{\mathscr{D}^*}$.

Lemma 3.25. *Given a closed* MSO*-formula* φ*, we have*

$$
\mathfrak{T} \models \varphi \qquad \textit{if and only if} \qquad \mathfrak{T} \models \varphi^\circ
$$

The main result on $(-)^\circ$ is the following. Its proof is deferred to Appendix A.4.

Theorem 3.26. *Given a closed* MSO*-formula* φ*,*

$$\mathsf{FSO} \vdash \varphi^\circ \qquad \textit{if and only if} \qquad \mathsf{MSO} \vdash \varphi$$

Theorem 3.26 can actually be extended to FSO formulae. This is essentially the content of the following result.

Proposition 3.27. *For a closed* FSO*-formula* φ*, we have the following.*

$$\mathsf{FSO} \vdash \quad \varphi \iff \langle\varphi\rangle^\circ \tag{3.6}$$

$$\mathsf{FSO} \vdash \varphi \quad \textit{iff} \quad \mathsf{MSO} \vdash \langle\varphi\rangle \tag{3.7}$$

$$\mathfrak{T} \models \varphi \quad \textit{iff} \quad \mathfrak{T} \models \langle\varphi\rangle \tag{3.8}$$

Remark 3.28. Theorem 3.26 and Proposition 3.27 will be used in our completeness argument (§8) in two different ways:

(1) We first obtain completeness of FSO (augmented with the Axiom $(PosDet)$ of §5.6) for MSO formulae via a usual translation of formulae to automata. From this result, completeness of $\mathsf{FSO} + (PosDet)$ follows by Proposition 3.27, while completeness of MSO (augmented with $\langle - \rangle$-translations of suitable instances of $(PosDet)$) follows by Theorem 3.26.
(2) In addition, we will use Proposition 3.27 in §7 in order to import the MSO-theory of $\mathbb{N}$ for the infinite paths of $\mathfrak{T}$. We rely on this for the version of the Büchi-Landweber Theorem (namely that FSO decides parity games on finite graphs) used in the completeness argument of §8, as well as for the Simulation Theorem in §9.

3.7. **Notations.** We now introduce some notation that we will use throughout our formalization of games and automata in FSO.

3.7.1. FSO *with Extended* HF*-Terms.* First, recall that the syntax of FSO formally disallows Functions in HF-terms. We propose here a notation system that allows them in some circumstances. For instance, assuming $(F : K)$, we can use the notation

$$(F(t) \mathrel{\dot\in} L) \;:=\; (\exists k \mathrel{\dot\in} K)\big(F(t) \doteq k \;\wedge\; k \mathrel{\dot\in} L\big)$$

More generally, consider an atomic formula

$$M \star N \qquad\qquad (\text{for } \star \in \{\doteq, \dot\in, \dot\subseteq\})$$

with M and N terms on the following grammar:

$$M, N \quad ::= \quad k \quad | \quad \dot\kappa \quad | \quad F(t) \quad | \quad \dot{\mathsf{g}}_{n,m}(L_1, \dots, L_n) \tag{3.9}$$

Such formulae can be interpreted in FSO, provided one assumes bounds for the Function variables occurring in them. Let M and N be as above, and assume their free Function variables to be among $\boldsymbol{F} = F_1, \dots, F_n$. Note that there are (proper) HF-terms M' and N' such that

$$M = M'[\boldsymbol{F}(\boldsymbol{t})/\boldsymbol{\ell}] \qquad \text{and} \qquad N = N'[\boldsymbol{F}(\boldsymbol{t})/\boldsymbol{\ell}]$$

for some HF-variables $\boldsymbol{\ell} = \ell_1, \dots, \ell_c$ and where $\boldsymbol{F}(\boldsymbol{t}) = F_{i_1}(t_1), \dots, F_{i_c}(t_c)$. Given proper HF-terms $K_1, \dots, K_n$, assuming $\boldsymbol{F} : \boldsymbol{K}$, one can let

$$M \star N \;:=\; \big(\exists \boldsymbol{\ell} \in \boldsymbol{L}\big)\,\big(\boldsymbol{F}(\boldsymbol{t}) \doteq \boldsymbol{\ell} \;\wedge\; M' \star N'\big)$$

where $\boldsymbol{L} = L_1, \dots, L_c$ is such that $L_j = K_i$ iff the jth element of $\boldsymbol{F}(\boldsymbol{t})$ is $F_i(t_j)$. Note however that the above defined formula $M \star N$ actually depends on the choice of $\boldsymbol{K}$, so we rather write it as:

$$(M \star N)_{\boldsymbol{F},\boldsymbol{K}}$$

Generalizing further we can, with the above method, interpret in FSO formulae build with HF-terms in the sense of (3.9). The interpretation in FSO of such a formula φ with free Function variables among $\boldsymbol{F} = F_1, \dots, F_n$ is defined by induction, and depends on a choice of proper HF-terms $\boldsymbol{K} = K_1, \dots, K_n$. Using notation as above, we arrive at the following definition:

$$\begin{array}{rcll}
(t \star u)_{\boldsymbol{F},\boldsymbol{K}} & := & (t \star u) & (\text{for } \star \in \{\dot{=}, \dot{\in}\}) \\
(M \star N)_{\boldsymbol{F},\boldsymbol{K}} & := & \left\{ \begin{array}{ll} G(u) \mathrel{\dot{=}} N & \text{if } (M \star N) = (G(u) \mathrel{\dot{=}} N) \\ & \text{with } N \text{ a proper HF-term} \\ (\exists \boldsymbol{\ell} \mathrel{\dot{\in}} \boldsymbol{L})\,(\boldsymbol{F}(\boldsymbol{t}) \mathrel{\dot{=}} \boldsymbol{\ell} \;\; \wedge \;\; M' \star N') & \\ & \text{otherwise} \end{array} \right. & (\text{for } \star \in \{\dot{=}, \dot{\in}, \dot{\subseteq}\}) \\
(\neg\varphi)_{\boldsymbol{F},\boldsymbol{K}} & := & \neg(\varphi_{\boldsymbol{F},\boldsymbol{K}}) & \\
(\varphi \vee \psi)_{\boldsymbol{F},\boldsymbol{K}} & := & \varphi_{\boldsymbol{F},\boldsymbol{K}} \vee \psi_{\boldsymbol{F},\boldsymbol{K}} & \\
((\exists x)\varphi)_{\boldsymbol{F},\boldsymbol{K}} & := & (\exists x)\varphi_{\boldsymbol{F},\boldsymbol{K}} & \\
((\exists m \star M)\varphi)_{\boldsymbol{F},\boldsymbol{K}} & := & (\exists \boldsymbol{\ell} \mathrel{\dot{\in}} \boldsymbol{L})(\exists m \star M')\,(\boldsymbol{F}(\boldsymbol{t}) \mathrel{\dot{=}} \boldsymbol{\ell} \;\; \wedge \;\; \varphi_{\boldsymbol{F},\boldsymbol{K}}) & (\text{for } \star \in \{\dot{\in}, \dot{\subseteq}\}) \\
((\exists G : M)\varphi)_{\boldsymbol{F},\boldsymbol{K}} & := & (\exists \boldsymbol{\ell} \mathrel{\dot{\in}} \boldsymbol{L})(\exists G : M')\,\big(\boldsymbol{F}(\boldsymbol{t}) \mathrel{\dot{=}} \boldsymbol{\ell} \;\; \wedge \;\; \varphi_{G\boldsymbol{F},M'\boldsymbol{K}}\big) &
\end{array}$$

Beware that $(\varphi)_{\boldsymbol{F},\boldsymbol{K}}$ only makes sense under the assumptions $\boldsymbol{F} : \boldsymbol{K}$. Keeping this in mind we may obtain, for instance, the following formulations of the Functional Choice Axioms of §3.4.5.

- HF-*Bounded Choice for Functions*.

$$(\forall x)(\exists k \mathrel{\dot{\in}} K)\varphi(x,k) \implies (\exists F : K)(\forall x)\varphi(x, F(x))$$

- *Iterated* HF-*Bounded Choice*.

$$(\forall k \mathrel{\dot{\in}} K)(\exists F : L)\varphi(k,F) \implies \big(\exists F : L^K\big)\,(\forall k \mathrel{\dot{\in}} K)\varphi(k, F(-,k))$$

3.7.2. *Notations for Products and Functions.* We now introduce notation for a form of *product type*, based on the function spaces and application functions of §3.4.4.(g). The main idea is to be able to manipulate a Function variable

$$F \;\; : \;\; K^L$$

as a function

$$F \;\; : \;\; \mathscr{D}^* \times L \;\; \longrightarrow \;\; K$$

Furthermore, it is convenient to allow such F to have a domain defined by an FSO formula $\psi(-)$, and to write

$$F \;\; : \;\; \psi(-) \times L \;\; \longrightarrow \;\; K \qquad \text{for} \qquad (\forall x)\big(\psi(x) \implies F(x) \mathrel{\dot{\in}} K^L\big)$$

We develop here a notation system for such "function" and "product types". In §3.7.3, we discuss formulations of the Functional Choice Axioms of §3.4.5 induced by this notation. In order not to overload the arrow symbol $\longrightarrow$ (which will be used with games later on), we will write typing declarations as

$$F : \mathscr{D}^* \times L \;\mathtt{to}\; K \qquad \text{instead of} \qquad F : \mathscr{D}^* \times L \longrightarrow K$$

Notation 3.29 (Product Types)**.** *Product types* are given by the following grammar, where $\psi(-)$ is an FSO formula of an individual variable (with possibly other free variables of any sort), and where K is an HF-term.

$$\Pi \quad ::= \quad \psi(-) \quad | \quad K \quad | \quad \Pi \times K$$

The *arity* of a product type Π is:

- $(1, n)$ if Π is of the form $\psi(-) \times K_1 \times \cdots \times K_n$,
- $(0, n)$ if Π is of the form $K_1 \times \cdots \times K_n$.

Product types are to be used with the following defined formulae.

$$\begin{array}{r c l} (t, \boldsymbol{K}) = (u, \boldsymbol{L}) & := & t \doteq u \ \wedge \ \boldsymbol{K} \doteq \boldsymbol{L} \\ (t, \boldsymbol{K}) \in \psi(-) \times \boldsymbol{L} & := & \psi(t) \ \wedge \ \boldsymbol{K} \mathrel{\dot\in} \boldsymbol{L} \\ \\ f : K \ \mathsf{to} \ L & := & f \mathrel{\dot\in} L^K \\ F : \psi(-) \ \mathsf{to} \ L & := & (\forall x)(\psi(x) \ \Rightarrow \ F(x) \mathrel{\dot\in} L) \\ \mathrm{F} : \Pi \times K \ \mathsf{to} \ L & := & \mathrm{F} : \Pi \ \mathsf{to} \ L^K \end{array}$$

Here F stands for a Function variable F if the arity of Π is of the form $(1, n)$, and for an HF-variable f if the arity of Π is of the form $(0, n)$. Moreover, for $\Pi = \psi(-) \times K_1 \times \cdots \times K_n$, we let

$$(\exists F : \Pi \ \mathsf{to} \ L)\varphi \ := \ \left(\exists F : L^{K_n \times \cdots \times K_1}\right) \left[F : \Pi \ \mathsf{to} \ L \ \wedge \ \varphi\right]$$

Remarks 3.30.

(1) Thanks to Rem. 3.17, using the Axioms of HF-Bounded Choice (§3.4.5), we have

$$\left(\mathrm{F} : \Pi \ \mathsf{to} \ L \ \wedge \ \varphi\right) \implies (\exists \mathrm{F} : \Pi \ \mathsf{to} \ L)\varphi$$

(2) Using Convention 3.16, for each product type Π we have

$$(\mathrm{F} : \Pi \times K_1 \times \cdots \times K_n \ \mathsf{to} \ K) \iff \left(\mathrm{F} : \Pi \ \mathsf{to} \ K^{K_n \times \cdots \times K_1}\right)$$

It follows that for each product type Π and each formula φ we have

$$(\exists \mathrm{F} : \Pi \times K_1 \times \cdots \times K_n \ \mathsf{to} \ K)\varphi \iff \left(\exists \mathrm{F} : \Pi \ \mathsf{to} \ K^{K_n \times \cdots \times K_1}\right) \varphi$$

Notation 3.31. In the following, given a product type Π, we use the notation $\tilde{t} : \Pi$, where $\tilde{t}$ stands for a tuple of the form $(t, K_1, \ldots, K_n)$ if Π has arity $(1, n)$, or of the form $(K_1, \ldots, K_n)$ if Π has arity $(0, n)$. When Π is clear from the context, we write $\tilde{t}$ instead of $\tilde{t} : \Pi$, and furthermore we may omit the overset tilde, writing t instead of $\tilde{t}$.

Write $\tilde{x}$ for tuples of variables of the form $(x, k_1, \ldots, k_n)$ or of the form $(k_1, \ldots, k_n)$.

(1) If $\Pi = \psi(-) \times K_1 \times \cdots \times K_n$ and $\tilde{t} = (t, L_1, \ldots, L_n)$, we write $F^{\Pi \to K}(\tilde{t})$ for the HF-term

$$@_{K_1 \times \cdots \times K_n, K}(F(t), (L_1, \ldots, L_n))$$

If $\Pi = K_1 \times \cdots \times K_n$ and $\tilde{t} = (L_1, \ldots, L_n)$, we write $f^{\Pi \to K}(\tilde{t})$ for the HF-term

$$@_{K_1 \times \cdots \times K_n, K}(f, (L_1, \ldots, L_n))$$

When Π and K are clear from the context, in either case above we write $\mathrm{F}(\tilde{t})$ for $\mathrm{F}^{\Pi \to K}(\tilde{t})$.

(2) We furthermore write $\tilde{t} \mathrel{\dot\in} \mathrm{F}$ or even $\mathrm{F}(\tilde{t})$ for the formula

$$\mathrm{F}(\tilde{t}) \doteq 1$$

(3) We extend product types as follows

$$\Pi \quad ::= \quad \dots \quad | \quad \mathscr{D}^* \quad | \quad X$$

where X is a Function variable. We let

$$\begin{array}{rcl} F : X \ \mathtt{to}\ L & := & (\forall x)(X(x) \ \Rightarrow \ F(x) \mathrel{\dot\in} L) \\ F : \mathscr{D}^* \ \mathtt{to}\ L & := & (\forall x)(F(x) \mathrel{\dot\in} L) \end{array}$$

$$\begin{array}{rcl} (t, \boldsymbol{K}) \in \mathscr{D}^* \times \boldsymbol{L} & := & \top \ \wedge \ \boldsymbol{K} \mathrel{\dot\in} \boldsymbol{L} \\ (t, \boldsymbol{K}) \in X \times \boldsymbol{L} & := & X(t) \doteq 1 \ \wedge \ \boldsymbol{K} \mathrel{\dot\in} \boldsymbol{L} \end{array}$$

Note that

$$\begin{array}{rcl} F : (\mathscr{D}^* \times K_1 \times \cdots \times K_n) \ \mathtt{to}\ L & \iff & F : (\top \times K_1 \times \cdots \times K_n) \ \mathtt{to}\ L \\ (\exists F : \mathscr{D}^* \ \mathtt{to}\ L)\varphi & \iff & (\exists F : L)\varphi \end{array}$$

3.7.3. *Choice and Comprehension.* We list here some important straightforward consequences of the Functional Choice Axioms of §3.4.5 pertaining to Product Types.

Theorem 3.32. $\mathsf{FSO}_{\mathscr{D}}$ *proves the following generalizations of the Functional Choice Axioms of §3.4.5:*

- HF-Bounded Choice.

$$(\forall \tilde{x} \mathrel{\dot\in} \Pi)(\exists k \mathrel{\dot\in} L)\varphi(\tilde{x}, k) \implies (\exists \mathrm{F} : \Pi \ \mathtt{to}\ L)(\forall \tilde{x} \mathrel{\dot\in} \Pi)\varphi(\tilde{x}, \mathrm{F}(\tilde{x}))$$

- Iterated HF-Bounded Choice.

$$(\forall k \mathrel{\dot\in} K)(\exists \mathrm{F} : \Pi \ \mathtt{to}\ L)\varphi(k, \mathrm{F}) \implies \left(\exists \mathrm{F} : \Pi \ \mathtt{to}\ L^K\right)(\forall k \mathrel{\dot\in} K)\varphi(k, \mathrm{F}(-, k))$$

Proof.

- *Iterated* HF-*Bounded Choice for* Π *of arity* $(0, n)$.
  Let $\Pi = K_1 \times \cdots \times K_n$. We have to prove

$$(\forall k \mathrel{\dot\in} K)(\exists f : \Pi \ \mathtt{to}\ L)\varphi(k, f) \implies \left(\exists f : \Pi \ \mathtt{to}\ L^K\right)(\forall k \mathrel{\dot\in} K)\varphi(k, f((-), k))$$

  which by Remark 3.30.(2) amounts to

$$(\forall k \mathrel{\dot\in} K)\left(\exists f \mathrel{\dot\in} L^{K_n \times \cdots \times K_1}\right)\varphi(k, f) \implies \left(\exists f \mathrel{\dot\in} L^{K \times K_n \times \cdots \times K_1}\right)(\forall k \mathrel{\dot\in} K)\varphi(k, f((-), k))$$

  But by the axioms on HF-Sets this is equivalent to

$$(\forall k \mathrel{\dot\in} K)\left(\exists f \mathrel{\dot\in} L^{K_n \times \cdots \times K_1}\right)\varphi(k, f) \implies \left(\exists f \mathrel{\dot\in} L^{K_n \times \cdots \times K_1 \times K}\left((\forall k \mathrel{\dot\in} K)\varphi(k, f(k, (-)))\right)\right.$$

  which follows from the axiom of HF-Bounded Choice for HF-Sets.
- HF-*Bounded Choice for* Π *of arity* $(0, n)$.
  Let $\Pi = K_1 \times \cdots \times K_n$. We have to prove

$$(\forall \boldsymbol{k} \mathrel{\dot\in} \boldsymbol{K})(\exists k \mathrel{\dot\in} L)\varphi(\boldsymbol{k}, k) \implies (\exists f : \Pi \ \mathtt{to}\ L)(\forall \boldsymbol{k} \mathrel{\dot\in} \boldsymbol{K})\varphi(\boldsymbol{k}, f(\boldsymbol{k}))$$

  which by Remark 3.30.(2) amounts to

$$(\forall \boldsymbol{k} \mathrel{\dot\in} \boldsymbol{K})(\exists k \mathrel{\dot\in} L)\varphi(\boldsymbol{k}, k) \implies \left(\exists f \mathrel{\dot\in} K^{K_n \times \cdots \times K_1}\right)(\forall \boldsymbol{k} \mathrel{\dot\in} \boldsymbol{K})\varphi(\boldsymbol{k}, f(\boldsymbol{k}))$$

  We can then conclude by Iterated HF-Bounded Choice for HF-Sets (*i.e.* for Π of arity $(0, n)$).

- HF-*Bounded Choice for* Π *of the form* $\Pi = \mathscr{D}^* \times K_1 \times \cdots \times K_n$.
  Using Remark 3.30.(2), we have to prove
  $$(\forall x)(\forall \boldsymbol{k} \mathrel{\dot\in} \boldsymbol{K})(\exists k \mathrel{\dot\in} L)\varphi(\boldsymbol{k},k) \implies \big(\exists F : L^{K_n\times\cdots\times K_1}\big)\,(\forall x)(\forall \boldsymbol{k} \mathrel{\dot\in} \boldsymbol{K})\varphi(x,\boldsymbol{k},F(x,\boldsymbol{k}))$$
  But by the axiom of Iterated HF-Bounded Choice for HF-Sets, we have
  $$(\forall x)(\forall \boldsymbol{k} \mathrel{\dot\in} \boldsymbol{K})(\exists k \mathrel{\dot\in} L)\varphi(\boldsymbol{k},k) \implies (\forall x)\,\big(\exists f \mathrel{\dot\in} L^{K_n\times\cdots\times K_1}\big)\,(\forall \boldsymbol{k} \mathrel{\dot\in} \boldsymbol{K})\varphi(x,\boldsymbol{k},f(\boldsymbol{k}))$$
  and we conclude by the axiom of HF-Bounded Choice for Functions.
- *Iterated* HF-*Bounded Choice for* Π *of the form* $\Pi = \mathscr{D}^* \times K_1 \times \cdots \times K_n$.
  Using Remark 3.30.(2), we have to prove
  $$(\forall k \mathrel{\dot\in} K)\,\big(\exists F : L^{K_n\times\cdots\times K_1}\big)\,\varphi(k,F) \implies \big(\exists F : L^{K_n\times\cdots\times K_1\times K}\big)\,(\forall k \mathrel{\dot\in} K)\varphi(k,F(-,k))$$
  and we conclude by Iterated HF-Choice for Functions.
- HF-*Bounded Choice for* Π *of the form* $\Pi = \psi(-) \times K_1 \times \cdots \times K_n$.
  Let $\Pi_0 := \mathscr{D}^* \times K_1 \times \cdots \times K_n$. We have to show
  $$(\forall \tilde{x} \mathrel{\dot\in} \Pi)(\exists k \mathrel{\dot\in} L)\varphi(\tilde{x},k) \quad\implies\quad (\exists F : \Pi \;\texttt{to}\; L)(\forall \tilde{x} \mathrel{\dot\in} \Pi)\varphi(\tilde{x},F(\tilde{x}))$$
  but this follows from
  $$\begin{aligned}&(\forall \tilde{x} \mathrel{\dot\in} \Pi_0)(\exists k \mathrel{\dot\in} L)\big(\tilde{x} \mathrel{\dot\in} \Pi \Rightarrow \varphi(\tilde{x},k)\big) \quad\implies\\ &\qquad\qquad (\exists F : \Pi_0 \;\texttt{to}\; L)(\forall \tilde{x} \mathrel{\dot\in} \Pi_0)\big(\tilde{x} \mathrel{\dot\in} \Pi \Rightarrow \varphi(\tilde{x},F(\tilde{x}))\big)\end{aligned}$$
- HF-*Bounded Choice for* Π *of the form* $\Pi = \psi(-) \times K_1 \times \cdots \times K_n$.
  Let $\Pi_0 := \mathscr{D}^* \times K_1 \times \cdots \times K_n$. We have to show
  $$(\forall k \mathrel{\dot\in} K)(\exists F : \Pi \;\texttt{to}\; L)\varphi(k,F) \implies \big(\exists F : \Pi \;\texttt{to}\; L^K\big)\,(\forall k \mathrel{\dot\in} K)\,\varphi(k,F(-,k))$$
  but this follows from
  $$\begin{aligned}&(\forall k \mathrel{\dot\in} K)(\exists F : \Pi_0 \;\texttt{to}\; L)\big(F : \Pi \;\texttt{to}\; L \;\wedge\; \varphi(k,F)\big) \quad\implies\\ &\qquad\qquad \big(\exists F : \Pi_0 \;\texttt{to}\; L^K\big)\,(\forall k \mathrel{\dot\in} K)\big(F(-,k) : \Pi \;\texttt{to}\; L \;\wedge\; \varphi(k,F(-,k))\big)\end{aligned}$$

□

Theorem 3.33 (Comprehension for Product Types)**.** $\mathsf{FSO}_{\mathscr{D}}$ *proves the following form of Comprehension, where V does not occur free in φ:*
$$(\exists V : \Pi \;\texttt{to}\; 2)(\forall \tilde{x} \mathrel{\dot\in} \Pi)\big(V(\tilde{x}) \iff \varphi(\tilde{x})\big)$$

Proof. We require
$$(\exists V : \Pi \;\texttt{to}\; 2)(\forall \tilde{x} \mathrel{\dot\in} \Pi)\big(V(\tilde{x}) \doteq 1 \iff \varphi(x)\big)$$
By excluded middle, bounded existentials and generalization we have,
$$(\forall \tilde{x} \mathrel{\dot\in} \Pi)(\exists k \mathrel{\dot\in} 2)\big(k \doteq 1 \iff \varphi(x)\big)$$
and we conclude by HF-Bounded Choice. □

Remarks 3.34.

(1) In the case of $\Pi = \mathscr{D}^*$, Theorem 3.33 gives Comprehension for characteristic functions:
$$(\exists X : \mathscr{D}^* \;\texttt{to}\; 2)(\forall x)\big(x \in X \iff \varphi(x)\big) \qquad (X \text{ not free in } \varphi)$$

(2) We have the following form of Comprehension for HF-Sets:
$$(\exists \ell \mathrel{\dot\subseteq} K)(\forall k \mathrel{\dot\in} K)\big(k \mathrel{\dot\in} \ell \iff \varphi(k)\big) \qquad (\ell \text{ not free in } \varphi)$$

Proof. By Theorem 3.33 we have

$$(\exists g : K \ \mathtt{to}\ 2)(\forall k \mathrel{\dot\in} K)\,[g(k) \doteq 1 \iff \varphi(k)] \qquad (g \text{ not free in } \varphi)$$

By the axiom on HF-Functions, we get that

$$\left(\exists g : 1 \ \mathtt{to}\ 2^K\right)(\forall k \mathrel{\dot\in} K)\,[g(0,k) \doteq 1 \iff \varphi(k)] \qquad (g \text{ not free in } \varphi)$$

and therefore

$$\left(\exists g \mathrel{\dot\in} 2^K\right)(\forall k \mathrel{\dot\in} K)\,[g(k) \doteq 1 \iff \varphi(k)] \qquad (g \text{ not free in } \varphi)$$

But now, we have

$$V_\omega \models \quad \left(\forall g \mathrel{\dot\in} 2^K\right)(\exists \ell \mathrel{\dot\subseteq} K)(\forall k \mathrel{\dot\in} K)\,[k \mathrel{\dot\in} \ell \iff g(k) \doteq 1]$$

It then follows from the Axioms on HF-Sets (Remark 3.15) that

$$(\exists \ell \mathrel{\dot\subseteq} K)(\forall k \mathrel{\dot\in} K)\,[k \mathrel{\dot\in} \ell \iff \varphi(k)] \qquad (\ell \text{ not free in } \varphi)$$

□

(3) Using Comprehension for HF-Sets, the well-orders on HF-Sets given by Remark 3.17 give the following Induction Scheme for HF-Sets:

$$(\forall k \mathrel{\dot\in} K)\Big[(\forall \ell \mathrel{\dot\in} K)\big(\ell \prec_K k \Rightarrow \varphi(\ell)\big) \implies \varphi(k)\Big] \quad \implies \quad (\forall k \mathrel{\dot\in} K)\varphi(k)$$

where $\preceq_K$ is the well-order on K given by Remark 3.17.

Proof. Assume toward a contradiction that there is some $k \mathrel{\dot\in} K$ such that $\neg\varphi(k)$. Then, by Comprehension for HF-Sets, for the set of all $k \mathrel{\dot\in} K$ such that $\neg\varphi(k)$. Since this set is non-empty, it has a $\preceq$-least element, say k_0. But then $\varphi(\ell)$ holds for all $\ell \prec k_0$, so we get $\varphi(k)$, a contradiction. □

4. Game Positions

This Section and the next one describe our setting for games. The games we consider ultimately aim at formalizing acceptance games of tree automata (§6), and thus must encompass acceptance games for *non-deterministic* tree automata. We shall therefore give a setting for infinite games, with players *Proponent* P (corresponding to *Automaton* or ∃loïse) and *Opponent* O (corresponding to *Pathfinder* or ∀bérlard). In the case of acceptance games, P plays for acceptance and O plays for rejection, and in the particular case of non-deterministic automata, P chooses transitions from the non-deterministic transition relation, while O chooses tree directions $d \in \mathscr{D}$, with the aim of building an infinite path. This leads to an inherent asymmetry in the very notion of games, where, from a game position with a given tree position $x \in \mathscr{D}^*$, P can only go to game positions with tree position x, while O must go to a game position with tree position a successor of x.

Due to the fact that we cannot access the usual primitive recursive codings in the monadic language, we will only consider games that are 'superposed' onto the infinite $\mathscr{D}$-tree, with only boundedly many positions associated with each tree node. Such a setting indeed suffices for the case of acceptance games arising from tree automata. Assume that we are given disjoint non-empty HF-Sets $\mathsf{P}_\mathcal{G}$ and $\mathsf{O}_\mathcal{G}$ of *Proponent* and *Opponent* labels respectively. Intuitively, *Proponent* will play from game positions of the form

$$\mathscr{D}^* \times \mathsf{P}_\mathcal{G}$$

while *Opponent* will play from positions of the form

$$\mathscr{D}^* \times \mathsf{O}_\mathcal{G}$$

A game will be given by specifying edge relations of the form

$$(x,k) \underset{\mathsf{P}}{\longrightarrow} (x,\ell) \quad \text{or} \quad (x,\ell) \underset{\mathsf{O}}{\longrightarrow} (x.d,k) \qquad \text{where } k \in \mathsf{P}_{\mathcal{G}},\, \ell \in \mathsf{O}_{\mathcal{G}} \text{ and } d \in \mathscr{D}.$$

So P can only move to a game position with the same underlying tree position, while O is forced to move to a game position with a successor underlying tree position. This induces a dag structure on game positions, whose underlying partial order $\trianglelefteq_{\mathcal{G}}$ is the lexicographic product of the usual tree order with the one setting P-labels smaller than all O-labels. The games we shall consider will all be subrelations of $\trianglelefteq_{\mathcal{G}}$.

This Section is devoted to the definition of this dag structure. We shall also prove some basic results related to induction in §4.2 and to infinite paths in §4.3. These will help proving some similar results for games in §5, for which arguments are more naturally given at the level of $\trianglelefteq_{\mathcal{G}}$.

4.1. **A Partial Order of Game Positions.** We first introduce the formal notion of labels of game positions.

Definition 4.1 (Labels of Game Positions)**.** *Labels of game positions* are pairs $(\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}})$ of HF-terms satisfying the following formula:

$$\mathrm{Labels}(\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}}) \quad := \quad \neg(\exists k \mathrel{\dot{\in}} \mathsf{P}_{\mathcal{G}} \cap \mathsf{O}_{\mathcal{G}}) \ \wedge \ (\exists k \mathrel{\dot{\in}} \mathsf{P}_{\mathcal{G}}) \ \wedge \ (\exists \ell \mathrel{\dot{\in}} \mathsf{O}_{\mathcal{G}})$$

We write $\mathsf{PO}_{\mathcal{G}}$ for $\mathsf{P}_{\mathcal{G}} \cup \mathsf{O}_{\mathcal{G}}$. When no ambiguity arises, we write P, O and PO for $\mathsf{P}_{\mathcal{G}}$, $\mathsf{O}_{\mathcal{G}}$ and $\mathsf{PO}_{\mathcal{G}}$ respectively.

Assume (P, O) are labels of game positions. Intuitively, game positions are pairs (x,k) with $x \in \mathscr{D}^*$ and $k \in \mathsf{PO}$, Proponent's positions are game positions with $k \in \mathsf{P}$ and Opponent's positions are game positions with $k \in \mathsf{O}$. To summarize, we have the informal correspondence:

$\mathscr{D}^* \times \mathsf{PO}$	Game positions
$\mathscr{D}^* \times \mathsf{P}$	Proponent's positions
$\mathscr{D}^* \times \mathsf{O}$	Opponent's positions

We will throughout the paper use the following notation to manipulate game positions and sets of game positions.

Notation 4.2 (Game Positions)**.** We introduce the following notation, assuming $\mathrm{Labels}(\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}})$.

(1) Variables, written $u, v, w,$ etc, range over game positions, that is over pairs (x,k) with x an Individual variable and k an HF-variable ranging over $\mathsf{PO}_{\mathcal{G}}$.

(2) Sets of game positions, written $U, V, W,$ etc, range over Functions $\mathscr{D}^* \times \mathsf{PO}_{\mathcal{G}}$ $\mathtt{to}$ 2. We will systematically use the following notation:

$$\begin{array}{lcll} V : \mathcal{G} \ \ \mathtt{to}\ 2 & := & V : \mathscr{D}^* \times \mathsf{PO}_{\mathcal{G}}\ \mathtt{to}\ 2 & \text{(sets of Game positions)} \\ V : \mathcal{G}_{\mathsf{P}}\ \mathtt{to}\ 2 & := & V : \mathscr{D}^* \times \mathsf{P}_{\mathcal{G}}\ \ \mathtt{to}\ 2 & \text{(sets of Proponent's positions)} \\ V : \mathcal{G}_{\mathsf{O}}\ \mathtt{to}\ 2 & := & V : \mathscr{D}^* \times \mathsf{O}_{\mathcal{G}}\ \ \mathtt{to}\ 2 & \text{(sets of Opponent's positions)} \end{array}$$

We often write $v \in V$ or $V(v)$ for $V(v) \doteq 1$.

(3) For a set of game positions V, we write V_{P} and V_{O} for the P and O subsets of V respectively. This amounts to the following abbreviations:

$$\begin{array}{lcl} v \in V_{\mathsf{P}} & := & v \in V \ \ \wedge \ \ v \in (\mathscr{D}^* \times \mathsf{P}_{\mathcal{G}}) \\ v \in V_{\mathsf{O}} & := & v \in V \ \ \wedge \ \ v \in (\mathscr{D}^* \times \mathsf{O}_{\mathcal{G}}) \end{array}$$

Intuitively, V_{P} represents $V \cap (\mathscr{D}^* \times \mathsf{P}_{\mathcal{G}})$ while V_{O} represents $V \cap (\mathscr{D}^* \times \mathsf{O}_{\mathcal{G}})$.

(4) In formulae we interpret quantifiers over (sets of) game positions as follows:

$$\begin{array}{rcl}(\exists v)\varphi & := & (\exists x)(\exists \ell \mathrel{\dot\in} \mathsf{PO}_{\mathcal{G}})\varphi[(x,\ell)/v] \\ (\exists V)\varphi & := & (\exists V : \mathcal{G} \mathrel{\mathtt{to}} 2)\varphi\end{array}$$

where, in the $\exists v$ case, we choose x, ℓ not free in φ.

We now introduce the partial order $\trianglelefteq$ on game positions.

Definition 4.3 (Partial Order on Game Positions)**.** The relations $\lhd_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})}$, $\trianglelefteq_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})}$ and $\mathsf{s}^{\lhd}_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})}$, where $\mathsf{P}_{\mathcal{G}}$ and $\mathsf{O}_{\mathcal{G}}$ are HF-variables, are defined as follows:

$$\begin{array}{rcl}(x,k)\ \lhd_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})}\ (y,\ell) & := & x \mathrel{\dot<} y \ \lor\ (x \mathrel{\dot=} y \ \land\ k \mathrel{\dot\in} \mathsf{P}_{\mathcal{G}} \ \land\ \ell \mathrel{\dot\in} \mathsf{O}_{\mathcal{G}}) \\ u\ \trianglelefteq_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})}\ v & := & u \lhd_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})} v \ \lor\ u = v \\ \mathsf{s}^{\lhd}_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})}(u,v) & := & u \lhd_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})} v \ \land\ \neg(\exists w)\big(u \lhd_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})} w \lhd_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})} v\big)\end{array}$$

When no ambiguity arises, we write $\lhd_{\mathcal{G}}$, $\trianglelefteq_{\mathcal{G}}$ and $\mathsf{s}^{\lhd}_{\mathcal{G}}$, or even $\lhd$, $\trianglelefteq$ and $\mathsf{s}^{\lhd}$ for $\lhd_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})}$, $\trianglelefteq_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})}$ and $\mathsf{s}^{\lhd}_{(\mathsf{P}_{\mathcal{G}},\mathsf{O}_{\mathcal{G}})}$ respectively.

Note that the formula $\mathsf{s}^{\lhd}(-,-)$ is actually bounded, since by Notation 4.2.(1), the variable w ranges over game positions, so that $(\exists w)$ stands for $(\exists w \in \mathscr{D}^* \times \mathsf{PO})$.
We note a number of useful properties of $\lhd$, in particular that it is a discrete partial order.

Proposition 4.4. $\mathsf{FSO}_{\mathscr{D}}$ *proves following, under the assumption* $\mathrm{Labels}(\mathsf{P},\mathsf{O})$.

(1) $u \lhd v \lhd w \implies u \lhd w$
(2) $\neg(u \lhd u)$
(3) $u \trianglelefteq v \trianglelefteq u \implies u = v$
(4) $u \lhd v \iff (\exists w \trianglelefteq v)(\mathsf{s}^{\lhd}(u,w)) \iff (\exists w' \trianglerighteq u)(\mathsf{s}^{\lhd}(w',v))$
(5) $(\forall k \in \mathsf{P})\big(u \trianglelefteq (\dot\varepsilon,k) \implies u = (\dot\varepsilon,k)\big)$

Proof.

(1) Assume $(x,k) \lhd (y,\ell) \lhd (z,m)$. Note that we must have $x \mathrel{\dot\leq} y \mathrel{\dot\leq} z$. If $x \mathrel{\dot<} z$, then $(x,k) \lhd (z,m)$ and we are done. Otherwise, $x \mathrel{\dot=} z$ and the Tree Axioms (§3.4.3) imply that $x \mathrel{\dot=} y \mathrel{\dot=} z$. But in this case, by definition of $\lhd$, we must have $\ell \in \mathsf{P} \cap \mathsf{O}$, contradicting the disjointness of P and O.
(2) If $(x,k) \lhd (x,k)$, since by the Tree Axioms $\neg(x \mathrel{\dot<} x)$, then we must have $k \in \mathsf{P} \cap \mathsf{O}$, a contradiction.
(3) $u \lhd v \lhd u$ would imply $u \lhd u$, a contradiction. So $u \trianglelefteq v \trianglelefteq u$ implies $u = v$.
(4) The implications $\exists w' \trianglerighteq u.\mathsf{s}^{\lhd}(w',v) \Rightarrow \exists w \trianglelefteq v.\mathsf{s}^{\lhd}(u,w) \Rightarrow u \lhd v$ are trivial.

As for the other implications, first note that $(x,k) \lhd (x,\ell)$ implies $\mathsf{s}^{\lhd}((x,k),(x,\ell))$. Indeed, using the Tree Axioms (§3.4.3), $(x,k) \lhd (y,m) \lhd (x,\ell)$ implies $y \mathrel{\dot=} x$, so that $m \in \mathsf{P} \cap \mathsf{O}$, contradicting $\mathsf{P} \cap \mathsf{O} = \varnothing$. Hence $\neg\exists w((x,k) \lhd w \lhd (x,\ell))$ and $\mathsf{s}^{\lhd}((x,k),(x,\ell))$.

- We show $u \lhd v \Rightarrow \exists w \trianglerighteq u.\mathsf{s}^{\lhd}(w,v)$.
Let $u = (x,k)$ and $v = (y,\ell)$. If $x \mathrel{\dot=} y$ then we are done. Otherwise, we must have $x \mathrel{\dot<} y$. If $\ell \in \mathsf{O}$, then for any $m \in \mathsf{P}$ we have $u \lhd (y,m) \lhd (y,\ell)$, so that $\mathsf{s}^{\lhd}((y,m),(y,\ell))$. Since P is assumed to be non-empty, we are done by taking (y,m) for w.
It remains the case of $\ell \in \mathsf{P}$. It follows from the Induction Scheme of $\mathsf{FSO}_{\mathscr{D}}$ (§3.4.2) that either $y \mathrel{\dot=} \dot\varepsilon$ or there is some z and some $d \in \mathscr{D}$ such that $y = \mathsf{S}_d(z)$. Moreover, it follows from Proposition 3.8 that $x \mathrel{\dot<} y$ implies $\neg(y \mathrel{\dot=} \dot\varepsilon)$. Then the Tree Axioms give $x \mathrel{\dot\leq} z$ from $x \mathrel{\dot<} \mathsf{S}_d(z)$. Let now $m \in \mathsf{O}$. We thus have $(x,k) \trianglelefteq (z,m) \lhd (y,\ell)$. Moreover, assuming $(z,m) \lhd (z',m') \lhd (y,\ell)$, then $z' \mathrel{\dot<} y$ implies $z' \mathrel{\dot\leq} z$ hence $z' \mathrel{\dot=} z$ and therefore $m \in \mathsf{P} \cap \mathsf{O}$, a contradiction. Hence $z' \mathrel{\dot=} y$, so that $\ell \in \mathsf{O} \cap \mathsf{P}$, again a contradiction.

- The case of $u \lhd v \;\Rightarrow\; \exists w \unlhd v.\mathsf{s}^{\lhd}(u,w)$ is dealt-with similarly, using the Tree Axioms (§3.4.3) and Proposition 3.8.
  Let $u = (x,k)$ and $v = (y,\ell)$. First, if $x = y$ then we have $\mathsf{s}^{\lhd}(u,v)$ as shown above. Assume now that $x \dot{<} y$ and that $k \in \mathsf{P}$. Consider any $m \in \mathsf{O}$ (recall that O is non-empty). Then, again as above we have $\mathsf{s}^{\lhd}((x,k),(x,m))$ and we are done since $x \dot{<} y$ implies $(x,m) \lhd (y,\ell)$. It remains the case of $k \in \mathsf{O}$. Since $x \dot{<} y$, by Proposition 3.8 there is some $d \in \mathscr{D}$ such that $\mathsf{S}_d(x) \unlhd y$. Take any $m \in \mathsf{P}$ (which is assumed to be non-empty). Then we have $(x,k) \lhd (\mathsf{S}_d(x),m) \lhd (y,\ell)$. Hence we are done as soon as we show $\mathsf{s}^{\lhd}((x,k),(\mathsf{S}_d(x),m))$. Given z and n with $(x,k) \lhd (z,n) \lhd (\mathsf{S}_d(x),m)$, the Tree Axioms imply that either $x \doteq z$ or $z \doteq \mathsf{S}_d(x)$. But $x \doteq z$ implies $k \in \mathsf{O} \cap \mathsf{P}$, a contradiction, while $z \doteq \mathsf{S}_d(x)$ implies $m \in \mathsf{P} \cap \mathsf{O}$, also a contradiction. Hence $\neg\exists w[(x,k) \lhd w \lhd (\mathsf{S}_d(x),m)]$, as required.

(5) Assume $(x,\ell) \unlhd (\dot{\varepsilon},k)$. By definition of $\unlhd$ we have $x \dot{\leq} \dot{\varepsilon}$. Then Proposition 3.8 implies $x = \dot{\varepsilon}$, so that $k \in \mathsf{P}$ implies $\ell = k$. □

4.2. **Induction and Recursion.** We now present some basic results on induction and recursion w.r.t. the partial order on game positions.

We can show that $\lhd$ satisfies well-founded induction from the induction principle on the underlying tree.

Theorem 4.5 ($\lhd$-Induction)**.** $\mathsf{FSO}_{\mathscr{D}}$ *proves the following, under the assumption* $\mathrm{Labels}(\mathsf{P},\mathsf{O})$.

$$(\forall V)\Big((\forall v)\big[(\forall u \lhd v)(u \in V) \implies v \in V\big] \implies (\forall v)(v \in V)\Big)$$

Proof. Let V be such that, for any game position v:

$$(\forall u \lhd v)\;\big(V(u) \implies V(v)\big) \tag{4.1}$$

We show that

$$(\forall x)(\forall y \dot{\leq} x)(\forall \ell \,\dot{\in}\, \mathsf{PO})\big((y,\ell) \in V\big)$$

by induction on x, whence the theorem will follow.

Suppose that $x = \dot{\varepsilon}$, and so $y = \dot{\varepsilon}$. We first prove the statement for arbitrary $\ell \,\dot{\in}\, \mathsf{P}$; in this case notice that there is no u such that $u \lhd (\dot{\varepsilon},\ell)$, and so we vacuously satisfy the LHS of (4.1) above. Therefore we have that $(\dot{\varepsilon},\ell) \in V$. Otherwise $\ell \,\dot{\in}\, \mathsf{O}$ and every $u \lhd (\dot{\varepsilon},\ell)$ is of the form $(\dot{\varepsilon},k)$ for some $k \,\dot{\in}\, \mathsf{P}$, and we have just shown that such u must be contained in V. Therefore we can conclude that $(\dot{\varepsilon},\ell) \in V$, again by (4.1), as required.

Now we consider the inductive step, assuming the statement above is already true for x and considering the case of $\mathsf{S}_d x$. If $y \dot{\leq} \mathsf{S}_d x$ then either $y \dot{\leq} x$ or $y \doteq \mathsf{S}_d x$. In the former case we have by the inductive hypothesis that, for any $\ell \,\dot{\in}\, \mathsf{PO}$, $(y,\ell) \in V$. So assume that $y \doteq \mathsf{S}_d x$. Again we distinguish when $\ell \,\dot{\in}\, \mathsf{P}$ and when $\ell \,\dot{\in}\, \mathsf{O}$ in order to exhibit the LHS of (4.1) above. In the former case, notice that any $(z,k) \lhd (y,\ell)$ is such that $z \dot{\leq} x$, and so we have that $(z,k) \in V$ by the inductive hypothesis; thus $(y,\ell) \in V$ by (4.1). In the latter case (when $\ell \,\dot{\in}\, \mathsf{O}$) we have for any $(z,k) \lhd (y,\ell)$ either $z \dot{\leq} x$ or ($z \doteq \mathsf{S}_d x$ and $k \,\dot{\in}\, \mathsf{P}$). In both cases we have seen that $(z,k) \in V$, and so again we have that $(y,\ell) \in V$ by (4.1). □

Since $\lhd$ is a partial order with induction, comprehension (Theorem 3.33) gives a *Recursion Theorem*, which allows us to define a set of game positions V by induction on game positions. This requires the value of V at a position v to be determined by its values at positions $u \lhd v$. Thus, if the value of V at v is given by a formula $\varphi(V, v)$, we assume that the following formula holds

$$\mathrm{Rec}(\varphi) \quad := \quad (\forall v)(\forall V, V')\Big[(\forall w \lhd v)\big(Vw \ \Leftrightarrow\ V'w\big) \ \implies\ \big(\varphi(V,v) \ \Leftrightarrow\ \varphi(V',v)\big)\Big]$$

The Recursion Theorem says that, assuming $\mathrm{Rec}(\varphi)$, the set of game positions V given by

$$Vv \quad \iff \quad (\forall U)\Big[(\forall u \unlhd v)\big(Uu \ \Leftrightarrow\ \varphi(U,u)\big) \ \implies\ Uv\Big]$$

is the unique set of game positions such that

$$Vv \quad \iff \quad \varphi(V, v)$$

Proposition 4.6 (Recursion Theorem). $\mathsf{FSO}_{\mathscr{D}}$ *proves that* $\mathrm{Labels}(\mathsf{P}, \mathsf{O}) \wedge \mathrm{Rec}(\varphi)$ *implies*

$$\begin{array}{ll} (\forall v)\left(Vv \iff (\forall U)\Big[(\forall u \unlhd v)\big(Uu \ \Leftrightarrow\ \varphi(U,u)\big) \ \Rightarrow\ Uv\Big]\right) \ \implies\ (\forall v)\,(Vv \ \Leftrightarrow\ \varphi(V,v)) \\ \wedge \qquad (\forall v)(Vv \ \Leftrightarrow\ \varphi(V,v)) \ \implies\ (\forall v)(Uv \ \Leftrightarrow\ \varphi(U,v)) \ \implies\ (\forall v)\,(Vv \ \Leftrightarrow\ Uv) \end{array}$$

Proof. Consider a formula $\varphi(V, v)$ and assume $\mathrm{Rec}(\varphi)$ and $\mathrm{Labels}(\mathsf{P}, \mathsf{O})$. We begin with the second part of the statement, namely the uniqueness part. Fix V, U. By $\lhd$-induction on v, we show that FSO proves the following formula $\psi(v) = \psi(V, U, v)$:

$$(\forall u \unlhd v)(Vu \Leftrightarrow \varphi(V,u)) \ \implies\ (\forall u \unlhd v)(Uu \Leftrightarrow \varphi(U,u)) \ \implies\ (\forall u \unlhd v)(Vu \Leftrightarrow Uu)$$

Let v and assume both premises of $\psi(v)$, as well as $\psi(w)$ for all $w \lhd v$. The premises of $\psi(v)$ imply those of $\psi(w)$ for $w \lhd v$, so that we have $(Vw \Leftrightarrow Uw)$ for all $w \lhd v$. Hence, given $u \unlhd v$, if $u \lhd v$ then we are done. It thus remains to show $(Vv \Leftrightarrow Uv)$. Thanks to the premises of $\psi(v)$, this amounts to showing $\varphi(V, v) \Leftrightarrow \varphi(U, v)$, which itself follows from $\mathrm{Rec}(\varphi)$, since $(Vw \Leftrightarrow Uw)$ for all $w \lhd v$.

We now turn to the first part of the statement. Let V such that

$$Vv \quad \iff \quad (\forall U)\Big[(\forall u \unlhd v)\big(Uu \ \Leftrightarrow\ \varphi(U,u)\big) \ \implies\ Uv\Big]$$

By $\lhd$-induction on v, we show that FSO proves the following formula

$$\theta(v) \quad := \quad (\forall u \unlhd v)\,\underbrace{\big(Vu \ \iff\ \varphi(V,u)\big)}_{\vartheta(u)}$$

So let v and assume $\theta(w)$ for all $w \lhd v$. Given $u \unlhd v$, if $u \lhd v$ then $\vartheta(u)$ follows from $\theta(u)$. It thus remains to show $\vartheta(v)$. We consider the two implications separately.

- *Case of* $\varphi(V, v) \implies Vv$. Assume $\varphi(V, v)$. By definition of V, we are done if we show

  $$(\forall U)\Big[(\forall u \unlhd v)\big(Uu \ \Leftrightarrow\ \varphi(U,u)\big) \ \implies\ Uv\Big]$$

  Given U such that $\big(Uu \ \Leftrightarrow\ \varphi(U, u)\big)$ for all $u \unlhd v$, we obtain Uv from $\varphi(U, v)$, which itself follows $\varphi(V, v)$ and $\mathrm{Rec}(\varphi)$. The premise of $\mathrm{Rec}(\varphi)$ follows from $(\forall w \lhd v)\psi(V, U, w)$, whose premises are in turn given by resp. $(\forall w \lhd v)\vartheta(w)$ and the assumption on U.
- *Case of* $Vv \implies \varphi(V, v)$. Assume Vv. By comprehension (Theorem 3.33) let U such that

  $$Uu \quad \iff \quad \Big[(u \lhd v \ \wedge\ Vu) \ \vee\ \big(u = v \ \wedge\ \varphi(V,v)\big)\Big]$$

  We obtain $\varphi(V, v)$ from Uv, which in turn by def. of V follows from $(\forall u \unlhd v)\big(Uu \ \Leftrightarrow\ \varphi(U, u)\big)$. In order to show the latter, note that by definition of U we have $(Uu \Leftrightarrow Vu)$ for all $u \lhd v$. Hence $\mathrm{Rec}(\varphi)$ gives $\varphi(U, v) \Leftrightarrow \varphi(V, v)$ and we get $(Uv \Leftrightarrow \varphi(U, v))$ from the definition of U. In the case of $u \lhd v$, namely $(Uu \Leftrightarrow \varphi(U,u))$, we have $(\forall w \unlhd u)(Uw \Leftrightarrow Vw)$ so that $\mathrm{Rec}(\varphi)$ implies $\varphi(U,u) \Leftrightarrow \varphi(V,u)$ and the result follows form $\vartheta(u)$. $\square$

4.3. **Infinite Paths.** We develop here a notion of infinite paths (*i.e.* unbounded linearly order sets) for the partial order $\unlhd$ on game positions. This material will be useful in Section 5.2 to handle properties of infinite plays in games which intrinsically rely on the particular structure of the relation $\unlhd$ on game positions. A typical example is the Predecessor Lemma 5.10.

Definition 4.7 (Game Paths)**.** Let P, O be HF-variables. Given a game position u and a set of game positions U, we say that U is a *path from* u if the following formula $\mathrm{Path}(\mathsf{P}, \mathsf{O}, u, U)$ holds:

$$\mathrm{Path}(\mathsf{P}, \mathsf{O}, u, U) \quad := \quad \begin{cases} & u \in U \\ \wedge & (\forall v \in U)(u \unlhd v) \\ \wedge & (\forall v \in U)(\exists w \in U)(\mathsf{s}^{\lhd}(v,w)) \\ \wedge & (\forall v, w \in U)(w \lhd v \ \vee \ v = w \ \vee \ v \lhd w) \end{cases}$$

We write $\mathrm{Path}(u, U)$ when P and O are clear from the context.

As a preparation to the Predecessor Lemma 5.10 for Infinite Plays, we prove here the analogous property for infinite paths.

Lemma 4.8 (Predecessor Lemma for Game Paths)**.** $\mathsf{FSO}_{\mathscr{D}}$ *proves the following. Assuming that* $\mathrm{Labels}(\mathsf{P}, \mathsf{O})$ *and that* $\mathrm{Path}(\mathsf{P}, \mathsf{O}, u_0, U)$ *hold for a game position* u_0 *and a set of game positions* U*, we have*

$$(\forall v \in U)\left[u_0 \lhd v \quad \Rightarrow \quad (\exists u \in U)(\mathsf{s}^{\lhd}(u,v))\right]$$

The proof of Lemma 4.8 relies on the following usual maximality principle for non-empty linearly-ordered bounded sets.

Lemma 4.9. $\mathsf{FSO}_{\mathscr{D}}$ *proves the following, assuming* $\mathrm{Labels}(\mathsf{P}, \mathsf{O})$*. Given a set of game positions* V*, if* V *is bounded (i.e.* $(\exists u)(\forall v \in V)(v \lhd u)$*), non-empty and linearly ordered, then* V *has a maximum element:* $(\exists u \in V)(\forall v \in V)(v \unlhd u)$*.*

Proof. By $\lhd$-induction, we prove the following property:

($\star$) For all u, for all V, if V is non-empty, linearly ordered by $\lhd$ and such that $\forall v \in V(v \unlhd u)$, then V has a $\lhd$-maximal element.

Let u and V satisfy the assumptions from ($\star$) above, and assume ($\star$) for all $c \lhd u$. First, if $u = v$ for some $v \in V$, then u is indeed the maximal element of V. So we can assume $v \lhd u$ for all $v \in V$.

By Comprehension for Product Types (Thm. 3.33), let U be the set of all w such that $\mathsf{s}^{\lhd}(w,u)$ and such that $v \unlhd w$ for some $v \in V$. For each $v \in V$, it follows from Proposition 4.4.(4) that there is some $w \in U$ such that $v \unlhd w$. In particular, U is non-empty since V is non-empty.

We claim the following:

Claim 4.9.1.

$$(\forall w \in U)(\exists! \tilde{w} \in V) \underbrace{(\forall v \in V)\Big(v \unlhd w \quad \Rightarrow \quad v \unlhd \tilde{w}\Big)}_{\vartheta(w,\tilde{w})}$$

Proof of Claim 4.9.1. Let $w \in U$. By Comprehension for Product Types (Thm. 3.33), let W be the set of all $v \in V$ such that $v \unlhd w$. Note that W is non-empty by definition of U. It is inherits the property of being linearly ordered from V, and by construction it is bounded by w with $w \lhd u$. By induction hypothesis, W has a maximal element, say $\tilde{w}$. We indeed have $\tilde{w} \in V$ and $v \unlhd \tilde{w}$ for all $v \in V$ with $v \unlhd w$. Since $\tilde{w} \unlhd w$, uniqueness follows from the antisymmetry of $\unlhd$. $\blacksquare$

The remainder of the argument relies on the particular structure of $\lhd$. Using Comprehension on HF-Sets, it follows from the definition of $\lhd$ that there is some $x \in \mathscr{D}^*$ and some HF-Set k such that U is exactly the set of all (x, ℓ) with $\ell \in k$. This observation allows us to show

Claim 4.9.2.

$$(\exists \tilde{w}_m \in V)\,\underbrace{(\forall w \in U)(\forall \tilde{w} \in V)\big(\vartheta(w,\tilde{w}) \quad\Rightarrow\quad \tilde{w} \unlhd \tilde{w}_m\big)}_{\varphi(\tilde{w}_m)}$$

Proof of Claim 4.9.2. Write $\preceq$ for the well-order on k given by Remark 3.17. By $\preceq$-Induction (Remark 3.34.(3)) we show the following:

$$(\forall \ell \in k)(\exists m \in k)\,\underbrace{(\forall n \preceq \ell)(\forall \tilde{w}_n, \tilde{w}_m \in V)\big(\vartheta((x,n),\tilde{w}_n) \quad\Rightarrow\quad \vartheta((x,m),\tilde{w}_m) \quad\Rightarrow\quad \tilde{w}_n \unlhd \tilde{w}_m\big)}_{\psi(\ell,m)}$$

Let $\ell \in k$ be such that the property holds for all $\ell' \prec \ell$. If ℓ is $\preceq$-minimal, the result follows from the existence of a unique $\tilde{w}$ such that $\vartheta((x,\ell),\tilde{w})$. Otherwise, let ℓ' be the $\preceq$-predecessor of ℓ, and let $m \in k$ such that $\psi(\ell', m)$ be given by induction hypothesis. By Claim 4.9.1, let $\tilde{w}_\ell, \tilde{w}_m$ be the unique elements of V such that $\vartheta((x,\ell),\tilde{w}_\ell)$ and $\vartheta((x,m),\tilde{w}_m)$. Since V is linearly ordered, we have either that $\tilde{w}_m \unlhd \tilde{w}_\ell$ or that $\tilde{w}_m \unlhd \tilde{w}_\ell$. In the former case, we take ℓ for the new m, and in the latter we keep the same m.

Since U is non-empty, there is a $\preceq$-maximal $\ell \in k$. Let $m \in k$ such that $\psi(\ell, m)$, and by Claim 4.9.1, let $\tilde{w}_m \in V$ such that $\vartheta((x,m),\tilde{w}_m)$. By definition of k, we do have $\tilde{w} \unlhd \tilde{w}_m$ for all $\tilde{w} \in V$ with $\vartheta(w,\tilde{w})$ for some $w \in U$. Hence we have that $\varphi(\tilde{w}_m)$. ■

Consider now $\tilde{w}_m \in V$ such that $\varphi(\tilde{w}_m)$. As noted above, for all $v \in V$ there is some $w \in U$ such that $v \lhd w$. But we also have $v \unlhd \tilde{w}$ where $\tilde{w}$ is unique such that $\vartheta(w,\tilde{w})$. It thus follows that $v \unlhd \tilde{w}_m$ for all $v \in V$.

This concludes the proof of Lemma 4.9. □

We can now prove Lemma 4.8.

Proof of Lemma 4.8. Fix $v \in U$ with $u_0 \lhd v$. By Comprehension for Product Types (Thm. 3.33), let W be the set of all $w \in U$ such that $w \lhd v$. Since $u_0 \lhd v$ and $\mathrm{Path}(u_0, U)$, the set W is non-empty, linearly ordered and bounded by v. By Lemma 4.9, it has a maximal element, say w. We have $u_0 \unlhd w$ and $w \lhd v$. Moreover, by $\mathrm{Path}(u_0, U)$ there is some $\tilde{w} \in U$ such that $\mathsf{s}^{\lhd}(w,\tilde{w})$. Again by $\mathrm{Path}(u_0, U)$, we have

$$(\tilde{w} \lhd v \quad\vee\quad \tilde{w} = v \quad\vee\quad v \lhd \tilde{w})$$

But $\tilde{w} \lhd v$ implies $\tilde{w} \unlhd w$, a contradiction, while $v \lhd \tilde{w}$ implies $w \lhd v \lhd \tilde{w}$, contradicting $\mathsf{s}^{\lhd}(w,\tilde{w})$. It thus follows that $\tilde{w} = v$ and we are done. □

5. Infinite Two-Player Games

This Section is devoted to definitions and basic properties relating to games, building on §4. We will use these games in §6 and §9 to formalize a basic theory of tree automata in FSO.

Our games are played on bipartite dags (with partial order $\unlhd_{\mathcal{G}}$) induced by labels of game positions $(\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}})$ in the sense of §4. Continuing §4, *Proponent* will play from positions of the form

$$\mathcal{G}_{\mathsf{P}} \quad=\quad \mathscr{D}^* \times \mathsf{P}_{\mathcal{G}}$$

while *Opponent* will play from positions of the form

$$\mathcal{G}_{\mathsf{O}} \quad=\quad \mathscr{D}^* \times \mathsf{O}_{\mathcal{G}}$$

A game will be given by specifying edge relations of the form

$$(x,k) \underset{\mathsf{P}}{\longrightarrow} (x,\ell) \quad \text{and} \quad (x,\ell) \underset{\mathsf{O}}{\longrightarrow} (x.d,k) \qquad \text{where } k \in \mathsf{P}_{\mathcal{G}},\ \ell \in \mathsf{O}_{\mathcal{G}} \text{ and } d \in \mathscr{D},$$

so that, for J either P or O,

$$u \underset{\mathsf{J}}{\longrightarrow} v \qquad \text{implies} \qquad u \lhd v$$

(actually even $\mathsf{s}^{\lhd}(u,v)$). We insist on the fact that P can only move to a game position with the same underlying tree position, while O is forced to move to a game position with a successor tree position.

We first give basic definitions and results on games (§5.1) and infinite plays (§5.2). Besides the above mentioned constraints on the shape of games, these notions are standard. Our notion of strategy is presented in §5.3. A crucial point here is that, w.r.t. our games, the monadic language imposes all strategies to be by construction *positional* in the usual sense (see e.g. [Tho97]). Finally, §5.4 briefly discusses our setting for *winning* in games, and §5.5 presents in more detail the important particular case of *parity* conditions. Parity conditions are one of the prominent formulations of winning conditions for ω-regular games. This is in particular due to the fact that they are *positionally* determined, *i.e.* the winner of a parity game can always win with a *positional* winning strategy [EJ91] (see also [Tho97, Wal02, PP04]). This is of crucial importance in our setting as all our strategies are inherently positional, due to the underlying limits on expressiveness in the language of MSO. Finally, the Axiom $(PosDet)$ of Positional Determinacy of Parity Games is formulated in §5.6.

5.1. **Games.** A game $\mathcal{G}$ will be given by labels of game positions $\mathsf{P}_{\mathcal{G}}$ and $\mathsf{O}_{\mathcal{G}}$ together with Functions

$$E_{\mathsf{P}} : \mathcal{G}_{\mathsf{P}} \ \mathtt{to}\ \mathcal{P}_*(\mathsf{O}_{\mathcal{G}}) \qquad \text{and} \qquad E_{\mathsf{O}} : \mathcal{G}_{\mathsf{O}} \ \mathtt{to}\ \mathcal{P}_*(\mathscr{D} \times \mathsf{P}_{\mathcal{G}})$$

where $\mathcal{P}_*(-)$ is the HF-Function of §3.4.4.(d). Such Functions $E_{\mathsf{P}}, E_{\mathsf{O}}$ induce edge relations $\longrightarrow_{\mathsf{P}_{\mathcal{G}}}$ and $\longrightarrow_{\mathsf{O}_{\mathcal{G}}}$ given by

$$\begin{array}{lll} (x,k) \longrightarrow_{\mathsf{P}_{\mathcal{G}}} (x,\ell) & \text{iff} & \ell \in E_{\mathsf{P}}(x,k) \\ (x,\ell) \longrightarrow_{\mathsf{O}_{\mathcal{G}}} (x.d,k) & \text{iff} & (d,k) \in E_{\mathsf{O}}(x,\ell) \end{array}$$

We make this formal in the following definition.

Definition 5.1 (Games and Edge Relations)**.**

(1) A *game* $\mathcal{G}$ is given by HF-terms $\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}}$ and Functions $E(\mathcal{G})_{\mathsf{P}}, E(\mathcal{G})_{\mathsf{O}}$ which satisfy the following formula

$$\mathrm{Game}(\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}}, E(\mathcal{G})_{\mathsf{P}}, E(\mathcal{G})_{\mathsf{O}}) \quad := \quad \left\{ \begin{array}{ll} & \mathrm{Labels}(\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}}) \\ \wedge & E(\mathcal{G})_{\mathsf{P}} : \mathcal{G}_{\mathsf{P}} \ \mathtt{to}\ \mathcal{P}_*(\mathsf{O}_{\mathcal{G}}) \\ \wedge & E(\mathcal{G})_{\mathsf{O}} : \mathcal{G}_{\mathsf{O}} \ \mathtt{to}\ \mathcal{P}_*(\mathscr{D} \times \mathsf{P}_{\mathcal{G}}) \end{array} \right.$$

We often write $\mathrm{Game}(\mathcal{G})$ for $\mathrm{Game}(\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}}, E(\mathcal{G})_{\mathsf{P}}, E(\mathcal{G})_{\mathsf{O}})$. Moreover, when no ambiguity arises, we abbreviate $\mathcal{G} = (\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}}, E(\mathcal{G})_{\mathsf{P}}, E(\mathcal{G})_{\mathsf{O}})$ as $\mathcal{G} = (\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}}, E_{\mathsf{P}}, E_{\mathsf{O}})$ or even $\mathcal{G} = (\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}}, E)$ or $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E)$.

(2) The *edge relations* induced by $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E_{\mathsf{P}}, E_{\mathsf{O}})$ are defined as follows:

$$\begin{array}{lll} (x,k) \longrightarrow_{\mathsf{P}_{\mathcal{G}}} (y,\ell) & := & k \mathrel{\dot\in} \mathsf{P} \ \wedge\ x \doteq y \ \wedge\ \ell \mathrel{\dot\in} E_{\mathsf{P}}(x,k) \\ (x,\ell) \longrightarrow_{\mathsf{O}_{\mathcal{G}}} (y,k) & := & \ell \mathrel{\dot\in} \mathsf{O} \ \wedge\ \bigvee_{d \in \mathscr{D}} \big(y \doteq \mathsf{S}_d(x) \ \wedge\ (d,k) \mathrel{\dot\in} E_{\mathsf{O}}(x,\ell) \big) \\ \quad u \longrightarrow_{\mathcal{G}} v & := & u \longrightarrow_{\mathsf{P}_{\mathcal{G}}} v \ \vee\ u \longrightarrow_{\mathsf{O}_{\mathcal{G}}} v \end{array}$$

When no ambiguity arises, we write $\longrightarrow_{\mathsf{P}}$, $\longrightarrow_{\mathsf{O}}$ and $\longrightarrow$, for $\longrightarrow_{\mathsf{P}_{\mathcal{G}}}$, $\longrightarrow_{\mathsf{O}_{\mathcal{G}}}$ and $\longrightarrow_{\mathcal{G}}$.

Note that $\mathrm{Game}(\mathcal{G})$ implies that the edge relation $\longrightarrow$ has no dead ends, *i.e.* that from any position, a move can always be made by one of the players. It follows that the edge relation $\longrightarrow$ induces an unbounded partial order. (Note that it already follows from the structure of $\longrightarrow$ that it induces a partial order.)

Lemma 5.2. $\mathsf{FSO}_{\mathscr{D}}$ *proves*

$$\mathrm{Game}(\mathcal{G}) \quad \Longrightarrow \quad (\forall u)(\exists v)\left(u \longrightarrow v\right)$$

Games are equipped with a natural notion of subgame. In this paper we will use subgames to ease some reasoning on automata (in particular in §9), and also to more easily define certain strategies that are more naturally seen as concepts at the game level (see §5.3). We only need the following weak notion of subgame.

Definition 5.3 (Subgame)**.** We say that $\mathcal{G}'$ is a *subgame* of $\mathcal{G}$ whenever the following formula holds

$$\mathrm{Sub}(\mathcal{G}', \mathcal{G}) \quad := \quad \mathsf{P}_{\mathcal{G}'} \doteq \mathsf{P}_{\mathcal{G}} \ \wedge \ \mathsf{O}_{\mathcal{G}'} \doteq \mathsf{O}_{\mathcal{G}} \ \wedge \ (\forall u, v)\left(u \underset{\mathcal{G}'}{\longrightarrow} v \ \Rightarrow \ u \underset{\mathcal{G}}{\longrightarrow} v\right)$$

Remark 5.4. Let $\mathcal{G} = (\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}}, E(\mathcal{G})_{\mathsf{P}}, E(\mathcal{G})_{\mathsf{O}})$ with $\mathrm{Game}(\mathcal{G})$. Then we have $\mathrm{Sub}(\mathcal{G}, \mathcal{G}(\trianglelefteq))$, where $\mathcal{G}(\trianglelefteq)$ stands for the game

$$(\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}}, E_{\mathsf{O}}, E_{\mathsf{P}})$$

in which by HF-Bounded Choice we let

$$E_{\mathsf{P}}(x, k) := \mathsf{O}_{\mathcal{G}} \qquad \text{and} \qquad E_{\mathsf{O}}(x, \ell) := (\mathscr{D} \times \mathsf{P}_{\mathcal{G}})$$

Note that the edge relation of $\mathcal{G}(\trianglelefteq)$ is precisely the relation $\trianglelefteq_{(\mathsf{P}_{\mathcal{G}}, \mathsf{O}_{\mathcal{G}})}$ of Definition 4.3, hence the notation.

The edge relation $\longrightarrow$ of a game $\mathcal{G}$ only specifies the *moves* of $\mathcal{G}$. In order to manipulate plays (*i.e.* sequences of moves) we define the reflexive-transitive closure $\longrightarrow^*$ and the transitive closure $\longrightarrow^+$ of $\longrightarrow$. As expected, these are second-order notions.[2]

Definition 5.5. Let $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E_{\mathsf{P}}, E_{\mathsf{O}})$ where P, O are HF-variables and $E_{\mathsf{P}}, E_{\mathsf{O}}$ are Function variables. We define the following formulae.

$$\begin{array}{rcll}
\mathrm{DC}_{\mathcal{G}}(V) & := & (\forall v \in V)(\forall u)\left(u \longrightarrow_{\mathcal{G}} v \ \Rightarrow \ u \in V\right) & \quad \textit{(V is downward-closed)} \\
u \longrightarrow^*_{\mathcal{G}} v & := & (\forall V)\left(\mathrm{DC}_{\mathcal{G}}(V) \ \Rightarrow \ v \in V \ \Rightarrow \ u \in V\right) & \\
u \longrightarrow^+_{\mathcal{G}} v & := & u \longrightarrow^*_{\mathcal{G}} v \ \wedge \ \neg(u = v) &
\end{array}$$

Whenever possible, we write $\longrightarrow^*$ and $\longrightarrow^+$ for $\longrightarrow^*_{\mathcal{G}}$ and $\longrightarrow^+_{\mathcal{G}}$.

The relations $\longrightarrow^*$ and $\longrightarrow^+$ satisfy properties analogous to those of Proposition 4.4:

Proposition 5.6 (Properties of Edge Relations)**.** $\mathsf{FSO}_{\mathscr{D}}$ *proves the following, under the assumption* $\mathrm{Game}(\mathcal{G})$.

(1) $u \longrightarrow v \ \Rightarrow \ \mathsf{s}^{\triangleleft}(u, v)$
(2) $\longrightarrow$ *is irreflexive and asymmetric.*
(3) $\longrightarrow^*$ *is reflexive and transitive.*
(4) $u \longrightarrow^* v \quad \Leftrightarrow \quad u = v \vee (\exists w)\,(u \longrightarrow^* w \longrightarrow v) \quad \Leftrightarrow \quad u = v \vee (\exists w)\,(u \longrightarrow w \longrightarrow^* v)$
(5) $u \longrightarrow^* v \ \Rightarrow \ u \trianglelefteq v$
(6) $\longrightarrow^*$ *is antisymmetric.*

[2] It is well known (see e.g. [Lib04, Chap. 4]) that transitive closure in graphs is not expressible in first-order logic over the edge relation.

(7) $u \longrightarrow^{+} v \;\;\Rightarrow\;\; u \lhd v$
(8) $\longrightarrow^{+}$ *is irreflexive and transitive.*
(9) $(\forall k \in \mathsf{P})\,(u \longrightarrow^{*} (\dot{\varepsilon}, k) \;\;\Rightarrow\;\; u = (\dot{\varepsilon}, k))$

Proof. (1) Assume $u \longrightarrow v$ and let $u = (x, k)$ and $v = (y, \ell)$. If $k \in \mathsf{P}$, then we must have $y \doteq x$ and $\ell \in \mathsf{O}$, so that $\mathsf{s}^{\lhd}(u, v)$. Otherwise, we must have $u \longrightarrow_{\mathsf{O}} v$ which implies $y = \mathsf{S}_d(x)$ for some $d \in \mathscr{D}$. Moreover, reasoning as in the proof of Proposition 4.4, we get $\mathsf{s}^{\lhd}(u, v)$ from the facts that $k \in \mathsf{O}$ and $\ell \in \mathsf{P}$.

(2) Inherited from the same properties for $\mathsf{s}^{\lhd}$.

(3) Reflexivity follows from reflexivity of implication and transitivity follows from transitivity of implication.

(4) Assume first $u \longrightarrow^{*} w \longrightarrow v$. Given $\mathrm{DC}(V)$ such that $v \in V$, we must have $w \in V$. But $u \longrightarrow^{*} w$ implies $u \in V$. Hence $u \longrightarrow^{*} v$. Similarly, if $u \longrightarrow w \longrightarrow^{*} v$, given $\mathrm{DC}(V)$ such that $v \in V$, we have $w \in V$ by definition of $\longrightarrow^{*}$ and we get $u \in V$ since $\mathrm{DC}(V)$.

Assume conversely that $u \longrightarrow^{*} v$ with $u \neq v$. We first show that $u \longrightarrow^{*} w \longrightarrow v$ for some $w \in G$. By Comprehension for Product Types (Theorem 3.33), let W be the set of all $\tilde{w}$ such that $\tilde{w} \longrightarrow^{*} w$ for some $w \longrightarrow v$, and let V be the union of W with $\{v\}$. We claim that V is downward closed. Indeed, assume given $w' \in V$ and $w'' \longrightarrow w'$. If $w' = v$, then (by reflexivity of $\longrightarrow^{*}$), $w'' \longrightarrow v$ implies $w'' \in W \subseteq V$. Otherwise, we must have $w' \in W$, so that $w' \longrightarrow^{*} \tilde{w}$ for some $\tilde{w} \longrightarrow w$. But by transitivity of $\longrightarrow^{*}$, we have $w'' \longrightarrow^{*} \tilde{w}$, so that $w'' \in W$. Since $v \in V$ and V is downward closed, $u \longrightarrow^{*} v$ implies $u \in V$, and $u \neq v$ implies $u \in W$, so that $u \longrightarrow^{*} w$ for some $w \longrightarrow v$.

We reason similarly in order to show $u \longrightarrow w \longrightarrow^{*} v$ for some $w \in G$. Again by Comprehension, let W be the set of all $\tilde{w}$ such that $\tilde{w} \longrightarrow w$ for some $w \longrightarrow^{*} v$, and let $V = W \cup \{v\}$. Again, V is downward closed, since $w \longrightarrow v$ implies $w \longrightarrow v \longrightarrow^{*} v$, and given $w' \longrightarrow \tilde{w} \in W$ with $\tilde{w} \longrightarrow w \longrightarrow^{*} v$ we have $w' \longrightarrow \tilde{w} \longrightarrow^{*} v$. Now, $u \longrightarrow^{*} v$ implies $u \in V$, but since $u \neq v$, we have $u \in W$ and we are done.

(5) For the right-to-left direction, assume $u \longrightarrow^{*} w \longrightarrow v$. Given $\mathrm{DC}(V)$ such that $v \in V$, we must have $w \in V$. But $u \longrightarrow^{*} w$ implies $u \in V$. Hence $u \longrightarrow^{*} v$.

For the left-to-right direction, assume $u \longrightarrow^{*} v$ with $\neg(u = v)$. By Comprehension for Product Types (Theorem 3.33), let W be the set of all $\tilde{w}$ such that $\tilde{w} \longrightarrow^{*} w$ for some $w \longrightarrow v$, and let V be the union of W with $\{v\}$. We claim that V is downward closed. Indeed, assume given $w' \in V$ and $w'' \longrightarrow w'$. If $w' = v$, then (by reflexivity of $\longrightarrow^{*}$), $w'' \longrightarrow v$ implies $w'' \in W \subseteq V$. Otherwise, we must have $w' \in W$, so that $w' \longrightarrow^{*} \tilde{w}$ for some $\tilde{w} \longrightarrow w$. But by transitivity of $\longrightarrow^{*}$, we have $w'' \longrightarrow^{*} \tilde{w}$, so that $w'' \in W$.

Since $v \in V$ and V is downward closed, $u \longrightarrow^{*} v$ implies $u \in V$, and $\neg(u = v)$ implies $u \in W$, so that $u \longrightarrow^{*} w$ for some $w \longrightarrow v$.

(6) Assume given u. By $\lhd$-induction, we show that $\forall v(u \longrightarrow^{*} v \Rightarrow u \unlhd v)$. So let v such that the property holds for all $w \lhd v$, and assume $u \longrightarrow^{*} v$. If $u = v$ then we are done. Otherwise, there is some $w \longrightarrow v$ such that $u \longrightarrow^{*} w$. But we have seen that $w \longrightarrow v$ implies $w \lhd v$, so that the induction hypothesis gives $u \unlhd w$, and we are done by transitivity of $\unlhd$.

(7) Inherited from the same property for $\unlhd$.

(8) If $u \longrightarrow^{+} v$ then $u \longrightarrow^{*} v$ and $u \neq v$, so that $u \unlhd v \wedge u \neq v$, which implies $u \lhd v$ by definition of $\unlhd$.

(9) Inherited from the same properties for $\lhd$.

(10) Inherited from the same property for $\unlhd$. $\square$

Induction for games (*i.e.* w.r.t. edge relations) is an immediate corollary to Theorem 4.5 and Proposition 5.6.

Corollary 5.7 (Game Induction)**.** $\mathsf{FSO}_{\mathscr{D}}$ *proves the following, under the assumption* $\mathrm{Game}(\mathcal{G})$.

$$(\forall V)\left((\forall v)\left[\left(\forall u \xrightarrow{+} v\right)(u \in V) \implies v \in V\right] \implies (\forall v)\big(v \in V\big)\right)$$

5.2. **Infinite Plays.** We now define our notion of infinite play. They are sets of game positions which are unbounded and linearly ordered w.r.t. $\longrightarrow$. Infinite plays will allow us to define winning in games (§5.4) and thus acceptance for tree automata (§6). Furthermore, we prove a number of basic properties on infinite plays on which we rely for the formalization of usual operations on tree automata.

In the following, given $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E)$, we write $\mathrm{Path}(\mathcal{G}, u, U)$ for $\mathrm{Path}(\mathsf{P}, \mathsf{O}, u, U)$, where Path is as in Definition 4.7.

Definition 5.8 (Infinite Plays)**.** Let $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E_{\mathsf{P}}, E_{\mathsf{O}})$ where P, O are HF-variables and $E_{\mathsf{P}}, E_{\mathsf{O}}$ are Function variables. Given a position u and a set of game positions U, we say that U is an *infinite play in* $\mathcal{G}$ *from* u when the following formula $\mathrm{Play}(\mathcal{G}, u, U)$ holds:

$$\mathrm{Play}(\mathcal{G}, u, U) \quad := \quad \begin{cases} & (u \in U) \\ \land & (\forall v \in U)\big(u \longrightarrow^*_{\mathcal{G}} v\big) \\ \land & (\forall v \in U)(\exists w \in U)\big(v \longrightarrow_{\mathcal{G}} w\big) \\ \land & (\forall v, w \in U)\big(v \longrightarrow^+_{\mathcal{G}} w \ \lor\ v = w \ \lor\ w \longrightarrow^+_{\mathcal{G}} v\big) \end{cases}$$

Note that $\mathrm{Play}(\mathcal{G}, u, U)$ is literally just the formula $\mathrm{Path}(\mathcal{G}, u, U)$ in which $\longrightarrow^*_{\mathcal{G}}$ replaces $\trianglelefteq$, $\longrightarrow_{\mathcal{G}}$ replaces $\mathsf{s}^{\triangleleft}(-,-)$ and $\longrightarrow^+_{\mathcal{G}}$ replaces $\triangleleft$. It follows from Proposition 5.6 that $\mathrm{Play}(\mathcal{G}, u, U)$ implies $\mathrm{Path}(\mathcal{G}, u, U)$. In other words, an infinite play in $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E)$ is simply an infinite path of the underlying partial order $\trianglelefteq_{(\mathsf{P},\mathsf{O})}$ which respects the transitions of $\mathcal{G}$ induced by E. Also, if $\mathcal{G}'$ is a subgame of $\mathcal{G}$, then $\mathrm{Play}(\mathcal{G}', u, U)$ implies $\mathrm{Play}(\mathcal{G}, u, U)$.

We now gather some basic properties on infinite plays. The first one will help to show that a set of game positions is linearly ordered.

Proposition 5.9. $\mathsf{FSO}_{\mathscr{D}}$ *proves the following, assuming* $\mathrm{Game}(\mathcal{G})$*. Let* V *and* $u_0 \in V$ *be such that*

$$\begin{cases} & (\forall v \in V)(u_0 \longrightarrow^* v) \\ \land & (\forall u \in V)(\exists! v \in V)(u \longrightarrow v) \\ \land & (\forall v \in V)\,[v \neq u_0 \ \Rightarrow\ (\exists u \in V)(u \longrightarrow v)] \end{cases}$$

Then

$$(\forall v, w \in V)\left(v \xrightarrow{+} w \quad \lor \quad v = w \quad \lor \quad w \xrightarrow{+} v\right)$$

Proof. First, it follows from Proposition 5.6.(6) that u_0 is unique such that $(\forall v \in V)(u_0 \longrightarrow^* v)$. By induction on the edge relation $\longrightarrow^+$ (cf. Corollary 5.7) we show

$$(\forall u \in V)\,\underbrace{(\forall v \in V)\left(u \xrightarrow{+} v \ \lor\ u = v \ \lor\ v \xrightarrow{+} u\right)}_{\theta(u)}$$

Let $u \in V$, and assume that $\theta(v)$ holds for all $v \in V$ such that $v \longrightarrow^+ u$. If $u = u_0$ then we are done since $u_0 \longrightarrow^* v$ for all $v \in V$. Otherwise, by assumption there is $v \in V$ with $v \longrightarrow u$, and moreover such that u is the unique $\longrightarrow$-successor of v in U.

Note that $v \longrightarrow u$ implies $v \longrightarrow^+ u$ (Proposition 5.6, (2) & (4)), so that $\theta(v)$ follows from the induction hypothesis. Given $w \in V$, if $w \longrightarrow^* v$ then we get $w \longrightarrow^* u$ and we are done. Otherwise, since $\theta(v)$ implies $v \longrightarrow^+ w$, we may appeal to the following.

Claim 5.9.1.

$$(\forall w \in V)\left(v \xrightarrow{+} w \quad\Rightarrow\quad u \xrightarrow{*} w\right)$$

Proof of Claim 5.9.1. We reason by induction on $\longrightarrow^+$. So let $w \in V$ with $v \longrightarrow^+ w$ and such that

$$(\forall w' \in V)\left(w' \xrightarrow{+} w \quad\Rightarrow\quad v \xrightarrow{+} w' \quad\Rightarrow\quad u \xrightarrow{*} w'\right)$$

Since $u_0 \longrightarrow^* v \longrightarrow^+ w$ we have $w \neq u_0$ by Proposition 5.6.(6), so that there is $w' \in V$ with $w' \longrightarrow w$. If $v \longrightarrow^+ w'$ then the induction hypothesis implies $u \longrightarrow^* w'$, so that $u \longrightarrow^+ w$ and we are done. Otherwise $\theta(v)$ implies $w' \longrightarrow^* v$. Assume for contradiction that $w' \longrightarrow^+ v$. We thus have

$$w' \xrightarrow{+} v \xrightarrow{+} w$$

Proposition 5.6.(7) then gives $w' \lhd v \lhd w$. But this contradicts $w' \longrightarrow w$ since the latter implies $\mathsf{s}^{\lhd}(w', w)$ by Proposition 5.6.(1). Hence $w' = v$. But then $v = w' \longrightarrow w \in V$ and, since u is the unique $\longrightarrow$-successor of v in V, we have $u = w$, as required. ■

This concludes the proof of Proposition 5.9. □

Proposition 5.9 is a useful tool to prove that given sets of game positions are infinite plays. Some constructions on automata (see §6, §9) furthermore require us to build plays in one game from plays in another game. To this end, we note here the following property, which we informally see as a partial converse to Proposition 5.9.

Lemma 5.10 (Predecessor Lemma for Infinite Plays)**.** $\mathsf{FSO}_{\mathscr{D}}$ *proves the following. Assuming* $\mathrm{Game}(\mathcal{G})$ *and* $\mathrm{Play}(\mathcal{G}, u_0, U)$*, we have*

$$(\forall v \in U)\left[u_0 \xrightarrow{+} v \quad\Longrightarrow\quad (\exists u \in U)\left(u \longrightarrow v\right)\right]$$

Proof. First, it follows from Proposition 5.6 that $\mathrm{Play}(\mathcal{G}, u_0, U)$ implies $\mathrm{Path}(\mathcal{G}, u_0, U)$. We invoke the Predecessor Lemma 4.8 for Game Paths. Assuming $u_0 \longrightarrow^+ v$, Proposition 5.6 implies $u_0 \lhd v$, so there is $u \in U$ such that $\mathsf{s}^{\lhd}(u, v)$. Since U is an infinite play, $u \in U$ has an $\longrightarrow$-successor in U, *i.e.* there is some $u' \in U$ such that $u \longrightarrow u'$. Again since U is an infinite play, we have

$$\left(v \xrightarrow{+} u' \quad\vee\quad u' = u \quad\vee\quad u' \xrightarrow{+} v\right)$$

But by Proposition 5.6 again, $v \longrightarrow^+ u'$ implies $u \lhd v \lhd u'$, contradicting $\mathsf{s}^{\lhd}(u, u')$, while $u' \longrightarrow^+ v$ implies $u \lhd u' \lhd v$, contradicting $\mathsf{s}^{\lhd}(u, v)$. Hence $u' = v$ and we are done. □

Next, we show that games have infinite plays from any position, relying on Remark 3.17.

Lemma 5.11. $\mathsf{FSO}_{\mathscr{D}}$ *proves that* $\mathrm{Game}(\mathcal{G})$ *implies*

$$(\forall v)(\exists U)\mathrm{Play}(\mathcal{G}, v, U)$$

Proof. Let $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E_{\mathsf{P}}, E_{\mathsf{O}})$. Fix $v \in V$. Using Remark 3.17, let $\preceq$ be a well-order on $\mathsf{P} \cup \mathsf{O}$. We extend the relation $\preceq$ to $\mathscr{D}^* \times (\mathsf{P} \cup \mathsf{O})$ by setting:

$$(x, k) \prec (y, \ell) \quad := \quad \begin{cases} (x \doteq y \ \wedge\ k \preceq \ell) & \vee \\ (\exists z) \bigvee_{d < d' \in \mathscr{D}} (x \doteq \mathsf{S}_d(z) \wedge y \doteq \mathsf{S}_{d'}(z)) & \end{cases}$$

Remark 3.17 implies that every non-empty W such that

$$(\forall (x,k) \in W)(x = \dot{\varepsilon}) \quad \vee \quad (\exists z)(\forall (x,k) \in W) \bigvee_{d \in \mathscr{D}} (x = \mathsf{S}_d(z))$$

has a $\preceq$-least element. By HF-Bounded Choice (Theorem 3.32), we define

$$E'_{\mathsf{P}} : \mathscr{D}^* \times \mathsf{P} \text{ to } \mathcal{P}_*(\mathsf{O}) \qquad \text{and} \qquad E'_{\mathsf{O}} : \mathscr{D}^* \times \mathsf{O} \text{ to } \mathcal{P}_*(\mathscr{D} \times \mathsf{P})$$

by setting, for J either P or O,

$$E'_{\mathsf{J}}(u) \quad := \quad \{\text{the } \preceq\text{-least element of } E_{\mathsf{J}}(u)\}$$

Let $\mathcal{G}' := (\mathsf{P}, \mathsf{O}, E')$. Note that we have $\mathrm{Game}(\mathcal{G}')$ and that

$$(\forall u, v) \left(u \underset{\mathcal{G}'}{\longrightarrow} v \ \ \Rightarrow \ \ u \underset{\mathcal{G}}{\longrightarrow} v \right) \tag{5.1}$$

By Comprehension for Product Types (Theorem 3.33), we then let

$$U \quad := \quad \{u \mid v \overset{*}{\underset{\mathcal{G}'}{\longrightarrow}} u\}$$

It remains to show

$$\mathrm{Play}(\mathcal{G}, v, U)$$

First, we have $v \in U$ by reflexivity of $\longrightarrow^*_{\mathcal{G}'}$ (Proposition 5.6.(3)), and $(\forall u \in U) \left(v \longrightarrow^*_{\mathcal{G}} u\right)$ follows from (5.1). Moreover, we have

$$(\forall u \in U)(\exists w \in U) \left(u \underset{\mathcal{G}}{\longrightarrow} w \right)$$

thanks to (5.1), since this property already holds for $\mathcal{G}'$. It remains to show that U is linearly ordered w.r.t. $\longrightarrow^*_{\mathcal{G}}$. We invoke Proposition 5.9: its first premise has already been discussed, its second follows from the definition of E', and its last one is Proposition 5.6.(4). $\square$

Finally, in some situations (typically for the *Simulation Theorem* in §9), it is convenient to build infinite plays from paths (in the sense of Definition 4.7).

Lemma 5.12 (Infinite Plays From Paths)**.** *Assume* $\mathrm{Game}(\mathcal{G})$ *and let* u_0 *and* U *be such that*

$$\mathrm{Path}(\mathcal{G}, u_0, U) \ \wedge \ (\forall u, v \in U) \left[\mathsf{s}^{\lhd}(u, v) \ \Rightarrow \ u \longrightarrow v \right]$$

Then $\mathsf{FSO}_{\mathscr{D}}$ *proves* $\mathrm{Play}(\mathcal{G}, u_0, U)$.

Proof. Thanks to Proposition 5.6, the result directly follows from the fact that

$$(\forall u, v \in U) \left(u \trianglelefteq v \ \ \Longrightarrow \ \ u \overset{*}{\longrightarrow} v \right)$$

Fix $u \in U$. By $\lhd$-induction we show $(\forall v \in U)(u \trianglelefteq v \ \ \Rightarrow \ \ u \longrightarrow^* v)$. So let $v \in U$ such that the property holds for all $w \lhd v$, and assume $u \trianglelefteq v$. If $u = v$ then we are done. Otherwise, by the Predecessor Lemma 4.8 for Paths, we have $\mathsf{s}^{\lhd}(w, v)$ for some $w \in U$ with $u \trianglelefteq w$. By induction hypothesis we get $u \longrightarrow^* w \longrightarrow v$ and we conclude by Proposition 5.6. $\square$

5.3. **Strategies.** We now turn to strategies. Our strategies are Functions from the positions of one player to the set of labels of the other player, which must respect the edge relations. This implies that all our strategies are, by definition, positional.

Definition 5.13 (Strategies)**.** Let $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E_\mathsf{P}, E_\mathsf{O})$ where P, O are HF-variables and where $E_\mathsf{P}, E_\mathsf{O}$ are Function variables.

(1) A P-*strategy on* $\mathcal{G}$ is a Function σ which satisfies the formula

$$\mathrm{Strat}_\mathsf{P}(\mathcal{G}, \sigma) \quad := \quad \sigma : \mathcal{G}_\mathsf{P} \ \mathtt{to}\ \mathsf{O} \ \wedge \ (\forall v)\,(\sigma(v) \in E_\mathsf{P}(v))$$

(2) An O-*strategy on* $\mathcal{G}$ is a Function σ which satisfies the formula

$$\mathrm{Strat}_\mathsf{O}(\mathcal{G}, \sigma) \quad := \quad \sigma : \mathcal{G}_\mathsf{O} \ \mathtt{to}\ \mathscr{D} \times \mathsf{P} \ \wedge \ (\forall v)\,(\sigma(v) \in E_\mathsf{O}(v))$$

Strategies naturally induce subgames in the sense of Definition 5.3. This will allow us to lift to strategies notions which are more naturally defined at the level of games.

Definition 5.14 (Subgame induced by a Strategy)**.** Given a player J (either P or O) and a J-strategy σ on $\mathcal{G}$, we let

$$\mathcal{G}{\restriction}\{\sigma\}_\mathsf{J} \quad := \quad (\mathsf{P}_\mathcal{G}\,,\, \mathsf{O}_\mathcal{G}\,,\, E(\mathcal{G}){\restriction}\{\sigma\}_\mathsf{J})$$

where

$$E(\mathcal{G}){\restriction}\{\sigma\}_\mathsf{P} \ := \ (\{\sigma\}_\mathsf{P}, E(\mathcal{G})_\mathsf{O}) \qquad \text{and} \qquad E(\mathcal{G}){\restriction}\{\sigma\}_\mathsf{O} \ := \ (E(\mathcal{G})_\mathsf{P}, \{\sigma\}_\mathsf{O})$$

and where $\{\sigma\}_\mathsf{J} \subseteq E(\mathcal{G})_\mathsf{J}$ is defined by HF-Bounded Choice to be the Function taking $u \in \mathscr{D}^* \times \mathcal{G}_\mathsf{J}$ to the singleton $\{\sigma(u)\}$.

Whenever possible, we write $\mathcal{G}{\restriction}\{\sigma\}$ or even just σ for $\mathcal{G}{\restriction}\{\sigma\}_\mathsf{J}$, when it is unambiguous.

Lemma 5.15. $\mathsf{FSO}_\mathscr{D}$ *proves the following, where* J *is a player (either* P *or* O*):*

$$\big(\mathrm{Game}(\mathcal{G}) \ \wedge \ \mathrm{Strat}_\mathsf{J}(\mathcal{G}, \sigma)\big) \quad \Longrightarrow \quad \mathrm{Game}(\sigma)$$

This in particular allows us to speak of the infinite plays of a strategy σ on $\mathcal{G}$ simply as infinite plays of the game $\mathcal{G}{\restriction}\{\sigma\}$.

5.4. **Winning.** In order to deal with acceptance for automata, we equip games with a notion of *winning*. Given a game $\mathcal{G}$, a *winning condition* on $\mathcal{G}$ is a formula $\mathcal{W}(U)$ where U is intended to range over the infinite plays of $\mathcal{G}$. As usual a P-strategy σ on $(\mathcal{G}, \mathcal{W})$ is winning from a position v whenever all the infinite plays U of σ from v satisfy $\mathcal{W}(U)$. Dually, an O-strategy is winning from v when all its infinite plays U from v satisfy $\neg\mathcal{W}(U)$.

We formally proceed as follows.

Definition 5.16. Let $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E_\mathsf{P}, E_\mathsf{O})$ where P, O are HF-variables and $E_\mathsf{P}, E_\mathsf{O}$ are Function variables. Let $\mathcal{W}(U)$ be a given FSO-formula where U is a Function variable.

(1) We define the following formulae.

$$\begin{array}{rcl} \mathrm{WonGame}_\mathsf{P}(\mathcal{G}, v, \mathcal{W}) & := & (\forall U)\big(\mathrm{Play}(\mathcal{G}, v, U) \ \Rightarrow \ \mathcal{W}(U)\big) \\ \mathrm{WonGame}_\mathsf{O}(\mathcal{G}, v, \mathcal{W}) & := & (\forall U)\big(\mathrm{Play}(\mathcal{G}, v, U) \ \Rightarrow \ \neg\mathcal{W}(U)\big) \end{array}$$

(2) Given a player J (either P or O), we say that a J-strategy σ is *winning in* $(\mathcal{G}, \mathcal{W})$ *from* v if the game $(\mathcal{G}{\restriction}\{\sigma\}_\mathsf{J}, \mathcal{W})$ is won by J from v, *i.e.* if the following formula holds

$$\mathrm{WinStrat}_\mathsf{J}(\mathcal{G}, \sigma, v, \mathcal{W}) \quad := \quad \mathrm{WonGame}_\mathsf{J}(\mathcal{G}{\restriction}\{\sigma\}_\mathsf{J}, v, \mathcal{W})$$

Strictly speaking, in Definition 5.16 above, $\mathrm{WonGame}_{\mathsf{J}}$ and $\mathrm{WinStrat}_{\mathsf{J}}$ are actually families of FSO formulae, parametrized by the choice of FSO-formula $\mathcal{W}$.

As expected, a game position cannot be winning for both players.

Lemma 5.17. $\mathsf{FSO}_{\mathscr{D}}$ *proves the following.*

$$\mathrm{Game}(\mathcal{G}) \quad\Longrightarrow\quad \mathrm{Strat}_{\mathsf{P}}(\mathcal{G},\sigma_{\mathsf{P}}) \quad\Longrightarrow\quad \mathrm{Strat}_{\mathsf{O}}(\mathcal{G},\sigma_{\mathsf{O}}) \quad\Longrightarrow$$
$$\neg(\exists v)\Big[\mathrm{WinStrat}_{\mathsf{P}}(\mathcal{G},\sigma_{\mathsf{P}},v,\mathcal{W}) \;\wedge\; \mathrm{WinStrat}_{\mathsf{O}}(\mathcal{G},\sigma_{\mathsf{O}},v,\mathcal{W})\Big]$$

Proof. Assume for contradiction that for some v we have

$$\mathrm{WinStrat}_{\mathsf{P}}(\mathcal{G},\sigma_{\mathsf{P}},v,\mathcal{W}) \;\wedge\; \mathrm{WinStrat}_{\mathsf{O}}(\mathcal{G},\sigma_{\mathsf{O}},v,\mathcal{W})$$

that is

$$(\forall U)\Big[\mathrm{Play}(\mathcal{G}\restriction\{\sigma_{\mathsf{P}}\},v,U) \;\Rightarrow\; \mathcal{W}(U)\Big] \;\wedge\; (\forall U)\Big[\mathrm{Play}(\mathcal{G}\restriction\{\sigma_{\mathsf{O}}\},v,U) \;\Rightarrow\; \neg\mathcal{W}(U)\Big]$$

Consider the game

$$\mathcal{G}' \quad:=\quad (\mathsf{P},\, \mathsf{O},\, \{\sigma_{\mathsf{P}}\}_{\mathsf{P}},\, \{\sigma_{\mathsf{O}}\}_{\mathsf{O}})$$

Note that $\mathcal{G}'$ is a subgame of both $\mathcal{G}\restriction\{\sigma_{\mathsf{P}}\}$ and $\mathcal{G}\restriction\{\sigma_{\mathsf{O}}\}$. We thus get

$$(\forall U)\Big[\mathrm{Play}(\mathcal{G}',v,U) \;\Rightarrow\; \mathcal{W}(U)\wedge\neg\mathcal{W}(U)\Big]$$

which implies that there is no U such that $\mathrm{Play}(\mathcal{G}',v,U)$, contradicting Lemma 5.11. $\square$

5.5. **Parity Conditions.** In this paper, we mostly consider winning conditions expressed as *parity conditions*. Parity conditions are defined from *colorings* of game positions by natural numbers from a given finite interval. We represent natural numbers and the operations and relations on them using the Functions on HF-Sets of FSO and the axioms of §3.4.4.

Convention 5.18. In order to conveniently manipulate colorings and parity conditions, we will use the following functions on finite ordinals (a.k.a. natural numbers), obtained from the *Axioms on* HF-*Functions* (see §3.4.4). We rely on the well-known fact that "*n is an ordinal*" can be expressed by an HF-formula $\mathrm{Ord}(n)$ (see e.g. [Jec06, Lemma 12.10]).

(1) We consider unary HF-Functions

$$[0,-]\,,\; [0,-)\,,\; (0,-] \;:\; V_\omega \;\longrightarrow\; V_\omega$$

such that for all finite ordinals n, we have

$$\mathbf{Sk}(\mathsf{ZFC}^-)\vdash \quad [0,n] \doteq \{0,\dots,n\} \;\wedge\; [0,n) \doteq \{0,\dots,n-1\} \;\wedge\; (0,n] \doteq \{1,\dots,n\}$$

(2) We consider binary HF-Functions

$$\dot{\mathsf{g}}_{\leq}\,,\; \dot{\mathsf{g}}_{<}\,,\; \dot{\mathsf{g}}_{\geq}\,,\; \dot{\mathsf{g}}_{>} \;:\; V_\omega\times V_\omega \;\longrightarrow\; 2$$

such that for finite ordinals n, m

$$\begin{array}{ll} \dot{\mathsf{g}}_{\leq}(n,m)=1 \text{ iff } n\leq m & \qquad \dot{\mathsf{g}}_{\geq}(n,m)=1 \text{ iff } n\geq m \\ \dot{\mathsf{g}}_{<}(n,m)=1 \text{ iff } n< m & \qquad \dot{\mathsf{g}}_{>}(n,m)=1 \text{ iff } n> m \end{array}$$

In FSO-formulae, we write $n\leq m$ for the formula $\dot{\mathsf{g}}_{\leq}(n,m)\doteq 1$, and so on.

(3) We consider a unary HF-Function
$$\dot{\mathsf{even}} \ : \ V_\omega \ \longrightarrow \ V_\omega$$
such that for each ordinal n, $\dot{\mathsf{even}}(n)$ is the set of ordinals $m \in [0,n]$ such that m represents an even number.

(4) We consider HF-Functions $\mathsf{max}(-,-)$ and $(-)+1$, computing respectively the maximum of two finite ordinals and the successor ordinal of an ordinal.

Remark 5.19. Even if "n *is an ordinal*" can be expressed by an HF-formula, quantification over all finite ordinals *cannot* be expressed in V_ω by an HF-formula, since for each finite ordinal $n > 0$ we have $n \in V_n \setminus V_{n-1}$. In particular, induction over finite HF-ordinals cannot be expressed by an HF-formula.

Definition 5.20 (Parity Conditions)**.** Let $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E_\mathsf{P}, E_\mathsf{O})$ where P, O are HF-variables and $E_\mathsf{P}, E_\mathsf{O}$ are Function variables.

(1) A *coloring* is given by a Function C and an HF-term n satisfying the following formula
$$\mathrm{Col}(\mathcal{G}, C, n) \quad := \quad \mathrm{Ord}(n) \ \wedge \ C : \mathcal{G} \ \mathsf{to} \ [0,n]$$

(2) We define the following formula:
$$\mathrm{Par}(\mathcal{G}, C, n, U) \quad := \quad (\exists m \mathrel{\dot\in} \dot{\mathsf{even}}(n)) \left[\begin{array}{cl} & (\forall u \in U)(\exists v \in U)(u \longrightarrow^+ v \ \wedge \ C(v) \doteq m) \\ \wedge & (\exists u \in U)(\forall v \in U)(u \longrightarrow^+ v \ \Rightarrow \ C(v) \geq m) \end{array} \right]$$

Remark 5.21. The formula $\mathrm{Par}(\mathcal{G}, C, n, U)$ will be used to say that an infinite play U satisfies the (min) parity condition induced by the coloring $C : \mathcal{G}$ to $[0,n]$. In the standard model $\mathfrak{T}$, if U is an infinite play in $\mathcal{G}$, then $\mathrm{Par}(\mathcal{G}, C, n, U)$ holds if and only if there is an even $m \leq n$ such that U has infinitely many positions colored by m, and U has only finitely many positions colored by any $k < m$. Also, notice that any U (not necessarily a play) satisfying $\mathrm{Par}(\mathcal{G}, C, n, U)$ in $\mathfrak{T}$ is infinite.

Remark 5.22. Assume that $\mathcal{G}'$ is a subgame of $\mathcal{G}$ (in the sense of Definition 5.3). Note that FSO proves
$$\mathrm{Col}(\mathcal{G}, C, n) \iff \mathrm{Col}(\mathcal{G}', C, n)$$
Furthermore, as noted earlier, every infinite play in $\mathcal{G}'$ is an infinite play in $\mathcal{G}$. It follows that FSO proves
$$\begin{array}{l} \mathrm{Game}(\mathcal{G}) \implies \mathrm{Col}(\mathcal{G}, C, n) \implies \mathrm{Game}(\mathcal{G}') \implies \mathrm{Sub}(\mathcal{G}', \mathcal{G}) \implies \\ \qquad (\forall U : \mathcal{G}' \ \mathsf{to} \ 2)(\forall v)\Big[\mathrm{Play}(\mathcal{G}', v, U) \implies \big(\mathrm{Par}(\mathcal{G}', C, n, U) \ \Leftrightarrow \ \mathrm{Par}(\mathcal{G}, C, n, U)\big)\Big] \end{array}$$

Remark 5.23. When considering parity automata in §6, it will actually be convenient to define acceptance via the formula Par for games of the form $\mathcal{G}(\trianglelefteq)$ in the sense of Remark 5.4. It follows from Remarks 5.4 and 5.22 that FSO proves
$$\begin{array}{l} \mathrm{Game}(\mathcal{G}) \implies \mathrm{Col}(\mathcal{G}(\trianglelefteq), C, n) \implies \\ \qquad (\forall U : \mathcal{G} \ \mathsf{to} \ 2)(\forall v)\Big[\mathrm{Play}(\mathcal{G}, v, U) \implies \Big(\mathrm{Par}(\mathcal{G}, C, n, U) \ \Leftrightarrow \ \mathrm{Par}(\mathcal{G}(\trianglelefteq), C, n, U)\Big)\Big] \end{array}$$

We use the following more succinct notation for winning in the case parity games.

Notation 5.24 (Winning in Parity Games)**.** Let $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E_{\mathsf{P}}, E_{\mathsf{O}})$ where P, O are HF-variables and $E_{\mathsf{P}}, E_{\mathsf{O}}$ are Function variables. Let C be a Function variable and n be an HF-variable. We write the following, where J is a player (either P or O).

$$\begin{array}{rcl} \mathrm{WonGame}_{\mathsf{J}}(\mathcal{G}, v, C, n) & := & \mathrm{WonGame}_{\mathsf{J}}(\mathcal{G}, v, \mathrm{Par}(\mathcal{G}, C, n, -)) \\ \mathrm{WinStrat}_{\mathsf{J}}(\mathcal{G}, \sigma, v, C, n) & := & \mathrm{WinStrat}_{\mathsf{J}}(\mathcal{G}, \sigma, v, \mathrm{Par}(\mathcal{G}, C, n, -)) \end{array}$$

5.6. **The Axiom of Positional Determinacy of Parity Games.** We now formulate the axiom scheme (PosDet), which states the (positional) determinacy of parity games. Intuitively (PosDet) should consist of all formulae of the form

$$\begin{array}{l} \mathrm{Game}(\mathcal{G}) \quad \Rightarrow \quad \mathrm{Col}(\mathcal{G}, C, n) \quad \Rightarrow \\ \qquad (\forall v \in \mathcal{G}) \left[\begin{array}{rl} (\exists \sigma_{\mathsf{P}} : \mathcal{G}_{\mathsf{P}} \text{ to } \mathsf{O}) & \left(\begin{array}{ll} & \mathrm{Strat}_{\mathsf{P}}(\mathcal{G}, \sigma_{\mathsf{P}}) \\ \wedge & \mathrm{WinStrat}_{\mathsf{P}}(\mathcal{G}, \sigma_{\mathsf{P}}, v, C, n) \end{array} \right) \\ \\ \vee \quad (\exists \sigma_{\mathsf{O}} : \mathcal{G}_{\mathsf{O}} \text{ to } \mathscr{D} \times \mathsf{P}) & \left(\begin{array}{ll} & \mathrm{Strat}_{\mathsf{O}}(\mathcal{G}, \sigma_{\mathsf{O}}) \\ \wedge & \mathrm{WinStrat}_{\mathsf{O}}(\mathcal{G}, \sigma_{\mathsf{O}}, v, C, n) \end{array} \right) \end{array} \right] \end{array}$$

But note that these formulae are open, and in particular

$$\mathcal{G} = (\mathsf{P}, \mathsf{O}, E_{\mathsf{P}}, E_{\mathsf{O}}) \quad \text{and} \quad C$$

contain free Function variables. On the other hand, when formulating our completeness results in §8, it will be interesting to have translations of instances of (PosDet) in MSO, based on the map $\langle - \rangle : \mathsf{FSO} \to \mathsf{MSO}$ of §3.6. However, the translation $\langle - \rangle$ only handles HF-closed formulae without free Function variables. We therefore officially let (PosDet) consist of all formulae $\mathrm{PosDet}(\mathsf{P}, \mathsf{O}, n)$, for P, O and n ranging over HF-terms (see §3.2), where $\mathrm{PosDet}(\mathsf{P}, \mathsf{O}, n)$ is the formula

$$\begin{array}{l} \mathrm{Labels}(\mathsf{P}, \mathsf{O}) \quad \Rightarrow \quad \mathrm{Ord}(n) \quad \Rightarrow \\ \qquad \Big(\forall E_{\mathsf{P}} : \mathcal{G}_{\mathsf{P}} \text{ to } \mathcal{P}_*(\mathsf{O})\Big)\Big(\forall E_{\mathsf{O}} : \mathcal{G}_{\mathsf{O}} \text{ to } \mathcal{P}_*(\mathscr{D} \times \mathsf{P})\Big)\Big(\forall C : \mathcal{G} \text{ to } [0, n]\Big) \\ \qquad (\forall v \in \mathcal{G}) \left[\begin{array}{rl} (\exists \sigma_{\mathsf{P}} : \mathcal{G}_{\mathsf{P}} \text{ to } \mathsf{O}) & \left(\begin{array}{ll} & \mathrm{Strat}_{\mathsf{P}}(\mathcal{G}, \sigma_{\mathsf{P}}) \\ \wedge & \mathrm{WinStrat}_{\mathsf{P}}(\mathcal{G}, \sigma_{\mathsf{P}}, v, C, n) \end{array} \right) \\ \\ \vee \quad (\exists \sigma_{\mathsf{O}} : \mathcal{G}_{\mathsf{O}} \text{ to } \mathscr{D} \times \mathsf{P}) & \left(\begin{array}{ll} & \mathrm{Strat}_{\mathsf{O}}(\mathcal{G}, \sigma_{\mathsf{O}}) \\ \wedge & \mathrm{WinStrat}_{\mathsf{O}}(\mathcal{G}, \sigma_{\mathsf{O}}, v, C, n) \end{array} \right) \end{array} \right] \end{array}$$

It follows from the positional determinacy of parity games [EJ91] (see also [Tho97, Wal02, PP04]) that all instances of (PosDet) hold in the standard model $\mathfrak{T}$ of FSO. We can thus extend Proposition 3.20 to the following.

Proposition 5.25. *For each closed* FSO*-formula* φ*,*

$$\mathfrak{T} \models \varphi \qquad \textit{whenever} \qquad \mathsf{FSO} + (\mathit{PosDet}) \vdash \varphi$$

5.6.1. *The Axiom of Positional Determinacy in* MSO. In order to obtain a complete axiomatization of $\mathsf{MSO}_{\mathscr{D}}$ from the completeness of $\mathsf{FSO}_{\mathscr{D}} + (\mathit{PosDet})$ (see §8), we extend the axioms of $\mathsf{MSO}_{\mathscr{D}}$ with sufficiently many translated instantiations $\langle \mathrm{PosDet}(\mathsf{P}, \mathsf{O}, n)\rangle$ for P, O and n *closed* HF-terms. However, in general these terms may contain arbitrary HF-Functions symbols, which make the translation $\langle \mathrm{PosDet}(\mathsf{P}, \mathsf{O}, n)\rangle$ in general uncomputable from P, O and n (see Remark 3.23 and §3.4.4). However, for each closed HF-*terms* P, O and n, there are constant symbols for HF-*sets* $\dot{\mathsf{P}}$, $\dot{\mathsf{O}}$ and $\dot{\mathsf{n}}$ such that the formulae $\langle \mathrm{PosDet}(\mathsf{P}, \mathsf{O}, n)\rangle$ and $\langle \mathrm{PosDet}(\dot{\mathsf{P}}, \dot{\mathsf{O}}, \dot{\mathsf{n}})\rangle$ are syntactically identical. We therefore officially take the following version of (PosDet) in $\mathsf{MSO}_{\mathscr{D}}$.

Definition 5.26 (The Axiom of Positional Determinacy in MSO)**.** We let $\langle \mathit{PosDet}\rangle$ consist of all formulae of the form $\langle \mathrm{PosDet}(\dot{\mathsf{P}}, \dot{\mathsf{O}}, \dot{\mathsf{n}})\rangle$, for $\dot{\mathsf{P}}$, $\dot{\mathsf{O}}$ and $\dot{\mathsf{n}}$ ranging over constant symbols for HF-sets.

6. ALTERNATING TREE AUTOMATA

We detail in this Section a representation of alternating tree automata in FSO. We closely follow the presentation of [Wal02]. Our main motivation to consider alternating automata is that when formulating acceptance with (parity) games (of the kind of §5), complementation follows from (positional) determinacy (*i.e.* in our setting from the Axiom (PosDet)). Let us recall the main ideas underlying alternating automata. The original formulation, as in e.g. [MS87, MS95], is for an automaton $\mathcal{A}$ with state set Q to have transitions with values in the free distributive lattice over $\mathscr{D} \times Q$ (in other words, transitions have positive Boolean formulae over $\mathscr{D} \times Q$ as values). Actually, following [Wal02] we can simply assume that transitions are of the form

$$\partial \quad : \quad Q \times \Sigma \quad \longrightarrow \quad \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times Q))$$

and we read $\partial(q, \mathsf{a})$ as the disjunctive normal form

$$\bigvee_{\gamma \in \partial(q,\mathsf{a})} \; \bigwedge_{(d,q') \in \gamma} (d, q')$$

This results in acceptance games where intuitively P plays from disjunctions while O plays from conjunctions. In the following we often call the $\gamma \in \partial(q, \mathsf{a})$ *conjunctions*.

We begin by giving basic definitions in 6.1. Because our setting is restricted to only describe *positional* strategies, and because parity games are positionally determined, we give a special emphasis to *parity* automata, whose acceptance conditions are parity conditions generated from a coloring of their states. We then present a series of operations on automata, on which we rely in §8.3 for the interpretation of MSO formulae as automata. We recapitulate them in Table 1. First, §6.2 and §6.3 present two simple constructions implementing respectively a substitution and a disjunction operation. We discuss in §6.4 and §6.5 the important special case of *non-deterministic* automata. Non-deterministic automata are important because they allow us, via the usual *projection* operation (§6.5), to interpret the existential quantifier of MSO (see §8.3). To this end, an important result of the theory of automata on infinite trees is the *Simulation Theorem* [EJ91, MS95], which states that each alternating automaton is equivalent to a non-deterministic one. The formalization of this result in FSO is deferred to §9. This is the only part of this paper where we shall (momentarily) use automata with acceptance conditions which are not parity conditions. This result moreover relies on the complete axiomatization of MSO on ω-words for paths of FSO (to be discussed in §7). Finally, in §6.6 we discuss complementation in the setting of FSO, and show that alternating automata can be complemented in FSO when we assume the Axiom (PosDet) of Positional Determinacy of Parity Games.

Name	**Notation**	**Requirements**	**Location in text**
Substitution	$\mathcal{A}[f] : \Gamma$	$\mathcal{A} : \Sigma$ and $f : \Gamma \ \mathtt{to}\ \Sigma$	§6.2 (Lem. 6.11)
Disjunction	$(\mathcal{A}_0 \oplus \mathcal{A}_1) : \Sigma$	$\mathcal{A}_i : \Sigma \quad (i = 0, 1)$	§6.3 (Lem. 6.13)
Complementation	$\sim\!\mathcal{A} : \Sigma$	$\mathcal{A} : \Sigma$ parity (FSO + Axiom $(PosDet)$)	§6.6 (Thm. 6.19)
Projection	$(\exists_\Gamma \mathcal{A}) : \Sigma$	$\mathcal{A} : \Sigma \times \Gamma$ non-deterministic	§6.5 (Prop. 6.18)
Simulation	$\mathrm{ND}(\mathcal{A}) : \Sigma$	$\mathcal{A} : \Sigma$ HF-closed	§9 (Thm. 9.1)

Table 1: Operations on Automata.

6.1. **Alternating Tree Automata in $\mathsf{FSO}_{\mathscr{D}}$.** We present here a representation of alternating tree automata in FSO.

Definition 6.1 (Alternating Tree Automata)**.** Given an HF-Set Σ, an *Alternating Tree Automaton* (or simply *Automaton*) $\mathcal{A}$ on Σ (notation $\mathcal{A} : \Sigma$) is given by HF-terms $Q_\mathcal{A}$, $q^\iota_\mathcal{A}$ and $\partial_\mathcal{A}$ together with an FSO-formula $\Omega_\mathcal{A}(U)$ of a Function variable U, which are required to satisfy the following formula:

$$\mathrm{Aut}(\Sigma, Q_\mathcal{A}, q^\iota_\mathcal{A}, \partial_\mathcal{A}) \quad := \quad (\exists \mathsf{a} \mathrel{\dot\in} \Sigma) \ \wedge\ q^\iota_\mathcal{A} \mathrel{\dot\in} Q_\mathcal{A} \ \wedge\ \partial_\mathcal{A} : Q_\mathcal{A} \times \Sigma \ \mathtt{to}\ \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times Q_\mathcal{A}))$$

where $\mathcal{P}_*(-)$ is the HF-Function of §3.4.4.(d). We write

$$\mathcal{A} : \Sigma \quad = \quad (Q_\mathcal{A},\, q^\iota_\mathcal{A},\, \partial_\mathcal{A},\, \Omega_\mathcal{A})$$

and adopt the following terminology: Σ is the *input alphabet* of $\mathcal{A}$, $Q_\mathcal{A}$ is its set of *states* (with $q^\iota_\mathcal{A}$ *initial*), $\partial_\mathcal{A}$ is the *transition function* of $\mathcal{A}$ and $\Omega_\mathcal{A}$ is its *acceptance condition*.

We often write $\mathrm{Aut}(\mathcal{A} : \Sigma)$ or even $\mathrm{Aut}(\mathcal{A})$ for $\mathrm{Aut}(\Sigma, Q_\mathcal{A}, q^\iota_\mathcal{A}, \partial_\mathcal{A})$.

An automaton $\mathcal{A} : \Sigma$ is intended to run over Σ-labeled $\mathscr{D}$-ary trees, represented as Functions $F : \Sigma$ (equivalently $F : \mathscr{D}^* \ \mathtt{to}\ \Sigma$, following §3.7). As usual, acceptance is modeled using games, which we formalize in the setting of §5.

Definition 6.2 (Acceptance Games)**.** Given an automaton $\mathcal{A} : \Sigma$ and a Function $F : \Sigma$ we define the *acceptance game* $\mathcal{G}(\mathcal{A}, F)$ as follows:

$$\mathsf{P}_{\mathcal{G}(\mathcal{A},F)} \quad := \quad Q_\mathcal{A} \qquad\qquad \mathsf{O}_{\mathcal{G}(\mathcal{A},F)} \quad := \quad Q_\mathcal{A} \times \mathcal{P}_*(\mathscr{D} \times Q_\mathcal{A})$$

and $E(\mathcal{G}(\mathcal{A}, F))_\mathsf{P}$, $E(\mathcal{G}(\mathcal{A}, F))_\mathsf{O}$ are defined by HF-Bounded Choice for Product Types (Theorem 3.32) and Comprehension for HF-Sets (Remark 3.34) as

$$\begin{array}{rrcl} & (q', \gamma) \in E(\mathcal{G}(\mathcal{A}, F))_\mathsf{P}(x, q) & \text{iff} & q' \doteq q \ \wedge\ \gamma \in \partial_\mathcal{A}(q, F(x)) \\ \text{and} & (d, q') \in E(\mathcal{G}(\mathcal{A}, F))_\mathsf{O}(x, (q, \gamma)) & \text{iff} & (d, q') \in \gamma \end{array}$$

Remark 6.3. Note that $\mathrm{Aut}(\mathcal{A})$ implies $\mathrm{Game}(\mathcal{G}(\mathcal{A}, F))$ for $F : \Sigma$. The edge relations of $\mathcal{G}(\mathcal{A}, F)$ (in the sense of Definition 5.1) are given by

$$\begin{array}{rcllcl} (x, q) & \longrightarrow_\mathsf{P} & (x, (q, \gamma)) & \text{iff} & \gamma \in \partial_\mathcal{A}(q, F(x)) \\ (x, (q, \gamma)) & \longrightarrow_\mathsf{O} & (\mathsf{S}_d(x), q') & \text{iff} & (d, q') \in \gamma \end{array}$$

Note also that an O-position $(x, (q, \gamma))$ is equipped with the information $(x, q) \in \mathcal{G}(\mathcal{A}, F)_\mathsf{P}$. It follows that an O-position has at most one predecessor. This is useful when complementing automata (§6.6).

Convention 6.4. It the rest of this paper, unless explicitly stated otherwise, when speaking of an infinite play in an acceptance game $\mathcal{G}(\mathcal{A}, F)$ (including infinite plays in strategies in such games), we always mean an infinite play from position $(\dot{\varepsilon}, q^{\iota}_{\mathcal{A}})$.

Given $\mathcal{A} : \Sigma$ and $F : \Sigma$, the acceptance condition $\Omega_{\mathcal{A}}(-)$ of $\mathcal{A}$ induces a winning condition in the sense of §5.4 in the game $\mathcal{G}(\mathcal{A}, F)$. This gives the following notions of tree acceptance and language generated by an automaton.

Definition 6.5 (Language of an Automaton)**.** Given an automaton $\mathcal{A} : \Sigma$, a winning condition $\Omega_{\mathcal{A}}$ (in the sense of Definition 5.16) and $F : \Sigma$, we say that $\mathcal{A}$ *accepts* F when the following formula $F \in \mathcal{L}(\mathcal{A})$ holds.

$$F \in \mathcal{L}(\mathcal{A}) \quad := \quad (\exists \sigma_{\mathsf{P}} : \mathcal{G}(\mathcal{A}, F)_{\mathsf{P}} \ \mathtt{to}\ \mathsf{O}) \left(\begin{array}{ll} & \mathrm{Strat}_{\mathsf{P}}(\mathcal{G}(\mathcal{A}, F), \sigma_{\mathsf{P}}) \\ \wedge & \mathrm{WinStrat}_{\mathsf{P}}(\mathcal{G}(\mathcal{A}, F), \sigma_{\mathsf{P}}, v, \Omega_{\mathcal{A}}) \end{array} \right)$$

Recall that the formulae Strat and $\mathrm{WinStrat}$ are defined in Def. 5.13 (§5.3) and Def. 5.16 (§5.4) respectively. In words, the formula $F \in \mathcal{L}(\mathcal{A})$ of Definition 6.5 states that P has a winning strategy from position $(\dot{\varepsilon}, q^{\iota}_{\mathcal{A}})$ in the game $\mathcal{G}(\mathcal{A}, F)$.

Except for the Simulation Theorem in §9, we shall only consider automata whose acceptance conditions are given by *parity conditions* in the sense of §5.5. Recall from Definition 5.20 that a parity condition on a game $\mathcal{G}$ is given by the formula

$$\mathrm{Par}(\mathcal{G}, C, n, U)$$

which depends on $\mathcal{G}$. However, it is desirable that automata come, as in Definition 6.1, with acceptance conditions which are independent from any particular acceptance game. Note that for a given automaton $\mathcal{A}$, all acceptance games $\mathcal{G}(\mathcal{A}, F)$ have the same sets of P and O labels and positions; the input trees F can only induce different edge relations. Recall now the games $\mathcal{G}(\trianglelefteq)$ from Remark 5.4. The game $\mathcal{G}(\trianglelefteq)$ has the same labels and positions as $\mathcal{G}$, but its edge relation is exactly the partial order $\trianglelefteq$ discussed in §4. It follows that for a fixed automaton $\mathcal{A} : \Sigma$, all acceptance games $\mathcal{G}(\mathcal{A}, F)$ for $F : \Sigma$ induce the same $\mathcal{G}(\mathcal{A}, F)(\trianglelefteq)$, that we shall write

$$\mathcal{G}(\mathcal{A})(\trianglelefteq) \tag{6.1}$$

Definition 6.6 (Parity Automata)**.** Let

$$\mathcal{A} : \Sigma \quad = \quad (Q_{\mathcal{A}}\,,\, q^{\iota}_{\mathcal{A}}\,,\, \partial_{\mathcal{A}}\,,\, \Omega_{\mathcal{A}})$$

We say that $\mathcal{A}$ is a *parity* automaton if $\mathcal{A}$ comes equipped with HF-terms $n_{\mathcal{A}}$ and $C_{\mathcal{A}}$ such that the two following conditions are satisfied.

(1) The following formula holds

$$\mathrm{PAut}(\mathcal{A}, C_{\mathcal{A}}, n_{\mathcal{A}}) \quad := \quad \mathrm{Aut}(\mathcal{A}) \ \wedge\ \mathrm{Ord}(n_{\mathcal{A}}) \ \wedge\ C_{\mathcal{A}} : Q_{\mathcal{A}} \ \mathtt{to}\ [0, n_{\mathcal{A}}]$$

(2) The formula $\Omega_{\mathcal{A}}(U)$ is $\mathrm{Par}(\mathcal{G}(\mathcal{A})(\trianglelefteq), \hat{C}_{\mathcal{A}}, n_{\mathcal{A}})$, where

$$\hat{C}_{\mathcal{A}}(x, k) \quad := \quad \left\{ \begin{array}{lll} C_{\mathcal{A}}(q) & \text{if } k = q \in Q_{\mathcal{A}} & (\mathsf{P}\text{-position}) \\ C_{\mathcal{A}}(q) & \text{if } k = (q, \gamma) \in Q_{\mathcal{A}} \times \mathcal{P}_*(\mathscr{D}^* \times Q_{\mathcal{A}}) & (\mathsf{O}\text{-position}) \end{array} \right.$$

We write

$$\mathcal{A} \quad = \quad (Q_{\mathcal{A}}\,,\, q^{\iota}_{\mathcal{A}}\,,\, \partial_{\mathcal{A}}\,,\, C_{\mathcal{A}}\,,\, n_{\mathcal{A}})$$

for a parity automaton $\mathcal{A}$ with $C_{\mathcal{A}}$ and $n_{\mathcal{A}}$ as above. Furthermore, we write $\mathrm{Par}(\mathcal{A}, \hat{C}_{\mathcal{A}}, n_{\mathcal{A}}, U)$ or even $\mathrm{Par}(\mathcal{A}, U)$ for the formula $\mathrm{Par}(\mathcal{G}(\mathcal{A})(\trianglelefteq), \hat{C}_{\mathcal{A}}, n_{\mathcal{A}}, U)$.

In Definition 6.6, the purpose of the coloring $\hat{C}_{\mathcal{A}}$ is to equip the game $\mathcal{G}(\mathcal{A})(\trianglelefteq)$ with a coloring in the sense of Def. 5.20 (§5.5), namely a coloring of the positions of the game, while the coloring $C_{\mathcal{A}}$ only colors the states of $\mathcal{A}$.

Note that it follows from Remarks 5.4 and 5.23 that FSO proves

$$\mathrm{PAut}(\mathcal{A} : \Sigma) \quad \Longrightarrow \quad (\forall F : \Sigma)\Big(\mathrm{Sub}\Big(\mathcal{G}(\mathcal{A}, F),\, \mathcal{G}(\mathcal{A})(\trianglelefteq)\Big)\Big)$$

where the formula $\mathrm{Sub}(\mathcal{G}, \mathcal{G}')$ (stating that $\mathcal{G}$ is a subgame of $\mathcal{G}'$) is defined in Def. 5.3 (§5.1), and

$$\begin{array}{l} \mathrm{PAut}(\mathcal{A} : \Sigma) \quad \Longrightarrow \quad (\forall F : \Sigma)(\forall U : \mathcal{G}(\mathcal{A}, F) \ \mathtt{to}\ 2) \\ \Big(\mathrm{Play}\big(\mathcal{G}(\mathcal{A}, F), (\dot{\varepsilon}, q^{\iota}_{\mathcal{A}}), U\big) \ \Rightarrow \ \Big[\mathrm{Par}\big(\mathcal{G}(\mathcal{A}, F), \hat{C}_{\mathcal{A}}, n_{\mathcal{A}}, U\big) \ \Leftrightarrow \ \mathrm{Par}\big(\mathcal{A}, \hat{C}_{\mathcal{A}}, n_{\mathcal{A}}, U\big)\Big]\Big) \end{array}$$

The following simple fact will be useful when proving the Simulation Theorem in §9.

Remark 6.7. Given two plays U and V in $\mathcal{G}(\mathcal{A})(\trianglelefteq)$, if $U_{\mathsf{P}} = V_{\mathsf{P}}$ then

$$\mathrm{Par}\big(\mathcal{G}(\mathcal{A})(\trianglelefteq), U\big) \ \Leftrightarrow \ \mathrm{Par}\big(\mathcal{G}(\mathcal{A})(\trianglelefteq), V\big)$$

Other than the Simulation Theorem in §9, all constructions we need on automata can be performed on automata $\mathcal{A} : \Sigma$ where Σ, $Q_{\mathcal{A}}$, $q^{\iota}_{\mathcal{A}}$, $\partial_{\mathcal{A}}$, $C_{\mathcal{A}}$ and $n_{\mathcal{A}}$ are given by arbitrary HF-terms. However, our completeness result (§8) ultimately relies, via Proposition 7.8, on the completeness of $\mathsf{FSO}[\dot{<}]^{\omega}$ over ω-words (§7) and requires automata to be given by *closed* HF-terms. In addition, our proof of the Simulation Theorem uses McNaughton's Theorem [McN66], and imports it into FSO by Proposition 7.8, which also requires automata to be closed objects. This leads to the following.

Definition 6.8. A parity automaton $\mathcal{A} : \Sigma$ is HF-*closed* if Σ, $Q_{\mathcal{A}}$, $q^{\iota}_{\mathcal{A}}$, $\partial_{\mathcal{A}}$, $C_{\mathcal{A}}$ and $n_{\mathcal{A}}$ are closed HF-terms.

Remark 6.9. For each of our constructions on automata (see Table 1), the alphabets, states and colorings of new automata will be obtained by composing simple Functions on HF-Sets from §3.4.4 and Convention 5.18. In particular this means that the obtained automata have HF-closed alphabets, states and coloring provided we started from HF-closed ones.

On the other hand, transition functions may be more complex (see §6.6 or §9), and we often present them in a way suggesting the use of the Axiom of HF-Bounded Choice for HF-Sets (§3.4.5). This is unproblematic when HF-closedness is not at issue. To preserve HF-closedness, starting from HF-closed automata, the transition functions of the newly built automata must always be read as being constructed from *concrete* HF-sets.

Convention 6.10. In the rest of this paper, whenever we speak of a (parity) automaton $\mathcal{A}$ in formal statements, we always mean that the formula $\mathrm{Aut}(\mathcal{A})$ (resp. $\mathrm{PAut}(\mathcal{A})$) holds. (By contrast, HF-closedness is an external notion.)

6.2. **Substitution.** Let $\mathcal{A} : \Sigma$ be an automaton and let Γ and $f : \Gamma \ \mathtt{to}\ \Sigma$ be HF-sets. The automaton $\mathcal{A}[f] : \Gamma$ is defined to have the same states and acceptance condition as $\mathcal{A} : \Sigma$, and its transitions are given by

$$(q, \mathtt{b}) \quad \longmapsto \quad \partial_{\mathcal{A}}(q, f(\mathtt{b}))$$

Note that $\mathrm{Aut}(\mathcal{A}) \land (\exists \mathtt{b} \mathrel{\dot{\in}} \Gamma)$ implies $\mathrm{Aut}(\mathcal{A}[f])$. Also, $\mathcal{A}[f]$ is a parity automaton whenever $\mathcal{A}$ is. Furthermore, it follows from Remark 6.9 that $\mathcal{A}[f] : \Gamma$ is HF-closed when $\mathcal{A} : \Gamma$ is HF-closed and in addition Γ and f are closed HF-terms. A typical use of substitution, on which we rely when translating formulae to automata in §8.3, is to enlarge the input alphabet of an automaton. For instance, given HF-closed $\Sigma_1, \dots, \Sigma_n$ and an HF-closed $\mathcal{A} : \Sigma_i$, we obtain an HF-closed

$$\mathcal{A}[\pi^n_i] : \Sigma_1 \times \dots \times \Sigma_n$$

where

$$\pi^n_i \quad : \quad \Sigma_1 \times \dots \times \Sigma_n \quad \longrightarrow \quad \Sigma_i$$

is a projection HF-Function of §3.4.4.(f).

Lemma 6.11. *Given Γ, f and $\mathcal{A}$ as above, $\mathsf{FSO}_{\mathscr{D}}$ proves the following.*

$$(\forall H : \Gamma)\left[H \in \mathcal{L}(\mathcal{A}[f]) \;\;\Leftrightarrow\;\; (\forall F : \Sigma)\Big((\forall x)\Big[F(x) = f(H(x))\Big] \;\Rightarrow\; F \in \mathcal{L}(\mathcal{A})\Big)\right]$$

Note that by HF-Bounded Choice, $\mathsf{FSO}_{\mathscr{D}}$ proves that

$$(\forall H : \Gamma)(\exists F : \Sigma)(\forall x)\Big(F(x) = f(H(x))\Big)$$

so the above Lemma could have equivalently been stated with an existentially bound F.

6.3. **Disjunction.** We use here the HF-Functions from §3.4.4.(h) and Convention 5.18.(4). Given parity automata $\mathcal{A}_0, \mathcal{A}_1 : \Sigma$, the parity automaton $\mathcal{A}_0 \oplus \mathcal{A}_1 : \Sigma$ has state set

$$Q_{\mathcal{A}_0} + Q_{\mathcal{A}_1} + \{q^\iota\}$$

with q^ι initial, transitions given by

$$\begin{array}{rcll} (q^\iota, \mathtt{a}) & \longmapsto & \partial_{\mathcal{A}_0}(q^\iota_{\mathcal{A}_0}, \mathtt{a}) + \partial_{\mathcal{A}_1}(q^\iota_{\mathcal{A}_1}, \mathtt{a}) & \quad (\text{modulo } Q_{\mathcal{A}_i} \hookrightarrow Q_{\mathcal{A}_0 \oplus \mathcal{A}_1}) \\ (q_{\mathcal{A}_i}, \mathtt{a}) & \longmapsto & \partial_{\mathcal{A}_i}(q_{\mathcal{A}_i}, \mathtt{a}) & \quad (\text{for } q_{\mathcal{A}_i} \in Q_{\mathcal{A}_i}) \end{array}$$

and coloring $C : Q_{\mathcal{A}_0 \oplus \mathcal{A}_1}$ to $[0, n]$ (where $n = \mathsf{max}(n_{\mathcal{A}_0}, n_{\mathcal{A}_1})$) given by

$$\begin{array}{rcll} C(q^\iota) & := & n & \\ C(q_{\mathcal{A}_i}) & := & C_{\mathcal{A}_i}(q_{\mathcal{A}_i}) & \quad (\text{for } q_{\mathcal{A}_i} \in Q_{\mathcal{A}_i}) \end{array}$$

We have

$$\mathrm{Aut}(\mathcal{A}_0) \quad \Longrightarrow \quad \mathrm{Aut}(\mathcal{A}_1) \quad \Longrightarrow \quad \mathrm{Aut}(\mathcal{A}_0 \oplus \mathcal{A}_1)$$

Moreover, if follows from Remark 6.9 that $\mathcal{A}_0 \oplus \mathcal{A}_1 : \Sigma$ is HF-closed whenever $\mathcal{A}_0 : \Sigma$ and $\mathcal{A}_1 : \Sigma$ are.

Remark 6.12. Even in our positional setting, strictly speaking the automaton $\mathcal{A}_0 \oplus \mathcal{A}_1$ does not require $\mathcal{A}_0$ and $\mathcal{A}_1$ to be *parity* automata (see Table 1). However, the acceptance condition of $\mathcal{A}_0 \oplus \mathcal{A}_1$ is actually simpler to define when both $\mathcal{A}_0$ and $\mathcal{A}_1$ are parity automata. Since we shall only need $\mathcal{A}_0 \oplus \mathcal{A}_1$ for parity automata, we only formally define disjunction in this setting.

Lemma 6.13. $\mathsf{FSO}_{\mathscr{D}}$ *proves the following.*

$$(F : \Sigma)\Big(F \in \mathcal{L}(\mathcal{A}_0 \oplus \mathcal{A}_1) \;\;\Longleftrightarrow\;\; \Big(F \in \mathcal{L}(\mathcal{A}_0) \;\vee\; F \in \mathcal{L}(\mathcal{A}_1)\Big)\Big)$$

Proof. Assume first that $F \in \mathcal{L}(\mathcal{A}_0 \oplus \mathcal{A}_1)$ for $F : \Sigma$, and consider a winning P-strategy σ in the acceptance game $\mathcal{G}(\mathcal{A}_0 \oplus \mathcal{A}_1, F)$. We first look at the move of σ on the initial position $(\dot{\varepsilon}, q^\iota)$. By definition of $\mathcal{A}_0 \oplus \mathcal{A}_1$ we have

$$\sigma(\dot{\varepsilon}, q^\iota) = (q, \gamma) \qquad \text{with} \qquad \gamma \in \Big(\partial_{\mathcal{A}_0}\big(q^\iota_{\mathcal{A}_0}, F(\dot{\varepsilon})\big) + \partial_{\mathcal{A}_1}\big(q^\iota_{\mathcal{A}_1}, F(\dot{\varepsilon})\big)\Big)$$

Assume $\gamma \in \partial_{\mathcal{A}_i}(q^\iota_{\mathcal{A}_i}, F(\dot{\varepsilon}))$. Then σ induces a P-strategy σ_i in $\mathcal{G}(\mathcal{A}_i, F)$. The strategy σ_i is defined using HF-Bounded Choice for Product Types (Theorem 3.32) by putting

$$\sigma_i(x, q_{\mathcal{A}_i}) \quad = \quad \begin{cases} \sigma(\dot{\varepsilon}, q^\iota) & \text{if } (x, q_{\mathcal{A}_i}) = (\dot{\varepsilon}, q^\iota) \\ \sigma(x, q_{\mathcal{A}_i}) & \text{otherwise} \end{cases}$$

It remains to show that σ_i is winning, that is

$$(\forall V : \mathcal{G}(\mathcal{A}_i, F) \texttt{ to } 2)\Big(\mathrm{Play}(\sigma_i, \iota, V) \;\; \Rightarrow \;\; \mathrm{Par}(\mathcal{A}_i, V)\Big)$$

for $\iota = (\dot{\varepsilon}, q^\iota_{\mathcal{A}_i})$. Consider an infinite play V of σ_i from ι. Then by Comprehension for Product Types (Theorem 3.33), define $U : \mathcal{G}(\mathcal{A}_0 \oplus \mathcal{A}_1, F) \texttt{ to } 2$ as the set of all (x, ℓ) such that either $(x, \ell) = (\dot{\varepsilon}, q^\iota)$ or $(x, \ell) \in V$. It is clear that

$$\mathrm{Par}(\mathcal{A}_i, V) \;\; \Leftrightarrow \;\; \mathrm{Par}(\mathcal{A}_1 \oplus \mathcal{A}_2, U)$$

The converse is proved similarly. $\square$

6.4. **Non-Deterministic Automata.** We turn to the important class of alternating automata known as *non-deterministic* automata. Non-deterministic automata are important because they allow us, via the usual *projection* operation (§6.5), to interpret the existential quantifier of MSO (see §8). An important result in the theory of automata on infinite trees is the *Simulation Theorem* [EJ91, MS95] (addressed in §9), stating that each alternating automata can be simulated by a non-deterministic one.

Intuitively, an automaton $\mathcal{A}$ is non-deterministic if in acceptance games O can only explicitly choose tree directions but not states.

Definition 6.14 (Non-Deterministic Automata)**.** An automaton $(\mathcal{A} : \Sigma)$ in the sense of Definition 6.1, with

$$\partial_{\mathcal{A}} : Q_{\mathcal{A}} \times \Sigma \texttt{ to } \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}}))$$

is *non-deterministic* if for every $q \in Q_{\mathcal{A}}$, every $\mathtt{a} \in \Sigma$, every $\gamma \in \partial_{\mathcal{A}}(q, \mathtt{a})$, and every tree direction $d \in \mathscr{D}$, there is at most one $q' \in Q_{\mathcal{A}}$ such that $(d, q') \in \gamma$.

The key property of non-deterministic automata is that in each play of a P-strategy σ in an acceptance game, the sequence of states is uniquely determined from the tree positions. We formally state this as follows.

Lemma 6.15. *Consider a non-deterministic automaton $\mathcal{A} : \Sigma$, and let $F : \Sigma$. Furthermore let σ be a P-strategy in $\mathcal{G}(\mathcal{A}, F)$. Then $\mathsf{FSO}_{\mathscr{D}}$ proves that for all $x \in \mathscr{D}^*$ and all infinite plays V and V' of σ, if*

$$(\exists q \in Q_{\mathcal{A}})(x, q) \in V \quad \wedge \quad (\exists q' \in Q_{\mathcal{A}})(x, q') \in V'$$

then for all $y \mathrel{\dot{\leq}} x$, all $q \in Q_{\mathcal{A}}$, and all $\gamma \in \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}})$, we have

$$\Big[(y, q) \in V \;\; \Leftrightarrow \;\; (y, q) \in V'\Big] \quad \wedge \quad \Big[(y, (q, \gamma)) \in V \;\; \Leftrightarrow \;\; (y, (q, \gamma)) \in V'\Big]$$

Proof. Fix σ and V, V' as in the statement of the Lemma and let $x \in \mathscr{D}^*$. First, note that for every $y \mathrel{\dot{\leq}} x$ we have

Claim 6.15.1.

$$(\exists q \in Q_{\mathcal{A}})\big((y, q) \in V\big) \quad \wedge \quad (\exists q \in Q_{\mathcal{A}})\big((y, q) \in V'\big)$$

Proof of Claim 6.15.1. We use the Induction Axiom of FSO (§3.4.2). The property holds for $\dot\varepsilon \mathrel{\dot\leq} x$ since $(\dot\varepsilon, q^\iota_{\mathcal{A}})$ belongs to both V and V'. Assume now the property for $y \mathrel{\dot\leq} x$, and consider some tree direction $d \in \mathscr{D}$ such that $\mathsf{S}_d(y) \mathrel{\dot\leq} x$. By assumption, we have some $q \in Q_{\mathcal{A}}$ such that $(y, q) \in V$, and by using $\mathrm{Game}(\sigma)$ twice, we get some $q' \in Q_{\mathcal{A}}$ and some $d' \in \mathscr{D}$ such that

$$\exists q' \Big((y, q) \underset{\sigma}{\longrightarrow}\underset{\sigma}{\longrightarrow} (\mathsf{S}_{d'}(y), q')\Big)$$

But since V is a play of σ, by Proposition 5.6 we must have $\mathsf{S}_{d'}(y) \trianglelefteq x$, so that $d' = d$ and we are done. The same reasoning gives the result for V'. ■

Using the Induction Axiom of FSO (§3.4.2), we now show that

$$(\forall y \mathrel{\dot\leq} x)(\forall q \in Q_{\mathcal{A}})\Big[(y, q) \in V \quad \Leftrightarrow \quad (y, q) \in V'\Big]$$

First, we have

$$(\dot\varepsilon, q) \in V \quad \Leftrightarrow \quad (\dot\varepsilon, q) \in V' \quad \Leftrightarrow \quad q = q^\iota_{\mathcal{A}}$$

Assume now the property for $y \mathrel{\dot\leq} x$ and let us prove it for $\mathsf{S}_d(y)$ with $\mathsf{S}_d(y) \mathrel{\dot\leq} x$. It follows from the induction hypothesis and Claim 6.15.1 that we have $(y, q) \in V$ and $(y, q) \in V'$ for some $q \in Q_{\mathcal{A}}$. Again by Claim 6.15.1, let $q', q'' \in Q_{\mathcal{A}}$ such that $(\mathsf{S}_d(y), q') \in V$ and $(\mathsf{S}_d(y), q'') \in V'$. Now since V and V' are plays of σ, there are γ, γ' such that $(y, (q, \gamma)) \in V$ and $(y, (q, \gamma')) \in V'$, and we necessarily have

$$(q, \gamma) \quad = \quad (q, \gamma') \quad = \quad \sigma(x, q)$$

so that $\gamma = \gamma'$. Moreover, we have $(d, q'), (d, q'') \in \gamma$, but this implies $q' = q''$ since $\mathcal{A}$ is non-deterministic.

This concludes the proof of Lemma 6.15. □

Corollary 6.16. *Given $\mathcal{A}$, F and σ as in Lemma 6.15,* $\mathsf{FSO}_{\mathscr{D}}$ *proves that for each $x \in \mathscr{D}^*$ there is at most one $q \in Q_{\mathcal{A}}$ such that*

$$(\exists U : \mathcal{G}(\mathcal{A}, F) \;\mathtt{to}\; 2)\Big(\mathrm{Play}(\sigma,\, (\dot\varepsilon, q^\iota_{\mathcal{A}}),\, U) \;\;\wedge\;\; (x, q) \in U\Big)$$

We now state the *Simulation Theorem* [EJ91, MS95]. Its proof in $\mathsf{FSO}_{\mathscr{D}}$, requiring HF-closedness of automata, is deferred to §9.

Theorem 6.17 (Simulation)**.** *For each* HF*-closed parity automaton $\mathcal{A} : \Sigma$ there is a non-deterministic* HF*-closed parity automaton* $\mathrm{ND}(\mathcal{A}) : \Sigma$ *such that*

$$\mathsf{FSO} \vdash \quad \mathcal{L}(\mathrm{ND}(\mathcal{A})) = \mathcal{L}(\mathcal{A})$$

6.5. **Projection.** We now discuss the usual operation of *projection*, which allows us to interpret (existential) quantification in MSO (see §8.3). This operation is defined on arbitrary alternating automata, but it only correctly computes the appropriate projection for non-deterministic ones.

Given an automaton $\mathcal{A} : \Sigma \times \Gamma$ as in Definition 6.1, we define its *projection* on Σ to be the automaton $\exists_\Gamma \mathcal{A} : \Sigma$ with

$$\exists_\Gamma \mathcal{A} \quad := \quad (Q_{\mathcal{A}},\, q^\iota_{\mathcal{A}},\, \partial_{\exists_\Gamma \mathcal{A}},\, C_{\mathcal{A}},\, n_{\mathcal{A}})$$

where

$$\partial_{\exists_\Gamma \mathcal{A}} \;\;:\;\; Q_{\mathcal{A}} \times \Sigma \;\;\longrightarrow\;\; \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}}))$$

is given by

$$\partial_{\exists_\Gamma \mathcal{A}}(q, \mathtt{a}) \quad := \quad \bigcup_{\mathtt{b} \in \Gamma} \partial_{\mathcal{A}}(q, (\mathtt{a}, \mathtt{b}))$$

Note that $\mathrm{Aut}(\mathcal{A} : \Sigma \times \Gamma)$ implies $\mathrm{Aut}(\exists_\Gamma \mathcal{A})$. Moreover, $\exists_\Gamma \mathcal{A} : \Sigma$ is an (HF-closed) parity automaton whenever so is $\mathcal{A} : \Sigma \times \Gamma$.

We shall now prove that $\exists_\Gamma \mathcal{A} : \Sigma$ indeed implements the projection of $\mathcal{A} : \Sigma \times \Gamma$. This involves a notion of pairing for trees. Given $F : \Sigma$ and $G : \Gamma$, we let $\langle F, G\rangle : \Gamma \times \Sigma$ be given (using the axiom of HF-Bounded Choice for HF-Functions (§3.4.5)) by

$$\langle F, G\rangle(x) \quad := \quad (F(x), G(x))$$

Proposition 6.18. *Consider a* non-deterministic $\mathcal{A} : \Sigma \times \Gamma$ *and let* $\exists_\Gamma \mathcal{A} : \Sigma$ *be as defined above. Then* $\mathsf{FSO}_{\mathscr{D}}$ *proves the following.*

$$(\forall F : \Sigma)\Big[F \in \mathcal{L}(\exists_\Gamma \mathcal{A}) \quad \Leftrightarrow \quad (\exists G : \Gamma)\Big(\langle F, G\rangle \in \mathcal{L}(\mathcal{A})\Big)\Big]$$

Proof. Given $G : \Gamma$ and a winning P-strategy σ on $\mathcal{G}(\mathcal{A}, \langle F, G\rangle)$, it is easy to see that σ is also a winning strategy on $\mathcal{G}(\exists_\Gamma \mathcal{A}, F)$.

Conversely, assume that σ is a winning P-strategy on $\mathcal{G}(\exists_\Gamma \mathcal{A}, F)$. We define a tree $G : \Gamma$ by HF-Bounded Choice for HF-Functions (§3.4.5) as follows:

- For $x \in \mathscr{D}^*$, if there is some infinite play U of σ such that $(x, q) \in U$ for some state $q \in Q_\mathcal{A}$, then we let $G(x)$ be some $\mathtt{b} \in \Gamma$ such that $\sigma(x, q) \in \partial_\mathcal{A}(q, (F(x), \mathtt{b}))$.
- Otherwise, we let $G(x)$ be any element of Γ.

We now define a P-strategy σ_G on $\mathcal{G}(\mathcal{A}, \langle F, G\rangle)$ as follows, again using HF-Bounded Choice for HF-Functions (§3.4.5).

- If $(x, q) \in U$ for some infinite play U of σ, then we let $\sigma_G(x, q) := \sigma(x, q)$.
- Otherwise, we let $\sigma_G(x, q) = (q, \gamma)$, where $\gamma \in \partial_\mathcal{A}(q, \langle F, G\rangle(x))$.

We first check that σ_G is indeed a strategy on $\mathcal{G}(\mathcal{A}, \langle F, G\rangle)$, namely that for all $(x, q) \in \mathscr{D}^* \times Q_\mathcal{A}$, if $\sigma_G(x, q) = (q, \gamma)$ then $\gamma \in \partial_\mathcal{A}(q, \langle F, G\rangle(x))$. If (x, q) belongs to no infinite play of σ, then the result follows by definition of σ_G. Otherwise, by Corollary 6.16, q is unique in $Q_\mathcal{A}$ such that (x, q) belongs to an infinite play of σ, and we are done since

$$\sigma_G(x, q) = \sigma(x, q) \in \partial_\mathcal{A}(q, \langle F, G\rangle(x))$$

In order to show that σ_G is winning, we show that any infinite play of σ_G is also an infinite play of σ. So let $U : \mathcal{G}(\mathcal{A}, \langle F, G\rangle)$ $\mathtt{to}$ 2 such that

$$\mathrm{Play}(\sigma_G,\ (\dot\varepsilon, q^\iota_\mathcal{A}),\ U)$$

We are done if we show that

$$(\forall (x, q) \in U)\,(\sigma(x, q) = \sigma_G(x, q))$$

which follows from the fact that

Claim 6.18.1.

$$(\forall (x, q) \in U)(\exists W : \mathcal{G}(\mathcal{A}, \langle F, G\rangle)\ \mathtt{to}\ 2)\Big(\mathrm{Play}(\sigma,\ (\dot\varepsilon, q^\iota_\mathcal{A}),\ W) \quad \wedge \quad (x, q) \in W\Big)$$

Proof of Claim 6.18.1. We apply the Induction Scheme of $\mathsf{FSO}_{\mathscr{D}}$ (§3.4.2). In the base case $x = \dot\varepsilon$, and we conclude by Lemma 5.11.

For the induction step consider the case of $\mathsf{S}_d(x)$, assuming the property for x. So let $q' \in Q_\mathcal{A}$ such that $(\mathsf{S}_d(x), q') \in U$. First, by applying twice the Predecessor Lemma 5.10 for Infinite Plays, we get some $q \in Q_\mathcal{A}$ such that $(x, q) \in U$, and by induction hypothesis, there is some infinite play W of σ such that $(x, q) \in W$. But then, by definition of σ_G, we have $\sigma(x, q) = \sigma_G(x, q)$. We thus have $(d, q') \in \gamma$, where $(q, \gamma) = \sigma(x, q)$. Using Lemma 5.11, let now W' be an infinite play of

σ from position $(\mathsf{S}_d(x), q')$. By Comprehension for Product Types (Theorem 3.33), we define an infinite play W'' of σ from position $(\dot{\varepsilon}, q^\iota_{\mathcal{A}})$ as follows:

- Given u a position of $\mathcal{G}(\mathcal{A}, \langle F, G\rangle)$, if $u \in W'$ then $u \in W''$. Otherwise, we let $u \in W''$ iff $u \in W$ and $u \longrightarrow^*_\sigma (\mathsf{S}_d(x), q')$.

It is then easy to check that W'' is an infinite play of σ. ■

This concludes the proof of Proposition 6.18. □

6.6. **Complementation.** It is known that, assuming the determinacy of acceptance games, alternating tree automata are closed under complement [MS87]. On the other hand, our setting only allows us to manipulate *positional* strategies on acceptance games, which leads us to formulate complementation for *parity* automata, since their acceptance games are always positionally determined. Thus, in this section, we formalize the fact that, assuming the axiom $(PosDet)$, each alternating parity automaton has a complement in FSO. More precisely, we prove the following.

Theorem 6.19 (Complementation of Tree Automata)**.** *For each (*HF*-closed) parity automaton $\mathcal{A} : \Sigma$, there is an (*HF*-closed) parity automaton $\sim\mathcal{A} : \Sigma$ such that*

$$\mathsf{FSO} + (PosDet) \vdash \quad (\forall F : \Sigma)\Big(F \in \mathcal{L}(\sim\mathcal{A}) \;\;\Leftrightarrow\;\; F \notin \mathcal{L}(\mathcal{A})\Big)$$

Alternating automata may be directly complemented in a locally syntactic fashion. For an automaton $\mathcal{A} : \Sigma$ we may define a complement automaton $\sim\mathcal{A} : \Sigma$ with the same states as $\mathcal{A}$, and such that P-strategies in acceptance games for $\sim\mathcal{A}$ correspond (w.r.t. the visited states in infinite plays) to O-strategies in acceptance games for $\mathcal{A}$, and vice-versa. Closely following [Wal02], the basic idea is to see the transition function of $\mathcal{A}$

$$\partial_{\mathcal{A}} \;\;:\;\; Q_{\mathcal{A}} \times \Sigma \;\;\longrightarrow\;\; \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}}))$$

as taking $(q, \mathtt{a})$ to the disjunctive normal form

$$\bigvee_{\gamma \in \partial_{\mathcal{A}}(q,\mathtt{a})} \; \bigwedge_{(d,q') \in \gamma} (d, q')$$

Then, for the complement $\sim\mathcal{A} : \Sigma$ of $\mathcal{A}$, we can let

$$\partial_{\sim\mathcal{A}} \;\;:\;\; Q_{\mathcal{A}} \times \Sigma \;\;\longrightarrow\;\; \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}}))$$

take $(q, \mathtt{a})$ to the De Morgan dual of $\partial_{\mathcal{A}}(q, \mathtt{a})$.

We now proceed to the formal definition.

Definition 6.20. Given a parity automaton $\mathcal{A} : \Sigma$, we define the parity automaton $\sim\mathcal{A} : \Sigma$ as follows. The automaton $\sim\mathcal{A}$ has the same states and initial state as $\mathcal{A}$. Its transitions are defined as

$$\partial_{\sim\mathcal{A}}(q, a) \quad := \quad \Big\{ \overline{\gamma} \in \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}}) \;\;\mid\;\; (\forall \gamma \in \partial_{\mathcal{A}}(q, a))\big(\overline{\gamma} \cap \gamma \neq \varnothing\big) \Big\}$$

Its coloring is given as follows, using Convention 5.18.(4):

$$C_{\sim\mathcal{A}}(q) \;\;:=\;\; C_{\mathcal{A}}(q) + 1$$

Note that by Remark 6.9, $\sim\mathcal{A} : \Sigma$ is HF-closed whenever so is $\mathcal{A} : \Sigma$. We are now going to prove Theorem 6.19. To this end, fix a parity automaton $\mathcal{A} : \Sigma$ and let $\sim\mathcal{A} : \Sigma$ be as in Definition 6.20. Fix also some $F : \Sigma$. We split Theorem 6.19 into the following statements.

Proposition 6.21. $\mathsf{FSO} + (PosDet) \vdash \;\; F \notin \mathcal{L}(\mathcal{A}) \;\;\Longrightarrow\;\; F \in \mathcal{L}(\sim\mathcal{A})$.

Proposition 6.22. $\mathsf{FSO} \vdash \quad F \in \mathcal{L}(\sim\mathcal{A}) \implies F \notin \mathcal{L}(\mathcal{A})$.

The key is that P-strategies on $\mathcal{G}(\sim\mathcal{A}, F)$ correspond to O-strategies on $\mathcal{G}(\mathcal{A}, F)$, and vice-versa. We make this formal in §6.6.1 and §6.6.2 below. First, notice that $Q_{\sim\mathcal{A}} = Q_{\mathcal{A}}$, so that the games $\mathcal{G}(\mathcal{A}, F)$ and $\mathcal{G}(\sim\mathcal{A}, F)$ have the same sets of labels

$$\mathsf{P} \;:=\; Q_{\mathcal{A}} \qquad \text{and} \qquad \mathsf{O} \;:=\; Q_{\mathcal{A}} \times \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}})$$

In the following, we let

$$\mathcal{G} \;:=\; \mathscr{D}^* \times \mathsf{PO}$$

be the set of positions of the games $\mathcal{G}(\mathcal{A}, F)$ and $\mathcal{G}(\sim\mathcal{A}, F)$, and we let $\iota := (\dot{\varepsilon}, q^{\iota}_{\mathcal{A}})$ be their (common) initial position.

6.6.1. *Proof of Proposition 6.21.* We are going to show that $\mathsf{FSO} + (PosDet)$ proves

$$F \notin \mathcal{L}(\mathcal{A}) \implies F \in \mathcal{L}(\sim\mathcal{A})$$

First, given an O-strategy σ_{O} on $\mathcal{G}(\mathcal{A}, F)$, we define a P-strategy σ_{P} on $\mathcal{G}(\sim\mathcal{A}, F)$. Assuming that σ_{O} satisfies $\mathrm{Strat}_{\mathsf{O}}(\mathcal{G}(\mathcal{A}, F), \sigma_{\mathsf{O}})$, the strategy σ_{P} will satisfy $\mathrm{Strat}_{\mathsf{P}}(\mathcal{G}(\sim\mathcal{A}, F), \sigma_{\mathsf{P}})$. Recall that this in particular means

$$\sigma_{\mathsf{O}} : \mathcal{G}_{\mathsf{O}} \text{ to } \mathscr{D} \times \mathsf{P} \qquad \text{and} \qquad \sigma_{\mathsf{P}} : \mathcal{G}_{\mathsf{P}} \text{ to } \mathsf{O}$$

By HF-Bounded Choice for Product Types (Theorem 3.32) we are going to define σ_{P} such that $\sigma_{\mathsf{P}}(x, q) \in \partial_{\sim\mathcal{A}}(q, F(x))$ for each $(x, q) \in \mathscr{D}^* \times Q_{\mathcal{A}}$. Assume fixed $(x, q) \in \mathscr{D}^* \times Q_{\mathcal{A}}$. For all $\gamma \in \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}})$ such that $\gamma \in \partial_{\mathcal{A}}(q, F(x))$, we have $\sigma_{\mathsf{O}}(x, (q, \gamma)) \in \gamma$. By HF-Comprehension (Remark 3.34), let

$$\overline{\gamma} \quad := \quad \{\sigma_{\mathsf{O}}(x, (q, \gamma)) \mid \gamma \in \partial_{\mathcal{A}}(q, F(x))\}$$

By construction, we thus have $\overline{\gamma} \in \partial_{\sim\mathcal{A}}(q, F(x))$, and we let

$$\sigma_{\mathsf{P}}(x, q) \;:=\; (q, \overline{\gamma})$$

We trivially have $\mathrm{Strat}_{\mathsf{P}}(\mathcal{G}(\sim\mathcal{A}, F), \sigma_{\mathsf{P}})$.

Lemma 6.23. *Consider σ_{O} and σ_{P} as above. For every infinite play V of σ_{P} in $\mathcal{G}(\sim\mathcal{A}, F)$ there is some infinite play U of σ_{O} in $\mathcal{G}(\mathcal{A}, F)$ with $V_{\mathsf{P}} = U_{\mathsf{P}}$.*

Proof. We define U by Comprehension for Product Types (Theorem 3.33) as follows.

- First, for $(x, k) \in \mathcal{G}_{\mathsf{P}}$, if $(x, k) \in V_{\mathsf{P}}$ then we let $(x, k) \in U_{\mathsf{P}}$.
- Consider $(x, (q, \gamma)) \in \mathcal{G}_{\mathsf{O}}$. Using Remark 3.17, let $\prec$ be a well-order on $\mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}})$. Then we let $(x, (q, \gamma)) \in U_{\mathsf{O}}$ iff $(x, q) \in V_{\mathsf{P}}$ and γ is $\preceq$-minimal in $\partial_{\mathcal{A}}(q, F(x))$ such that $(\mathsf{S}_d(x), q') \in V_{\mathsf{P}}$ for $(d, q') = \sigma_{\mathsf{O}}(x, (q, \gamma))$.

Note that consecutive P-positions in U_{P} are indeed connected by the edge relation of $\mathcal{G}(\mathcal{A}, F)$:

Claim 6.23.1.

$$(x, q), (\mathsf{S}_d(x), q') \in U_{\mathsf{P}} \;\Rightarrow\; (\exists! u \in U_{\mathsf{O}}) \left((x, q) \underset{\sigma_{\mathsf{O}}}{\longrightarrow} u \underset{\sigma_{\mathsf{O}}}{\longrightarrow} (\mathsf{S}_d(x), q') \right)$$

Proof of Claim 6.23.1. We first show uniqueness. Let $(y_0, (q_0, \gamma_0)), (y_1, (q_1, \gamma_1)) \in U_{\mathsf{O}}$ be between (x, q) and $(\mathsf{S}_d(x), q')$. Then we must have $y_0 = y_1 = x$ and $q_0 = q_1 = q$. Hence, γ_0 and γ_1 are both $\preceq$-minimal in $\partial_{\mathcal{A}}(q, F(x))$ such that $\sigma_{\mathsf{O}}(x, (q, \gamma_0)) = \sigma_{\mathsf{O}}(x, (q, \gamma_1)) = (d, q')$, yielding $\gamma_0 = \gamma_1$ as required.

We now show the existence of an appropriate $(x, (q, \gamma)) \in U_{\mathsf{O}}$. Since $\mathrm{Play}(\sigma_{\mathsf{P}}, \iota, V)$, we have $(d, q') \in \overline{\gamma}$ with $(\ell, \overline{\gamma}) \in \sigma_{\mathsf{P}}(y, \ell)$ for some $(y, \ell) \in V_{\mathsf{P}}$. But $\mathrm{Play}(\sigma_{\mathsf{P}}, \iota, V)$ moreover implies that either $(y, \ell) \lhd (x, q)$ or $(x, q) \unlhd (y, \ell)$, from which follows that $(y, \ell) = (x, q)$ and $(q, \overline{\gamma}) \in \sigma_{\mathsf{P}}(x, q)$. Since

$$\overline{\gamma} \quad := \quad \{\sigma_{\mathsf{O}}(x, \gamma) \mid \gamma \in \partial_{\mathcal{A}}(q, F(x))\}$$

it follows that $(d, q') \in \sigma_{\mathsf{O}}(x, \gamma)$ for some $\gamma \in \partial_{\mathcal{A}}(q, F(x))$, and we are done. ■

We now check that U is indeed an infinite play of σ_{O}, *i.e.* that $\mathrm{Play}(\sigma_{\mathsf{O}}, \iota, U)$ holds. First, we have $\iota \in U$. Moreover,

Claim 6.23.2.

$$(\forall u \in U) \left(\iota \xrightarrow[\sigma_{\mathsf{O}}]{*} u\right)$$

Proof of Claim 6.23.2. We reason by induction on $\longrightarrow_{\sigma_{\mathsf{O}}}$ (Corollary 5.7). First, if $u \in U_{\mathsf{O}}$, then u is of the form $(x, (q, \gamma))$. By definition of U_{O} we have $(x, q) \in U_{\mathsf{P}}$ with $(x, q) \longrightarrow_{\sigma_{\mathsf{O}}} (x, (q, \gamma))$ and we conclude by induction hypothesis.

Consider now the case of $u \in U_{\mathsf{P}} = V_{\mathsf{P}}$. In this case, u of the form (x, q). We apply Proposition 3.8.(5), stating that either $x \doteq \dot{\varepsilon}$ or $x = \mathsf{S}_d(y)$ for some d and y. In the former case, since V is a play, we have $\iota \longrightarrow^*_{\sigma_{\mathsf{P}}} (x, q)$, and Proposition 5.6.(9) implies $u = \iota$. In the latter case, assume x is $\mathsf{S}_d(y)$. We apply twice the Predecessor Lemma 5.10 for Infinite Plays, which gives some $(y, q') \in V_{\mathsf{P}}$ such that

$$(y, q') \xrightarrow[\sigma_{\mathsf{P}}]{+} (\mathsf{S}_d(y), q)$$

By induction hypothesis we get $\iota \longrightarrow^*_{\sigma_{\mathsf{O}}} (y, q')$ and we conclude by Claim 6.23.1. ■

Also,

Claim 6.23.3.

$$(\forall u \in U)(\exists! v \in U) \left(u \xrightarrow[\sigma_{\mathsf{O}}]{} v\right)$$

Proof of Claim 6.23.3. The case of $u \in U_{\mathsf{P}} = V_{\mathsf{P}}$ follows directly from the definition of U_{O} and the fact that $\sigma_{\mathsf{O}} : \mathcal{G}_{\mathsf{O}} \ \mathtt{to}\ \mathscr{D} \times \mathsf{P}$ and $\mathrm{Play}(\sigma_{\mathsf{P}}, \iota, V)$. Consider now the case of $u \in U_{\mathsf{O}}$. By definition of U_{O} there is some $v \in U_{\mathsf{P}}$ such that $u \longrightarrow_{\sigma_{\mathsf{O}}} v$. Uniqueness follows from the fact that $U_{\mathsf{P}} = V_{\mathsf{P}}$ and $\mathrm{Play}(\sigma_{\mathsf{P}}, \iota, V)$. ■

In order to obtain $\mathrm{Play}(\sigma_{\mathsf{O}}, \iota, U)$, we invoke Proposition 5.9 and it remains to show:

Claim 6.23.4.

$$(\forall u \in U) \left[u \neq \iota \quad \Rightarrow \quad (\exists v \in U) \left(v \xrightarrow[\sigma_{\mathsf{O}}]{} u\right)\right]$$

Proof of Claim 6.23.4. The case of $u \in U_{\mathsf{O}}$ follows from the definition of U_{O}. The case of $u \in U_{\mathsf{P}}$ directly follow from Claim 6.23.1 (together with Proposition 5.6.(9)) and $\mathrm{Play}(\sigma_{\mathsf{P}}, \iota, V)$. ■

This concludes the proof of Lemma 6.23. □

We use the following simple fact in order to obtain from Lemma 6.23 that σ_{P} is winning in $\mathcal{G}({\sim}\mathcal{A}, F)$ whenever σ_{O} is winning in $\mathcal{G}(\mathcal{A}, F)$.

Lemma 6.24. *Given plays $U, V : \mathcal{G}\ \mathtt{to}\ 2$ as in Lemma 6.23, we have* $\mathrm{Par}(\mathcal{A}, U) \Leftrightarrow \neg\mathrm{Par}({\sim}\mathcal{A}, V)$.

We now have everything we need to obtain Proposition 6.21, namely

$$\mathsf{FSO} + (\mathit{PosDet}) \vdash \quad F \notin \mathcal{L}(\mathcal{A}) \implies F \in \mathcal{L}({\sim}\mathcal{A})$$

Assume $F \notin \mathcal{L}(\mathcal{A})$. By Definition 6.5, there is no winning P-strategy in $\mathcal{G}(\mathcal{A}, F)$. By the axiom of positional determinacy of parity games (PosDet) there is a winning O-strategy σ_{O} in $\mathcal{G}(\mathcal{A}, F)$, so that

$$(\forall U : \mathcal{G} \ \mathfrak{to}\ 2)\Big(\mathrm{Play}(\sigma_{\mathsf{O}}, \iota, U) \ \Rightarrow\ \neg\mathrm{Par}(\mathcal{A}, U)\Big) \tag{6.2}$$

Consider now the P-strategy σ_{P} on $\mathcal{G}({\sim}\mathcal{A}, F)$ as defined above. We claim that σ_{P} is winning, that is

Claim 6.25.

$$(\forall V : \mathcal{G} \ \mathfrak{to}\ 2)\Big(\mathrm{Play}(\sigma_{\mathsf{P}}, \iota, V) \ \Rightarrow\ \mathrm{Par}({\sim}\mathcal{A}, V)\Big)$$

Proof of Claim 6.25. Given an infinite play V of σ_{P}, by Lemma 6.23 we can build an infinite play U of σ_{O}, which by (6.2) satisfies $\neg\mathrm{Par}(\mathcal{A}, -)$, so that V satisfies $\mathrm{Par}({\sim}\mathcal{A}, -)$ thanks to Lemma 6.24. ☐

We thus have $F \in \mathcal{L}({\sim}\mathcal{A}, F)$. This concludes the proof of Proposition 6.21.

6.6.2. *Proof of Proposition 6.22.* We are now going to show that FSO proves

$$F \in \mathcal{L}({\sim}\mathcal{A}) \implies F \notin \mathcal{L}(\mathcal{A})$$

We associate a (winning) O-strategy σ_{O} on $\mathcal{G}(\mathcal{A}, F)$ to each (winning) P-strategy σ_{P} on $\mathcal{G}({\sim}\mathcal{A}, F)$. Assuming that the P-strategy satisfies $\mathrm{Strat}_{\mathsf{P}}(\mathcal{G}({\sim}\mathcal{A}, F), \sigma_{\mathsf{P}})$, the O-strategy will satisfy $\mathrm{Strat}_{\mathsf{O}}(\mathcal{G}(\mathcal{A}, F), \sigma_{\mathsf{O}})$. Note that

$$\sigma_{\mathsf{P}} : \mathcal{G}_{\mathsf{P}} \ \mathfrak{to}\ \mathsf{O} \qquad \text{and} \qquad \sigma_{\mathsf{O}} : \mathcal{G}_{\mathsf{O}} \ \mathfrak{to}\ \mathscr{D} \times \mathsf{P}$$

We define $\sigma_{\mathsf{O}}(x, (q, \gamma))$ for each position

$$(x, (q, \gamma)) \in \mathscr{D}^* \times (Q_{\mathcal{A}} \times \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}}))$$

By definition of $\partial_{\sim\mathcal{A}}(q, F(p))$, we have $\sigma_{\mathsf{P}}(p, q) = (q, \overline{\gamma})$ where $\overline{\gamma}$ intersects all $\gamma \in \partial_{\mathcal{A}}(q, F(p))$. So if $\gamma \in \partial_{\mathcal{A}}(q, F(p))$, by HF-Bounded Choice for Product Types (Theorem 3.32) we let $\sigma_{\mathsf{O}}(p, (q, \gamma))$ be some (d, q') such that $(d, q') \in \gamma \cap \overline{\gamma}$. Otherwise, since $\gamma \neq \varnothing$, we let $\sigma_{\mathsf{O}}(p, (q, \gamma))$ be some (d, q') such that $(d, q') \in \gamma$.

We also trivially have that $\mathrm{Strat}_{\mathsf{O}}(\mathcal{G}(\mathcal{A}, F), \sigma_{\mathsf{O}})$.

Lemma 6.26. *Consider a P-strategy σ_{P} and an O-strategy σ_{O} as in above. For every infinite play V of σ_{O} on $\mathcal{G}(\mathcal{A}, F)$ there is some infinite play U of σ_{P} on $\mathcal{G}({\sim}\mathcal{A}, F)$ with $V_{\mathsf{P}} = U_{\mathsf{P}}$.*

Proof. We define U by Comprehension for Product Types (Theorem 3.33) as follows.

- *Definition of U.* For $(x, k) \in \mathcal{G}_{\mathsf{P}}$, if $(x, k) \in V_{\mathsf{P}}$ then we let $(x, k) \in U_{\mathsf{P}}$, and for $(x, (q, \overline{\gamma})) \in \mathcal{G}_{\mathsf{O}}$, we let $(x, (q, \overline{\gamma})) \in U_{\mathsf{O}}$ iff $(q, \overline{\gamma}) = \sigma_{\mathsf{P}}(x, q)$ for $(x, q) \in U_{\mathsf{P}}$.

Similarly as in Lemma 6.23, we have

Claim 6.26.1.

$$(x, q), (\mathsf{S}_d(x), q') \in U_{\mathsf{P}} \ \Rightarrow\ (\exists! u \in U_{\mathsf{O}}) \left((x, q) \xrightarrow[\sigma_{\mathsf{P}}]{} u \xrightarrow[\sigma_{\mathsf{P}}]{} (\mathsf{S}_d(x), q')\right)$$

Proof of Claim 6.26.1. Uniqueness directly follows from the fact that $u = (x, \sigma_{\mathsf{P}}(x,q))$. As for existence, we directly have $(x,q) \longrightarrow_{\sigma_{\mathsf{P}}} u$, so it remains to show $u \longrightarrow_{\sigma_{\mathsf{P}}} (\mathsf{S}_d(x), q')$, which amounts to $(d,q') \in \overline{\gamma}$ for $(q,\overline{\gamma}) = \sigma_{\mathsf{P}}(x,q)$. But $(\mathsf{S}_d(x), q') \in V_{\mathsf{P}}$ with $\mathrm{Play}(\sigma_{\mathsf{O}}, \iota, V)$ imply that $(d,q') = \sigma_{\mathsf{O}}(x,(\ell,\gamma))$ for some ℓ such that $(x,\ell) \in V_{\mathsf{P}}$ and some $\gamma \in \partial_{\mathcal{A}}(\ell, F(x))$. Moreover, $\mathrm{Play}(\sigma_{\mathsf{O}}, \iota, V)$ implies $\ell = q$. By definition of σ_{O}, we thus have $(d,q') \in \gamma \cap \overline{\gamma}$ and we are done. ■

We now check that $\mathrm{Play}(\sigma_{\mathsf{P}}, \iota, U)$. Note that $\iota \in U$. Moreover, proceeding as in Lemma 6.23, we have

Claim 6.26.2.

$$(\forall u \in U)\left(\iota \xrightarrow[\sigma_{\mathsf{P}}]{*} u\right)$$

Proof of Claim 6.26.2. By induction on $\longrightarrow_{\sigma_{\mathsf{P}}}$ (Corollary 5.7). The case of $u \in U_{\mathsf{O}}$ follows directly form the induction hypothesis and the definition of U_{O}. As for $u \in U_{\mathsf{P}}$, we proceed as in Lemma 6.23, using Claim 6.26.1 and Lemma 5.10. ■

Continuing as in Lemma 6.23, we now invoke Proposition 5.9 and we are left with showing

Claim 6.26.3.

$$(\forall u \in U)(\exists! v \in U)\left(u \underset{\sigma_{\mathsf{P}}}{\longrightarrow} v\right) \quad \wedge \quad (\forall u \in U)\left[u \neq \iota \quad \Rightarrow \quad (\exists v \in U)\left(v \underset{\sigma_{\mathsf{P}}}{\longrightarrow} u\right)\right]$$

Proof of Claim 6.26.3. The cases of $u \in U_{\mathsf{P}}$ follow from the definition of U_{O}, and from Claim 6.23.1 (together with Proposition 5.6.(9)) and $\mathrm{Play}(\sigma_{\mathsf{O}}, \iota, V)$. Consider now $u \in U_{\mathsf{O}}$. The predecessor property follows from the definition of U_{O}. The unique successor property is obtained from Claim 6.26.1 together with $\mathrm{Play}(\sigma_{\mathsf{O}}, \iota, V)$. ■

This concludes the proof of Lemma 6.26. □

Similarly as in §6.6.1, we use the following simple fact.

Lemma 6.27. *Given plays $U, V : \mathcal{G} \;\mathtt{to}\; 2$ as in Lemma 6.26, we have* $\mathrm{Par}(\sim\!\mathcal{A}, U) \Leftrightarrow \neg\mathrm{Par}(\mathcal{A}, V)$

It is now easy to obtain Proposition 6.22, namely

$$\mathsf{FSO} \vdash \quad F \in \mathcal{L}(\sim\!\mathcal{A}) \quad \Longrightarrow \quad F \notin \mathcal{L}(\mathcal{A})$$

Assume that $F \in \mathcal{L}(\sim\!\mathcal{A})$. By Definition 6.5, we thus have a winning P-strategy σ_{P} in $\mathcal{G}(\sim\!\mathcal{A}, F)$, so that

$$(\forall U : \mathcal{G} \;\mathtt{to}\; 2)\Big(\mathrm{Play}(\sigma_{\mathsf{P}}, \iota, U) \quad \Rightarrow \quad \mathrm{Par}(\sim\!\mathcal{A}, U)\Big)$$

Consider now the O-strategy σ_{O} on $\mathcal{G}(\mathcal{A}, F)$ as defined above. Reasoning as in the case $F \notin \mathcal{L}(\mathcal{A})$ (§6.6.1), Lemmas 6.26 and 6.27 imply

$$(\forall V : \mathcal{G} \;\mathtt{to}\; 2)\Big(\mathrm{Play}(\sigma_{\mathsf{O}}, \iota, V) \quad \Rightarrow \quad \neg\mathrm{Par}(\mathcal{A}, V)\Big)$$

It then follows from Lemma 5.17 that there is no winning P-strategy on $\mathcal{G}(\mathcal{A}, F)$, so that $F \notin \mathcal{L}(\mathcal{A})$.

This concludes the proof of Proposition 6.22.

$$(\forall x)\,\big(\dot\varepsilon \mathrel{\dot\leq} x\big) \qquad \neg(\exists x)\,(x \mathrel{\dot<} x) \qquad (\forall x)(\forall y)(\forall z)\,(x \mathrel{\dot<} y \;\Rightarrow\; y \mathrel{\dot<} z \;\Rightarrow\; x \mathrel{\dot<} z)$$

$$(\forall x)(\exists y)\,(x \mathrel{\dot<} y) \qquad (\forall x)(\forall y)\,[x \mathrel{\dot<} y \;\vee\; x \doteq y \;\vee\; y \mathrel{\dot<} x]$$

$$(\forall x)\Big[(\exists y \mathrel{\dot<} x) \;\Rightarrow\; (\exists y \mathrel{\dot<} x)\neg(\exists z)\big(y \mathrel{\dot<} z \mathrel{\dot<} x\big)\Big]$$

Figure 6: Axioms on the relation $\dot<$ of $\mathsf{FSO}[\dot<]^\omega$.

7. MSO on Infinite Words in Paths of $\mathsf{FSO}_{\mathscr{D}}$

We discuss here the theory of MSO over ω-words for the *infinite paths* of $\mathsf{FSO}_{\mathscr{D}}$. Since MSO on ω-words admits a complete axiomatization [Sie70], this will allow us to freely import results on MSO over ω-words for the paths of $\mathsf{FSO}_{\mathscr{D}}$. In particular, our completeness argument (§8) relies on a version of the Büchi-Landweber's Theorem [BL69] formulated with MSO over ω-words, that we lift for free to $\mathsf{FSO}_{\mathscr{D}}$. Also, to prove the Simulation Theorem 6.17 in §9, we use McNaughton's Theorem [McN66], and similarly obtain it for free in $\mathsf{FSO}_{\mathscr{D}}$.

An obvious way to obtain MSO over ω-words is to consider the system MSO_1 (that is $\mathsf{MSO}_{\mathscr{D}}$ for $\mathscr{D} = 1$). However, recall that we want to see each path of $\mathsf{FSO}_{\mathscr{D}}$ (in the sense of (7.3) below) as a model of MSO on ω-words. This is technically simpler if, following [Rib12], one uses a version of MSO on ω-words over a purely relational vocabulary with only the strict order $\dot<$ on numbers as atomic relation (besides equality $\doteq$).

Definition 7.1 (The Theory $\mathsf{FSO}[\dot<]^\omega$)**.** The language of $\mathsf{FSO}[\dot<]^\omega$ is the language of $\mathsf{FSO}_{\mathscr{D}}$ with the following restriction:

- the only *Individual terms* of $\mathsf{FSO}[\dot<]^\omega$ are the constant $\dot\varepsilon$ and the individual variables (x, y, z etc.)

The deduction rules of $\mathsf{FSO}[\dot<]^\omega$ are the same as the rules of $\mathsf{FSO}_{\mathscr{D}}$. The axioms of $\mathsf{FSO}[\dot<]^\omega$ are the Equality Axioms of §3.4.1, the Axioms on HF-Sets of §3.4.4, the Functional Choice Axioms of §3.4.5, together with the axioms displayed on Figure 6, stating that $\dot<$ is a discrete unbounded strict linear order with $\dot\varepsilon$ as its minimal element (see e.g. [Rib12]), and with the following induction scheme.

- *Well-Founded Induction.* For each formula φ, the axiom

$$(\forall x)\big[(\forall y \mathrel{\dot<} x)(\varphi(y)) \implies \varphi(x)\big] \implies (\forall x)\varphi(x)$$

Remark 7.2. Note that all Individuals of $\mathsf{FSO}[\dot<]^\omega$ are Individuals of $\mathsf{FSO}_{\mathscr{D}}$, but not conversely. As a consequence, all HF-terms of $\mathsf{FSO}[\dot<]^\omega$ are HF-terms of $\mathsf{FSO}_{\mathscr{D}}$, but not conversely. Also, note that it may have seemed more natural not to include the individual constant $\dot\varepsilon$ in the language of $\mathsf{FSO}[\dot<]^\omega$. We have included it because this eases our concrete uses of $\mathsf{FSO}[\dot<]^\omega$ in §8.4 and §9.2.

Similarly to the case of $\mathscr{D}$-ary trees (§2), the theory $\mathsf{FSO}[\dot<]^\omega$ is intended to be interpreted in a theory $\mathsf{MSO}[\dot<]^\omega$. Intuitively, $\mathsf{MSO}[\dot<]^\omega$ is to $\mathsf{FSO}[\dot<]^\omega$ what $\mathsf{MSO}_{\mathscr{D}}$ is to $\mathsf{FSO}_{\mathscr{D}}$.

Definition 7.3 (The Theory $\mathsf{MSO}[\dot<]^\omega$)**.** The language of $\mathsf{MSO}[\dot<]^\omega$ is the language of $\mathsf{MSO}_{\mathscr{D}}$ with the following restriction:

- the only *Individual terms* of $\mathsf{MSO}[\dot<]^\omega$ are individual variables (x, y, z etc.).

The axioms of $\mathsf{MSO}[\dot<]^\omega$ are the equality axioms and the comprehension scheme of $\mathsf{MSO}_{\mathscr{D}}$ (§2.2), together with the induction scheme and the axioms on $\dot<$ of $\mathsf{FSO}[\dot<]^\omega$ displayed in Figure 6.

We write $\mathfrak{N}$ both for the standard model of $\mathsf{FSO}[\dot<]^\omega$ and for the standard model of $\mathsf{MSO}[\dot<]^\omega$. In the case of $\mathsf{MSO}[\dot<]^\omega$, formulae are interpreted in $\mathfrak{N}$ as expected: individual variables range over $\mathbb{N}$, monadic predicate variables range over $\mathcal{P}(\mathbb{N})$ and $\dot<$ is the standard order $<$ on $\mathbb{N}$. The interpretation of $\mathsf{FSO}[\dot<]^\omega$-formulae in $\mathfrak{N}$ is similar, with the obvious changes w.r.t. §3.5 for the interpretation of terms, and where Functions range over

$$\bigcup_{\kappa \in V_\omega} (\mathbb{N} \quad\longrightarrow\quad \kappa)$$

The key property of $\mathsf{MSO}[\dot<]^\omega$ we rely on is that it completely axiomatizes the theory of the standard model $\mathfrak{N}$ of ω-words [Sie70] (see also [Rib12]).

Theorem 7.4 ([Sie70]). *For every closed* $\mathsf{MSO}[\dot<]^\omega$*-formula* φ*,*

$$\mathfrak{N} \models \varphi \qquad \textit{if and only if} \qquad \mathsf{MSO}[\dot<]^\omega \vdash \varphi$$

The formula translation from $\mathsf{FSO}_{\mathscr{D}}$ to $\mathsf{MSO}_{\mathscr{D}}$ of §3.6 restricts to a translation of $\mathsf{FSO}[\dot<]^\omega$-formulae to $\mathsf{MSO}[\dot<]^\omega$-formulae. This easily extends to theories, and we get the following version of Proposition 3.27.

Proposition 7.5. *For every closed* $\mathsf{FSO}[\dot<]^\omega$*-formula* φ*,*

$$\mathsf{FSO}[\dot<]^\omega \vdash \varphi \qquad \textit{if and only if} \qquad \mathsf{MSO}[\dot<]^\omega \vdash \langle\varphi\rangle \tag{7.1}$$

$$\mathfrak{N} \models \varphi \qquad \textit{if and only if} \qquad \mathfrak{N} \models \langle\varphi\rangle \tag{7.2}$$

Thanks to (7.2), the completeness of $\mathsf{MSO}[\dot<]^\omega$ directly gives the completeness of $\mathsf{MSO}[\dot<]^\omega$ w.r.t. the translation closed of $\mathsf{FSO}[\dot<]^\omega$-formulae φ:

$$\mathsf{FSO}[\dot<]^\omega \vdash \varphi \qquad \text{if and only if} \qquad \mathfrak{N} \models \langle\varphi\rangle$$

Our goal now is to prove that if a closed $\mathsf{FSO}[\dot<]^\omega$ formula holds in the standard model $\mathfrak{N}$ of ω-words, then $\mathsf{FSO}_{\mathscr{D}}$ proves its relativization to any rooted tree path. Given a formula φ of $\mathsf{FSO}[\dot<]^\omega$ and a Function variable P, write φ^P for the $\mathsf{FSO}_{\mathscr{D}}$ formula obtained from φ by relativizing all individual quantifications to P and by replacing all Function quantifications $F : K$ by $F : P \texttt{ to } K$. Moreover, we say that $P : \mathscr{D}^* \texttt{ to } 2$ is a *rooted path* when the following formula $\mathrm{TPath}(P)$ holds:

$$\mathrm{TPath}(P) \quad := \quad \left\{ \begin{array}{ll} & (\dot\varepsilon \in P) \\ \wedge & (\forall x, y \in P)\,(x \mathrel{\dot<} y \;\;\vee\;\; x \doteq y \;\;\vee\;\; y \mathrel{\dot<} x) \\ \wedge & (\forall x \in P)(\exists y \in P)(\mathrm{S}(x,y)) \end{array} \right. \tag{7.3}$$

where $\mathrm{S}(x,y)$ stands for

$$x \mathrel{\dot<} y \;\;\wedge\;\; \neg(\exists z)\big[x \mathrel{\dot<} z \mathrel{\dot<} y\big]$$

We can now formally state the property we are targeting:

$$\mathsf{FSO}_{\mathscr{D}} \vdash (\forall P : 2)\,\big(\mathrm{TPath}(P) \;\;\Rightarrow\;\; \varphi^P\big) \qquad \text{whenever} \qquad \mathfrak{N} \models \varphi \tag{7.4}$$

The proof of (7.4) is deferred to Proposition 7.8. It relies on two lemmas. The first one is an adaptation of Lemma 4.8 (§4.3) to rooted tree paths, which will give the last axiom of Figure 6 for rooted tree paths. The second one is a weakening of (7.4) where $\mathsf{FSO}[\dot<]^\omega \vdash \varphi$ is assumed instead of $\mathfrak{N} \models \varphi$.

Lemma 7.6. $\mathsf{FSO}_{\mathscr{D}}$ *proves the following, assuming* $P : \mathscr{D}^* \texttt{ to } 2$ *and* $\mathrm{TPath}(P)$*:*

$$(\forall x \in P)\Big[(\exists y \in P)(y \mathrel{\dot<} x) \;\;\Rightarrow\;\; (\exists y \in P)\big(y \mathrel{\dot<} x \;\wedge\; \neg(\exists z \in P)(y \mathrel{\dot<} z \mathrel{\dot<} x)\big)\Big]$$

Proof. The argument is exactly the same as in the proof of Lemma 4.8, as soon as we have an analogue to Lemma 4.9. This amounts to the following, whose proof is considerably simpler than that of Lemma 4.9.

Claim 7.6.1. *Assume that* $(X : 2)$ *is bounded (i.e.* $(\exists x)(\forall y \in X)(y \dot{<} x)$*), non-empty and linearly ordered. Then* X *has a maximal element:* $(\exists z \in X)(\forall y \in X)(y \dot{\leq} z)$.

Proof of Claim 7.6.1. Let X be as in the statement and let $y \dot{<} x$ for all $y \in X$. By Well-Founded Induction (Theorem 3.9) we can assume x to be minimal with this property, in the sense that

$$z \dot{<} x \quad \Rightarrow \quad (\exists y \in X)\neg(y \dot{<} z)$$

Now, by Proposition 3.8, x is either $\dot{\varepsilon}$ or a successor. If $x \doteq \dot{\varepsilon}$, then Proposition 3.8 leads to a contradiction since X is assumed to be non-empty. So assume $x = \mathsf{S}_d(z)$. It then follows from the Tree axioms of $\mathsf{FSO}_{\mathscr{D}}$ (Figure 3) that $y \dot{\leq} z$ for all $y \in X$, and the minimality of x implies $z \in X$. ■

This concludes the proof of Lemma 7.6. □

Lemma 7.7. *For all closed* $\mathsf{FSO}[\dot{<}]^\omega$*-formula* φ*, we have*

$$\mathsf{FSO}_{\mathscr{D}} \vdash \forall P : 2 \left(\mathrm{TPath}(P) \;\Rightarrow\; \varphi^P\right) \qquad \textit{whenever} \qquad \mathsf{FSO}[\dot{<}]^\omega \vdash \varphi$$

Proof. The proof is by induction on derivations of $\mathsf{FSO}[\dot{<}]^\omega$-formulae. For formulae $\boldsymbol{\psi}, \varphi$ with free Function variables $\boldsymbol{F} = F_1, \ldots, F_p$ and free Individual variables $\boldsymbol{x} = x_1, \ldots, x_q$ (and possibly further free HF-variables), we show that for all HF-terms $\boldsymbol{K} = K_1, \ldots, K_p$ of $\mathsf{FSO}[\dot{<}]^\omega$ we have

$$\boldsymbol{F} : \boldsymbol{K} \,,\, \boldsymbol{\psi} \vdash_{\mathsf{FSO}[\dot{<}]^\omega} \varphi \qquad \text{implies} \qquad \mathrm{TPath}(P) \,,\, \boldsymbol{F} : \boldsymbol{P} \texttt{ to } \boldsymbol{K} \,,\, \boldsymbol{x} \dot{\in} P \,,\, \boldsymbol{\psi}^{\boldsymbol{P}} \vdash_{\mathsf{FSO}_{\mathscr{D}}} \varphi^P$$

The cases for each inference rule are immediate from their respective induction hypothesis, and we also easily obtain the Equality Axioms (§3.4.1), the Axioms of HF-Sets (§3.4.4) and the Axiom of HF-Bounded Choice for HF-Sets (§3.4.5). We resort on Theorem 3.32 for the axioms of HF-Bounded Choice for Functions and of Iterated HF-Bounded Choice. Moreover, the Induction axiom of $\mathsf{FSO}[\dot{<}]^\omega$ on the formula $\varphi(x)$ directly follows from Well-Founded Induction in $\mathsf{FSO}_{\mathscr{D}}$ (Theorem 3.9) on the formula

$$\psi(x) \quad := \quad (x \in P \;\Rightarrow\; \varphi(x))$$

It remains to deal with the $\dot{<}$-axioms of Figure 6. The first five axioms (stating that $\dot{<}$ is an unbounded linear order) directly follow from the Tree axioms of $\mathsf{FSO}_{\mathscr{D}}$ (Figure 3) and from relativization to P with $\mathrm{TPath}(P)$. Finally, we have to show that $\mathsf{FSO}_{\mathscr{D}}$ proves that the translation of the predecessor axiom holds within P whenever $\mathrm{TPath}(P)$ is assumed:

$$\mathrm{TPath}(P) \quad \Rightarrow \quad \forall x \in P\,[\exists y \in P(y \dot{<} x) \;\Rightarrow\; \exists y \in P\,(y \dot{<} x \,\wedge\, \neg\exists z \in P(y \dot{<} z \dot{<} x))]$$

This is handled by Lemma 7.6. □

We have now everything we need to prove (7.4).

Proposition 7.8. *Consider a closed formula* φ *of* $\mathsf{FSO}[\dot{<}]^\omega$*. Then*

$$\mathsf{FSO}_{\mathscr{D}} \vdash (\forall P : 2)\left(\mathrm{TPath}(P) \;\Rightarrow\; \varphi^P\right) \qquad \textit{whenever} \qquad \mathfrak{N} \models \varphi$$

Proof. Assume $\mathfrak{N} \models \varphi$. By (7.2) we have $\mathfrak{N} \models \langle\varphi\rangle$. Theorem 7.4 then implies $\mathsf{MSO}[\dot{<}]^\omega \vdash \langle\varphi\rangle$, and by (7.1) we have that $\mathsf{FSO}[\dot{<}]^\omega \vdash \varphi$. We conclude by Lemma 7.7. □

$\mathsf{FSO} \vdash \varphi \Longleftrightarrow \langle\varphi\rangle^\circ$			Proposition 3.27, (3.6)
$\mathsf{FSO} \vdash \varphi$	if and only if	$\mathsf{MSO} \vdash \langle\varphi\rangle$	Proposition 3.27, (3.7)
$\mathsf{FSO} \vdash \varphi^\circ$	if and only if	$\mathsf{MSO} \vdash \varphi$	Theorem 3.26
$\mathfrak{T} \models \varphi^\circ$	if and only if	$\mathfrak{T} \models \varphi$	Lemma 3.25

Table 2: Mutual Interpretability of FSO and MSO (§3.6).

8. Completeness

This Section is devoted to the proof of our main result, the completeness of $\mathsf{FSO} + (PosDet)$.

Theorem 8.1 (Main Theorem)**.** *For each closed formula φ of FSO,*

$$\mathsf{FSO} + (PosDet) \vdash \varphi \qquad \textit{or} \qquad \mathsf{FSO} + (PosDet) \vdash \neg\varphi$$

8.1. **Overview.** The two main ingredients of Theorem 8.1 are the following.

(1) The translations

$$\langle - \rangle \ : \ \mathsf{FSO} \longrightarrow \mathsf{MSO} \qquad \text{and} \qquad (-)^\circ \ : \ \mathsf{MSO} \longrightarrow \mathsf{FSO}$$

providing faithful mutual interpretations of FSO and MSO (§3.6, recapitulated in Table 2).

(2) The translation of MSO-formulae to automata, that we detail in §8.2 and §8.3 below. This translation relies on the correctness of the constructions on automata of §6, which are recapitulated in Table 1. In particular, we require the Axiom $(PosDet)$ of positional determinacy of parity games (§5.6) for the complementation of tree automata (Theorem 6.19).

The mutual interpretability results of Table 2 also allows us to obtain a completeness result for MSO. Recall that $\langle PosDet\rangle$ is defined in Definition 5.26, §5.6.1. We then get the following corollary to Theorem 8.1.

Corollary 8.2. *For each closed formula φ of MSO,*

$$\mathsf{MSO} + \langle PosDet\rangle \vdash \varphi \qquad \textit{or} \qquad \mathsf{MSO} + \langle PosDet\rangle \vdash \neg\varphi$$

Proof. Consider a closed MSO-formula φ. Assume $\mathsf{FSO} + (PosDet) \vdash \varphi^\circ$. Let $\mathrm{PosDet}(\mathsf{P}_i, \mathsf{O}_i, n_i)$ $(i = 1, \dots, k)$ be the instances of $(PosDet)$ used in the proof, so that

$$\mathsf{FSO} \vdash \quad \wedge_{1\leq i\leq k}\mathrm{PosDet}(\mathsf{P}_i, \mathsf{O}_i, n_i) \implies \varphi^\circ$$

By (3.6) (Proposition 3.27), we get

$$\mathsf{FSO} \vdash \quad \wedge_{1\leq i\leq k}\langle\mathrm{PosDet}(\mathsf{P}_i, \mathsf{O}_i, n_i)\rangle^\circ \implies \varphi^\circ$$

and since $(-)^\circ$ commutes over propositional connectives, by Theorem 3.26 we obtain

$$\mathsf{MSO} \vdash \quad \wedge_{1\leq i\leq k}\langle\mathrm{PosDet}(\mathsf{P}_i, \mathsf{O}_i, n_i)\rangle \implies \varphi$$

Moreover, since φ° is HF-closed, we can assume the HF-terms P_i, O_i and n_i to be closed. It follows that there are constants for HF-sets $\dot{\mathsf{P}}_i$, $\dot{\mathsf{O}}_i$ and $\dot{\mathsf{n}}_i$ $(i = 1\dots,k)$ such that each formula $\langle\mathrm{PosDet}(\mathsf{P}_i, \mathsf{O}_i, n_i)\rangle$ is syntactically identical to $\langle\mathrm{PosDet}(\dot{\mathsf{P}}_i, \dot{\mathsf{O}}_i, \dot{\mathsf{n}}_i)\rangle$. We thus obtain

$$\mathsf{MSO} \vdash \quad \wedge_{1\leq i\leq k}\langle\mathrm{PosDet}(\dot{\mathsf{P}}_i, \dot{\mathsf{O}}_i, \dot{\mathsf{n}}_i)\rangle \implies \varphi$$

which implies that $\mathsf{MSO} + \langle PosDet\rangle$ proves φ.

If $\mathsf{FSO} + (PosDet)$ does not prove φ°, Theorem 8.1 gives $\mathsf{FSO} + (PosDet) \vdash \neg(\varphi^\circ)$ and we conclude similarly. □

In particular, it follows from Proposition 5.25 that $\mathsf{FSO} + (PosDet)$ completely axiomatizes the standard model $\mathfrak{T}$ of $\mathscr{D}$-ary trees.

Corollary 8.3.

- *For each closed formula φ of* FSO*,*

$$\mathfrak{T} \models \varphi \qquad \text{if and only if} \qquad \mathsf{FSO} + (PosDet) \vdash \varphi$$

- *For each closed formula φ of* MSO*,*

$$\mathfrak{T} \models \varphi \qquad \text{if and only if} \qquad \mathsf{MSO} + \langle PosDet \rangle \vdash \varphi$$

Remark 8.4. Note that it follows from Remark 3.12 that Theorem 8.1 together with Corollary 8.3 implies the decidability of FSO over its standard model $\mathfrak{T}$. By Lemma 3.25 (see Table 2) we thus obtain a proof of Rabin's Tree Theorem [Rab69], namely the decidability of MSO over $\mathfrak{T}$. However, even if provability in FSO is semi-recursive, the axiom set of FSO is not recursive and the interpretation of HF-Functions is not computable (see Remarks 3.12 and 3.14 in §3.4.4, as well as Remark 3.23 in §3.6.1). We further elaborate on this in §8.5.

We will actually deduce Theorem 8.1 via Proposition 3.27, (3.6) (see Table 2) from the following.

Theorem 8.5. *For each closed formula φ of* MSO*,*

$$\mathsf{FSO} + (PosDet) \vdash \varphi^\circ \qquad \textit{or} \qquad \mathsf{FSO} + (PosDet) \vdash \neg\varphi^\circ$$

The proof of Theorem 8.5 proceeds as expected via a translation of MSO-formulae to automata. As usual, such translations are easier to define when one starts from a version of MSO with a purely relational and individual-free language. We perform a translation of MSO to such a language in §8.2. Then, the translation of formulae to automata is presented in §8.3. It relies on the constructions of §6. We thus arrive at Proposition 8.9, namely that for each closed formula φ of MSO there is an HF-closed parity automaton $\mathcal{A}$ over the singleton alphabet 1 such that

$$\mathsf{FSO} + (PosDet) \vdash \quad \varphi^\circ \iff (\exists F : 1)\big(F \in \mathcal{L}(\mathcal{A})\big)$$

In order to obtain Theorem 8.5, it remains to show that FSO actually *decides* the emptiness of such automata:

$$\mathsf{FSO} \vdash (\exists F : 1)\big(F \in \mathcal{L}(\mathcal{A})\big) \qquad \text{or} \qquad \mathsf{FSO} \vdash \neg(\exists F : 1)\big(F \in \mathcal{L}(\mathcal{A})\big)$$

This is Proposition 8.10. Its proof relies on the fact that the acceptance games of $(\mathcal{A} : 1)$ are actually generated from closed HF-Sets. We call such games *reduced parity games*. Section 8.4 is devoted to defining reduced parity games and to showing that FSO decides winning for them (Theorem 8.22). This essentially amounts to a version of the Büchi-Landweber Theorem [BL69] (see also e.g. [Tho97, PP04]), the effective determinacy of parity games on finite graphs, which is obtained thanks to the completeness of $\mathsf{FSO}[\dot{<}]^\omega$ (§7). Theorem 8.22 then follows from the lifting of $\mathsf{FSO}[\dot{<}]^\omega$ to the paths of FSO (Proposition 7.8).

8.2. **Restricted Languages for $\mathsf{MSO}_{\mathscr{D}}$.** For the translation of formulae to automata, it is useful and customary to work with formulae in a slightly different syntax, based on a purely relational, individual-free vocabulary.

8.2.1. *Restriction to a Relational Language.* We first restrict to a purely relational vocabulary, based on the defined formulae

$$\mathrm{S}_d(x, y) \quad := \quad (\mathsf{S}_d(x) \doteq y) \qquad\qquad \text{(for each } d \in \mathscr{D}\text{)}$$

The *relational formulae* $\varphi, \psi \in \mathbf{\Lambda}^R_{\mathscr{D}}$ are built from atomic formulae Xy and $\mathrm{S}_d(x, y)$ by means of $\neg$, $\vee$, $\exists x$ and $\exists X$. To each MSO-formula $\varphi \in \mathbf{\Lambda}$ we associate a formula φ^R as follows. For t a term of MSO, define the formula $(z \coloneqq t)$ by structural induction on t:

$$\begin{array}{rcl} (z \coloneqq y) & := & (z \doteq y) \\ (z \coloneqq \dot{\varepsilon}) & := & \neg(\exists z') \bigvee_{d \in \mathscr{D}} \mathrm{S}_d(z', z) \\ (z \coloneqq \mathsf{S}_d(t)) & := & (\exists z')\big(z' \coloneqq t \wedge \mathrm{S}_d(z', z)\big) \end{array}$$

Note that

$$\mathsf{MSO} \vdash \quad (z \coloneqq t) \quad \Leftrightarrow \quad (z \doteq t)$$

Then, φ^R is obtained from φ by replacing each atomic formula Xt, where t is not a variable, by $(\exists z)[(z \coloneqq t) \wedge Xz]$, where z is a fresh variable.

Lemma 8.6. *For every* MSO*-formula* φ*, we have* $\mathsf{MSO} \vdash \varphi \Leftrightarrow \varphi^R$.

8.2.2. *Restriction to an Individual-Free Language.* The next step is to get rid of individual quantifiers. Consider the defined formulae:

$$\begin{array}{rcl} (X \mathrel{\dot\subseteq} Y) & := & (\forall x)(Xx \;\; \Rightarrow \;\; Yx) \\ \mathrm{S}_d(X, Y) & := & (\exists x)(\exists y)\,[Xx \;\; \wedge \;\; Yy \;\; \wedge \;\; \mathrm{S}_d(x, y)] \end{array}$$

The *individual-free formulae* $\varphi, \psi \in \mathbf{\Lambda}^{IF}_{\mathscr{D}}$ are built from atomic formulae $(X \mathrel{\dot\subseteq} Y)$ and $\mathrm{S}_d(X, Y)$ by means of negation, disjunction and *second-order monadic* quantification $\exists X$ only. Let $\varphi \in \mathbf{\Lambda}^R$ with free variables among $x_1, \ldots, x_p, Y_1, \ldots, Y_q$. We inductively associate to φ a formula $\varphi^{IF} \in \mathbf{\Lambda}^{IF}$ with free variables among $X_1, \ldots, X_p, Y_1, \ldots, Y_q$ as follows. Let

$$((\exists x_{p+1})\varphi)^{IF} \quad := \quad (\exists X_{p+1})\left[\mathrm{Sing}(X_{p+1}) \;\; \wedge \;\; \varphi^{IF}\right]$$

where

$$\begin{array}{rcl} \mathrm{Sing}(X) & := & \neg(X \doteq \varnothing) \;\; \wedge \;\; (\forall Y)\left[Y \mathrel{\dot\subseteq} X \;\; \Rightarrow \;\; \big(Y \doteq \varnothing \;\; \vee \;\; X \mathrel{\dot\subseteq} Y\big)\right] \\ (X \doteq \varnothing) & := & (\forall Y)(X \mathrel{\dot\subseteq} Y) \end{array}$$

The other inductive cases are given as follows:

$$\begin{array}{rclrcl} (Y_j(x_i))^{IF} & := & X_i \mathrel{\dot\subseteq} Y_j & \qquad (\mathrm{S}_d(x_i, x_j))^{IF} & := & \mathrm{S}_d(X_i, X_j) \\ (\neg\varphi)^{IF} & := & \neg\varphi^{IF} & (\varphi \vee \psi)^{IF} & := & \varphi^{IF} \;\vee\; \psi^{IF} \\ ((\exists Y_{q+1})\varphi)^{IF} & := & (\exists Y_{q+1})\varphi^{IF} & & & \end{array}$$

Lemma 8.7. *For every formula* $\varphi \in \mathbf{\Lambda}^R$ *with free variables among* $\boldsymbol{x}, \boldsymbol{Y}$*, we have*

$$\boldsymbol{X}\boldsymbol{x}\,,\, \mathrm{Sing}(\boldsymbol{X}) \vdash_{\mathsf{MSO}} \quad \varphi \;\; \Leftrightarrow \;\; \varphi^{IF}$$

By composing the translations $(-)^R : \mathbf{\Lambda} \to \mathbf{\Lambda}^R$ and $(-)^{IF} : \mathbf{\Lambda}^R \to \mathbf{\Lambda}^{IF}$, we obtain:

Corollary 8.8. *For every closed* MSO*-formula* φ*, there is a closed formula* $\psi \in \mathbf{\Lambda}^{IF}$ *such that* $\mathsf{MSO} \vdash \varphi \Leftrightarrow \psi$.

8.3. **From Formulae to Automata.** We are now going to associate to each formula $\varphi \in \mathbf{\Lambda}^{IF}$ with free variables among $X_1, \ldots, X_p$ an HF-closed parity automaton $\mathcal{A}(\varphi) : 2^p$ such that

$$\mathsf{FSO} + (PosDet) \vdash \quad (\forall F_{X_1} : 2) \ldots (\forall F_{X_p} : 2)\Big(\langle F_{X_1}, \ldots, F_{X_p}\rangle \in \mathcal{L}(\mathcal{A}(\varphi)) \quad \Leftrightarrow \quad \varphi^\circ\Big)$$

Note that the correctness of $\mathcal{A}(\varphi)$ w.r.t. φ is proved in FSO using the translation $(-)^\circ : \mathsf{MSO} \to \mathsf{FSO}$ of §3.6.2. Recall that $(-)^\circ$ replaces each monadic variable X_i of φ by a Function variable $(F_{X_i} : 2)$. The construction of $\mathcal{A}(\varphi)$ from φ is done by induction on φ using the operations on automata devised in §6 (see Table 1). The base cases are provided by the automata $\mathcal{A}(X_i \mathrel{\dot{\subseteq}} X_j)$ and $\mathcal{A}(\mathrm{S}_d(X_i, X_j))$ discussed in §8.3.1 below for the atomic formulae of $\mathbf{\Lambda}^{IF}$. The inductive cases are performed as follows, where we implicitly apply substitutions (cf. §6.2) when necessary:

$$\begin{array}{rcll} \mathcal{A}(\varphi \vee \psi) & := & \mathcal{A}(\varphi) \oplus \mathcal{A}(\psi) & \text{(Lemma 6.13)} \\ \mathcal{A}(\neg\varphi) & := & \sim\!\mathcal{A}(\varphi) & \text{(Theorem 6.19)} \\ \mathcal{A}((\exists X_{p+1})\varphi) & := & \exists_2 \mathrm{ND}(\mathcal{A}(\varphi)) & \text{(Proposition 6.18 \& Theorem 6.17)} \end{array}$$

In particular, if φ is closed then $\mathcal{A}(\varphi)$ is an automaton over the singleton alphabet 1, whence by Corollary 8.8 we have:

Proposition 8.9. *For each closed formula φ of* MSO *there is an* HF*-closed parity automaton* $(\mathcal{A} : 1)$ *such that*

$$\mathsf{FSO} + (PosDet) \vdash \quad \varphi^\circ \quad \Leftrightarrow \quad (\exists F : 1)\big(F \in \mathcal{L}(\mathcal{A})\big)$$

In order to obtain Theorem 8.5 from Proposition 8.9, it remains to show that FSO actually *decides* the emptiness of $\mathcal{L}(\mathcal{A})$ for an HF-closed parity automaton $\mathcal{A}$ over the singleton alphabet 1.

Proposition 8.10. *Given an* HF*-closed parity automaton* $(\mathcal{A} : 1)$*,*

$$\mathsf{FSO} \vdash (\exists F : 1)\big(F \in \mathcal{L}(\mathcal{A})\big) \qquad or \qquad \mathsf{FSO} \vdash \neg(\exists F : 1)\big(F \in \mathcal{L}(\mathcal{A})\big)$$

Proposition 8.10 is proved in §8.4 below.

8.3.1. *Automata for Atomic Formulae.* We provide HF-closed parity automata for the atomic formulae $(X_1 \mathrel{\dot{\subseteq}} X_2)$ and $\mathrm{S}_d(X_1, X_2)$ of the individual-free syntax $\mathbf{\Lambda}^{IF}$ of MSO.

- The automaton $\mathcal{A}(X_1 \mathrel{\dot{\subseteq}} X_2)$ over 2×2 has state set $\mathbb{B} = \{\mathtt{tt}, \mathtt{ff}\}$, with $\mathtt{tt}$ initial, transitions given by

$$\begin{array}{rcll} (\mathtt{tt}, (i,j)) & \longmapsto & \{\{(d, \mathtt{ff}) \mid d \in \mathscr{D}\}\} & \text{if } i = 1 \text{ and } j = 0 \\ (\mathtt{tt}, (i,j)) & \longmapsto & \{\{(d, \mathtt{tt}) \mid d \in \mathscr{D}\}\} & \text{otherwise} \\ (\mathtt{ff}, (-,-)) & \longmapsto & \{\{(d, \mathtt{ff}) \mid d \in \mathscr{D}\}\} & \end{array}$$

  and coloring $C : \mathbb{B}\ \mathtt{to}\ 2$ given by

$$C(\mathtt{tt}) \quad := \quad 0 \qquad \text{and} \qquad C(\mathtt{ff}) \quad := \quad 1$$

- For $d \in \mathscr{D}$, the automaton $\mathcal{A}(\mathrm{S}_d(X_1, X_2))$ over 2×2 has state set $Q_\mathrm{S} := \mathbb{B} + \{\mathtt{w}\}$, with $\mathtt{ff}$ initial, transitions given by

$$\begin{array}{rcl} (\mathtt{ff}, (0,-)) & \longmapsto & \{\{(d', \mathtt{ff})\} \mid d' \in \mathscr{D}\} \\ (\mathtt{ff}, (1,-)) & \longmapsto & \{\{(d, \mathtt{w})\}\} \\ (\mathtt{w}, (-,1)) & \longmapsto & \{\{(d', \mathtt{tt}) \mid d' \in \mathscr{D}\}\} \\ (\mathtt{w}, (-,0)) & \longmapsto & \{\{(d', \mathtt{ff})\} \mid d' \in \mathscr{D}\} \\ (\mathtt{tt}, (-,-)) & \longmapsto & \{\{(d', \mathtt{tt}) \mid d' \in \mathscr{D}\}\} \end{array}$$

and with coloring given by

$$C(\mathtt{tt}) \quad := \quad 0 \qquad\qquad C(\mathtt{ff}) \quad := \quad 1 \qquad\qquad C(\mathtt{w}) \quad := \quad 0$$

Remark 8.11. Recall from §8.2.2 that the formula $\mathrm{S}_d(X, Y)$ of the individual-free syntax $\mathbf{\Lambda}^{IF}$ amounts in MSO to the formula $(\exists x)(\exists y)\,[Xx \wedge Yy \wedge y \doteq \mathsf{S}_d(x)]$. So the automaton $\mathcal{A}(\mathrm{S}_d(X, Y))$ only looks *for some* $x \in X$ and $y \in Y$ such that y is the d-successor of x, but is does not check whether X and Y are singletons.

Lemma 8.12. FSO *proves that*

$$\begin{array}{l l l} (\forall F_{X_1} : 2)(\forall F_{X_2} : 2)\Big(\langle F_{X_1}, F_{X_2}\rangle \in \mathcal{L}(\mathcal{A}(X_1, X_2)) & \Leftrightarrow & (X_1 \dot{\subseteq} X_2)^\circ\Big) \\ (\forall F_{X_1} : 2)(\forall F_{X_2} : 2)\Big(\langle F_{X_1}, F_{X_2}\rangle \in \mathcal{L}(\mathcal{A}(\mathrm{S}_d(X_1, X_2))) & \Leftrightarrow & (\mathrm{S}_d(X_1, X_2))^\circ\Big) \end{array}$$

8.4. **Reduced Parity Games.** The goal of this Section is to prove Proposition 8.10, namely that for an HF-closed parity automaton $\mathcal{A}$ over the singleton alphabet 1,

$$\mathsf{FSO} \vdash (\exists F : 1)\big(F \in \mathcal{L}(\mathcal{A})\big) \qquad \text{or} \qquad \mathsf{FSO} \vdash \neg(\exists F : 1)\big(F \in \mathcal{L}(\mathcal{A})\big)$$

Consider an HF-closed automaton $\mathcal{A}$ over the singleton alphabet $1 = \{0\}$. Then for any $(F : 1)$ the game $\mathcal{G} := \mathcal{G}(\mathcal{A}, F)$ has edge relations induced by functions

$$e_{\mathsf{P}} : \mathsf{P}_{\mathcal{G}} \ \mathtt{to}\ \mathcal{P}_*(\mathsf{O}_{\mathcal{G}}) \qquad \text{and} \qquad e_{\mathsf{O}} : \mathsf{O}_{\mathcal{G}} \ \mathtt{to}\ \mathcal{P}_*(\mathscr{D} \times \mathsf{P}_{\mathcal{G}}) \tag{8.1}$$

given (following Remark 6.9) by

$$\begin{array}{lll} (q', \gamma) \mathrel{\dot{\in}} e_{\mathsf{P}}(q) & \text{iff} & q' \doteq q \ \wedge \ \gamma \in \partial_{\mathcal{A}}(q, 0) \\ (d, q') \mathrel{\dot{\in}} e_{\mathsf{O}}(q, \gamma) & \text{iff} & (d, q') \mathrel{\dot{\in}} \gamma \end{array}$$

So in particular the edge relations of $\mathcal{G}(\mathcal{A}, F)$ are independent from F. But also, since

$$\mathsf{P}_{\mathcal{G}} \ := \ Q_{\mathcal{A}} \qquad \text{and} \qquad \mathsf{O}_{\mathcal{G}} \ := \ Q_{\mathcal{A}} \times \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}})$$

the whole game $\mathcal{G}(\mathcal{A}, F)$ is actually generated from HF-Sets.

In this Section, we discuss games generated from HF-Sets, that we call *reduced games*. We show that for reduced parity games, winning can actually be defined within $\mathsf{FSO}[\dot{<}]^\omega$. Thanks to the completeness of $\mathsf{FSO}[\dot{<}]^\omega$ w.r.t. its standard model (§7), this implies that $\mathsf{FSO}[\dot{<}]^\omega$ itself decides winning in such games. This essentially amounts to a version of the Büchi-Landweber Theorem [BL69] using the completeness of $\mathsf{MSO}[\dot{<}]^\omega$ over its standard model. Using Proposition 7.8 we can then lift this result to FSO.

In §8.4.1 and §8.4.2 we repeat some material of §5, but for the slightly different setting of reduced games. We then obtain that $\mathsf{FSO}[\dot{<}]^\omega$ decides winning in reduced parity games, and we lift this to FSO in Theorem 8.22, §8.4.3. This directly entails Proposition 8.10.

8.4.1. *Reduced Games as* HF*-Sets.* The purpose of this Section is to give adaptations of the notions of §5 to those parity games which are entirely generated from HF-Sets. All the formulae of this Section are HF-formulae in the sense of Definition 3.11. Hence, thanks to the Axioms of HF-Sets (Remark 3.15, §3.4.4) their closed instances are provable (both in FSO and $\mathsf{FSO}[\dot{<}]^\omega$) if and only if they hold in V_ω.

Definition 8.13 (Reduced Games)**.** A *reduced game* G is given by HF-terms $\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}}$ which satisfy the following formula

$$\mathrm{Game}_0(\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}}) \quad := \quad \Big(\mathrm{Labels}(\mathsf{P}, \mathsf{O}) \ \wedge \ e_{\mathsf{P}} : \mathsf{P} \ \mathsf{to} \ \mathcal{P}_*(\mathsf{O}) \ \wedge \ e_{\mathsf{O}} : \mathsf{O} \ \mathsf{to} \ \mathcal{P}_*(\mathscr{D} \times \mathsf{P})\Big)$$

We often write $\mathrm{Game}_0(G)$ for $\mathrm{Game}_0(\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}})$. Moreover, when no ambiguity arises, we abbreviate $G = (\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}})$ as $G = (\mathsf{P}, \mathsf{O}, e(G))$.

Definition 8.14 (Reduced Subgame)**.** We say that $G' = (\mathsf{P}', \mathsf{O}', e'_{\mathsf{P}}, e'_{\mathsf{O}})$ is a *reduced subgame* of $G = (\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}})$ whenever the following formula holds

$$\mathrm{Sub}_0(G', G) \quad := \quad \left\{ \begin{array}{ll} & \mathsf{P}' \doteq \mathsf{P} \ \wedge \ \mathsf{O}' \doteq \mathsf{O} \\ \wedge & (\forall k \in \mathsf{P}')\big(e'_{\mathsf{P}}(k) \mathrel{\dot{\subseteq}} e_{\mathsf{P}}(k)\big) \\ \wedge & (\forall \ell \in \mathsf{O}')\big(e'_{\mathsf{O}}(\ell) \mathrel{\dot{\subseteq}} e_{\mathsf{O}}(\ell)\big) \end{array} \right.$$

Definition 8.15 (Reduced Strategies)**.** Let $G = (\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}})$ where $\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}}$ are HF-variables.

(1) A *reduced* P*-strategy on* G is an HF-set s which satisfies the formula

$$\mathrm{Strat}^0_{\mathsf{P}}(G, s) \quad := \quad s : \mathsf{P} \ \mathsf{to} \ \mathsf{O} \ \wedge \ (\forall k \in \mathsf{P})\,(s(k) \in e_{\mathsf{P}}(k))$$

(2) A *reduced* O*-strategy on* G is an HF-set s which satisfies the formula

$$\mathrm{Strat}^0_{\mathsf{O}}(G, s) \quad := \quad s : \mathsf{O} \ \mathsf{to} \ \mathscr{D} \times \mathsf{P} \ \wedge \ (\forall \ell \in \mathsf{O})\,(s(\ell) \in e_{\mathsf{O}}(\ell))$$

Definition 8.16 (Reduced Subgame induced by a Reduced Strategy)**.** Given a player J (either P or O) and a J-strategy s on G, we let

$$G{\restriction}\{s\}_{\mathsf{J}} \quad := \quad \big(\mathsf{P}_{\mathcal{G}}\,,\ \mathsf{O}_{\mathcal{G}}\,,\ e(G){\restriction}\{\sigma\}_{\mathsf{J}}\big)$$

where

$$e(G){\restriction}\{\sigma\}_{\mathsf{P}} \quad := \quad \big(\{s\}_{\mathsf{P}}, e(\mathcal{G})_{\mathsf{O}}\big) \qquad \text{and} \qquad e(G){\restriction}\{\sigma\}_{\mathsf{O}} \quad := \quad \big(e(\mathcal{G})_{\mathsf{P}}, \{s\}_{\mathsf{O}}\big)$$

and where $\{s\}_{\mathsf{J}} \subseteq e_{\mathsf{J}}$ is defined (following the method of Remark 6.9) to be the function taking $k \in G_{\mathsf{J}}$ to the singleton $\{s(k)\}$.

Whenever possible, we write $G{\restriction}\{s\}$ or even just s for $G{\restriction}\{s\}_{\mathsf{J}}$.

8.4.2. *Reduced Games in* $\mathsf{FSO}[\dot{<}]^\omega$. In §8.4.1 we gave notions of reduced parity games and reduced strategies. In this Section, we work within $\mathsf{FSO}[\dot{<}]^\omega$ and show that this setting suffices to define *winning* for reduced parity games. Thanks to the completeness of $\mathsf{FSO}[\dot{<}]^\omega$ w.r.t. the standard model of ω-words (§7), we obtain that $\mathsf{FSO}[\dot{<}]^\omega$ *decides* winning in such games. This is essentially the Büchi-Landweber Theorem [BL69].

We use the following $\mathsf{FSO}[\dot{<}]^\omega$-formula:

$$\mathrm{S}(x, y) \quad := \quad x \mathrel{\dot{<}} y \ \wedge \ \neg(\exists z)\big[x \mathrel{\dot{<}} z \mathrel{\dot{<}} y\big]$$

Definition 8.17 (Infinite Plays in Reduced Games)**.** Working in $\mathsf{FSO}[\dot{<}]^\omega$, let $G = (\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}})$, where $\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}}$ are HF-variables. Given an HF set K and a Function $(\tilde{V} : \mathsf{P})$, we say that $\tilde{V}$ is an *infinite play in G from K* when the following formula $\mathrm{Play}[\dot{<}](G, K, \tilde{V})$ holds:

$$\tilde{V}(\dot{\varepsilon}) \doteq K \ \wedge \ (\forall x)(\forall y)\Big(\mathrm{S}(x, y) \ \Rightarrow \ \big(\exists \ell \mathrel{\dot{\in}} e_{\mathsf{P}}(\tilde{V}(x))\big)\big(\exists d \mathrel{\dot{\in}} \mathscr{D}\big)\Big[(d, \tilde{V}(y)) \in e_{\mathsf{O}}(\ell)\Big]\Big)$$

Note that in Definition 8.17 above, we use the notation $\tilde{V}$ for a play in a reduced games, to mark the difference with the notion of plays (and more generally sets of game positions) in the setting of §4.

Definition 8.18 (Parity Conditions for Reduced Games)**.** Working in $\mathsf{FSO}[\dot{<}]^\omega$, let $G = (\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}})$, where $\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}}$ are HF-variables.

(1) A *coloring* is given by Function C and an HF-set n satisfying the following formula

$$\mathrm{Col}_0(G, C, n) \quad := \quad \mathrm{Ord}(n) \ \wedge \ C : \mathsf{P} \ \mathtt{to} \ [0, n]$$

(2) We define the following formula:

$$\mathrm{Par}[\dot{<}](C, n, \tilde{V}) \quad := \quad \big(\exists m \in \dot{\mathsf{even}}(n)\big) \left[\begin{array}{ll} & (\forall x)(\exists y)\big(x \mathrel{\dot{<}} y \ \wedge \ C(\tilde{V}(y)) \doteq m\big) \\ \wedge & (\exists x)(\forall y)\big(x \mathrel{\dot{<}} y \ \Rightarrow \ C(\tilde{V}(y)) \geq m\big) \end{array} \right]$$

Definition 8.19 (Winning of Reduced Parity Games)**.** Working in $\mathsf{FSO}[\dot{<}]^\omega$, let $G = (\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}})$, where $\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}}$ are HF-variables. Furthermore let C be a Function variable and n be an HF-variable.

(1) We define the following formulae.

$$\begin{array}{rcl} \mathrm{WonGame}[\dot{<}]_{\mathsf{P}}(G, K, C, n) & := & (\forall \tilde{V} : \mathsf{P})\big(\mathrm{Play}[\dot{<}](G, K, \tilde{V}) \ \Rightarrow \ \mathrm{Par}[\dot{<}](C, n, \tilde{V})\big) \\ \mathrm{WonGame}[\dot{<}]_{\mathsf{O}}(G, K, C, n) & := & (\forall \tilde{V} : \mathsf{P})\big(\mathrm{Play}[\dot{<}](G, K, \tilde{V}) \ \Rightarrow \ \neg\mathrm{Par}[\dot{<}](C, n, \tilde{V})\big) \end{array}$$

(2) Given a player J (either P or O), we say that a J-strategy s is *winning in* (G, C, n) *from* K if the game $(G\{s\}_{\mathsf{J}}, \mathrm{Par}[\dot{<}](C, n, -))$ is won by J from K, *i.e.* if the following formula holds

$$\mathrm{WinStrat}[\dot{<}]_{\mathsf{J}}(G, s, K, C, n) \quad := \quad \mathrm{WonGame}[\dot{<}]_{\mathsf{J}}(G{\restriction}\{s\}_{\mathsf{J}}, K, C, n)$$

Note that in Definition 8.19, we have denoted strategies in reduced games with a lower case roman s. This notation contrasts with our notation σ for games in the sense of §5 in order to insist on the fact that strategies on reduced games are HF-sets.

Consider now $G = (\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}})$ where P, O, e_{P} and e_{O} are closed HF-terms such that

$$V_\omega \models \mathrm{Game}_0(G)$$

Assume also given closed HF-terms n and C such that

$$V_\omega \models \mathrm{Col}_0(G, C, n)$$

Then the positional determinacy of parity games (cf. [EJ91]) implies that for every HF-set $\kappa \in \mathsf{P}$, the following holds in the standard model $\mathfrak{N}$ of $\mathsf{FSO}[\dot{<}]^\omega$:

- For some player J (either P or O) there an HF-set s such that

$$\mathfrak{N} \models \quad \mathrm{Strat}^0_{\mathsf{J}}(G, \mathsf{s}) \ \wedge \ \mathrm{WinStrat}[\dot{<}]_{\mathsf{J}}(G, \mathsf{s}, \kappa, C, n)$$

Thanks to the completeness of $\mathsf{FSO}[\dot{<}]^\omega$ w.r.t. $\mathfrak{N}$ (Theorem 7.4 and Proposition 7.5), we obtain the following result, that may be viewed as a formulation of the Büchi-Landweber Theorem [BL69] (see also e.g. [Tho97, PP04]). Recall that $\mathrm{Strat}^0_{\mathsf{J}}(G, \mathsf{s})$ holds in $\mathfrak{N}$ (resp. $\mathsf{FSO}[\dot{<}]^\omega$, FSO) if and only if it holds in V_ω.

Proposition 8.20. *Assume given closed* HF*-terms* $G = (\mathsf{P}, \mathsf{O}, e_{\mathsf{P}}, e_{\mathsf{O}})$*,* n *and* C *such that*

$$V_\omega \models \quad \mathrm{Game}_0(G) \ \wedge \ \mathrm{Col}_0(G, C, n)$$

Then for every $\kappa \in \mathsf{P}$*, there is a player* J *(either* P *or* O*) and an* HF*-set* s *such that*

$$\mathsf{FSO}[\dot{<}]^\omega \vdash \quad \mathrm{WinStrat}[\dot{<}]_{\mathsf{J}}(G, \mathsf{s}, \kappa, C, n)$$

Proposition 7.8, namely

$$\mathsf{FSO}_{\mathscr{D}} \vdash (\forall P : 2)\ \big(\mathrm{TPath}(P) \ \Rightarrow \ \varphi^P\big) \qquad \text{whenever} \qquad \mathfrak{N} \models \varphi$$

(for φ a closed $\mathsf{FSO}[\dot{<}]^\omega$-formula) moreover gives the following.

Proposition 8.21. *Assume given closed* HF*-terms* $G = (\mathsf{P}, \mathsf{O}, e_\mathsf{P}, e_\mathsf{O})$*,* n *and* C *such that*

$$V_\omega \models \quad \mathrm{Game}_0(G) \ \wedge\ \mathrm{Col}_0(G, C, n)$$

Then for every $\kappa \in \mathsf{P}$*, there is a player* J *(either* P *or* O*) and an* HF*-set* $\mathtt{s}$ *such that*

$$\mathsf{FSO} \vdash \quad (\forall X : 2)\Big(\mathrm{TPath}(X) \ \Rightarrow\ \mathrm{WinStrat}[\dot{<}]^X_\mathsf{J}(G, \mathtt{s}, \kappa, C, n)\Big)$$

8.4.3. *Reduced Games in* FSO. We now come back to FSO. In this Section, we show, using Proposition 8.21, that FSO decides winning for parity games induced from reduced parity games (Theorem 8.22). This directly gives Proposition 8.10.

A reduced game $G = (\mathsf{P}, \mathsf{O}, e_\mathsf{P}, e_\mathsf{O})$ induces a game $\mathcal{G} = (\mathsf{P}, \mathsf{O}, E_\mathsf{P}, E_\mathsf{O})$ in the sense of Definition 5.1, where

$$E_\mathsf{P} : \mathscr{D}^* \times \mathsf{P} \ \mathtt{to}\ \mathcal{P}_*(\mathsf{O}) \qquad \text{and} \qquad E_\mathsf{O} : \mathscr{D}^* \times \mathsf{O} \ \mathtt{to}\ \mathcal{P}_*(\mathscr{D} \times \mathsf{P})$$

are defined using HF-Bounded Choice for Product Types (Theorem 3.32) as

$$E_\mathsf{P}(x, k) \quad := \quad e_\mathsf{P}(k) \qquad \text{and} \qquad E_\mathsf{O}(x, \ell) \quad := \quad e_\mathsf{O}(\ell)$$

Similarly, a strategy s in a reduced game G induces a strategy σ in $\mathcal{G}$ in the sense of Definition 5.13, with

$$\sigma(x, k) \quad := \quad s(k)$$

As for colorings, from $(C : \mathsf{P}\ \mathtt{to}\ [0, n])$ we define $\hat{C} : \mathcal{G}\ \mathtt{to}\ [0, n]$ as in Definition 6.6:

$$\hat{C}(x, k) \quad := \quad \begin{cases} C(k) & \text{if } k \in \mathsf{P} \qquad (\mathsf{P}\text{-position}) \\ n_\mathcal{A} & \text{if } k \in \mathsf{O} \qquad (\mathsf{O}\text{-position}) \end{cases}$$

We clearly have the following:

$$\begin{array}{lll} \mathsf{FSO} \vdash \mathrm{Game}(\mathcal{G}) & \text{whenever} & V_\omega \models \mathrm{Game}_0(G) \\ \mathsf{FSO} \vdash \mathrm{Strat}_\mathsf{J}(\mathcal{G}, \sigma) & \text{whenever} & V_\omega \models \mathrm{Strat}^0_\mathsf{J}(G, s) \\ \mathsf{FSO} \vdash \mathrm{Col}(\mathcal{G}, \hat{C}, n) & \text{whenever} & V_\omega \models \mathrm{Col}_0(G, C, n) \end{array}$$

Theorem 8.22. *Assume given closed* HF*-terms* $G = (\mathsf{P}, \mathsf{O}, e_\mathsf{P}, e_\mathsf{O})$*,* n *and* C *such that*

$$V_\omega \models \quad \mathrm{Game}_0(G) \ \wedge\ \mathrm{Col}_0(G, C, n)$$

Then for every $\kappa \in \mathsf{P}$*,*

$$\begin{array}{lll} \textit{either} & \mathsf{FSO} \vdash & (\exists \sigma_\mathsf{P} : \mathcal{G}_\mathsf{P}\ \mathtt{to}\ \mathsf{O}) \left(\begin{array}{ll} & \mathrm{Strat}_\mathsf{P}(\mathcal{G}, \sigma_\mathsf{P}) \\ \wedge & \mathrm{WinStrat}_\mathsf{P}(\mathcal{G}(\trianglelefteq), \sigma_\mathsf{P}, \kappa, C, n) \end{array} \right) \\ \\ \textit{or} & \mathsf{FSO} \vdash & (\exists \sigma_\mathsf{O} : \mathcal{G}_\mathsf{O}\ \mathtt{to}\ \mathscr{D} \times \mathsf{P}) \left(\begin{array}{ll} & \mathrm{Strat}_\mathsf{O}(\mathcal{G}, \sigma_\mathsf{O}) \\ \wedge & \mathrm{WinStrat}_\mathsf{O}(\mathcal{G}(\trianglelefteq), \sigma_\mathsf{O}, \kappa, C, n) \end{array} \right) \end{array}$$

In the statement of Theorem 8.22, $\mathcal{G}(\trianglelefteq)$ refers to the the game of Remark 5.4 (see also Remark 5.23).

Remark 8.23. The crucial differences between Theorem 8.22 and the axiom $(PosDet)$ are the following. On one hand, Theorem 8.22 allows us to derive $(PosDet)$ for games on *finite* graphs only, while $(PosDet)$ speaks about arbitrary FSO-definable games (in the sense of §5). On the other hand, Theorem 8.22 says that FSO *decides* winning for games on finite graphs, while $(PosDet)$ is a statement of *determinacy*, i.e. that one of the players wins, but not *which* player wins.

Proof of Theorem 8.22. Fix G, n, C and κ as in the statement. Let J and $\mathtt{s}$ be given by Proposition 8.21, and let σ be induced from $\mathtt{s}$ as above. We are going to show that σ is winning in $\mathcal{G}$ from position κ:

$$(\forall V : \mathcal{G} \ \mathtt{to}\ 2)\Big(\mathrm{Play}(\sigma, \kappa, V) \ \Rightarrow\ \mathrm{Par}(\mathcal{G}, \hat{C}, n, V)\Big)$$

So let $V : \mathcal{G}\ \mathtt{to}\ 2$ be an infinite play of σ from κ. Our plan is to obtain $\mathrm{Par}(\mathcal{G}, \hat{C}, n, V)$ from Proposition 8.21. By Comprehension for Product Types (Theorem 3.33), let $|V| : \mathscr{D}^*\ \mathtt{to}\ 2$ be the set of all $x \in \mathscr{D}^*$ such that $(x, k) \in V$ for some $k \in \mathsf{P}$. Note that $\mathrm{TPath}(|V|)$ holds in FSO. Proposition 8.21 then gives

$$\mathsf{FSO} \vdash \quad \mathrm{WinStrat}[\dot{<}]^{|V|}_{\mathsf{J}}(G, \mathtt{s}, \kappa, C, n)$$

Note that

$$\mathrm{WinStrat}[\dot{<}]^{|V|}_{\mathsf{P}}(G, \mathtt{s}, \kappa, C, n) \iff (\forall \tilde{V} : |V|\ \mathtt{to}\ \mathsf{P})\Big(\mathrm{Play}[\dot{<}]^{|V|}(\mathtt{s}, \kappa, \tilde{V}) \ \Rightarrow\ \mathrm{Par}[\dot{<}]^{|V|}(C, n, \tilde{V})\Big)$$

and similarly for $\mathrm{WinStrat}[\dot{<}]^{|V|}_{\mathsf{O}}(G, \mathtt{s}, \kappa, C, n)$. By HF-Bounded Choice for Functions (§3.4.5), let $\tilde{V} : |V|\ \mathtt{to}\ \mathsf{P}$ take $x \in |V|$ to the unique $k \in \mathsf{P}$ such that $(x, k) \in V$. Then we are done as soon as we show

Claim 8.23.1.

$$\mathrm{Play}[\dot{<}]^{|V|}(\mathtt{s}, \kappa, \tilde{V}) \quad \wedge \quad \Big(\mathrm{Par}(\mathcal{G}, \hat{C}, n, V) \ \Leftrightarrow\ \mathrm{Par}[\dot{<}]^{|V|}(C, n, \tilde{V})\Big)$$

Proof of Claim 8.23.1. The property on parity conditions follows from the fact that for all $m \in [0, n]$ we have

$$\begin{aligned} &\left[\begin{array}{ll} & (\forall x \in |V|)(\exists y \in |V|)\big(x \mathrel{\dot{<}} y \ \wedge\ C(\tilde{V}(y)) \doteq m\big) \\ \wedge & (\exists x \in |V|)(\forall y \in |V|)\big(x \mathrel{\dot{<}} y \ \Rightarrow\ C(\tilde{V}(y)) \geq m\big) \end{array}\right] \\ \iff\ & \left[\begin{array}{ll} & (\forall u \in V)(\exists v \in V)(u \lhd v \ \wedge\ \hat{C}(v) \doteq m) \\ \wedge & (\exists u \in V)(\forall v \in V)(u \lhd v \ \Rightarrow\ \hat{C}(v) \geq m) \end{array}\right] \end{aligned}$$

As for $\mathrm{Play}[\dot{<}]^{|V|}(\mathtt{s}, \kappa, \tilde{V})$, note that it unfolds to

$$\begin{aligned} &\tilde{V}(\dot{\varepsilon}) \doteq \kappa \quad \wedge \\ &(\forall x \in |V|)(\forall y \in |V|)\Big(\mathrm{S}^{|V|}(x, y) \ \Rightarrow\ \big(\exists \ell \mathrel{\dot{\in}} e(\mathtt{s})_{\mathsf{P}}(\tilde{V}(x))\big)\big(\exists d \mathrel{\dot{\in}} \mathscr{D}\big)\Big[(d, \tilde{V}(y)) \in e(\mathtt{s})_{\mathsf{O}}(\ell)\Big]\Big) \end{aligned}$$

where

$$\mathrm{S}^{|V|}(x, y) \quad = \quad (x \mathrel{\dot{<}} y) \ \wedge\ \neg(\exists z \in |V|)\big[x \mathrel{\dot{<}} z \mathrel{\dot{<}} y\big]$$

But this directly follows from the definition of σ from $\mathtt{s}$ together with the fact that V is a play of σ from κ. ■

This concludes the proof of Theorem 8.22. □

We are now ready to prove Proposition 8.10, thus completing the proof of Theorem 8.5.

Proof of Proposition 8.10. We have to show that for an HF-closed parity automaton $(\mathcal{A} : 1)$,

$$\mathsf{FSO} \vdash (\exists F : 1)\big(F \in \mathcal{L}(\mathcal{A})\big) \qquad \text{or} \qquad \mathsf{FSO} \vdash \neg(\exists F : 1)\big(F \in \mathcal{L}(\mathcal{A})\big)$$

For any $(F : 1)$, the game $\mathcal{G}(\mathcal{A}, F)$ is generated as above from the edge relations (8.1). Moreover, recall from Definition 6.6 that the winning condition of $\mathcal{G}(\mathcal{A}, F)$ is generated, as in the statement of

Theorem 8.22, by the game $\mathcal{G}(\mathcal{A}, F)(\trianglelefteq)$ of Remark 5.4. We then conclude by Theorem 8.22, and this completes the proof of Proposition 8.10. □

8.5. **Remarks on Recursiveness.** We noted in Remark 8.4 that the completeness of $\mathsf{FSO}+(PosDet)$ indeed allows us to decide FSO and MSO formulae in the standard model $\mathfrak{T}$ of §3.5. This however comes with two apparent defects. The first one is that the interpretation $[\![-]\!]$ of HF-terms fixed in Convention 3.13 is not computable (see Remarks 3.14 and 3.21), because provability in $\mathbf{Sk}(\mathsf{ZFC}^-)$ is not decidable (as this theory contains the Π^0_1 fragment of arithmetic). The second one is that, although the axiom set $\mathsf{MSO}+\langle PosDet\rangle$ is even polynomial-time recognizable (recall that $\langle PosDet\rangle$ is defined in Definition 5.26, §5.6.1), the interpretation $\langle - \rangle$ for HF-terms relies on Convention 3.13 (fixing the interpretation of HF-Functions), and is thus not computable. We discuss here a workaround for this involving a slightly different setting for FSO. We chose to not officially work in that setting because we found it less uniform and elegant than the current presentation of FSO, which nonetheless still allows us to derive Rabin's Tree Theorem [Rab69].

Rather than taking all the axioms on HF-sets of §3.4.4, in particular considering the whole theory $\mathbf{Sk}(\mathsf{ZFC}^-)$ there, we may work in systems parametrized by chosen sets of HF-Functions. A way to implement this would be to consider systems $\mathsf{FSO}(\mathbf{SK})$, where the parameter $\mathbf{SK}$ specifies *some* interpretations $\mathsf{g}_{n,m}$ for constants $\dot{\mathsf{g}}_{n,m}$ such that (3.2) is assumed to hold. Concretely, a specification $\mathbf{SK}$ consists of a set $\mathrm{SK} \subseteq \mathbb{N} \times \mathbb{N}$ together with functions

$$\mathsf{g}_{n,m} \quad : \quad V_\omega^n \quad \longrightarrow \quad V_\omega \qquad\qquad \text{(for each } (n,m) \in \mathrm{SK})$$

Given a set $\mathrm{SK} \subseteq \mathbb{N} \times \mathbb{N}$, we let $\mathsf{ZFC}^-(\mathrm{SK})$ consist of ZFC^- augmented with the axioms

$$(\forall k_1, \dots, k_n)(\exists! \ell)(\varphi_{n,m}) \quad \Rightarrow \quad (\forall k_1, \dots, k_n)\varphi_{n,m}[\dot{\mathsf{g}}_{n,m}(k_1, \dots, k_n)/\ell] \quad \text{(for each } (n,m) \in \mathrm{SK})$$

We say that $\mathbf{SK}$ is a *specification* if

$$\mathbf{SK} \quad = \quad (\mathrm{SK}, (\mathsf{g}_{n,m})_{(n,m)\in\mathrm{SK}})$$

where, for each $(n,m) \in \mathrm{SK}$,

- $\mathsf{g}_{n,m}$ is a computable function $V_\omega^n \to V_\omega$, and
- for each each $\dot{\mathsf{g}}_{n',m'}$ occurring in $\varphi_{n,m}$, we have $(n',m') \in \mathrm{SK}$, and
- $\mathsf{ZFC}^-(\mathrm{SK}) \vdash (\forall k_1, \dots, k_n)(\exists! \ell)\varphi_{n,m}$, and
- $V_\omega \models (\forall k_1, \dots, k_n)\varphi_{n,m}[\mathsf{g}_{n,m}(k_1, \dots, k_n)/\ell]$

Given a specification $\mathbf{SK}$, one can fix the interpretation of all constants $(\dot{\mathsf{g}}_{n,m})_{n,m\in\mathbb{N}}$ by taking for $\dot{\mathsf{g}}_{n,m}$ with $(n,m) \notin \mathrm{SK}$ the function $V_\omega^n \to V_\omega$ with constant value $\varnothing$.

For the formal definition of $\mathsf{FSO}(\mathbf{SK})$, instead of the Axioms on HF-Sets of §3.4.4, one has the following.

- For each $(n,m) \in \mathrm{SK}$, and for all HF-terms $\boldsymbol{K} = K_1, \dots, K_n$, the axiom

$$\varphi_{n,m}[\boldsymbol{K}/\boldsymbol{k}][\dot{\mathsf{g}}_{n,m}(\boldsymbol{K})/\ell]$$

- For each closed HF-formula φ such that $V_\omega \models \varphi$, the axiom

$$\varphi$$

Given a specification $\mathbf{SK}$, the interpretations $[\![-]\!]$ and $\langle - \rangle$ are computable. All results of this paper hold for sufficiently large specifications.

Theorem 8.24. *Let* $\mathbf{SK}$ *be a specification defining all the* HF*-Functions of (a)–(h), §3.4.4, as well as those of Convention 5.18, §5.5. Then all the results stated in §8 hold for* $\mathsf{FSO}(\mathbf{SK})$ *instead of* FSO.

9. The Simulation Theorem

This Section is devoted to the proof of the *Simulation Theorem*, cf. [EJ91, MS95].

Theorem 9.1 (Simulation Theorem 6.17)**.** *For each* HF*-closed parity automaton* $\mathcal{A} : \Sigma$ *there is a non-deterministic* HF*-closed parity automaton* $\mathrm{ND}(\mathcal{A}) : \Sigma$ *such that*

$$\mathsf{FSO} \vdash \quad \mathcal{L}(\mathrm{ND}(\mathcal{A})) = \mathcal{L}(\mathcal{A})$$

We assume that $\mathcal{A}$ is HF-closed in Theorem 9.1 because we rely on McNaughton's Theorem [McN66], in the standard model for ω-words, which we import into FSO thanks to Proposition 7.8.

Before a detailed exposition, let us explain the main idea behind Theorem 9.1. We momentarily work in the usual mathematical universe (*i.e.* not in the formal theory FSO). Recall that in a non-deterministic automaton $\mathcal{N}$, O can only explicitly choose tree directions, since for each possible $\gamma_{\mathcal{N}}$ in the image of $\partial_{\mathcal{N}}$, if $(d, q), (d, q') \in \gamma_{\mathcal{N}}$ then $q = q'$, by definition. In order to obtain a non-deterministic automaton $\mathcal{N}$ from an alternating automaton $\mathcal{A}$, the idea is to perform a subset construction, such that each $\gamma_{\mathcal{N}}$ in the image of $\partial_{\mathcal{N}}$ is of the form

$$\gamma_{\mathcal{N}} \quad = \quad \{(d, S'_d) \mid d \in \mathscr{D}\}$$

where each S'_d gathers states q such that $(d, q) \in \gamma_{\mathcal{A}}$ with $\gamma_{\mathcal{A}}$ in the image of $\partial_{\mathcal{A}}$.

More precisely, assuming $S \in \mathcal{P}_*(Q_{\mathcal{A}})$, one may consider functions

$$\begin{array}{ccccc} f & : & S & \longrightarrow & \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}}) \\ & & q & \longmapsto & \gamma_q \in \partial_{\mathcal{A}}(q, \mathtt{a}) \end{array} \tag{9.1}$$

Each such f induces

$$\gamma_{\mathcal{N}}(f) = \{(d, S'_d(f)) \mid d \in \mathscr{D}\} \quad \text{where} \quad S'_d(f) = \{q \mid (d, q) \in f(q)\}$$

and we can let

$$\partial_{\mathcal{N}}(S, \mathtt{a}) \quad := \quad \{\gamma_{\mathcal{N}}(f) \mid f \text{ is as in (9.1)}\}$$

Then, for each $\gamma_{\mathcal{N}}(f)$ in the image of $\partial_{\mathcal{N}}$ and for each tree direction $d \in \mathscr{D}$, the set S'_d is unique such that $(d, S'_d) \in \gamma_{\mathcal{N}}(f)$, and $\mathcal{N}$ satisfies the property asked in Definition 6.4 to non-deterministic automata.

There is however a difficulty in the definition of the acceptance condition of $\mathcal{N}$. We follow here the construction of [Wal02] where the states of $\mathcal{N}$, rather than being simply sets of states, are sets of pairs of states $S \in \mathcal{P}(Q_{\mathcal{A}} \times Q_{\mathcal{A}})$. Then an infinite sequence of states $S_0, S_1, \ldots \in Q_{\mathcal{N}}$ induces a set of *traces* $q_0, q_1, \ldots \in Q_{\mathcal{A}}$ with $(q_i, q_{i+1}) \in S_{i+1}$. For $(S_n)_{n \in \mathbb{N}} \in Q_{\mathcal{N}}^{\omega}$ to be accepting, one may then require all its traces $(q_n)_{n \in \mathbb{N}} \in Q_{\mathcal{A}}^{\omega}$ to be accepting. We may obtain a parity condition for $\mathcal{N}$ by noticing that its acceptance condition is ω-regular (*i.e.* definable in MSO over ω-words). This allows us to apply McNaughton's Theorem [McN66], and to obtain a deterministic ω-word parity automaton $\mathcal{D}$ whose language is the set of accepting sequences $(S_n)_{n \in \mathbb{N}} \in Q_{\mathcal{N}}^{\omega}$. A suitable product of $\mathcal{N}$ with $\mathcal{D}$ then gives a non-deterministic parity automaton equivalent to $\mathcal{A}$.

The organization of this Section follows the above construction. Working in FSO, consider a parity automaton $\mathcal{A} : \Sigma$. We will build a *non-deterministic* automaton $\mathrm{ND}(\mathcal{A}) : \Sigma$ with the same language. The automaton $\mathrm{ND}(\mathcal{A})$ will be defined in three steps:

(1) We first define in §9.1 a non-deterministic automaton $!\mathcal{A}$ in the sense of Definition 6.1. The acceptance condition of $!\mathcal{A}$ will be given by an FSO-formula with a free Function variable (intended to be range over infinite plays) rather than a parity condition.

(2) For an HF-*closed* parity automaton $\mathcal{A}$, the formula describing the acceptance condition of $!\mathcal{A}$ is then transformed in §9.2 to a $\mathsf{FSO}[\dot{<}]^\omega$ formula relativized to infinite rooted tree paths (see §7). This construction relies, via Proposition 7.8, on Proposition 7.5 (*i.e.* Proposition 3.27) which requires the manipulation of closed (and in particular HF-closed) objects.
(3) Using the tools of §7, and relying on McNaughton's Theorem [McN66] (see also e.g. [Tho97, PP04]), in §9.3 we will then turn $!\mathcal{A}$ into an equivalent non-deterministic *parity* automaton $\mathrm{ND}(\mathcal{A})$, in the sense of Definition 6.6.

In this Section, it is convenient to work with the following games.

Definition 9.2. Given an automaton $\mathcal{A} : \Sigma$, we let $\mathcal{G}(\mathcal{A})$ be the game with

$$\mathsf{P}_{\mathcal{G}(\mathcal{A})} \;:=\; Q_{\mathcal{A}} \qquad \text{and} \qquad \mathsf{O}_{\mathcal{G}(\mathcal{A})} \;:=\; Q_{\mathcal{A}} \times \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}})$$

and with transitions defined by HF-Bounded Choice for Product Types (Theorem 3.32) and Comprehension for HF-Sets (Remark 3.34) as

$$\begin{array}{rrcl} & (q', \gamma) \in E(\mathcal{G}(\mathcal{A}))_{\mathsf{P}}(x, q) & \text{iff} & q' \doteq q \;\wedge\; (\exists \mathsf{a} \in \Sigma)\, [\gamma \in \partial_{\mathcal{A}}(q, \mathsf{a})] \\ \text{and} & (d, q') \in E(\mathcal{G}(\mathcal{A}))_{\mathsf{O}}(x, (q, \gamma)) & \text{iff} & (d, q') \in \gamma \end{array}$$

As for winning, we will consider the game $\mathcal{G}(\mathcal{A})$ as being equipped with the winning condition $\Omega_{\mathcal{A}}$ in the sense of §5.4. Note that for $F : \Sigma$, the acceptance game $\mathcal{G}(\mathcal{A}, F)$ is a subgame of $\mathcal{G}(\mathcal{A})$ in the sense of Def. 5.3.

Remark 9.3. Note that if $\mathrm{Aut}(\mathcal{A})$ then Σ is non-empty, so we indeed have $\mathrm{Game}(\mathcal{G}(\mathcal{A}))$. For each $F : \Sigma$, the acceptance game $\mathcal{G}(\mathcal{A}, F)$ is a subgame of $\mathcal{G}(\mathcal{A})$ (in the sense of Def. 5.3). In particular infinite plays in $\mathcal{G}(\mathcal{A}, F)$ are infinite plays in $\mathcal{G}(\mathcal{A})$. Moreover, it is easy to see that (winning) strategies on $\mathcal{G}(\mathcal{A}, F)$ are (winning) strategies on $\mathcal{G}(\mathcal{A})$.

Furthermore, note that the game $\mathcal{G}(\mathcal{A})(\trianglelefteq)$ induced by Remark 5.3 from Definition 9.2 is precisely the game $\mathcal{G}(\mathcal{A})(\trianglelefteq)$ of (6.1). It follows that in the case of a parity automaton $\mathcal{A}$, we unambiguously extend the notation of Definition 6.6 and write $\mathrm{Par}(\mathcal{A}, \hat{C}_{\mathcal{A}}, n_{\mathcal{A}}, U)$ or $\mathrm{Par}(\mathcal{A}, U)$ for the formula $\mathrm{Par}(\mathcal{G}(\mathcal{A})(\trianglelefteq), \hat{C}_{\mathcal{A}}, n_{\mathcal{A}}, U)$.

9.1. **The Construction of $!\mathcal{A}$.** Consider an alternating parity automaton $\mathcal{A}$, in the sense of Definition 6.6. So we have $\mathcal{A} = (Q_{\mathcal{A}},\, q^\iota_{\mathcal{A}},\, \partial_{\mathcal{A}},\, C_{\mathcal{A}},\, n_{\mathcal{A}})$ where

$$\partial_{\mathcal{A}} \;:\; Q_{\mathcal{A}} \times \Sigma \;\longrightarrow\; \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}})) \qquad \text{and} \qquad C_{\mathcal{A}} : Q_{\mathcal{A}} \;\mathtt{to}\; [0, n_{\mathcal{A}}]$$

We define the state set and the initial state of $!\mathcal{A}$ as:

$$Q_{!\mathcal{A}} \;:=\; \mathcal{P}_*(Q_{\mathcal{A}} \times Q_{\mathcal{A}}) \qquad \text{and} \qquad q^\iota_{!\mathcal{A}} \;:=\; (q^\iota_{\mathcal{A}}, q^\iota_{\mathcal{A}})$$

The transition function of $!\mathcal{A}$ is defined as follows, using Remark 6.9. For $\mathsf{a} \in \Sigma$ and $S \in Q_{!\mathcal{A}}$ we let $!\gamma \in \partial_{!\mathcal{A}}(S, \mathsf{a})$ if and only if there is some HF-set

$$\begin{array}{rcrcl} f & : & S & \longrightarrow & \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}}) \\ & & q & \longmapsto & \gamma_q \in \partial_{\mathcal{A}}(q, \mathsf{a}) \end{array}$$

such that $!\gamma = \{(d, S'_d) \mid d \in \mathscr{D} \wedge S'_d \neq \varnothing\}$, where

$$S'_d \;=\; \{(q, q') \mid q \in \pi_2(S) \;\wedge\; (d, q') \in f(q)\} \tag{9.2}$$

and where π_2 is a projection HF-Function of §3.4.4.(f).

Remark 9.4. We indeed have

$$\partial_{!\mathcal{A}} \quad : \quad Q_{!\mathcal{A}} \times \Sigma \quad \longrightarrow \quad \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times Q_{!\mathcal{A}}))$$

since for $S \in Q_{!\mathcal{A}} = \mathcal{P}_*(Q_\mathcal{A} \times Q_\mathcal{A})$, by HF-Bounded Choice for HF-Sets (§3.4.5) there is always some $f \in \mathcal{P}_*(\mathscr{D} \times Q_\mathcal{A})^{\pi_2(S)}$ with $\forall q \in \pi_2(S)(f(q) \in \partial_\mathcal{A}(q, \mathsf{a}))$, and moreover such that S'_d is non-empty for at least one $d \in \mathscr{D}$.

Note our unusual choice of taking *non-empty* sets as states of $!\mathcal{A}$. It would have been more natural to allow the empty set as a state, in particular because it would have allowed us to strengthen Corollary 6.16 to an "exists unique" statement. This could also have worked in our setting where games are assumed to have no dead ends, and in which transitions of alternating automata range over non-empty sets of non-empty subsets of $\mathscr{D} \times Q_\mathcal{A}$. However, the empty state would have appeared in the transitions of $!\mathcal{A}$ only in case there is some tree direction $d \in \mathscr{D}$ which is not available to O at some stage. Since the empty state of $!\mathcal{A}$ would have been unconditionally winning for P, this would have lead to an additional case to handle in the proof of completeness of $!\mathcal{A}$ (Proposition 9.7 below).

So far we have defined for $!\mathcal{A}$ a state set (with an initial state) and a transition function. As explained above, we will not directly equip it with a parity condition. Instead, we will define its acceptance condition via an FSO-formula $\mathcal{W}_{!\mathcal{A}}$, in the sense of Definition 6.1. Consider first $V : \mathcal{G}(!\mathcal{A})\ \mathtt{to}\ 2$ and $T : \mathcal{G}(\mathcal{A})\ \mathtt{to}\ 2$. We say that T is a *trace* in V if the following formula $\mathrm{Trace}(T, V)$ holds,

$$\begin{aligned}
&\mathrm{Path}(\mathcal{G}(\mathcal{A}), (\dot{\varepsilon}, q^\iota_\mathcal{A}), T) \\
\wedge\ &\big(\forall (x, q) \in T_\mathsf{P}\big)\big(\exists S \in Q_{!\mathcal{A}}\big)\Big[(x, S) \in V_\mathsf{P} \ \wedge\ q \in \pi_2(S)\Big] \\
\wedge\ &\big(\forall (x, q), (y, q') \in T_\mathsf{P}\big)\big(\forall S \in Q_{!\mathcal{A}}\big)\ \Big[(x, q) \lhd^{\mathsf{P};\mathsf{O}}_{\mathcal{G}(\mathcal{A})} (y, q') \ \Rightarrow\ (y, S) \in V_\mathsf{P} \ \Rightarrow\ (q, q') \in S\Big]
\end{aligned}$$

where we use the following formula:

$$u \lhd^{\mathsf{P};\mathsf{O}}_{\mathcal{G}(\mathcal{A})} v \quad := \quad \big(\exists w \in \mathcal{G}(\mathcal{A})_\mathsf{O}\big)\Big(\mathsf{s}^{\lhd}_{\mathcal{G}(\mathcal{A})}(u, w) \ \wedge\ \mathsf{s}^{\lhd}_{\mathcal{G}(\mathcal{A})}(w, v)\Big)$$

The formula $\mathcal{W}_{!\mathcal{A}}(V)$ is defined to be:

$$\mathcal{W}_{!\mathcal{A}}(V) \quad := \quad \big(\forall T : \mathcal{G}(\mathcal{A})\ \mathtt{to}\ 2\big)\Big[\mathrm{Trace}(T, V) \ \Rightarrow\ \mathrm{Par}(\mathcal{A}, \hat{C}_\mathcal{A}, n_\mathcal{A}, T)\Big]$$

Recall our notation $\mathrm{Par}(\mathcal{A}, \hat{C}_\mathcal{A}, n_\mathcal{A}, -)$ from Remark 9.3. Note that $\mathcal{W}_{!\mathcal{A}}$ requires no condition w.r.t. the *transitions* of $\mathcal{G}(\mathcal{A})$. We are now going to show that $!\mathcal{A}$ has the same language as $\mathcal{A}$.

Theorem 9.5. *Fix a parity automaton $\mathcal{A} : \Sigma$ and consider the automaton $!\mathcal{A} : \Sigma$ as defined above. Then $\mathsf{FSO}_\mathscr{D}$ proves that for all $F : \Sigma$, $!\mathcal{A}$ accepts F if and only if $\mathcal{A}$ accepts F.*

The proof of Theorem 9.5 is split into Propositions 9.7 and 9.8 below.

Convention 9.6. In Propositions 9.7 and 9.8, for fixed automata $\mathcal{A}$ and $!\mathcal{A}$, we let

$$\iota \ := \ (\dot{\varepsilon}, q^\iota_\mathcal{A}) \qquad \text{and} \qquad \iota_! \ := \ (\dot{\varepsilon}, q^\iota_{!\mathcal{A}})$$

Proposition 9.7. *Fix a parity automaton $\mathcal{A} : \Sigma$ and consider the automaton $!\mathcal{A} : \Sigma$ as defined above. Then $\mathsf{FSO}_\mathscr{D}$ proves that for all $F : \Sigma$, if $\mathcal{A}$ accepts F then $!\mathcal{A}$ accepts F.*

Proof. Let σ be a winning P-strategy in $\mathcal{G}(\mathcal{A}, F)$. We define a winning P-strategy τ in $\mathcal{G}(!\mathcal{A}, F)$. Note that

$$\sigma : \mathcal{G}(\mathcal{A})_\mathsf{P}\ \mathtt{to}\ \mathsf{O}_{\mathcal{G}(\mathcal{A})} \qquad \text{and} \qquad \tau : \mathcal{G}(!\mathcal{A})_\mathsf{P}\ \mathtt{to}\ \mathsf{O}_{\mathcal{G}(!\mathcal{A})}$$

First, given a P-position (x, S) in $\mathcal{G}(!\mathcal{A}, F)$, we define a conjunction

$$\gamma_{(x,S)} \in \partial_{!\mathcal{A}}(S, F(x)) \subseteq \mathcal{P}_*(\mathscr{D} \times Q_{!\mathcal{A}})$$

as follows.

- *Definition of* $\gamma_{(x,S)}$. For each $q \in \pi_2(S)$, $\sigma(x, q)$ gives some $\gamma_q \in \partial_{\mathcal{A}}(q, F(x))$. HF-Bounded HF-Choice (§3.4.5) then gives

$$\begin{array}{rcccl} f & : & \pi_2(S) & \longrightarrow & \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}}) \\ & & q & \longmapsto & \gamma_q \in \partial_{\mathcal{A}}(q, F(x)) \end{array}$$

  By HF-Comprehension (Remark 3.34), we then let $\gamma_{(x,S)}$ be $\{(d, S'_d) \mid d \in \mathscr{D} \wedge S'_d \neq \varnothing\}$ where each S'_d is defined as in (9.2).

We now define the P-strategy τ on $\mathcal{G}(!\mathcal{A}, F)$. By HF-Bounded Choice for Functions (§3.4.5), we let

$$\tau(x, S) \quad := \quad (S, \gamma_{(x,S)}) \qquad \text{for each } (x, S) \in \mathcal{G}(!\mathcal{A})_{\mathsf{P}}$$

We have $\mathrm{Strat}_{\mathsf{P}}(\mathcal{G}(!\mathcal{A}, F), \tau)$ directly by definition of $\partial_{!\mathcal{A}}$. It remains to check that τ is winning in $\mathcal{G}(!\mathcal{A}, F)$. Consider an infinite play of τ, that is some $V : \mathcal{G}(!\mathcal{A})\ \mathtt{to}\ 2$ such that $\mathrm{Play}(\tau, \iota_!, V)$. Since σ is winning in $\mathcal{G}(\mathcal{A}, F)$, by definition of $\mathcal{W}_{!\mathcal{A}}$ and by Remarks 6.7 and 9.3, we are done if we show that:

$$\big(\forall T : \mathcal{G}(\mathcal{A})\ \mathtt{to}\ 2\big)\Big(\mathrm{Trace}(T, V) \;\Rightarrow\; \big(\exists U : \mathcal{G}(\mathcal{A})\ \mathtt{to}\ 2\big)\big[U_{\mathsf{P}} = T_{\mathsf{P}} \;\wedge\; \mathrm{Play}(\sigma, \iota, U)\big]\Big)$$

Assume $\mathrm{Trace}(T, V)$. By HF-Comprehension for Product Types, we let $U : \mathcal{G}(\mathcal{A})\ \mathtt{to}\ 2$ be such that $U_{\mathsf{P}} = T_{\mathsf{P}}$ and such that U_{O} consists of the $\{(x, \sigma(x, q))\}$ for $(x, q) \in U_{\mathsf{P}}$. Note that we actually have $U : \sigma\ \mathtt{to}\ 2$. It remains to check that

$$\mathrm{Play}(\sigma,\ \iota,\ U)$$

We apply Lemma 5.12, whence it remains to show:

$$\mathrm{Path}(\mathcal{G}(\mathcal{A}), \iota, U) \tag{9.3}$$

$$\big(\forall u, u' \in U\big)\left[\mathsf{s}^{\lhd}_{\mathcal{G}(\mathcal{A})}(u, u') \;\Rightarrow\; u \underset{\sigma}{\longrightarrow} u'\right] \tag{9.4}$$

Note that $\mathrm{Path}(\mathcal{G}(\mathcal{A}), \iota, T)$ since $\mathrm{Trace}(T, V)$.

- *Proof of* (9.3). We obviously have $\iota \in U_{\mathsf{P}} = T_{\mathsf{P}}$. Also, given $u \in U$, if $u \in U_{\mathsf{P}} = T_{\mathsf{P}}$ then $\iota \trianglelefteq_{\mathcal{G}(\mathcal{A})} u$, and if $u \in U_{\mathsf{O}}$, then $v \longrightarrow_\sigma u$ for some $v \in U_{\mathsf{P}} = T_{\mathsf{P}}$, so that $\iota \trianglelefteq_{\mathcal{G}(\mathcal{A})} v \lhd_{\mathcal{G}(\mathcal{A})} u$. Moreover, for each $u \in U_{\mathsf{P}}$, we have $u \longrightarrow_\sigma v$ for some $v \in U_{\mathsf{O}}$, and we get $\mathsf{s}^{\lhd}_{\mathcal{G}(\mathcal{A})}(u, v)$ by Proposition 5.6.

  It remains to show that U is linearly ordered w.r.t. $\trianglelefteq_{\mathcal{G}(\mathcal{A})}$. For U_{P} this follows from the same property for T_{P}. Now let $u \in U_{\mathsf{P}}$ and $v' \in U_{\mathsf{O}}$. Hence v' is of the form $(x, \sigma(x, q))$ with $v := (x, q) \in U_{\mathsf{P}} = T_{\mathsf{P}}$. If $u \trianglelefteq_{\mathcal{G}(\mathcal{A})} v$ then $u \lhd_{\mathcal{G}(\mathcal{A})} v'$ and we are done. Otherwise, $v \lhd_{\mathcal{G}(\mathcal{A})} u$. But by definition of $\trianglelefteq_{\mathcal{G}(\mathcal{A})}$, this implies $u = (y, q')$ with $x \dot{<} y$, so that $v' \lhd_{\mathcal{G}(\mathcal{A})} u$. Consider now $u', v' \in U_{\mathsf{O}}$ and let $u, v \in U_{\mathsf{P}}$ be their immediate predecessors. If $u \lhd_{\mathcal{G}(\mathcal{A})} v$ then $u' \lhd_{\mathcal{G}(\mathcal{A})} v'$ and we are done. Otherwise, without loss of generality we have that $u = v$. But then $u' = v'$ by definition of U. ■
- *Proof of* (9.4). Assume first $u \in U_{\mathsf{P}}$. In this case, u is of the form (x, q) with $(x, q) \in T_{\mathsf{P}}$, and u' is of the form $(x, \sigma(x, q'))$ with $(x, q') \in T_{\mathsf{P}}$. But T is linearly ordered w.r.t. $\trianglelefteq_{\mathcal{G}(\mathcal{A})}$, so that $q = q'$. It follows that $u \longrightarrow_\sigma u'$.

Assume now that $u \in U_{\mathsf{O}}$. In this case, u is of the form $(x, (q, \gamma_{\mathcal{A}}))$ with $(x, q) \in U_{\mathsf{P}} = T_{\mathsf{P}}$ and $(q, \gamma_{\mathcal{A}}) = \sigma(x, q)$. Moreover, $u' \in U_{\mathsf{P}} = T_{\mathsf{P}}$ is of the form $(\mathsf{S}_d(x), q')$. We thus get $u \longrightarrow_{\sigma} u'$ as soon as

$$(d, q') \in \gamma_{\mathcal{A}}$$

Since $\mathrm{Trace}(T, V)$ and since V is a play of τ, there are unique S, S' with $(x, S), (\mathsf{S}_d(x), S') \in V_{\mathsf{P}}$ and such that $q \in \pi_2(S)$ and $q' \in \pi_2(S')$. Moreover, we necessarily have $(d, S') \in \gamma_{(x,S)}$ for $(S, \gamma_{(x,S)}) = \tau(x, S)$. But $\mathrm{Trace}(T, V)$ implies $(q, q') \in S'$, and it follows that $(d, q') \in \gamma_{\mathcal{A}}$ by definition of $\gamma_{(x,S)}$. ■

This concludes the proof of Proposition 9.7. □

Proposition 9.8. *Fix a parity automaton $\mathcal{A} : \Sigma$ and consider the automaton $!\mathcal{A} : \Sigma$ as defined above. Then $\mathsf{FSO}_{\mathscr{D}}$ proves that for all $F : \Sigma$, if $!\mathcal{A}$ accepts F then $\mathcal{A}$ accepts F.*

Proof. Let τ be a winning P-strategy in $\mathcal{G}(!\mathcal{A}, F)$. We will define a winning P-strategy σ in $\mathcal{G}(\mathcal{A}, F)$. To this end, we invoke Corollary 6.16, which tells us that since $!\mathcal{A}$ is non-deterministic, for each $x \in \mathscr{D}^*$ there is at most one $S \in Q_{!\mathcal{A}}$ such that (x, S) belongs to an infinite play of τ. Moreover, using Remark 3.17, for each $S \in Q_{!\mathcal{A}}$ we fix a well-order $\preceq$ on $\mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}})^{\pi_2(S)}$.

We now define the strategy σ.

- *Definition of* σ. We apply HF-Bounded Choice for Product Types (Theorem 3.32). Consider $(x, q) \in \mathcal{G}(\mathcal{A}, F)_{\mathsf{P}}$. We first assign to (x, q) an $S \in Q_{!\mathcal{A}}$ such that $q \in \pi_2(S)$. If there exists such an S where furthermore (x, S) belongs to an infinite play of τ, then this S is unique and we choose that one. Otherwise, by Comprehension for HF-Sets (Remark 3.34), we define an ad hoc $S \in Q_{!\mathcal{A}}$ with $q \in \pi_2(S)$.

  Let now $(S, \gamma_{(x,S)}) := \tau(x, S)$. By definition of $\gamma_{(x,S)}$ there is some

$$\begin{array}{rcccl} f & : & \pi_2(S) & \longrightarrow & \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}}) \\ & & q & \longmapsto & \gamma_q \in \partial_{\mathcal{A}}(q, F(x)) \end{array}$$

  such that $\gamma_{(x,S)} = \{(d, S'_d) \mid d \in \mathscr{D} \wedge S'_d \neq \varnothing\}$ where each S'_d is as in (9.2). Consider the $\prec$-least such f. We let

$$\sigma(x, q) \quad := \quad (q, f(q))$$

It remains to show that σ is winning. To this end, given an infinite play T of σ, we will define an infinite play V of τ such that:

$$\mathrm{Trace}(T, V)$$

Since τ is assumed to be winning, thanks to Remarks 6.7 and 9.3, this will imply that σ is also winning. Assume $\mathrm{Play}(\sigma, \iota, T)$. We define V using the Recursion Theorem (Proposition 4.6). Let $\varphi(V, v)$ be a FSO-formula stating that:

- either $v = \iota_!$,
- or $v = (x, \tau(x, S))$ with $(x, S) \in V$,
- or $v = (\mathsf{S}_d(x), S'_d)$ and
  - for some $q' \in Q_{\mathcal{A}}$ we have $(\mathsf{S}_d(x), q') \in T$,
  - and for some $S \in Q_{!\mathcal{A}}$, we have $(x, S) \in V$ and $\tau(x, S) = (S, \gamma_{(x,S)})$ with $(d, S'_d) \in \gamma_{(x,S)}$.

Note that $\varphi(V, v)$ indeed satisfies the assumptions of the Recursion Theorem (Proposition 4.6), since

- in the second clause we always have $(x, S) \lhd_{\mathcal{G}(!\mathcal{A})} (x, \tau(x, S))$, and $\tau(x, S)$ is uniquely determined from (x, S);
- in the last clause, we always have $(x, S) \lhd_{\mathcal{G}(!\mathcal{A})} (\mathsf{S}_d(x), S'_d)$.

Note also that since T is a play of σ, there is at most one $d \in \mathscr{D}$ such that $(\mathsf{S}_d(x), q) \in T$ for some $q \in Q_{\mathcal{A}}$, and S'_d is uniquely determined from d and $\tau(x, S)$ by construction of $!\mathcal{A}$. So by the Recursion Theorem (Proposition 4.6) we indeed let $V : \mathcal{G}(!\mathcal{A})$ $\mathtt{to}$ 2 be unique such that

$$\big(\forall v \in \mathcal{G}(!\mathcal{A})\big)\Big[v \in V \quad\Leftrightarrow\quad \varphi(V, v)\Big]$$

We begin with a series of easy claims on V.

Claim 9.8.1. *For every $x \in \mathscr{D}^*$, there is at most one $S \in Q_{!\mathcal{A}}$ such that $(x, S) \in V$.*

Proof of Claim 9.8.1. We apply the Induction Axiom of FSO (§3.4.2). The property holds for $\dot{\varepsilon}$, since $(\dot{\varepsilon}, S) \in V$ implies $S = q^{\iota}_{!\mathcal{A}}$ by definition of V. Now assume the property for x and let us show it for $\mathsf{S}_d(x)$. So assume $(\mathsf{S}_d(x), S'_d), (\mathsf{S}_d(x), \tilde{S}'_d) \in V$. By definition of V, there are $(x, S), (x, \tilde{S}) \in V$ such that $(d, S'_d) \in \gamma_{(x,S)}$ and $(d, \tilde{S}'_d) \in \gamma_{(x,\tilde{S})}$ where $(S, \gamma_{(x,S)}) = \tau(x, S)$ and $(\tilde{S}, \gamma_{(x,\tilde{S})}) = \tau(x, \tilde{S})$. But by induction hypothesis we get $S = \tilde{S}$, which implies $\gamma_{(x,S)} = \gamma_{(x,\tilde{S})}$. This in turn implies $S'_d = \tilde{S}'_d$ by construction of $!\mathcal{A}$. ■

Claim 9.8.2. *For every $u \in V$, the set $\{v \in V \mid v \trianglelefteq_{\mathcal{G}(!\mathcal{A})} u\}$ is linearly ordered w.r.t. $\longrightarrow^*_\tau$.*

Proof of Claim 9.8.2. We reason by $\triangleleft$-Induction (Theorem 4.5). So let $u \in V$ be such that the property holds for all $w \triangleleft_{\mathcal{G}(!\mathcal{A})} u$.

Assume first that $u \in V_{\mathsf{O}}$. In this case, we must have $u = (x, \tau(x, S))$ with $(x, S) \in V$. By induction hypothesis, the set $\{v \in V \mid v \trianglelefteq_{\mathcal{G}(!\mathcal{A})} (x, S)\}$ is linearly ordered w.r.t. $\longrightarrow^*_\tau$. On the other hand, it follows from Claim 9.8.1 that (x, S) is the only immediate $\longrightarrow_\tau$-predecessor of u in V. Since $(x, S) \longrightarrow_\tau u$, we get the result by Proposition 5.6.

Assume now that $u \in V_{\mathsf{P}}$. If $u = \iota_!$ then the result is trivial. Otherwise, u is of the form $(\mathsf{S}_d(x), S'_d)$ and its membership to V is given by the last clause defining V. Let S be such that $(x, S) \in V$ and such that $\tau(x, S) = (S, \gamma_{(x,S)})$ with $(d, S'_d) \in \gamma_{(x,S)}$. Since $(x, \tau(x, S)) \longrightarrow_\tau u$ with $(x, \tau(x, S)) \in V$, by induction hypothesis the set $\{v \mid v \trianglelefteq_{\mathcal{G}(!\mathcal{A})} (x, \tau(x, S))\}$ is linearly ordered w.r.t. $\longrightarrow^*_\tau$. In order to obtain the result for $\{v \mid v \trianglelefteq_{\mathcal{G}(!\mathcal{A})} (\mathsf{S}_d(x), S'_d)\}$ we need to show that $(x, \tau(x, S))$ is the unique immediate $\longrightarrow_\tau$-predecessor of $(\mathsf{S}_d(x), S'_d)$ in V. But if $(x, \tau(x, \tilde{S})) \in V$ then we should have $(x, \tilde{S}) \in V$, so that $\tilde{S} = S$ by Claim 9.8.1. ■

Claim 9.8.3. *For every $u \in V$, there is an infinite play U of τ such that:*

$$\big(\forall v \trianglelefteq_{\mathcal{G}(!\mathcal{A})} u\big)\Big(v \in V \quad\Leftrightarrow\quad u \in U\Big)$$

Proof of Claim 9.8.3. Let $u \in V$. First, by Lemma 5.11 there is an infinite play U_0 in the game $\mathcal{G}(!\mathcal{A})\restriction\{\tau\}$ such that $u \in U_0$ and $u \longrightarrow^*_\tau v$ for all $v \in U_0$. By Comprehension for Product Types (Theorem 3.33) we let

$$U \quad := \quad U_0 \cup \{v \in V \mid v \trianglelefteq_{\mathcal{G}(!\mathcal{A})} u\}$$

We then get $\mathrm{Play}(\tau, \iota_!, U)$ from Claim 9.8.2 and $\mathrm{Play}(\tau, u, U_0)$. ■

Claim 9.8.4. *Let $(x, S) \in V$, and assume $(x, q), (\mathsf{S}_d(x), q') \in T$ with $q \in \pi_2(S)$. Then there is some $S'_d \in Q_{!\mathcal{A}}$ such that $(\mathsf{S}_d(x), S'_d) \in V$ and $(q, q') \in S'_d$. Moreover, we have $(d, S'_d) \in \gamma_{(x,S)}$ for $(S, \gamma_{(x,S)}) = \tau(x, S)$.*

Proof of Claim 9.8.4. Since T is a play of σ, we have $(d, q') \in \gamma$ for $(q, \gamma) = \sigma(x, q)$. Moreover, by Claim 9.8.3, (x, S) belongs to an infinite play of τ. Since $q \in \pi_2(S)$, by definition of σ this implies that there is some S'_d such that $(d, S'_d) \in \gamma_{(x,S)}$ for $(S, \gamma_{(x,S)}) = \tau(x, S)$ and $(q, q') \in S'_d$. We then obtain $(\mathsf{S}_d(x), S'_d) \in V$ by definition of V. ■

We now proceed to show:

$$\mathrm{Play}(\tau,\, \iota_!,\, V) \quad \wedge \quad \mathrm{Trace}(T, V)$$

We begin with $\mathrm{Trace}(T, V)$. First, we have $\mathrm{Path}(\mathcal{G}(\mathcal{A}), \iota, T)$ since T is a play of σ. Moreover

Claim 9.8.5.

$$\big(\forall (x,q) \in T_{\mathsf{P}}\big)\big(\exists S \in Q_{!\mathcal{A}}\big)\Big((x,S) \in V_{\mathsf{P}} \;\; \wedge \;\; q \in \pi_2(S)\Big)$$

Proof of Claim 9.8.5. Using the Induction Axiom of FSO (§3.4.2), we show

$$(\forall x)\big(\forall q \in Q_{\mathcal{A}}\big)\Big((x,q) \in T_{\mathsf{P}} \;\; \Rightarrow \;\; \big(\exists S \in Q_{!\mathcal{A}}\big)\Big[(x,S) \in V_{\mathsf{P}} \; \wedge \; q \in \pi_2(S)\Big]\Big)$$

For the base case $x = \dot{\varepsilon}$, if $(x,q) \in T$ then we must have $q = q^{\iota}_{\mathcal{A}}$, so $q \in \pi_2(q^{\iota}_{!\mathcal{A}})$. Assume now the property for x, and consider $\mathsf{S}_d(x)$ and $q, q' \in Q_{\mathcal{A}}$ such that $(x,q) \in T$ and $(\mathsf{S}_d(x), q') \in T$. Furthermore, by induction hypothesis, let $S \in Q_{!\mathcal{A}}$ such that $(x,S) \in V$ and $q \in \pi_2(S)$. By Claim 9.8.4, we then get $(\mathsf{S}_d(x), S'_d) \in V$ for some $S'_d \in Q_{!\mathcal{A}}$ with $q' \in \pi_2(S'_d)$. ■

We can now show the last required property for $\mathrm{Trace}(T, V)$, namely:

Claim 9.8.6.

$$\big(\forall (x,q),(y,q') \in T_{\mathsf{P}}\big)\big(\forall S \in Q_{!\mathcal{A}}\big)\Big[(x,q) \lhd^{\mathsf{P};\mathsf{O}}_{\mathcal{G}(\mathcal{A})} (y,q') \;\; \Rightarrow \;\; (y,S) \in V_{\mathsf{P}} \;\; \Rightarrow \;\; (q,q') \in S\Big]$$

Proof of Claim 9.8.6. Let $(x,q),(y,q') \in T$ and $S' \in Q_{!\mathcal{A}}$ such that $(x,q) \lhd^{\mathsf{P};\mathsf{O}}_{\mathcal{G}(\mathcal{A})} (y,q')$ and $(y,S') \in V$. Then by definition of $\lhd_{\mathcal{G}(\mathcal{A})}$ we must have $y = \mathsf{S}_d(x)$ for some $d \in \mathscr{D}$. Moreover, by Claim 9.8.5 there is some $S \in Q_{!\mathcal{A}}$ such that $(x,S) \in V$ and $q \in \pi_2(S)$. By Claim 9.8.4, we then have $(\mathsf{S}_d(x), S'_d) \in V$ for some $S'_d \in Q_{!\mathcal{A}}$ with $(q,q') \in S'_d$. It follows from Claim 9.8.1 that $S' = S'_d$ so that $(q,q') \in S'$ and we are done. ■

We now turn to showing $\mathrm{Play}(\tau, \iota_!, V)$. Since $\iota_! \in V$, thanks to Proposition 5.9 it remains to show:

$$\left\{\begin{array}{ll} & (\forall u \in V)\big(\iota_! \longrightarrow^*_\tau u\big) \\ \wedge & (\forall u \in V)(\exists! v \in V)\big(u \longrightarrow_\tau v\big) \\ \wedge & (\forall v \in V)\Big(v \neq \iota_! \;\; \Rightarrow \;\; (\exists u \in V)\big(u \longrightarrow_\tau v\big)\Big) \end{array}\right.$$

First, we easily have:

Claim 9.8.7.

$$(\forall v \in V)\Big(v \neq \iota_! \;\; \Rightarrow \;\; (\exists u \in V)\big(u \underset{\tau}{\longrightarrow} v\big)\Big)$$

Proof of Claim 9.8.7. The result follows from Claim 9.8.3, but it can be proved directly, without the inductions underlying Claim 9.8.3. Indeed, if $v = (x, \tau(x,S))$, with $(x,S) \in V$, then the result directly follows from the definitions of V and of the game $\mathcal{G}(!\mathcal{A})\restriction\{\tau\}$. Otherwise, we have $v = (\mathsf{S}_d(x), S'_d)$, and there is $(x,S) \in V$ such that $\tau(x,S) = (S, \gamma_{(x,S)})$ with $(d, S'_d) \in \gamma_{(x,S)}$. But $(x,S) \in V$ implies $(x, \tau(x,S)) \in V$, and again the result directly follows from the definition of $\mathcal{G}(!\mathcal{A})\restriction\{\tau\}$. ■

It then easily follows that:

Claim 9.8.8.

$$(\forall u \in V)\left(\iota_! \xrightarrow[\tau]{*} u\right)$$

Proof of Claim 9.8.8. First, we have $\iota_! \in V$ by definition of V. Moreover, given $u \in V$ we have either $\iota_! \longrightarrow^*_\tau u$ or $u \longrightarrow^*_\tau \iota_!$ by Claim 9.8.2. The result then follows from Proposition 5.6. ■

It remains to show:

$$(\forall u \in V)(\exists! v \in V)\left(u \underset{\tau}{\longrightarrow} v\right) \tag{9.5}$$

To this end, we first show:

Claim 9.8.9.

$$\big(\forall (x, S) \in V_{\mathsf{P}}\big)\big(\exists q \in Q_{\mathcal{A}}\big)\Big((x, q) \in T_{\mathsf{P}} \;\; \wedge \;\; q \in \pi_2(S)\Big)$$

Proof of Claim 9.8.9. Using the Induction Axiom of FSO (§3.4.2), we show

$$(\forall x)(\forall S \in Q_{!\mathcal{A}})\Big((x, S) \in V_{\mathsf{P}} \;\; \Rightarrow \;\; (\exists q \in Q_{\mathcal{A}})\big((x, q) \in T_{\mathsf{P}} \;\; \wedge \;\; q \in \pi_2(S)\big)\Big)$$

For the base case $x = \dot{\varepsilon}$, if $(x, S) \in V$ then we must have $S = q^{\iota}_{!\mathcal{A}}$. Then we are done since $\iota \in T$ and $q^{\iota}_{\mathcal{A}} \in \pi_2(S)$. Assume the property for x, and consider $\mathsf{S}_d(x)$ and $S, S' \in Q_{!\mathcal{A}}$ such that $(x, S), (\mathsf{S}_d(x), S') \in V$. Furthermore, by induction hypothesis, let $q \in Q_{\mathcal{A}}$ such that $(x, q) \in T$ and $q \in \pi_2(S)$. By definition of V, we have $(\mathsf{S}_d(x), q') \in T$ for some $q' \in Q_{\mathcal{A}}$. It then follows from Claim 9.8.4 that $q' \in \pi_2(S'_d)$ for some $S'_d \in Q_{!\mathcal{A}}$ such that $(\mathsf{S}_d(x), S'_d) \in V$. But Claim 9.8.1 implies $S' = S'_d$ so that $q' \in \pi_2(S')$ and we are done. ■

We can now prove (9.5).

Proof of (9.5). If $u = (x, S) \in V$, then $v = (x, \tau(x, S)) \in V$ and is unique such that $u \longrightarrow_{\tau} v$. Otherwise, $u = (x, \tau(x, S))$ for some $(x, S) \in V$, and we have to show that there are some unique $d \in \mathscr{D}$ and $S'_d \in Q_{!\mathcal{A}}$ such that $(\mathsf{S}_d(x), S'_d) \in V$. First, by Clim 9.8.9 there is some $q \in Q_{\mathcal{A}}$ such that $(x, q) \in T$ and $q \in \pi_2(S)$. Moreover, since T is a play of σ, with have $(\mathsf{S}_d(x), q') \in T$ for some unique $d \in \mathscr{D}$ and $q' \in Q_{\mathcal{A}}$. It then follows from Claim 9.8.4 that there is some $S'_d \in Q_{!\mathcal{A}}$ such that $(\mathsf{S}_d(x), S'_d) \in V$ and $u \longrightarrow_{\tau} (\mathsf{S}_d(x), S'_d)$. The uniqueness of S'_d follows from Claim 9.8.1. ■

This concludes the proof of Proposition 9.8. □

In the proof of Proposition 9.8 above, we have used Claim 9.8.9 in order to show that V is a play of σ. Let us state here for the record that this has a more general converse: Claim 9.8.9 holds whenever T is a trace in V for V a play in $\mathcal{G}(!\mathcal{A})$:

Lemma 9.9. *Given* $V : \mathcal{G}(!\mathcal{A})$ $\mathtt{to}$ 2 *and* $T : \mathcal{G}(\mathcal{A})$ $\mathtt{to}$ 2*, in* $\mathsf{FSO}_{\mathscr{D}}$ *we have*

$$\big(\forall (x, S) \in V_{\mathsf{P}}\big)\big(\exists q \in Q_{\mathcal{A}}\big)\Big[(x, q) \in T_{\mathsf{P}} \;\; \wedge \;\; q \in \pi_2(S)\Big]$$

whenever

$$\mathrm{Play}\big(\mathcal{G}(!\mathcal{A}),\, (\dot{\varepsilon}, q^{\iota}_{!\mathcal{A}}),\, V\big) \;\; \wedge \;\; \mathrm{Trace}(T, V)$$

Proof. Using the Induction Axiom of FSO (§3.4.2), we show

$$(\forall x)\big(\forall S \in Q_{!\mathcal{A}}\big)\Big((x, S) \in V_{\mathsf{P}} \;\; \Rightarrow \;\; (\exists q \in Q_{\mathcal{A}})\Big[(x, q) \in T_{\mathsf{P}} \;\; \wedge \;\; q \in \pi_2(S)\Big]\Big)$$

For the base case $x = \dot{\varepsilon}$, if $(x, S) \in V$, since $\mathrm{Play}(\mathcal{G}(!\mathcal{A}), (\dot{\varepsilon}, q^{\iota}_{!\mathcal{A}}), V)$ we must have $S = q^{\iota}_{!\mathcal{A}}$, so $q^{\iota}_{\mathcal{A}} \in \pi_2(S)$. Assume now the property for x, and consider $d \in \mathscr{D}$ and $S, S' \in Q_{!\mathcal{A}}$ such that $(x, S) \in V$ and $(\mathsf{S}_d(x), S') \in V$. Furthermore, by induction hypothesis, let $q \in Q_{\mathcal{A}}$ such that $(x, q) \in T$ and $q \in \pi_2(S)$. It follows from $\mathrm{Path}(\mathcal{G}(\mathcal{A}), (\dot{\varepsilon}, q^{\iota}_{\mathcal{A}}), T)$ that $(\mathsf{S}_{d'}(x), q') \in T$ for some $d' \in \mathscr{D}$ and some $q' \in Q_{\mathcal{A}}$. Moreover, $\mathrm{Trace}(T, V)$ implies $q' \in \pi_2(S'')$ for some S'' such that $(\mathsf{S}_{d'}(x), S'') \in V$. But $\mathrm{Play}(\mathcal{G}(!\mathcal{A}), (\dot{\varepsilon}, q^{\iota}_{!\mathcal{A}}), V)$ implies $d' = d$ and $S'' = S'$ and we are done. □

9.2. **Reformulating the Acceptance Condition of $!\mathcal{A}$.** For an automaton $\mathcal{A}$ which we now assume to be HF-closed (in the sense of Definition 6.8), we are going to formulate the FSO-formula $\mathcal{W}_{!\mathcal{A}}$ as a parity condition, which will allow us to obtain a parity automaton $\mathrm{ND}(\mathcal{A})$ in §9.3. In order to obtain a parity condition from $\mathcal{W}_{!\mathcal{A}}$ we note (following [Wal02]) that (when read in the standard model) it defines an ω-regular condition, which can thus by McNaughton's Theorem [McN66] (see also e.g. [Tho97, PP04]) be formulated with a deterministic parity automaton on ω-words. We are actually not going to formalize McNaughton's Theorem in our setting. Rather, we will apply Proposition 7.8, which allows us to import in FSO any true FSO-formula on the infinite paths of $\mathscr{D}^*$. Our way to the application of Proposition 7.8 proceeds with constructions similar to some of those in the proof of Theorem 8.22.

Consider some $V : \mathcal{G}(!\mathcal{A})$ $\mathtt{to}$ 2 such that:

$$\mathrm{Play}(\mathcal{G}(!\mathcal{A}),\, (\dot\varepsilon, q^\iota_{!\mathcal{A}}),\, V)$$

By Comprehension for Product Types (Theorem 3.33), let $|V| : \mathscr{D}^*$ $\mathtt{to}$ 2 be the set of all $x \in \mathscr{D}^*$ such that $(x, S) \in V$ for some $S \in Q_{!\mathcal{A}}$. Note that $\mathrm{TPath}(|V|)$ (recall that TPath is defined in (7.3)). Furthermore, by HF-Bounded Choice for Functions (§3.4.5), let $\tilde V : |V|$ $\mathtt{to}$ $Q_{!\mathcal{A}}$ take $x \in |V|$ to the unique $S \in Q_{!\mathcal{A}}$ such that $(x, S) \in V$.

In FSO we have that $\mathcal{W}_{!\mathcal{A}}$ is equivalent to the following formula $\mathcal{W}[\dot<]^{|V|}_{!\mathcal{A}}(\tilde V)$:

$$\Big(\forall \tilde T : |V| \ \mathtt{to}\ Q_{\mathcal{A}}\Big)\Big[\mathrm{Trace}[\dot<]^{|V|}(\tilde T, \tilde V) \ \ \Rightarrow\ \ \mathrm{Par}[\dot<]^{|V|}(C_{\mathcal{A}}, n_{\mathcal{A}}, \tilde T)\Big]$$

where

- the formula $\mathrm{Trace}[\dot<]^{|V|}(\tilde T, \tilde V)$ is

$$\left\{\begin{array}{ll} & (\forall x \in |V|)\Big[\tilde T(x) \in \pi_2(\tilde V(x))\Big] \\ \wedge & (\forall x, y \in |V|)\Big[\mathsf{S}_{\dot<}(x,y) \ \ \Rightarrow\ \ (\tilde T(x), \tilde T(y)) \in \tilde V(y)\Big] \end{array}\right.$$

  with

$$\mathsf{S}_{\dot<}(x,y) \quad := \quad x \mathrel{\dot<} y \ \wedge\ \neg\exists z\,(x \mathrel{\dot<} z \mathrel{\dot<} y)$$

- and, for $C : Q_{\mathcal{A}}$ $\mathtt{to}$ $[0,n]$, the formula $\mathrm{Par}[\dot<]^{|V|}(C, n, \tilde T)$ is (using Convention 5.18):

$$\big(\exists m \in \dot{\mathsf{even}}(n)\big)\left[\begin{array}{ll} & (\forall x \in |V|)(\exists y \in |V|)\big(x \mathrel{\dot<} y \ \wedge\ C(\tilde T(y)) \doteq m\big) \\ \wedge & (\exists x \in |V|)(\forall y \in |V|)\big(x \mathrel{\dot<} y \ \Rightarrow\ C(\tilde T(y)) \geq m\big) \end{array}\right]$$

Let us first note the following simple property. Recall from Definition 6.6 that $C_{\mathcal{A}} : Q_{\mathcal{A}}$ $\mathtt{to}$ $[0, n_{\mathcal{A}}]$ is a coloring of the states of $\mathcal{A}$, while $\hat C_{\mathcal{A}}$ colors the positions of $\mathcal{G}(\mathcal{A})$, by taking for P-positions (x, q) the color given by $C_{\mathcal{A}}$ to q and for O-positions the maximal color $n_{\mathcal{A}}$.

Lemma 9.10. *Assume given V and $|V|$ as above. Let $T : \mathcal{G}(\mathcal{A})$ $\mathtt{to}$ 2 and $\tilde T : |V|$ $\mathtt{to}$ $Q_{\mathcal{A}}$ such that*

$$\mathrm{Path}\big(\mathcal{G}(\mathcal{A}), (\dot\varepsilon, q^\iota_{\mathcal{A}}), T\big) \quad \wedge \quad \big(\forall (x,q) \in \mathcal{G}(\mathcal{A})_{\mathsf{P}}\big)\Big[(x,q) \in T \ \ \Leftrightarrow\ \ \Big(x \in |V| \ \wedge\ \tilde T(x) = q\Big)\Big]$$

Then:

$$\mathrm{Par}\big(\mathcal{A}, \hat C_{\mathcal{A}}, n_{\mathcal{A}}, T\big) \ \ \Leftrightarrow\ \ \mathrm{Par}[\dot<]^{|V|}\big(C_{\mathcal{A}}, n_{\mathcal{A}}, \tilde T\big)$$

Lemma 9.11. *Given V, $|V|$ and $\tilde V$ as above, $\mathsf{FSO}_{\mathscr{D}}$ proves that*

$$\mathcal{W}_{!\mathcal{A}}(V) \ \ \Leftrightarrow\ \ \mathcal{W}[\dot<]^{|V|}_{!\mathcal{A}}(\tilde V)$$

Proof. Recall that the formula $\mathcal{W}_{!\mathcal{A}}$ requires no condition w.r.t. the *transitions* of $\mathcal{G}(\mathcal{A})$. We proceed as follows:

- Assume first $\mathcal{W}_{!\mathcal{A}}(V)$, and let $\tilde{T} : |V| \ \mathtt{to}\ Q_{\mathcal{A}}$ such that $\mathrm{Trace}[\dot{<}]^{|V|}(\tilde{T}, \tilde{V})$. Using Remark 3.17, let $\prec$ be a well-order on $\mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}})$. By Comprehension for Product Types (Theorem 3.33), let $T : \mathcal{G}(\mathcal{A}) \ \mathtt{to}\ 2$ such that for all $(x, q) \in \mathcal{G}(\mathcal{A})_{\mathsf{P}}$, we have

$$(x,q) \in T_{\mathsf{P}} \quad \Leftrightarrow \quad \Big(x \in |V| \ \wedge \ \tilde{T}(x) = q\Big)$$

and such that $(x, \gamma) \in T_{\mathsf{O}}$ iff $\gamma \in \mathcal{P}_*(\mathscr{D} \times Q_{\mathcal{A}})$ is $\preceq$-minimal such that $\tilde{T}(\mathsf{S}_d(x)) = q$ for some $(d, q) \in \gamma$. Since V is an infinite play of $\mathcal{G}(!\mathcal{A})$ from $(\dot{\varepsilon}, q^{\iota}_{!\mathcal{A}})$, we may conclude by Lemma 9.10 as soon as we show:

$$\mathrm{Trace}(T, V)$$

We obviously have $\mathrm{Path}(\mathcal{G}(\mathcal{A}), T)$ as well as $(\dot{\varepsilon}, q^{\iota}_{\mathcal{A}}) \trianglelefteq_{\mathcal{G}(\mathcal{A})} u$ for all $u \in T$. Moreover we have:

$$\big(\forall (x,q) \in T_{\mathsf{P}}\big)\big(\exists S \in Q_{!\mathcal{A}}\big)\Big[(x,S) \in V_{\mathsf{P}} \ \ \wedge \ \ q \in \pi_2(S)\Big]$$

To see this, let $(x, q) \in T$, so that $\tilde{T}(x) = q$. So by assumption we have $q \in \pi_2(\tilde{V}(x))$, and we are done since $(x, \tilde{V}(x)) \in V(x)$.

Finally we have

$$\big(\forall (x,q),(y,q') \in T_{\mathsf{P}}\big)\big(\forall S \in Q_{!\mathcal{A}}\big)\Big[(x,q) \lhd^{\mathsf{P};\mathsf{O}}_{\mathcal{G}(\mathcal{A})} (y,q') \ \ \Rightarrow \ \ (y,S) \in V_{\mathsf{P}} \ \ \Rightarrow \ \ (q,q') \in S\Big]$$

To see this, given $(x, q), (y, q') \in T$ such that $(x, q) \lhd^{\mathsf{P};\mathsf{O}}_{\mathcal{G}(\mathcal{A})} (y, q')$ we necessarily have $\mathsf{S}_{\dot{<}}(x, y)$, so that $(q, q') \in \tilde{V}(y)$ since $\tilde{T}(x) = q$ and $\tilde{T}(y) = q'$. But then we are done since $\tilde{V}(y)$ is the unique $S \in Q_{!\mathcal{A}}$ such that $(y, S) \in V$.

- Conversely, assume $\mathcal{W}[\dot{<}]^{|V|}_{!\mathcal{A}}(\tilde{V})$ and let $T : \mathcal{G}(\mathcal{A}) \ \mathtt{to}\ 2$ such that $\mathrm{Trace}(T, V)$. Since V is an infinite play of $\mathcal{G}(!\mathcal{A})$ from $(\dot{\varepsilon}, q^{\iota}_{!\mathcal{A}})$, Lemma 9.9 implies:

$$\big(\forall (x,S) \in V_{\mathsf{P}}\big)\big(\exists q \in Q_{\mathcal{A}}\big)\Big[(x,q) \in T_{\mathsf{P}} \ \ \wedge \ \ q \in \pi_2(S)\Big]$$

It follows that for all $x \in |V|$ there is $q \in Q_{\mathcal{A}}$ such that $(x, q) \in T$, and this defines $\tilde{T} : |V| \ \mathtt{to}\ Q_{\mathcal{A}}$ by HF-Bounded Choice for Functions (§3.4.5). Note that we have:

$$(\forall x)(\forall q \in Q_{\mathcal{A}})\ \Big[(x,q) \in T_{\mathsf{P}} \ \ \Leftrightarrow \ \ \Big(x \in |V| \ \wedge \ \tilde{T}(x) = q\Big)\Big]$$

We can then conclude by Lemma 9.10 as soon as we show:

$$\mathrm{Trace}[\dot{<}]^{|V|}(\tilde{T}, \tilde{V})$$

To see this, first, for all $x \in |V|$, we have $(x, \tilde{T}(x)) \in T$, so that $T(x) \in \pi_2(\tilde{V}(x))$ by definition of $\tilde{V}$. Moreover, given $x, y \in |V|$ with $\mathsf{S}_{\dot{<}}(x, y)$, we have

$$(x, \tilde{T}(x)) \lhd^{\mathsf{P};\mathsf{O}}_{\mathcal{G}(\mathcal{A})} (y, \tilde{T}(y))$$

so that $(\tilde{T}(x), \tilde{T}(y)) \in \tilde{V}(y)$ since $\mathrm{Trace}(T, V)$.

This concludes the proof of Lemma 9.11. □

We are now going to show that $\mathcal{W}[\dot{<}]^{|V|}_{!\mathcal{A}}(\tilde{V})$ is equivalent in FSO to a *parity* automaton on ω-words. This relies on McNaughton's Theorem [McN66] applied in the usual standard model $\mathfrak{N}$ of ω-words, and, via Proposition 7.8, on the completeness of $\mathsf{FSO}[\dot{<}]^{\omega}$. In order to apply Proposition 7.8, we rewrite $\mathcal{W}[\dot{<}]^{|V|}_{!\mathcal{A}}(\tilde{V})$ as the relativization to $|V|$ of the $\mathsf{FSO}[\dot{<}]^{\omega}$-formula

$$\mathcal{W}[\dot{<}]_{!\mathcal{A}}(\tilde{V}) \quad := \quad \big(\forall \tilde{T} : Q_{\mathcal{A}}\big)\ \Big[\mathrm{Trace}[\dot{<}](\tilde{V}, \tilde{T}) \ \ \Rightarrow \ \ \mathrm{Par}[\dot{<}](C_{\mathcal{A}}, n_{\mathcal{A}}, \tilde{T})\Big]$$

where $\mathrm{Par}[\dot{<}](C, n, \tilde{T})$ is the formula of Definition 8.18, and where

$$\mathrm{Trace}[\dot{<}](\tilde{V}, \tilde{T}) \quad := \quad \begin{cases} & (\forall x)\left[\tilde{T}(x) \in \pi_2(\tilde{V}(x))\right] \\ \wedge & (\forall x)(\forall y)\left[\mathsf{S}_{\dot{<}}(x,y) \;\; \Rightarrow \;\; (\tilde{T}(x), \tilde{T}(y)) \in \tilde{V}(y)\right] \end{cases}$$

Note that $\mathcal{W}[\dot{<}]^{|V|}_{!\mathcal{A}}(\tilde{V})$ is the relativization to $|V|$ of $\mathcal{W}[\dot{<}]_{!\mathcal{A}}(\tilde{V})$:

$$\mathcal{W}[\dot{<}]^{|V|}_{!\mathcal{A}}(\tilde{V}) \quad = \quad (\mathcal{W}[\dot{<}]_{!\mathcal{A}})^{|V|}(\tilde{V})$$

Since $\mathcal{A}$ is HF-closed, the formula $\mathcal{W}[\dot{<}]_{!\mathcal{A}}(\tilde{V})$ is also HF-closed, and we can look at it in the standard model $\mathfrak{N}$ of ω-words (see §7). By McNaughton's Theorem [McN66] (see also e.g. [Tho97, PP04]), there is a deterministic parity ω-word automaton $\mathcal{D} = (Q_{\mathcal{D}},\, q^{\iota}_{\mathcal{D}},\, \partial_{\mathcal{D}},\, c_{\mathcal{D}})$ over $Q_{!\mathcal{A}}$, which accepts $\tilde{V}$ exactly when:

$$\mathfrak{N} \models \mathcal{W}[\dot{<}]_{!\mathcal{A}}(\tilde{V})$$

It then follows that in $\mathfrak{N}$, for all $\tilde{V} : Q_{!\mathcal{A}}$, the formula $\mathcal{W}[\dot{<}]_{!\mathcal{A}}(\tilde{V})$ is equivalent to

$$\big(\forall \tilde{R} : Q_{\mathcal{D}}\big)\Bigg(\tilde{R}(\dot{\varepsilon}) = q^{\iota}_{\mathcal{D}} \;\; \Rightarrow \\ (\forall x)(\forall y)\left[\mathsf{S}_{\dot{<}}(x,y) \;\; \Rightarrow \;\; \tilde{R}(y) = \partial_{\mathcal{D}}\big(\tilde{R}(x), \tilde{V}(x)\big)\right] \;\; \Rightarrow \;\; \mathrm{Par}[\dot{<}]\big(C_{\mathcal{D}}, n_{\mathcal{D}}, \tilde{R}\big)\Bigg)$$

Proposition 7.8 then implies that FSO proves that for $\tilde{V} : |V| \;\mathtt{to}\; Q_{!\mathcal{A}}$, the formula $\mathcal{W}[\dot{<}]^{|V|}_{!\mathcal{A}}(\tilde{V})$ is equivalent to

$$\big(\forall \tilde{R} : |V| \;\mathtt{to}\; Q_{\mathcal{D}}\big)\Bigg(\tilde{R}(\dot{\varepsilon}) = q^{\iota}_{\mathcal{D}} \;\; \Rightarrow \\ (\forall x \in |V|)(\forall y \in |V|)\left[\mathsf{S}_{\dot{<}}(x,y) \;\; \Rightarrow \;\; \tilde{R}(y) = \partial_{\mathcal{D}}\big(\tilde{R}(x), \tilde{V}(x)\big)\right] \;\; \Rightarrow \\ \mathrm{Par}[\dot{<}]^{|V|}\big(C_{\mathcal{D}}, n_{\mathcal{D}}, \tilde{R}\big)\Bigg)$$

9.3. **Definition of the Parity Automaton** $\mathrm{ND}(\mathcal{A})$**.** Consider an alternating parity tree automaton $\mathcal{A} : \Sigma$ as in the beginning of §9, and assume it to be HF-closed. Let $!\mathcal{A} : \Sigma$ be defined as in §9.1. Moreover, let $\mathcal{D} : Q_{!\mathcal{A}}$ be the parity deterministic ω-word automaton of §9.2. We then let

$$\mathrm{ND}(\mathcal{A}) \quad := \quad (Q_{!\mathcal{A}} \times Q_{\mathcal{D}}\;,\; (q^{\iota}_{!\mathcal{A}}, q^{\iota}_{\mathcal{D}})\;,\; \partial_{\mathrm{ND}(\mathcal{A})}\;,\; C_{\mathrm{ND}(\mathcal{A})}\;,\; n_{\mathcal{D}})$$

where:

- the transition function

$$\partial_{\mathrm{ND}(\mathcal{A})} \quad : \quad (Q_{!\mathcal{A}} \times Q_{\mathcal{D}}) \times \Sigma \quad \longrightarrow \quad \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times (Q_{!\mathcal{A}} \times Q_{\mathcal{D}})))$$

  takes $((S,q), \mathtt{a})$ to the set of all $\gamma \in \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times (Q_{!\mathcal{A}} \times Q_{\mathcal{D}})))$ such that for some $\gamma_{!\mathcal{A}} \in \partial_{!\mathcal{A}}(S, \mathtt{a})$,

$$\gamma \quad = \quad \Big\{\big(d\;,\; \big(S'_d\;,\; \partial_{\mathcal{D}}(q,S)\big)\big) \;\;\mid\;\; (d, S'_d) \in \gamma_{!\mathcal{A}}\Big\}$$

- the coloring $C_{\mathrm{ND}(\mathcal{A})} : Q_{!\mathcal{A}} \times Q_{\mathcal{D}} \;\mathtt{to}\; [0, n_{\mathcal{D}}]$ takes (S,q) to $C_{\mathcal{D}}(q)$.

Note that $\mathrm{ND}(\mathcal{A}) : \Sigma$ is HF-closed by Remark 6.9. We shall now show that $\mathrm{ND}(\mathcal{A})$ is equivalent to $\mathcal{A}$, thus completing the proof of the Simulation Theorem 9.1. The proof is split into Propositions 9.13 and 9.14. As expected, we invoke Theorem 9.5, that $\mathsf{FSO}_{\mathscr{D}}$ proves the equivalence of $!\mathcal{A}$ and $\mathcal{A}$.

Convention 9.12. In Propositions 9.13 and 9.14, for fixed automata $!\mathcal{A}$ and $\mathrm{ND}(\mathcal{A})$, we let

$$\iota_! \;:=\; (\dot{\varepsilon}, q^{\iota}_{!\mathcal{A}}) \qquad \text{and} \qquad \iota_{\mathrm{ND}} \;:=\; (\dot{\varepsilon}, q^{\iota}_{\mathrm{ND}(\mathcal{A})})$$

Proposition 9.13. *Fix an* HF*-closed automaton* $\mathcal{A} : \Sigma$ *and consider* $\mathrm{ND}(\mathcal{A}) : \Sigma$ *as defined above. Then* $\mathsf{FSO}_{\mathscr{D}}$ *proves that for all* $F : \Sigma$*, if* $\mathcal{A}$ *accepts* F *then* $\mathrm{ND}(\mathcal{A})$ *accepts* F*.*

Proof. Thanks to Theorem 9.5, we are done if we show that $\mathrm{ND}(\mathcal{A})$ accepts F whenever $!\mathcal{A}$ accepts F. Let $\sigma : \mathcal{G}(!\mathcal{A}, F)_{\mathsf{P}} \;\mathtt{to}\; \mathsf{O}_{\mathcal{G}(!\mathcal{A},F)}$ be the winning P-strategy in $\mathcal{G}(!\mathcal{A}, F)$. We define a strategy $\tau : \mathcal{G}(\mathrm{ND}(\mathcal{A}), F)_{\mathsf{P}} \;\mathtt{to}\; \mathsf{O}_{\mathcal{G}(\mathrm{ND}(\mathcal{A}),F)}$ as follows.

- *Definition of* τ. By HF-Bounded Choice for Functions (§3.4.5), we let $\tau(x, (S, q_{\mathcal{D}}))$ be $((S, q_{\mathcal{D}}), \gamma)$, where $\gamma \in \mathcal{P}_*(\mathcal{P}_*(\mathscr{D} \times (Q_{!\mathcal{A}} \times Q_{\mathcal{D}})))$ is defined by Comprehension for HF-Sets (Remark 3.34) as the set of all $(d, (S'_d, \partial_{\mathcal{D}}(q_{\mathcal{D}}, S)))$ such that $(d, S'_d) \in \gamma_{!\mathcal{A}}$, where $\sigma(x, S) = (S, \gamma_{!\mathcal{A}})$.

It remains to show that τ is winning. So let T such that

$$\mathrm{Play}(\tau,\, \iota_{\mathrm{ND}},\, T)$$

By Comprehension for Product Types (Theorem 3.33), let $||U|| : \mathscr{D}^* \times Q_{\mathcal{A}} \;\mathtt{to}\; 2$ consist of the $(x, q_{!\mathcal{A}})$ for which there is $q_{\mathcal{D}} \in Q_{\mathcal{D}}$ such that $(x, (q_{!\mathcal{A}}, q_{\mathcal{D}})) \in T$. By HF-Bounded Choice for Functions (§3.4.5), now let $\tilde{U} : ||U|| \;\mathtt{to}\; Q_{\mathcal{D}}$ take $(x, q_{!\mathcal{A}}) \in ||U||$ to (the necessarily unique) $q_{\mathcal{D}}$ such that $(x, (q_{!\mathcal{A}}, q_{\mathcal{D}})) \in T$. We have:

$$\mathrm{Par}\big(\mathrm{ND}(\mathcal{A}), \hat{C}_{\mathrm{ND}(\mathcal{A})}, n_{\mathrm{ND}(\mathcal{A})}, T\big) \quad\Leftrightarrow\quad \mathrm{Par}[\dot{<}]^{||U||}\big(C_{\mathcal{D}}, n_{\mathcal{D}}, \tilde{U}\big)$$

It then follows from Lemma 9.11 that we are done if we show that $||U||$ is the set of all $(x, q_{\mathcal{A}}) \in V_{\mathsf{P}}$ for some $V : \mathcal{G}(!\mathcal{A}) \;\mathtt{to}\; 2$ such that:

$$\mathrm{Play}(\sigma, \iota_!, V)$$

But this is immediate from Comprehension for Product Types (Theorem 3.33) by letting V be the union of $||U||$ with the set of all $(x, \sigma(x, q_{!\mathcal{A}}))$ for $(x, q_{!\mathcal{A}}) \in ||U||$. $\square$

When proving that $\mathcal{L}(\mathrm{ND}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{A})$ in Proposition 9.14 below, in order to apply Proposition 9.8, we have to extract a P-strategy on $\mathcal{G}(!\mathcal{A}, F)$ from a P-strategy on $\mathcal{G}(\mathrm{ND}(\mathcal{A}), F)$. But $\mathrm{ND}(\mathcal{A})$ has more states than $!\mathcal{A}$, so we have to resort to Corollary 6.16, stating that in plays of strategies on *non-deterministic automata*, states are uniquely determined from tree positions.

Proposition 9.14. *Fix an* HF*-closed automaton* $\mathcal{A} : \Sigma$ *and consider* $\mathrm{ND}(\mathcal{A}) : \Sigma$ *as defined above. Then* $\mathsf{FSO}_{\mathscr{D}}$ *proves that for all* $F : \Sigma$*, if* $\mathrm{ND}(\mathcal{A})$ *accepts* F *then* $\mathcal{A}$ *accepts* F*.*

Proof. Thanks to Theorem 9.5, we are done if we show $!\mathcal{A}$ accepts F whenever $\mathrm{ND}(\mathcal{A})$ accepts F. Let $\tau : \mathcal{G}(\mathrm{ND}(\mathcal{A}), F)_{\mathsf{P}} \;\mathtt{to}\; \mathsf{O}_{\mathcal{G}}(\mathrm{ND}(\mathcal{A}), F)$ be the winning P-strategy in $\mathcal{G}(\mathrm{ND}(\mathcal{A}), F)$. We are going to define a winning strategy $\sigma : \mathcal{G}(!\mathcal{A}, F)_{\mathsf{P}} \;\mathtt{to}\; \mathsf{O}_{\mathcal{G}}(!\mathcal{A}, F)$ in $\mathcal{G}(!\mathcal{A}, F)$. Note that $\mathrm{ND}(\mathcal{A})$ has more states than $!\mathcal{A}$ and that,

$$\tau : \mathscr{D}^* \times (Q_{!\mathcal{A}} \times Q_{\mathcal{D}}) \;\mathtt{to}\; 2$$

whereas we need to define:

$$\sigma : \mathscr{D}^* \times Q_{!\mathcal{A}} \;\mathtt{to}\; 2$$

As mentioned, we resort to Corollary 6.16. The strategy σ is defined by HF-Bounded Choice for Functions (§3.4.5) as follows. Let $(x, S) \in \mathscr{D}^* \times Q_{!\mathcal{A}}$.

- Assume that there is a play U of τ such that

$$(\exists q_{\mathcal{D}} \in Q_{\mathcal{D}})\Big((x, (S, q_{\mathcal{D}})) \in U\Big)$$

Then it follows from Corollary 6.16 there is a unique $q_{\mathcal{D}}$ such that

$$(\exists U)\Big(\mathrm{Play}(\tau,\, \iota_{\mathrm{ND}},\, U) \;\;\wedge\;\; (x, (S, q_{\mathcal{D}})) \in U\Big)$$

In this case, we let $\sigma(x, S)$ be $(S, \gamma_{!\mathcal{A}})$ where, by Comprehension for HF-Sets (Remark 3.34), $\gamma_{!\mathcal{A}}$ is the set of all (d, S'_d) such that there is some $q'_{\mathcal{D}} \in Q_{\mathcal{D}}$ with $(d, (S'_d, q'_{\mathcal{D}})) \in \gamma_{\mathrm{ND}(\mathcal{A})}$ for $((S, q_{\mathcal{D}}), \gamma_{\mathrm{ND}(\mathcal{A})}) = \tau(x, (S, q_{\mathcal{D}}))$.

- Otherwise, we let $\sigma(x, S)$ be $(S, \gamma_{!\mathcal{A}})$ where, by Comprehension for HF-Sets (Remark 3.34), $\gamma_{!\mathcal{A}}$ is the set of all (d, S'_d) such that there is some $q_{\mathcal{D}} \in Q_{\mathcal{D}}$ with $(d, (S'_d, q_{\mathcal{D}})) \in \gamma_{\mathrm{ND}(\mathcal{A})}$ for $((S, q^{\iota}_{\mathcal{D}}), \gamma_{\mathrm{ND}(\mathcal{A})}) = \tau(x, (S, q^{\iota}_{\mathcal{D}}))$.

We are now going to show that σ is winning. To this end, consider an infinite play of σ, that is some $V : \mathcal{G}(!\mathcal{A})$ $\mathfrak{to}$ 2 such that

$$\mathrm{Play}(\sigma,\, \iota_{!},\, V)$$

We are going to define an infinite play of τ, that is some $U : \mathcal{G}(\mathrm{ND}(\mathcal{A}))$ $\mathfrak{to}$ 2 with

$$\mathrm{Play}(\tau,\, \iota_{\mathrm{ND}},\, U)$$

First, note that we are done if U satisfies the following property:

$$(\forall (x, S) \in V)(\exists q_{\mathcal{D}} \in Q_{\mathcal{D}})\big((x, (S, q_{\mathcal{D}})) \in U\big) \tag{9.6}$$

Indeed, by Comprehension (Theorem 3.33), let $|V| : \mathscr{D}^*$ $\mathfrak{to}$ 2 be the set of all $x \in \mathscr{D}^*$ such that $(x, S) \in V$ for some (necessarily unique) $S \in Q_{!\mathcal{A}}$. Moreover, by HF-Bounded Choice for Functions (§3.4.5), let $\tilde{V} : |V|$ $\mathfrak{to}$ $Q_{!\mathcal{A}}$ take $x \in |V|$ to the unique $S \in Q_{!\mathcal{A}}$ such that $(x, S) \in V$. By HF-Bounded Choice for Functions (§3.4.5), let now $\tilde{U} : |V|$ $\mathfrak{to}$ $Q_{\mathcal{D}}$ take $x \in |V|$ to the unique $q_{\mathcal{D}} \in Q_{\mathcal{D}}$ such that $(x, (\tilde{V}(x), q_{\mathcal{D}})) \in U$. Since $\mathrm{Par}(\mathcal{G}(\mathrm{ND}(\mathcal{A})), \hat{C}_{\mathrm{ND}(\mathcal{A})}, n_{\mathrm{ND}(\mathcal{A})}, U)$, we have $\mathrm{Par}[\dot{<}]^{|V|}(C_{\mathcal{D}}, n_{\mathcal{D}}, \tilde{U})$, so that $\mathcal{W}[\dot{<}]^{|V|}_{!\mathcal{A}}(\tilde{V})$ and we conclude by Lemma 9.11.

We now define an infinite play U of τ satisfying (9.6), for which we appeal to the Recursion Theorem (Proposition 4.6). Let $\varphi(U, u)$ be a FSO formula stating the disjunction of the following:

- $u = \iota_{\mathrm{ND}}$; or
- $u = (x, \tau(x, (S, q_{\mathcal{D}})))$ with $(x, (S, q_{\mathcal{D}})) \in U$; or
- $u = (\mathsf{S}_d(x), (S'_d, q'_{\mathcal{D}}))$, where
  - $(\mathsf{S}_d(x), S'_d) \in V$, and
  - for some $q_{\mathcal{D}} \in Q_{\mathcal{D}}$ and some $S \in Q_{!\mathcal{A}}$ such that $(x, S) \in V$ and $(x, (S, q_{\mathcal{D}})) \in U$, we have $q'_{\mathcal{D}} = \partial_{\mathcal{D}}(q_{\mathcal{D}}, S)$.

By the Recursion Theorem (Proposition 4.6) we let $U : \mathcal{G}(\mathrm{ND}(\mathcal{A}))$ $\mathfrak{to}$ 2 be unique such that:

$$\big(\forall u \in \mathcal{G}(\mathrm{ND}(\mathcal{A}))\big)\Big[u \in U \quad\Leftrightarrow\quad \varphi(U, u)\Big]$$

We need to show (9.6) and:

$$\mathrm{Play}(\tau,\, \iota_{\mathrm{ND}},\, U)$$

We first show that U is a play of τ. Since $\iota_{\mathrm{ND}} \in U$, by Proposition 5.9 it suffices to show:

$$\left\{\begin{array}{ll} & (\forall u \in U)\big(\iota_{\mathrm{ND}} \longrightarrow^*_\tau u\big) \\ \wedge & (\forall u \in U)(\exists! v \in U)\big(u \longrightarrow_\tau v\big) \\ \wedge & (\forall v \in U)\Big(v \neq \iota_{\mathrm{ND}} \quad\Rightarrow\quad (\exists u \in U)\big[u \longrightarrow_\tau v\big]\Big) \end{array}\right.$$

We proceed similarly to the proof of Proposition 9.8. First, we prove:

Claim 9.14.1.
$$(\forall v \in U)\left(v \neq \iota_{\mathrm{ND}} \;\Rightarrow\; (\exists u \in U)\left[u \underset{\tau}{\longrightarrow} v\right]\right)$$

Proof of Claim 9.14.1. The result directly follows from the definition of φ and the definition of $\mathcal{G}(\mathrm{ND}(\mathcal{A}))\restriction\{\tau\}$ (Definition 5.13). If $v = (x, \tau(x, (S, q_{\mathcal{D}})))$, with $(x, (S, q_{\mathcal{D}})) \in U$, then the result is trivial. Otherwise, we have $v = (\mathsf{S}_d(x), (S'_d, q'_{\mathcal{D}}))$, and there is $(x, (S, q_{\mathcal{D}})) \in U$ such that $(x, S) \in V$ and $q'_{\mathcal{D}} = \partial_{\mathcal{D}}(q_{\mathcal{D}}, S)$. Note that $(x, (S, q_{\mathcal{D}})) \in U$ implies $(x, \tau(x, (S, q_{\mathcal{D}}))) \in U$, and similarly, that $(x, S) \in V$ implies $(s, \sigma(x, S)) \in V$. By definition of σ, we have $\sigma(x, S) = (S, \gamma_{!\mathcal{A}})$ where $\gamma_{!\mathcal{A}}$ is the set of all $(\tilde{d}, S'_{\tilde{d}})$ such that $(\tilde{d}, (S'_{\tilde{d}}, q'_{\mathcal{D}})) \in \gamma_{\mathrm{ND}(\mathcal{A})}$, where $((S, q_{\mathcal{D}}), \gamma_{\mathrm{ND}(\mathcal{A})}) = \tau(x, (S, q_{\mathcal{D}}))$. But then we are done since we indeed have:
$$\big(x\ ,\ \tau(x, (S, q_{\mathcal{D}}))\big) \quad \underset{\tau}{\longrightarrow} \quad \big(\mathsf{S}_d(x)\ , (S'_d, q'_{\mathcal{D}})\big)$$
■

Now we prove:

Claim 9.14.2.
$$(\forall u \in U)\left(\iota_{\mathrm{ND}} \overset{*}{\underset{\tau}{\longrightarrow}} u\right)$$

Proof of Claim 9.14.2. We proceed by $\lhd$-Induction (Theorem 4.5). So let $u \in U$ s.t. $\iota_{\mathrm{ND}} \longrightarrow^*_\tau v$ for all $v \lhd u$ with $v \in U$. The result is trivial if $u = \iota_{\mathrm{ND}}$. Otherwise, by Claim 9.14.1, there is $v \in U$ such that $v \longrightarrow_\tau u$. But $v \lhd u$ by Proposition 5.6, so we have $\iota_{\mathrm{ND}} \longrightarrow^*_\tau v$ by induction hypothesis and we conclude by Proposition 5.6, again. ■

It remains to show
$$(\forall u \in U)(\exists! v \in U)\left(u \underset{\tau}{\longrightarrow} v\right) \tag{9.7}$$

We first prove:

Claim 9.14.3.
$$\big(\forall (x, (S, q_{\mathcal{D}})) \in U\big)\left[(x, S) \in V\right]$$

Proof of Claim 9.14.3. The property follows from a case analysis according to the following usual consequence of Induction (see Proposition 3.8, §3.4.3):
$$(\forall x)\left(x \doteq \varepsilon \ \ \vee\ \ (\exists y) \bigvee_{d \in \mathscr{D}} x \doteq \mathsf{S}_d(y)\right)$$

In the case of $x \doteq \varepsilon$, if $(x, (S, q_{\mathcal{D}})) \in U$ then we must have $S = q^\iota_{!\mathcal{A}}$, so that $(x, S) \in V$. Consider now the case of $x \doteq \mathsf{S}_d(y)$. If $u = (x, (S, q_{\mathcal{D}})) \in U$, then it follows from $\varphi(U, u)$ that we have $(x, S) \in V$ and we are done. ■

We can now prove (9.7).

Proof of (9.7). If $u = (x, (S, q_{\mathcal{D}}))$, then $(x, \tau(x, (S, q_{\mathcal{D}})))$ is the unique successor of u in U. Assume $u = (x, \tau(x, (S, q_{\mathcal{D}})))$. It then follows from Claim 9.14.1 that $(x, (S, q_{\mathcal{D}})) \in U$, and by Claim 9.14.3 we also get $(x, S) \in V$. Since $(x, S) \in V$, we have $(\mathsf{S}_d(x), S'_d) \in V$ for some unique $d \in \mathscr{D}$ and $S'_d \in Q_{!\mathcal{A}}$. It follows from $\varphi(U, u)$ that $v = (\mathsf{S}_d(x), (S'_d, q'_{\mathcal{D}})) \in U$, where $q'_{\mathcal{D}} = \partial_{\mathcal{D}}(q_{\mathcal{D}}, S)$. It remains to show that v is unique such :
$$u \underset{\tau}{\longrightarrow} v$$

Uniqueness follows from $\varphi(U, u)$ and the fact that V is a play, so it remains to show $u \longrightarrow_\tau v$. Note that $(x, (S, q_\mathcal{D})) \in U$ implies $(x, \tau(x, (S, q_\mathcal{D}))) \in U$, and similarly, that $(x, S) \in V$ implies $(s, \sigma(x, S)) \in V$. By definition of σ, we have $\sigma(x, S) = (S, \gamma_{!\mathcal{A}})$ where $\gamma_{!\mathcal{A}}$ is the set of all $(\tilde{d}, S'_{\tilde{d}})$ such that $(\tilde{d}, (S'_{\tilde{d}}, q'_\mathcal{D})) \in \gamma_{\mathrm{ND}(\mathcal{A})}$, where $((S, q_\mathcal{D}), \gamma_{\mathrm{ND}(\mathcal{A})}) = \tau(x, (S, q_\mathcal{D}))$. This finishes the proof since we indeed have:

$$\big(x\ ,\ \tau(x, (S, q_\mathcal{D}))\big) \underset{\tau}{\longrightarrow} \big(\mathsf{S}_d(x)\ ,\ (S'_d, q'_\mathcal{D})\big) \qquad \blacksquare$$

Finally, we prove (9.6), that is:

$$\big(\forall (x, S) \in V\big)\big(\exists q_\mathcal{D} \in Q_\mathcal{D}\big)\Big[(x, (S, q_\mathcal{D})) \in U\Big]$$

Proof of (9.6). Using the Induction Axiom of FSO (§3.4.2), we show

$$(\forall x)(\forall S \in Q_{!\mathcal{A}})\Big((x, S) \in V \quad \Rightarrow \quad (\exists q_\mathcal{D} \in Q_\mathcal{D})\big((x, (S, q_\mathcal{D})) \in U\big)\Big)$$

For the base case $x \doteq \dot{\varepsilon}$, if $(x, S) \in V$ then we must have $S = q^\iota_{!\mathcal{A}}$, and we indeed obtain $(x, (S, q^\iota_\mathcal{D})) \in U$. Assume now the property for x, and consider some $d \in \mathscr{D}$ and S'_d such that $(\mathsf{S}_d(x), S'_d) \in V$. Since V is a play, it follows from the Predecessor Lemma 5.10 for Infinite Plays (applied twice) that $(x, S) \in V$ for some $S \in Q_{!\mathcal{A}}$. It follows by induction hypothesis that $(x, (S, q_\mathcal{D})) \in U$ for some $q \in Q_\mathcal{D}$. But now, taking $q'_\mathcal{D} = \partial_\mathcal{D}(q_\mathcal{D}, S)$, we have $(\mathsf{S}_d(x), (S'_d, q'_\mathcal{D})) \in U$ and we are done. $\blacksquare$

This concludes the proof of Proposition 9.14. $\square$

10. Conclusion

In this paper, we proposed for each non-empty (hereditarily) finite set $\mathscr{D}$ the theory $\mathsf{FSO}_\mathscr{D}$ of *Functional (Monadic) Second-Order Logic* on the full (infinite) $\mathscr{D}$-ary tree. The theory $\mathsf{FSO}_\mathscr{D}$ (henceforth FSO) is a uniform extension of MSO on the full $\mathscr{D}$-ary tree with hereditarily finite sets. We formalized in FSO a basic theory of (alternating) tree automata and (acceptance) games. This allowed us, in the theory of FSO augmented with an axiom $(PosDet)$ of positional determinacy of parity games, to formalize a translation of MSO-formulae to automata adapted from [Wal02]. We then deduced the completeness of $\mathsf{FSO} + (PosDet)$ thanks to a variant of the Büchi-Landweber Theorem [BL69], stating that MSO decides winning for (definable) games of finite graphs (and obtained thanks to the completeness of MSO over ω-words [Sie70]). By naive proof enumeration, this gives a proof of Rabin's Tree Theorem [Rab69], the decidability of MSO on infinite trees. Moreover, since the formal theory FSO is conservative (w.r.t. the faithful translation $(-)^\circ : \mathsf{MSO} \to \mathsf{FSO}$) over a natural set of axioms for MSO, we also get a complete axiomatization of MSO on infinite trees, namely $\mathsf{MSO}_\mathscr{D} + \langle PosDet \rangle$ (cf. Definition 5.26, §5.6.1).

10.1. **Proof theoretic strength of complementation.** The present paper does not discuss proof theoretic strength. In the context of second-order arithmetic (in the sense of [Sim10]), it is known that complementation of tree automata is between Π^1_3 and Δ^1_3-comprehension [KM16]. As far as only games are concerned (as opposed to proving the correctness of an internal function for complementation), only Π^1_2-comprehension is required for the positional determinacy of (each level of) parity games [KM15, Lemma 4.6].

10.2. **Clarifying the status of $\langle PosDet \rangle$.** A problem arising from this work is whether the axiom schema $\langle PosDet \rangle$ is indeed independent of MSO. The latter may be seen as the monadic fragment of $\mathsf{PA2}$ (over the appropriate language) and, as we have mentioned, is complete when restricted to infinite words. While it might therefore be natural to suspect that MSO is already complete without $\langle PosDet \rangle$, we point out that the axiomatization of Weak MSO over infinite trees given in [Sie78] also augments the natural fragment of Peano arithmetic by an axiom of induction over finite trees. As we mentioned in the Introduction, the completeness of $\mathsf{MSO}_{\mathscr{D}}$ was erroneously claimed in the preliminary version of this work [DR15].

10.3. **On the notion of proof for MSO.** One outcome of this work is that our complete deduction system for MSO gives a new decision algorithm. Of course, the naive decision algorithm by proof enumeration is not very sophisticated, and it is worth restating that its correctness is itself driven by the usual automata-theoretic argument. Such an algorithm, nonetheless, makes no mention of automata and so can be adapted and improved purely in the setting of proof theory. In this sense, the algorithm is the first of its kind: a decision procedure for MSO on infinite trees that remains internal to the language, rather than requiring intermediate translations to automata.

A basic motivation for such algorithms is that, even if Rabin's Tree Theorem proves the existence of decision procedures for MSO on infinite trees, there is (as far as we know) no working implementation of such procedures.[3] Our axiomatization instead allows the targeting of (semi) automatic approaches, for instance based on proof assistants. As we mentioned in the Introduction, our axiomatization is polynomial-time recognizable and so indeed yields a meaningful notion of 'proof certificate': a proof of a theorem may be easily checked, without having to reprove the theorem again.

10.4. **Constructive systems and proof interpretations.** A further direction of research is to look for constructive interpretations of MSO. In the case of ω-words, preliminary steps were made in [PR19a]. The general idea is to proceed along the following steps:

(1) Determine the relevant computational information one should be able to extract from constructive proofs. In the case of MSO on ω-words, the approach taken in [PR17, PR18, PR19b] (and specializing [Rib20]) was to consider the provably total *causal* (or 1-Lipschitz) functions of MSO.
(2) Devise constructive variants of MSO (together with suitable proof interpretations), which are correct and complete for the chosen class of functions w.r.t. to their provable $\forall\exists$-sentences.

A realizability model for MSO has been proposed in [Rib20], in which the underlying logic is not only constructive but also *linear* (in the sense of [Gir87]).

Of course, similar approaches may also be considered in more traditional settings for constructive interpretations of proofs [TvD88, Koh08]. In particular, it is not clear (at least to us) what usual computational interpretations of Comprehension, following either Girard's *System F* [Gir72] or Spector's *bar-recursion* (see e.g. [TvD88, Koh08]), could say in the context of MSO. It is not yet clear what in this context should be the correct analogue in MSO of the quantifier free formulae of arithmetic. Regarding the quantifier-free formulae as those formulae with trivial realizers w.r.t. usual proof interpretations, the model of [Rib20] suggests in the case of ω-word that for languages based on linear logic, correct analogues of quantifier formulae are formulae which are both negative and positive. These formulae, called *deterministic* in [PR18, PR19b] may contain unbounded quantifiers, but these must be guarded by exponential modalities of suitable polarity ($!$ for $\forall$ and $?$ for $\exists$).

[3] To our knowledge, the MONA tool (`https://www.brics.dk/mona/`) only handles *Weak* MSO.

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APPENDIX A. PROOFS OF §3.6 (MUTUAL INTERPRETABILITY OF FSO AND MSO)

This appendix is devoted to the proofs of §3.6, namely the proofs that both translations

$$\langle - \rangle \;:\; \mathsf{FSO} \longrightarrow \mathsf{MSO} \qquad \text{and} \qquad (-)^\circ \;:\; \mathsf{MSO} \longrightarrow \mathsf{FSO}$$

preserve as well as reflect provability. Proofs concerning $(-)^\circ$ (namely Theorem 3.26 and Proposition 3.27) are given in §A.4. The main point is the translation

$$\langle - \rangle \quad : \quad \mathsf{FSO} \quad \longrightarrow \quad \mathsf{MSO}$$

and the proof of Theorem 3.24. Let us first give its full statement.

Theorem A.1 (Theorem 3.24)**.** *Consider* FSO*-formulae* $\varphi_1, \ldots, \varphi_m$ *and* φ*, with free* HF*-variables among* $\boldsymbol{\ell} = \ell_1, \ldots, \ell_p$*, free Function variables among* $\boldsymbol{F} = F_1, \ldots, F_n$*. Assume that*

$$\varphi_1, \ldots, \varphi_m \vdash_{\mathsf{FSO}} \varphi$$

Then, for all closed HF*-terms* $\boldsymbol{K} = K_1, \ldots, K_n$*, and for all* HF*-sets* $\boldsymbol{\lambda} = \lambda_1, \ldots, \lambda_p$*, we have*

$$\vdash_{\mathsf{MSO}} \quad \langle (\forall \boldsymbol{F} : \boldsymbol{K})\, (\varphi_1[\boldsymbol{\lambda}/\boldsymbol{\ell}] \quad \Longrightarrow \quad \cdots \quad \Longrightarrow \quad \varphi_m[\boldsymbol{\lambda}/\boldsymbol{\ell}] \quad \Longrightarrow \quad \varphi[\boldsymbol{\lambda}/\boldsymbol{\ell}]) \rangle \tag{A.1}$$

The proof of Theorem A.1 is split into §A.2 and §A.3.

Notation A.2. In the following, we write Γ for contexts of formulae in derivations.

A.1. Models and Henkin Completeness.

We first recall here Henkin models and completeness in the context of $\mathsf{MSO}_{\mathscr{D}}$. Henkin completeness is a useful tool to reason on provability, and we rely on it in §A.3 when showing that $\mathsf{MSO}_{\mathscr{D}}$ proves the $\langle - \rangle$-translation of the Functional Choices Axioms of $\mathsf{FSO}_{\mathscr{D}}$ (§3.4.5).

A Henkin structure for the language of $\mathsf{MSO}_{\mathscr{D}}$ is a tuple

$$\mathfrak{M} \quad := \quad (\mathfrak{M}^\iota,\, \varepsilon_{\mathfrak{M}},\, (\mathsf{S}_{\mathfrak{M},d})_{d \in \mathscr{D}},\, \mathfrak{M}^o,\, <_{\mathfrak{M}})$$

where $\mathfrak{M}^\iota$ is a set of *Individuals*, $\varepsilon_{\mathfrak{M}}$ is an element of $\mathfrak{M}^\iota$, $\mathsf{S}_{\mathfrak{M},d}$ is a function from $\mathfrak{M}^\iota$ to $\mathfrak{M}^\iota$ (for each $d \in \mathscr{D}$), $\mathfrak{M}^o \subseteq \mathcal{P}(\mathfrak{M}^\iota)$ is a set of (monadic) *Predicates*, and $<_{\mathfrak{M}}$ is a binary relation on $\mathfrak{M}^\iota$. In particular, *the standard model* of $\mathsf{MSO}_{\mathscr{D}}$ is

$$\mathfrak{T} \quad := \quad (\mathscr{D}^*,\, \varepsilon,\, (\mathsf{S}_d)_{d \in \mathscr{D}},\, \mathcal{P}(\mathscr{D}^*),\, <)$$

where ε is interpreted by the empty sequence $\varepsilon \in \mathscr{D}^*$, and where S_d is the map taking $p \in \mathscr{D}^*$ to $p.d \in \mathscr{D}^*$ and $<$ is the strict prefix order on $\mathscr{D}^*$. The formulae of $\mathsf{MSO}_{\mathscr{D}}$ are interpreted in Henkin models $\mathfrak{M}$ as usual: Individuals range over $\mathfrak{M}^\iota$, Predicates range over $\mathfrak{M}^o$, equality on Individuals is interpreted as equality in $\mathfrak{M}^\iota$, and $\dot{<}$ is interpreted as $<_{\mathfrak{M}}$. A Henkin structure $\mathfrak{M}$ is a *model of* $\mathsf{MSO}_{\mathscr{D}}$ if all the axioms of $\mathsf{MSO}_{\mathscr{D}}$ hold in $\mathfrak{M}$. As usual (see e.g. [Sha91]), $\mathsf{MSO}_{\mathscr{D}}$ is complete w.r.t. its Henkin models:

Theorem A.3 (Henkin Completeness)**.** *Given a closed* $\mathsf{MSO}_{\mathscr{D}}$*-formula* φ*, if* φ *holds in all models of* $\mathsf{MSO}_{\mathscr{D}}$*, then* $\mathsf{MSO}_{\mathscr{D}} \vdash \varphi$*.*

A.2. **Interpretation of FSO-Derivations in MSO.** We discuss here the part of Theorem A.1 concerning the logical rules of FSO (Figure 1 and Figure 4). Note that (A.1) amounts to showing

$$\Gamma_{\boldsymbol{F},\boldsymbol{K}} \quad , \quad \langle\varphi_1^\star[\boldsymbol{\lambda}/\boldsymbol{\ell}]\rangle \quad , \quad \dots \quad , \quad \langle\varphi_m^\star[\boldsymbol{\lambda}/\boldsymbol{\ell}]\rangle \quad \vdash_{\mathsf{MSO}} \quad \langle\varphi^\star[\boldsymbol{\lambda}/\boldsymbol{\ell}]\rangle$$

where $\Gamma_{\boldsymbol{F},\boldsymbol{K}}$ consists of formulae of the form $\Upsilon_{\boldsymbol{c}}(\boldsymbol{X})$ induced by the interpretation of the universal quantifications $\forall\boldsymbol{F} : \boldsymbol{K}$, and where $\varphi^\star$ and the $\varphi_j^\star$'s are corresponding *extended* $\mathsf{FSO}_{\mathscr{D}}$-formulae (without free Function variables). Explicitly, for $i = 1, \dots, n$, let $\boldsymbol{\kappa}_i = \kappa_{i,1}, \dots, \kappa_{i,c_i}$ enumerate $[\![K_i]\!]$ (so that $[\![K_i]\!] = \{\kappa_{i,1}, \dots, \kappa_{i,c_i}\}$). With these notations, we let

$$\begin{array}{rcl} \Gamma_{\boldsymbol{F},\boldsymbol{K}} & = & \Upsilon_{c_1}(X_{1,1}, \dots, X_{1,c_1}) \ , \ \dots \ , \ \Upsilon_{c_n}(X_{n,1}, \dots, X_{n,c_n}) \\ \varphi_j^\star & = & \varphi_j\left[|\boldsymbol{X}_1(t) \doteq_{\boldsymbol{\kappa}_1} L| \ / \ F_1(t) \doteq L \ , \ \dots \ , \ |\boldsymbol{X}_n(t) \doteq_{\boldsymbol{\kappa}_n} L| \ / \ F_n(t) \doteq L\right] \\ \varphi^\star & = & \varphi\left[|\boldsymbol{X}_1(t) \doteq_{\boldsymbol{\kappa}_1} L| \ / \ F_1(t) \doteq L \ , \ \dots \ , \ |\boldsymbol{X}_n(t) \doteq_{\boldsymbol{\kappa}_n} L| \ / \ F_n(t) \doteq L\right] \end{array}$$

The argument is then by induction on derivations. Since the interpretation $\langle - \rangle$ commutes with $\neg$, $\vee$ and $(\exists x)(-)$, the cases of the rules for these connectives follow from the induction hypothesis. The quantifier rules for HF-variables directly follow from the $\vee$-rules of MSO. It remains to deal with the quantifier rules for Function variables and with the Substitution rule.

A.2.1. $\exists F$-*Introduction.*

$$\frac{\Gamma \vdash \varphi[G/F] \qquad \Gamma \vdash (\forall x)(\exists k \mathrel{\dot\in} K)(G(x) \doteq k)}{\Gamma \vdash (\exists F : K)\varphi}$$

Let $\boldsymbol{F} = F_1, \dots, F_n$ such that the free Function variables of $\Gamma, (\exists F : K)\varphi$ are among $G\boldsymbol{F}$. Let $\boldsymbol{g} = g_1, \dots, g_p$ such that the free HF-variables of $\Gamma, (\exists F : K)\varphi$ are among $\boldsymbol{g}$. We further assume given closed HF-terms and HF-sets as in the statement of Theorem A.1:

- HF-sets $\boldsymbol{\gamma} = \gamma_1, \dots, \gamma_p$ to interpret $\boldsymbol{g} = g_1, \dots, g_p$,
- closed HF-terms $\boldsymbol{K} = K_1, \dots, K_n$ to bound $\boldsymbol{F} = F_1, \dots, F_n$,
- a closed HF-term L to bound G.

We adopt the following notational conventions:

- $K' := K[\boldsymbol{\gamma}/\boldsymbol{g}]$
- $[\![K']\!]$ is enumerated by $\boldsymbol{\kappa} = \kappa_1, \dots, \kappa_c$.
- $[\![L]\!]$ is enumerated by $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_d\}$.
- Each $[\![K_i]\!]$ (for $1 \le i \le n$) is enumerated by $\boldsymbol{\kappa}_i = \{\kappa_{i,1}, \dots, \kappa_{i,c_i}\}$.

We show

$$\Gamma_{G\boldsymbol{F},L\boldsymbol{K}} \ , \ \langle\Gamma^\star[\boldsymbol{\gamma}/\boldsymbol{g}]\rangle \quad \vdash_{\mathsf{MSO}} \quad (\exists\boldsymbol{X})\Big(\Upsilon_c(\boldsymbol{X}) \ \wedge$$
$$\Big\langle\varphi[\boldsymbol{\gamma}/\boldsymbol{g}]\Big[|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}} M| \ / \ F(t) \doteq M \ , \ |\boldsymbol{Y}(t) \doteq_{\boldsymbol{\lambda}} M| \ / \ G(t) \doteq M \ , \ \overrightarrow{|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}} M| \ / \ F(t) \doteq M}\Big]\Big\rangle\Big)$$

where

$$\Gamma_{G\boldsymbol{F},L\boldsymbol{K}} \quad = \quad \Upsilon_d(Y_1, \dots, Y_d) \ , \ \Upsilon_{c_1}(X_{1,1}, \dots, X_{1,c_1}) \ , \ \dots \ , \ \Upsilon_{c_n}(X_{n,1}, \dots, X_{n,c_n})$$

By induction hypothesis, we have

$$\Gamma_{G\boldsymbol{F},L\boldsymbol{K}} \ , \ \langle\Gamma^\star[\boldsymbol{\gamma}/\boldsymbol{g}]\rangle \quad \vdash_{\mathsf{MSO}} \quad (\forall x) \bigvee_{\kappa \in [\![K']\!]} \langle|\boldsymbol{Y}(x) \doteq_{\boldsymbol{\lambda}} \kappa|\rangle \tag{A.2}$$

and

$$\Gamma_{\boldsymbol{GF},\boldsymbol{LK}}\ ,\ \langle \Gamma^\star[\boldsymbol{\gamma}/\boldsymbol{g}]\rangle \quad \vdash_{\mathsf{MSO}}$$
$$\langle \varphi[\boldsymbol{\gamma}/\boldsymbol{g}]\left[|\boldsymbol{Y}(t) \doteq_{\boldsymbol{\lambda}} M| \;/\; F(t) \doteq M\ ,\ |\boldsymbol{Y}(t) \doteq_{\boldsymbol{\lambda}} M| \;/\; G(t) \doteq M\ ,\ \overrightarrow{|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}} M| \;/\; F(t) \doteq M}\right]\rangle \tag{A.3}$$

Note that (A.2) unfolds to

$$\Gamma_{\boldsymbol{GF},\boldsymbol{LK}}\ ,\ \langle \Gamma^\star[\boldsymbol{\gamma}/\boldsymbol{g}]\rangle \quad \vdash_{\mathsf{MSO}} \quad (\forall x)\left[\bigvee_{\kappa \in [\![K']\!]} \bigvee_{\substack{1 \leq j \leq d \\ \lambda_j = \kappa}} Y_j(x)\right]$$

By applying MSO's Comprehension scheme c times, we can therefore show that

$$\Gamma_{\boldsymbol{GF},\boldsymbol{LK}}\ ,\ \langle \Gamma^\star[\boldsymbol{\gamma}/\boldsymbol{g}]\rangle \quad \vdash_{\mathsf{MSO}} \quad (\exists X_1,\dots,X_c)\left[\Upsilon_c(\boldsymbol{X}) \ \wedge\ (\forall x) \bigvee_{\substack{1 \leq i \leq c \\ 1 \leq j \leq d \\ \kappa_i = \lambda_j}} \big(X_i(x) \wedge Y_j(x)\big)\right]$$

Since on the other hand

$$\Gamma_{\boldsymbol{GF},\boldsymbol{LK}}\ ,\ \langle \Gamma^\star[\boldsymbol{\gamma}/\boldsymbol{g}]\rangle \quad \vdash_{\mathsf{MSO}} \quad \Upsilon_d(\boldsymbol{Y})$$

it follows that for all HF-set κ we have

$$\Gamma_{\boldsymbol{GF},\boldsymbol{LK}}\ ,\ \langle \Gamma^\star[\boldsymbol{\gamma}/\boldsymbol{g}]\rangle\ ,\ \Upsilon_c(\boldsymbol{X})\ ,\ (\forall x)\left[\bigvee_{\substack{1 \leq i \leq c \\ 1 \leq j \leq d \\ \kappa_i = \lambda_j}} X_i(x) \wedge Y_j(x)\right] \quad \vdash_{\mathsf{MSO}}$$
$$\bigvee_{\substack{1 \leq i \leq c \\ \lambda_i = \kappa}} X_i(t) \quad \Leftrightarrow \quad \bigvee_{\substack{1 \leq j \leq d \\ \kappa_j = \kappa}} Y_j(t)$$

We then get from (A.3) that

$$\Gamma_{\boldsymbol{GF},\boldsymbol{LK}}\ ,\ \langle \Gamma^\star[\boldsymbol{\gamma}/\boldsymbol{g}]\rangle\ ,\ \Upsilon_c(\boldsymbol{X})\ ,\ (\forall x)\left[\bigvee_{\substack{1 \leq i \leq c \\ 1 \leq j \leq d \\ \kappa_i = \lambda_j}} X_i(x) \wedge Y_j(x)\right] \quad \vdash_{\mathsf{MSO}}$$
$$\langle \varphi[\boldsymbol{\gamma}/\boldsymbol{g}]\left[|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}} M| \;/\; F(t) \doteq M\ ,\ |\boldsymbol{Y}(t) \doteq_{\boldsymbol{\lambda}} M| \;/\; G(t) \doteq M\ ,\ \overrightarrow{|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}} M| \;/\; F(t) \doteq M}\right]\rangle$$

and the result follows.

A.2.2. *$\exists F$-Elimination.*

$$\frac{\Gamma \vdash (\exists F : K)\varphi \qquad \Gamma \,,\, (\forall x)(\exists k \mathrel{\dot\in} K)(F(x) \doteq k) \,,\, \varphi \vdash \psi}{\Gamma \vdash \psi} \ (F \text{ not free in } \Gamma, \psi)$$

Assume that Γ, ψ have free HF-variables among $\boldsymbol{k} = k_1, \ldots, k_p$, and free Function variables among $\boldsymbol{F} = F_1, \ldots, F_n$. Consider HF-sets $\boldsymbol{\kappa} = \kappa_1, \ldots, \kappa_p$ and closed HF-terms $\boldsymbol{M} = M_1, \ldots, M_n$ with $[\![M_i]\!] = \{\boldsymbol{\mu}_i\} = \{\mu_{i,1}, \ldots, \mu_{i,c_i}\}$. We have to show

$$\Gamma_{\boldsymbol{F},\boldsymbol{M}} \,,\, \langle \Gamma^\star[\boldsymbol{\kappa}/\boldsymbol{k}] \rangle \quad \vdash_{\mathsf{MSO}} \quad \langle \psi^\star[\boldsymbol{\kappa}/\boldsymbol{k}] \rangle$$

Let $\boldsymbol{G} = G_1, \ldots, G_m$ be the free Function variables (resp. $\boldsymbol{\ell} = \ell_1, \ldots, \ell_q$ the free HF-variables) of $(\exists F : K.\varphi)$ which are not among $\boldsymbol{F}$ (resp. not among $\boldsymbol{k}$). Furthermore, consider HF-Sets κ and $\boldsymbol{\lambda} = \lambda_1, \ldots, \lambda_q$ as well as closed HF-terms $\boldsymbol{N} = N_1, \ldots, N_m$ with $[\![N_j]\!] = \boldsymbol{\nu}_j = \nu_{j,1}, \ldots, \nu_{j,d_j}$ *non-empty*. Write $[\![K]\!] = \boldsymbol{\kappa}' = \kappa'_1, \ldots, \kappa'_c$ where $K' = K[\boldsymbol{\kappa}/\boldsymbol{k}, \boldsymbol{\lambda}/\boldsymbol{\ell}]$.

By induction hypothesis we have

$$\Gamma_{\boldsymbol{FG},\boldsymbol{MN}} \,,\, \langle \Gamma^\star[\boldsymbol{\kappa}/\boldsymbol{k}] \rangle \quad \vdash_{\mathsf{MSO}} \quad (\exists \boldsymbol{X})\big[\Upsilon_c(\boldsymbol{X}) \ \wedge \ \langle \phi^\star \rangle\big]$$

where $\phi = \varphi[\boldsymbol{\kappa}/\boldsymbol{k}, \boldsymbol{\lambda}/\boldsymbol{\ell}]$ and

$$\phi^\star \quad = \quad \phi\Big[|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}'} L| \;/\; F(t) \doteq L \,,\, \overrightarrow{|\boldsymbol{Y}(t) \doteq_{\boldsymbol{\nu}} L| \;/\; G(t) \doteq L} \,,\, \overrightarrow{|\boldsymbol{X}(t) \doteq_{\boldsymbol{\mu}} L| \;/\; F(t) \doteq L}\Big]$$

as well as (modulo some propositional reasoning)

$$\Gamma_{\boldsymbol{FG},\boldsymbol{MN}} \,,\, \langle \Gamma^\star[\boldsymbol{\kappa}/\boldsymbol{k}] \rangle \,,\, (\forall x) \bigvee_{1 \le i \le c} X_i(x) \,,\, \langle \phi^\star \rangle \quad \vdash_{\mathsf{MSO}} \quad \langle \psi^*[\boldsymbol{\kappa}/\boldsymbol{k}] \rangle$$

We thus get

$$\Gamma_{\boldsymbol{FG},\boldsymbol{MN}} \,,\, \langle \Gamma^\star[\boldsymbol{\kappa}/\boldsymbol{k}] \rangle \quad \vdash_{\mathsf{MSO}} \quad \langle \psi^*[\boldsymbol{\kappa}/\boldsymbol{k}] \rangle$$

It remains to eliminate $\Gamma_{\boldsymbol{G},\boldsymbol{N}}$. But this is easy by using the comprehension axiom of MSO together with the assumption that each $[\![N_j]\!]$ is non-empty.

A.2.3. *Substitution.*

$$\frac{\Gamma \vdash \varphi}{\Gamma[F(t)/k] \vdash \varphi[F(t)/k]} \ \text{(where } \Gamma[F(t)/k], \varphi[F(t)/k] \text{ are } \mathsf{FSO}\text{-formulae)}$$

Assume that Γ, φ have free HF-variables among $k, \boldsymbol{k}$ (with $\boldsymbol{k} = k_1, \ldots, k_p$), and free Function variables among $F, \boldsymbol{F}$ (with $\boldsymbol{F} = F_1, \ldots, F_n$). Consider HF-sets $\boldsymbol{\kappa} = \kappa_1, \ldots, \kappa_p$ and closed HF-terms $M, \boldsymbol{M}$ with $\boldsymbol{M} = M_1, \ldots, M_n$ and $[\![M]\!] = \{\boldsymbol{\mu}\} = \{\mu_1, \ldots, \mu_c\}$ and $[\![M_i]\!] = \{\boldsymbol{\mu}_i\} = \{\mu_{i,1}, \ldots, \mu_{i,c_i}\}$.

We have to show

$$\Gamma_{\boldsymbol{F},\boldsymbol{M}} \,,\, \Upsilon_c(X_1, \ldots, X_c) \,,\, \langle \Gamma^\star[\boldsymbol{\kappa}/\boldsymbol{k}]\,[|\boldsymbol{X}(t) \doteq_{\boldsymbol{\mu}} L| \;/\; (k \doteq L)] \rangle \quad \vdash_{\mathsf{MSO}} \qquad \langle \varphi^\star[\boldsymbol{\kappa}/\boldsymbol{k}]\,[|\boldsymbol{X}(t) \doteq_{\boldsymbol{\mu}} L| \;/\; (k \doteq L)] \rangle$$

For all $i \in \{1, \ldots, c\}$, by induction hypothesis we have

$$\Gamma_{\boldsymbol{F},\boldsymbol{K}} \,,\, \langle \Gamma^\star[\boldsymbol{\kappa}/\boldsymbol{k}][(\mu_i \doteq L)/(k \doteq L)] \rangle \quad \vdash_{\mathsf{MSO}} \quad \langle \varphi^\star[\boldsymbol{\kappa}/\boldsymbol{k}][(\mu_i \doteq L)/(k \doteq L)] \rangle$$

On the other hand recall that

$$\langle |\boldsymbol{X}(t) \doteq_\mu L| \rangle \quad = \quad \bigvee_{\substack{1 \le i \le c \\ \mu_i = [\![L]\!]}} X_i(t)$$

so that

$$\Upsilon_c(X_1, \dots, X_c) \, , \, X_i(t) \quad \vdash_{\mathsf{MSO}} \quad \langle |\boldsymbol{X}(t) \doteq_\mu L| \rangle \;\; \Leftrightarrow \;\; \langle \mu_i \doteq L \rangle$$

and thus

$$\Gamma_{\boldsymbol{F},\boldsymbol{M}} \, , \, \Upsilon_c(X_1, \dots, X_c) \, , \, X_i(t) \, , \, \langle \Gamma^\star[\boldsymbol{\kappa}/\boldsymbol{k}] \, [|\boldsymbol{X}(t) \doteq_{\boldsymbol{\mu}} L| \; / \; (k \doteq L)] \rangle \quad \vdash_{\mathsf{MSO}} \quad \langle \varphi^\star[\boldsymbol{\kappa}/\boldsymbol{k}] \, [|\boldsymbol{X}(t) \doteq_{\boldsymbol{\mu}} L| \; / \; (k \doteq L)] \rangle$$

The result then follows from the fact that

$$\Upsilon_c(X_1, \dots, X_c) \quad \vdash_{\mathsf{MSO}} \quad \bigvee_{1 \leq i \leq c} X_i(t)$$

A.3. **Interpretation of the Axioms of** $\mathsf{FSO}_\mathscr{D}$ **in** $\mathsf{MSO}_\mathscr{D}$**.** We now check that all the axioms of FSO (§3.4) are interpreted by $\langle - \rangle$ as probable formulae of MSO.

A.3.1. *Equality.*

- *Equality Axioms on Individuals.* The reflexivity axiom on individuals $(\forall x)(x \doteq x)$ is interpreted by the corresponding axiom of MSO. For Leibniz's scheme
$$(\forall x)(\forall y)\big(x \doteq y \quad \Longrightarrow \quad \varphi[x/z] \quad \Longrightarrow \quad \varphi[y/z]\big)$$
following §A.2, we only have to consider the case where φ is of the form of $\psi^\star$, that is, where φ is an HF-closed extended FSO-formula without free Function variables. In this case, the axiom is interpreted as
$$(\forall x)(\forall y)\big(x \doteq y \quad \Longrightarrow \quad \langle\varphi\rangle[x/z] \quad \Longrightarrow \quad \langle\varphi\rangle[y/z]\big)$$
and is an instance of the corresponding axiom of MSO.
- *Equality Axioms on* HF*-Sets.* Each HF-closed instance $(K \doteq K)$ of the reflexivity axiom is interpreted by the formula $\top$ since $[\![K]\!] = [\![K]\!]$ in V_ω. Consider now an HF-closed instance of Leibniz's scheme
$$(K \doteq L \quad \Longrightarrow \quad \varphi[K/m] \quad \Longrightarrow \quad \varphi[L/m])$$
Similarly as above, we can assume φ to be an extended FSO-formula without free Function variable, and with at most m as free HF-variable. In this case, if $[\![K]\!] = [\![L]\!]$, then the axiom is interpreted as
$$\langle\varphi[K/m]\rangle \quad \Longrightarrow \quad \langle\varphi[L/m]\rangle$$
which follows by induction on φ. If $[\![K]\!] \neq [\![L]\!]$, then $(K \doteq L)$ is interpreted as $\bot$ and the interpretation of the axiom trivially holds.

A.3.2. *Tree Axioms.* Each Tree axiom of $\mathsf{FSO}_\mathscr{D}$ is interpreted by the corresponding Tree axiom of $\mathsf{MSO}_\mathscr{D}$.

A.3.3. *Induction Scheme.* Consider an instance of induction:

$$\varphi(\dot{\varepsilon}) \quad \Longrightarrow \quad (\forall x)\left(\varphi(x) \implies \bigwedge_{d \in \mathscr{D}} \varphi(\mathsf{S}_d(x))\right) \quad \Longrightarrow \quad (\forall x)\varphi(x)$$

Similarly as above, we can assume φ to be an HF-closed extended FSO-formula without free Function variables, so that the axiom is translated to

$$\langle\varphi\rangle(\dot{\varepsilon}) \quad \Longrightarrow \quad (\forall x)\left(\langle\varphi\rangle(x) \implies \bigwedge_{d \in \mathscr{D}} \langle\varphi\rangle(\mathsf{S}_d(x))\right) \quad \Longrightarrow \quad (\forall x)\langle\varphi\rangle(x)$$

The latter follows by combining the Comprehension Scheme with the Induction Axiom of $\mathsf{MSO}_{\mathscr{D}}$.

A.3.4. *Axioms on* HF*-Sets.* We now show that MSO proves the translation of each closed instance of the Axioms on HF-Sets of §3.4.4. This trivially follows from the following easy fact:

Lemma A.4. *For each closed* HF*-formula φ,* $\mathsf{MSO} \vdash \langle\varphi\rangle$ *if and only if* $V_\omega \models \varphi$.

Proof. By induction on φ. □

Consider now a closed HF-axiom of the form

$$\varphi_{n,m}[\boldsymbol{K}/\boldsymbol{k}][\dot{\mathrm{g}}_{n,m}(\boldsymbol{K})/\ell]$$

Since

$$V_\omega \models \varphi_{n,m}[\boldsymbol{K}/\boldsymbol{k}][\mathrm{g}(\boldsymbol{K})/\ell]$$

where g is the HF-Function associated to $\varphi_{n,m}$ in Convention 3.13, by Lemma A.4 we get

$$\mathsf{MSO} \vdash \langle\varphi_{n,m}[\boldsymbol{K}/\boldsymbol{k}][\dot{\mathrm{g}}_{n,m}(\boldsymbol{K})/\ell]\rangle$$

A.3.5. HF*-Bounded Choice for* HF*-Sets.*

Lemma A.5. *Consider an instance of the axiom of* HF*-bounded choice for* HF*-sets*

$$\theta \quad := \quad (\forall k \mathrel{\dot{\in}} K)(\exists \ell \mathrel{\dot{\in}} L)\varphi(k,\ell) \implies \left(\exists f \mathrel{\dot{\in}} L^K\right)(\forall k \mathrel{\dot{\in}} K)\varphi(k, f(k))$$

where θ is an HF*-closed extended* FSO*-formula without free Function variables. Then we have* $\mathsf{MSO} \vdash \langle\theta\rangle$.

Proof. Note that $\langle\theta\rangle$ is the formula

$$\bigwedge_{\kappa \in [\![K]\!]} \bigvee_{\lambda \in [\![L]\!]} \langle\varphi(\kappa,\lambda)\rangle \quad \Longrightarrow \quad \bigvee_{f \in [\![L]\!]^{[\![K]\!]}} \bigwedge_{\kappa \in [\![K]\!]} \langle\varphi(\kappa, f(\kappa))\rangle$$

which holds by propositional logic. □

A.3.6. HF-*Bounded Choice for Functions.*

Lemma A.6. *Consider an instance of the axiom of* HF-*bounded choice for Functions*

$$\theta \quad := \quad (\forall x)(\exists k \mathrel{\dot\in} K)\varphi(x,k) \implies (\exists F : K)(\forall x)(\exists k \mathrel{\dot\in} K)\big(F(x) \doteq k \;\; \land \;\; \varphi(x,k)\big)$$

where θ is an HF-*closed extended* FSO-*formula without free Function variables. Then we have* $\mathsf{MSO} \vdash \langle\theta\rangle$.

Proof. Using Henkin Completeness (Theorem A.3) we show that θ holds in every model $\mathfrak{M}$ of $\mathsf{MSO}_{\mathscr{D}}$. By writing $\theta = \theta_1 \Rightarrow \theta_2$, we have

$$\langle\theta_1\rangle \quad = \quad (\forall x) \bigvee_{\kappa \in [\![K]\!]} \langle\varphi(x,\kappa)\rangle$$

Assume $\mathfrak{M} \models \langle\theta_1\rangle$, so that for all $\mathfrak{a} \in \mathfrak{M}^\iota$ there is $\kappa \in [\![K]\!]$ such that

$$\mathfrak{M} \models \langle\varphi(\mathfrak{a},\kappa)\rangle$$

On the other hand, consider an enumeration $\boldsymbol{\lambda} = \lambda_1, \dots, \lambda_c$ of $[\![K]\!]$, so that $\langle\theta_2\rangle$ expands to

$$(\exists X_1, \dots, X_c)\left(\Upsilon_c(X_1,\dots,X_c) \quad \land \quad (\forall x) \bigvee_{1 \le i \le c} \Big[\langle |X_1 \cdots X_c(x)| \doteq_{\boldsymbol{\lambda}} \lambda_i\rangle \;\; \land \;\; \langle\varphi(x,\lambda_i)\rangle\Big]\right)$$

that is

$$(\exists X_1, \dots, X_c)\left(\Upsilon_c(X_1,\dots,X_c) \quad \land \quad (\forall x) \bigvee_{1 \le i \le c} \Big[X_i(x) \;\; \land \;\; \langle\varphi(x,\lambda_i)\rangle\Big]\right)$$

For each $i \in \{1,\dots,c\}$ define by Comprehension a predicate $A_i \in \mathfrak{M}^o$ such that

$$\mathfrak{M} \models \quad (\forall x)\left[x \in A_i \quad \Longleftrightarrow \quad \left(\langle\varphi(x,\lambda_i)\rangle \;\; \land \;\; \bigwedge_{j<i} \neg A_j(x)\right)\right]$$

Assume now that $\mathfrak{M} \models \theta_1$. For all $\mathfrak{a} \in \mathfrak{M}^\iota$, there is some $i \in \{1,\dots,c\}$ such that $\mathfrak{M} \models \langle\varphi(\mathfrak{a},\lambda_i)\rangle$. Let $i_\mathfrak{a}$ be the least such i, so that $\mathfrak{a} \in A_{i_\mathfrak{a}}$ and $\mathfrak{a} \notin A_i$ if $i \neq i_\mathfrak{a}$. It follows that

$$\mathfrak{M} \models \quad \left(\Upsilon_c(A_1,\dots,A_c) \quad \land \quad (\forall x) \bigvee_{1 \le i \le c} \Big[A_i(x) \;\; \land \;\; \langle\varphi(x,\lambda_i)\rangle\Big]\right)$$

and we are done. $\square$

A.3.7. *Iterated* HF-*Bounded Choice.*

Lemma A.7. *Consider an instance of the axiom of Iterated* HF-*Choice*

$$\theta \quad := \quad (\forall k \mathrel{\dot\in} K)(\exists F : L)\varphi(k,F) \implies \big(\exists G : L^K\big)(\forall k \mathrel{\dot\in} K)\varphi(k)[G(k) \mathbin{/\!\!/} F]$$

where θ is an HF-*closed extended* FSO-*formula without free Function variables. Then we have* $\mathsf{MSO} \vdash \langle\theta\rangle$.

Proof. Using Henkin Completeness (Theorem A.3) we show that θ holds in every model $\mathfrak{M}$. Write $\theta = \theta_1 \Rightarrow \theta_2$ and assume that $\mathfrak{M} \models \langle\theta_1\rangle$. Let $\mathsf{g} = \mathsf{g}_1, \dots, \mathsf{g}_m$ be the enumeration of $[\![L]\!]^{[\![K]\!]}$ given by Convention 3.22. We have to define predicates $B_1, \dots, B_m \in \mathfrak{M}^o$ such that for all $\kappa \in [\![K]\!]$,

$$\mathfrak{M} \quad \models \quad \Upsilon_m(\boldsymbol{B}) \quad \wedge \quad \langle\varphi(\kappa)[G(k) /\!\!/ F][|B_1 \dots B_m(t) \doteq_{\mathsf{g}} M| \,/\, (G(t) \doteq M)]\rangle \tag{A.4}$$

Let $\boldsymbol{\lambda} = \lambda_1, \dots, \lambda_n$ be the enumeration of $[\![L]\!]$. By assumption, for all $\kappa \in [\![K]\!]$, there are $A_1^\kappa, \dots, A_n^\kappa \in \mathfrak{M}^o$ such that

$$\mathfrak{M} \models \quad \Upsilon_n(\boldsymbol{A}^\kappa) \quad \wedge \quad \langle\varphi[|\boldsymbol{A}^\kappa(t) \doteq_{\boldsymbol{\lambda}} N| \,/\, (F(t) \doteq N)]\rangle$$

Given $\kappa \in [\![K]\!]$, since $\mathfrak{M} \models \Upsilon_n(\boldsymbol{A}^\kappa)$, for all $\mathfrak{a} \in \mathfrak{M}^\iota$ there is a unique $i_{\mathfrak{a},\kappa} \in \{1, \dots, n\}$ such that $\mathfrak{a} \in A^\kappa_{i_{\mathfrak{a},\kappa}}$. Note that for each subformula $F(t) \doteq N$ of φ, for all $\kappa \in [\![K]\!]$ and all $\mathfrak{a} \in \mathfrak{M}^\iota$ we have

$$\mathfrak{M} \models \quad \bigvee_{\substack{1 \le i \le n \\ \lambda_i = [\![N']\!]}} A_i^\kappa(\mathfrak{a}) \quad \Longleftrightarrow \quad \langle\lambda_{i_{\mathfrak{a},\kappa}} \doteq N'\rangle$$

where N' is a closed instance of N.

We define by Comprehension, for $j \in \{1, \dots, m\}$, a predicate B_j such that

$$\mathfrak{M} \models \quad (\forall x) \left[x \in B_j \quad \Longleftrightarrow \quad \left(\bigwedge_{\substack{\kappa \in [\![K]\!] \\ 1 \le i \le n}} \left[A_i^\kappa(x) \Rightarrow \langle\lambda_i \doteq \mathsf{g}_j(\kappa)\rangle\right] \quad \wedge \quad \bigwedge_{j' < j} \neg B_{j'}(x) \right) \right]$$

We have to show (A.4). We begin with $\Upsilon_m(\boldsymbol{B})$. First note that the B_j's are disjoint by construction. We moreover have to show that each $\mathfrak{a} \in \mathfrak{M}^\iota$ belongs to some B_j. So let $\mathfrak{a} \in \mathfrak{M}^\iota$. Note that there is $\mathsf{g} \in [\![L]\!]^{[\![K]\!]}$ such that for each $\kappa \in [\![K]\!]$ we have $\mathsf{g}(\kappa) = \lambda_{i_{\mathfrak{a},\kappa}}$. Since $i_{\mathfrak{a},\kappa}$ is unique such that $\mathfrak{a} \in A^\kappa_{i_{\mathfrak{a},\kappa}}$, we indeed have

$$\mathfrak{M} \models \quad \bigwedge_{\substack{\kappa \in [\![K]\!] \\ 1 \le i \le n}} \left[A_i^\kappa(\mathfrak{a}) \quad \Longrightarrow \quad \lambda_i \doteq \mathsf{g}(\kappa)\right]$$

For $\mathfrak{a} \in \mathfrak{M}^\iota$ let $j_{\mathfrak{a}}$ be the unique $j \in \{1, \dots, m\}$ such that $\mathfrak{a} \in B_j$. It remains to show that for all $\kappa \in [\![K]\!]$,

$$\mathfrak{M} \quad \models \quad \langle\varphi(\kappa)[@_{K,L}(G,k) /\!\!/ F][|B_1 \dots B_m(t) \doteq_{\mathsf{g}} M| \,/\, (G(t) \doteq M)]\rangle$$

Recalling that

$$(F(t) \doteq N)[@_{K,L}(G,k) /\!\!/ F] \quad = \quad \exists f \mathrel{\dot\in} L^K \,(G(t) \doteq f \;\wedge\; @_{K,L}(f,k) \doteq N)$$

this follows from the fact that for all subformula $F(t) \doteq N$ of φ, for all $\kappa \in [\![K]\!]$ we have

$$\mathfrak{M} \models \quad |\boldsymbol{A}^{\boldsymbol{\kappa}}(t) \doteq_{\boldsymbol{\lambda}} N'| \quad \Longleftrightarrow \quad \bigvee_{1 \le j \le m} \big(|\boldsymbol{B}(t) \doteq_{\mathsf{g}} \mathsf{g}_j| \;\wedge\; \langle @_{K,L}(\mathsf{g}_j, \kappa) \doteq N'\rangle\big) \tag{A.5}$$

where N' is a closed instance of N. Then (A.5) amounts to showing that for $\kappa \in [\![K]\!]$ and $\mathfrak{a} \in \mathfrak{M}^\iota$ we have

$$\lambda_{i_{\mathfrak{a},\kappa}} = @_{K,L}(\mathsf{g}_{j_{\mathfrak{a}}}, \kappa)$$

But by definition, $j_{\mathfrak{a}}$ is the least $j \in \{1, \dots, m\}$ such that

$$\mathfrak{M} \models \quad \bigwedge_{\kappa \in [\![K]\!] \;\&\; 1 \le i \le n} [A_i^\kappa(\mathfrak{a}) \quad \Longrightarrow \quad \langle\lambda_i \doteq @_{K,L}(\mathsf{g}_j, \kappa)\rangle]$$

and we are done since $\mathfrak{a} \in A^{\kappa}_{i_{\mathfrak{a},\kappa}}$. ☐

A.4. **From** MSO **to** FSO (§**3.6.2).** We give here the remaining proofs of §3.6, which concern the translation

$$(-)^\circ \quad : \quad \mathsf{MSO} \quad \longrightarrow \quad \mathsf{FSO}$$

We repeat here its definition. Assume given a FSO-Function variable F_X for each monadic MSO-variable X. The map $(-)^\circ$ is inductively defined as follows:

$$\begin{array}{rclcrcl}
(X(t))^\circ & := & F_X(t) \doteq 1 & \qquad & (\varphi \lor \psi)^\circ & := & \varphi^\circ \lor \psi^\circ \\
(t \doteq u)^\circ & := & t \doteq u & & (\neg\varphi)^\circ & := & \neg(\varphi^\circ) \\
(t \mathrel{\dot{\leq}} u)^\circ & := & t \mathrel{\dot{\leq}} u & & ((\exists x)\varphi)^\circ & := & (\exists x)\varphi^\circ \\
 & & & & ((\exists X)\varphi)^\circ & := & (\exists F_X : 2)\varphi^\circ
\end{array}$$

We first check that the translation is correct w.r.t. the theories of MSO and FSO.

Proposition A.8. *For each closed* MSO*-formula* φ*,*

$$\mathsf{MSO} \vdash \varphi \qquad \textit{implies} \qquad \mathsf{FSO} \vdash \varphi^\circ$$

Proof. Consider Γ, φ, with free monadic variables among $\boldsymbol{Y} = Y_1, \dots, Y_n$. Then we prove by induction on MSO-derivations that

$$\Gamma \vdash_{\mathsf{MSO}} \varphi \qquad \text{implies} \qquad \boldsymbol{F_Y} : \mathbf{2},\, \Gamma^\circ \vdash_{\mathsf{FSO}} \varphi^\circ$$

We first discuss the axioms of MSO (§2). The Equality axioms of MSO translate to

$$(\forall x)(x \doteq x) \qquad \text{and} \qquad (\forall x)(\forall y)\,(x \doteq y \implies \varphi^\circ[x/z] \implies \varphi^\circ[y/z])$$

and directly follow from the Equality axioms of FSO. Similarly, the Tree axioms of MSO are translated to themselves and follow from the corresponding axioms of FSO. The Comprehension scheme $(\exists X)(\forall y)\,[X(y) \Leftrightarrow \varphi]$ translates to

$$(\exists F_X : 2)(\forall y)\big[F_X(y) \iff \varphi^\circ\big]$$

and directly follows from Theorem 3.33 (Comprehension for Product Types). The Induction axiom of MSO translates to

$$(\forall F_X : 2)\Big(F_X(\dot{\varepsilon}) \quad \implies \quad (\forall y)\bigwedge\nolimits_{d \in \mathscr{D}}\Big[F_X(y) \implies F_X(\mathsf{S}_d(y))\Big] \quad \implies \quad (\forall y)F_X(y)\Big)$$

which itself follows from the Induction scheme of FSO (§3.4.2).

Since the translation $(-)^\circ$ commutes over $\lor, \neg, (\exists x)(-)$, the cases of the corresponding rules follow directly from the induction hypothesis. The rules for $(\exists X)(-)$ translate to

$$\frac{\boldsymbol{F_Y} : \mathbf{2},\, \Gamma^\circ \vdash (\exists F_X : 2)\varphi^\circ \qquad \boldsymbol{F_Y} : \mathbf{2},\, F_X : 2,\, \Gamma^\circ,\, \varphi^\circ \vdash \psi^\circ}{\boldsymbol{F_Y} : \mathbf{2},\, \Gamma^\circ \vdash \psi^\circ}$$

$$\frac{\boldsymbol{F_Y} : \mathbf{2},\, \Gamma^\circ \vdash \varphi^\circ[F_Y/F_X]}{\boldsymbol{F_Y} : \mathbf{2},\, \Gamma^\circ \vdash (\exists F_X : 2)\varphi^\circ}$$

which are both derivable in FSO (for the second one, use that $\boldsymbol{F_Y} : \mathbf{2}, \Gamma^\circ \vdash F_Y : 2$). ☐

We are now going to translate φ° back to an MSO-formula. But note that φ° may contain free Function variables F_X (one for each free monadic variable X of the original formula ψ), so in order to apply the translation $\langle - \rangle$, we need first to replace in φ° all atomic formulae of the form $F_X(t) \doteq 1$ by suitable *extended* atomic formulae.

Definition A.9. Given an MSO-formula φ with free monadic variables among $\boldsymbol{Y} = Y_1, \dots, Y_n$, we let $\varphi^{\bullet}_{\boldsymbol{Y}}$ be the following MSO-formula

$$\varphi^{\bullet}_{\boldsymbol{Y}} \quad := \quad \left\langle \varphi^{\circ} \left[\overrightarrow{|X_0 X_1(t) \doteq_{0,1} 1| / F_Y(t) \doteq 1} \right] \right\rangle$$

Note that the extended FSO-formula $\varphi^{\circ}[\overrightarrow{|X_0 X_1(t) \doteq_{0,1} 1| / F_Y(t) \doteq 1}]$ contains no free Function variables. We write $\varphi^{\bullet}$ for $\varphi^{\bullet}_{\boldsymbol{Y}}$ when $\boldsymbol{Y}$ is exactly the set of free monadic variables of φ.

It remains to show that $\varphi^{\bullet}$ and φ are MSO-equivalent. This is in essence given by the following observation:

$$\mathsf{MSO} \vdash \quad \langle |X_0 X_1(t) \doteq_{0,1} 1| \rangle \iff X_1(t) \tag{A.6}$$

which is easily generalized by substitution.

Lemma A.10. *For all* MSO*-formula* φ *with free monadic variables among* $\boldsymbol{Y} = Y_1, \dots, Y_n$*, we have*

$$\mathsf{MSO} \vdash \quad \varphi[\overrightarrow{X_1(t)/Y(t)}] \iff \varphi^{\bullet}_{\boldsymbol{Y}}$$

Proof. The proof is by induction on φ. We reason by cases on φ.

- If φ is $Y_i(t)$, then we are done since $\mathsf{MSO} \vdash X_1^i(t) \Leftrightarrow \langle |X_0^i X_1^i(t) \doteq_{0,1} 1| \rangle$.
- If φ is $t \star u$ for $\star \in \{\doteq, \dot{\leq}\}$, then we are done since $\langle \varphi^{\circ} \rangle = \varphi$.
- If φ is $\psi_0 \vee \psi_1$, $\neg\psi$ or $(\exists x)\psi$, then we are done by induction hypothesis.
- Otherwise φ is $(\exists X)\psi$. Assume w.l.o.g. that X is Y_{n+1}. Then by induction hypothesis we have

  $$\mathsf{MSO} \vdash \quad \psi[\overrightarrow{X_1(t)/Y(t)}, X_1^{n+1}(t)/Y_{n+1}(t)] \iff \psi^{\bullet}_{\boldsymbol{Y}Y_{n+1}}$$

  Now, by (A.6), the formula $\psi^{\bullet}_{\boldsymbol{Y}Y_{n+1}}$, *i.e.*

  $$\left\langle \psi^{\circ} \left[\overrightarrow{|X_0 X_1(t) \doteq_{0,1} 1| / F_Y(t) \doteq 1}\ ,\ |X_0^{n+1} X_1^{n+1}(t) \doteq_{0,1} 1| / F_{Y_{n+1}}(t) \doteq 1 \right] \right\rangle$$

  is MSO-equivalent to a formula in which X_0^{n+1} does not occur. Hence, by MSO-Comprehension, we have

  $$\mathsf{MSO} \vdash \quad (\exists X_0) \left(\Upsilon_2(X_0^{n+1}, X_1^{n+1}) \ \wedge\ \psi^{\bullet}_{\boldsymbol{Y}Y_{n+1}} \right) \iff \psi^{\bullet}_{\boldsymbol{Y}Y_{n+1}}$$

  so that

  $$\begin{aligned} \mathsf{MSO} \vdash \quad (\exists X_1^{n+1})\psi[\overrightarrow{X_1(t)/Y(t)}\ ,\ X_1^{n+1}(t)/Y_{n+1}(t)] \iff & \\ (\exists X_0^{n+1} X_1^{n+1}) \left(\Upsilon_2(X_0^{n+1}, X_1^{n+1}) \ \wedge\ \psi^{\bullet}_{\boldsymbol{Y}Y_{n+1}} \right) & \end{aligned}$$

  and we are done. □

In case φ is closed, we have $\varphi^{\bullet} = \langle \varphi^{\circ} \rangle$, so that Lemma A.10 implies

$$\mathsf{MSO} \vdash \quad \varphi \iff \langle \varphi^{\circ} \rangle$$

We have now everything we need to prove Theorem 3.26.

Theorem A.11 (Theorem 3.26)**.** *Given a closed* MSO*-formula* φ*,*

$$\mathsf{FSO} \vdash \varphi^{\circ} \quad \textit{iff} \quad \mathsf{MSO} \vdash \varphi$$

Proof. If $\mathsf{FSO} \vdash \varphi^{\circ}$, then Theorem A.1 implies that $\mathsf{MSO} \vdash \langle \varphi^{\circ} \rangle$, and we conclude by Lemma A.10. The converse direction is given by Proposition A.8. □

Theorem A.11 can actually be extended to FSO formulae. The remaining key inductive argument is given by the following, which corresponds to (3.6) in Proposition 3.27.

Lemma A.12 (Proposition 3.27, (3.6))**.** *Given a closed* FSO*-formula* φ*, we have*

$$\mathsf{FSO} \vdash \quad \varphi \iff \langle \varphi \rangle^{\circ}$$

Proof. The proof is by induction on formulae. Similarly as in the proof of Theorem A.1, consider a FSO formula φ with free HF-variables among $\boldsymbol{\ell} = \ell_1, \dots, \ell_p$ and with free Function variables among $\boldsymbol{F} = F_1, \dots, F_n$. We show that for all $\boldsymbol{\lambda} = \lambda_1, \dots, \lambda_p$ and all closed HF-terms $\boldsymbol{K} = K_1, \dots, K_n$, we have

$$\boldsymbol{F} : \boldsymbol{K} \ , \ \boldsymbol{F}_{\boldsymbol{X_1}} : \boldsymbol{2} \ , \ \dots \ . \ \boldsymbol{F}_{\boldsymbol{X_n}} : \boldsymbol{2} \ , \ (\Gamma^{\Upsilon}_{\overrightarrow{\boldsymbol{X}}})^{\circ} \ , \ \Gamma_{\boldsymbol{F}, \overrightarrow{\boldsymbol{F}_{\boldsymbol{X}}}} \ \vdash_{\mathsf{FSO}} \quad \varphi[\boldsymbol{\lambda}/\boldsymbol{\ell}] \quad \iff \quad \langle \varphi^{\star}[\boldsymbol{\lambda}/\boldsymbol{\ell}] \rangle^{\circ}$$

where, for $[\![K_i]\!] = \{\kappa_{i,1}, \dots, \kappa_{i,c_i}\}$, we let

$$\begin{array}{rcl} \Gamma^{\Upsilon}_{\overrightarrow{\boldsymbol{X}}} & = & \Upsilon_{c_1}(X_{1,1}, \dots, X_{1,c_1}) \ , \ \dots \ , \ \Upsilon_{c_n}(X_{n,1}, \dots, X_{n,c_n}) \\ \Gamma_{\boldsymbol{F}, \overrightarrow{\boldsymbol{F}_{\boldsymbol{X}}}} & = & \bigwedge_{1 \le i \le n} \bigwedge_{1 \le j \le c_i} (\forall x) \big(F_i(x) \doteq \kappa_{i,j} \iff F_{X_{i,j}}(x) \doteq 1 \big) \\ \varphi^{\star} & = & \varphi \left[|\boldsymbol{X}_1(t) \doteq_{\boldsymbol{\kappa_1}} L| \ / \ F_1(t) \doteq L \ , \ \dots \ , \ |\boldsymbol{X}_n(t) \doteq_{\boldsymbol{\kappa_n}} L| \ / \ F_n(t) \doteq L \right] \end{array}$$

The cases of atomic formulae of the form $t \star u$ are trivial since

$$t \star u \quad = \quad \langle (t \star u)^{\star}[\boldsymbol{\lambda}/\boldsymbol{\ell}] \rangle^{\circ}$$

Furthermore, the cases of atomic formulae of the form $K \dot{\star} L$ directly follow from the Axioms on HF-Sets (Remark 3.15, §3.4.4). Assume now that φ is $F_i(t) \doteq L$. Then we have to show that

$$\boldsymbol{F} : \boldsymbol{K} \ , \ \overrightarrow{\boldsymbol{F}_{\boldsymbol{X}} : \boldsymbol{2}} \ , \ (\Gamma^{\Upsilon}_{\overrightarrow{\boldsymbol{X}}})^{\circ} \ , \ \Gamma_{\boldsymbol{F}, \overrightarrow{\boldsymbol{F}_{\boldsymbol{X}}}} \ \vdash_{\mathsf{FSO}} \quad F_i(t) \doteq L[\boldsymbol{\lambda}/\boldsymbol{\ell}] \quad \iff \quad \bigvee_{\substack{1 \le j \le c_i \\ \kappa_j = [\![L[\boldsymbol{\lambda}/\boldsymbol{\ell}]]\!]}} F_{X_{i,j}}(t) \doteq 1$$

- *Proof.* For the left-to-right direction, note that

$$\boldsymbol{F} : \boldsymbol{K} \ \vdash_{\mathsf{FSO}} \quad F_i(t) \doteq L[\boldsymbol{\lambda}/\boldsymbol{\ell}] \quad \Longrightarrow \quad L[\boldsymbol{\lambda}/\boldsymbol{\ell}] \mathrel{\dot{\in}} K_i$$

so that the Axioms on HF-Sets (Remark 3.15, §3.4.4) give

$$\boldsymbol{F} : \boldsymbol{K} \ \vdash_{\mathsf{FSO}} \quad F_i(t) \doteq L[\boldsymbol{\lambda}/\boldsymbol{\ell}] \quad \Longrightarrow \quad \bigvee_{1 \le j \le c_i} [\![L[\boldsymbol{\lambda}/\boldsymbol{\ell}]]\!] \doteq \kappa_j$$

and we conclude by $\Gamma_{\boldsymbol{F}, \overrightarrow{\boldsymbol{F}_{\boldsymbol{X}}}}$.

Conversely,

$$\Gamma_{\boldsymbol{F}, \overrightarrow{\boldsymbol{F}_{\boldsymbol{X}}}} \ \vdash_{\mathsf{FSO}} \quad \left(\bigvee_{\substack{1 \le j \le c_i \\ \kappa_j = [\![L[\boldsymbol{\lambda}/\boldsymbol{\ell}]]\!]}} F_{X_{i,j}}(t) \doteq 1 \right) \quad \Longrightarrow \quad F_i(t) \doteq [\![L[\boldsymbol{\lambda}/\boldsymbol{\ell}]]\!]$$

and we conclude using the Axioms on HF-Sets (Remark 3.15, §3.4.4). $\square$

As for the inductive cases, that of $\vee, \neg, (\exists x)(-)$ are trivial from the inductive hypotheses since all the involved translations commute over these connective. Consider now the case of $(\exists k \star L)\varphi$. We have to show that

$$\boldsymbol{F} : \boldsymbol{K} \ , \ \overrightarrow{\boldsymbol{F}_{\boldsymbol{X}} : \boldsymbol{2}} \ , \ (\Gamma^{\Upsilon}_{\overrightarrow{\boldsymbol{X}}})^{\circ} \ , \ \Gamma_{\boldsymbol{F}, \overrightarrow{\boldsymbol{F}_{\boldsymbol{X}}}} \ \vdash_{\mathsf{FSO}} \quad (\exists k \star L[\boldsymbol{\lambda}/\boldsymbol{\ell}])\varphi[\boldsymbol{\lambda}/\boldsymbol{\ell}] \ \Leftrightarrow \ \bigvee_{\kappa \star [\![L[\boldsymbol{\lambda}/\boldsymbol{\ell}]]\!]} \langle \varphi^{\star}[\boldsymbol{\lambda}/\boldsymbol{\ell}][\kappa/k] \rangle^{\circ}$$

But this directly follows from the induction hypothesis and the axioms on HF-Sets (Remark 3.15, §3.4.4) since

$$V_\omega \models \quad (\forall k \star L[\boldsymbol{\lambda}/\boldsymbol{\ell}])\Big(k \star [\![L[\boldsymbol{\lambda}/\boldsymbol{\ell}]]\!]\Big) \;\;\wedge\;\; (\forall k \star [\![L[\boldsymbol{\lambda}/\boldsymbol{\ell}]]\!])\Big(k \star L[\boldsymbol{\lambda}/\boldsymbol{\ell}]\Big)$$

It remains to deal with $(\exists F : K)\varphi$. We have to show that under $\boldsymbol{F} : \boldsymbol{K}$, $\overrightarrow{\boldsymbol{F_X} : \boldsymbol{2}}$, $(\Gamma^{\Upsilon}_{\overrightarrow{\boldsymbol{X}}})^\circ$, $\Gamma_{\boldsymbol{F},\overrightarrow{\boldsymbol{F_X}}}$, FSO proves that $(\exists F : K')\varphi[\boldsymbol{\lambda}/\boldsymbol{\ell}]$ is equivalent to

$$\Big[(\exists X_1,\ldots,X_c)\Big(\Upsilon_c(\boldsymbol{X}) \;\;\wedge\;\; \langle \varphi^*[|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}} L| \;/\; (F(t) \doteq L)][\boldsymbol{\lambda}/\boldsymbol{\ell}]\rangle\Big)\Big]^\circ$$

where $K' := K[\boldsymbol{\lambda}/\boldsymbol{\ell}]$ where $\boldsymbol{\kappa} = \kappa_1,\ldots,\kappa_c$ enumerates $[\![K']\!]$. By induction hypothesis we have

$$F : K' \,,\; \boldsymbol{F} : \boldsymbol{K} \,,\; \boldsymbol{F_X} : \boldsymbol{2} \,,\; \overrightarrow{\boldsymbol{F_X} : \boldsymbol{2}} \,,\; \Upsilon_c(\boldsymbol{X})^\circ \,,\; (\Gamma^{\Upsilon}_{\overrightarrow{\boldsymbol{X}}})^\circ \,,\; \Gamma_{F\boldsymbol{F},\to F_X \overrightarrow{\boldsymbol{F_X}}} \;\;\vdash_{\mathsf{FSO}}$$
$$\varphi[\boldsymbol{\lambda}/\boldsymbol{\ell}] \quad\Longleftrightarrow\quad \langle \varphi^*[|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}} L| \;/\; (F(t) \doteq L)][\boldsymbol{\lambda}/\boldsymbol{\ell}]\rangle^\circ$$

We first show

$$\boldsymbol{F} : \boldsymbol{K} \,,\; \overrightarrow{\boldsymbol{F_X} : \boldsymbol{2}} \,,\; (\Gamma^{\Upsilon}_{\overrightarrow{\boldsymbol{X}}})^\circ \,,\; \Gamma_{\boldsymbol{F},\overrightarrow{\boldsymbol{F_X}}} \;\;\vdash_{\mathsf{FSO}}\quad (\exists F : K')\varphi[\boldsymbol{\lambda}/\boldsymbol{\ell}] \quad\Longrightarrow$$
$$\Big[(\exists X_1,\ldots,X_c)\Big(\Upsilon_c(\boldsymbol{X}) \;\;\wedge\;\; \langle \varphi^*[|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}} L| \;/\; (F(t) \doteq L)][\boldsymbol{\lambda}/\boldsymbol{\ell}]\rangle\Big)\Big]^\circ$$

- *Proof.* Given $(F : 2)$ such that $\varphi[\boldsymbol{\lambda}/\boldsymbol{\ell}]$, for each $i \in \{1,\ldots,c\}$ define by Comprehension (Theorem 3.33) $(F_{X_i} : 2)$ as

  $$(\forall x)\,(x \in F_{X_i} \quad\Longleftrightarrow\quad F(x) \doteq \kappa_i)$$

  We then easily get $\Upsilon_c(\boldsymbol{X})^\circ$, and we conclude by induction hypothesis. □

We finally show

$$\boldsymbol{F} : \boldsymbol{K} \,,\; \overrightarrow{\boldsymbol{F_X} : \boldsymbol{2}} \,,\; \Gamma^{\Upsilon}_{\overrightarrow{\boldsymbol{X}}} \,,\; \Gamma_{\boldsymbol{F},\overrightarrow{\boldsymbol{F_X}}} \;\;\vdash_{\mathsf{FSO}}$$
$$\Big[(\exists X_1,\ldots,X_c)\Big(\Upsilon_c(\boldsymbol{X}) \;\wedge\; \langle \varphi^*[|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}} L| \;/\; (F(t) \doteq L)][\boldsymbol{\lambda}/\boldsymbol{\ell}]\rangle\Big)\Big]^\circ \;\;\Rightarrow\;\; (\exists F : K')\varphi[\boldsymbol{\lambda}/\boldsymbol{\ell}]$$

- *Proof.* Assume given $F_{X_1},\ldots,F_{X_c}$ such that $\Upsilon_c(\boldsymbol{X})^\circ$ and

  $$\langle \varphi^*[|\boldsymbol{X}(t) \doteq_{\boldsymbol{\kappa}} L| \;/\; (F(t) \doteq L)][\boldsymbol{\lambda}/\boldsymbol{\ell}]\rangle^\circ$$

  Using the assumption $\Upsilon_x(\boldsymbol{X})^\circ$, by HF-Bounded Choice for Functions, let $(F : K')$ such that

  $$(\forall x) \bigwedge_{1\le i\le c} (F(x) \doteq \kappa_i \quad\Longleftrightarrow\quad F_{X_i}(x) \doteq 1)$$

  and we are done by induction hypothesis. □

This concludes the proof of Lemma A.12. □

We then easily obtain the missing part of Proposition 3.27.

Proposition A.13 (Proposition 3.27)**.** *For a closed* FSO*-formula* φ*, we have the following.*

$$\mathsf{FSO} \vdash \varphi \quad \textit{iff} \quad \mathsf{MSO} \vdash \langle\varphi\rangle \tag{A.7}$$
$$\mathfrak{T} \models \varphi \quad \textit{iff} \quad \mathfrak{T} \models \langle\varphi\rangle \tag{A.8}$$

Proof. Consider first (A.7). Theorem A.11 gives

$$\mathsf{MSO} \vdash \langle\varphi\rangle \quad \text{iff} \quad \mathsf{FSO} \vdash \langle\varphi\rangle^\circ$$

and we conclude with Lemma A.12. As for (A.8), Lemma 3.25 gives

$$\mathfrak{T} \models \langle\varphi\rangle \quad \text{iff} \quad \mathfrak{T} \models \langle\varphi\rangle^\circ$$

and we conclude by combining Lemma A.12 with Proposition 3.20 (correctness of FSO w.r.t. $\mathfrak{T}$). $\square$

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