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Abstract. In this note we propose an approach to some questions about the birational geometry of smooth cubic fourfolds through the theory of Chow motives. We introduce the transcendental part $t(X)$ of the motive of X and prove that it is isomorphic to the (twisted) transcendental part $h_2^{tr}(F(X))$ in a suitable Chow-Künneth decomposition for the motive of the Fano variety of lines $F(X)$. Then we explain the relation between $t(X)$ and the motives of some special surfaces of lines contained in $F(X)$. If X is a special cubic fourfold in the sense of Hodge theory, and $F(X) \simeq S^{[2]}$, with S a K3 surface associated to X, then we show that $t(X) \simeq t_2(S)(1)$. Moreover we relate the existence of an isomorphism between the transcendental motive $t(X)$ and the (twisted) transcendental motive of a K3 surface to conjectures by Hassett and Kuznetsov on the rationality of a special cubic fourfold. Finally we give examples of cubic fourfolds such that the motive $t(X)$ is finite dimensional and of abelian type.

1. Introduction

We will work over the complex field. Cubic fourfolds are among the most mysterious objects in algebraic geometry. Despite the simplicity of the definition of such classically flavoured objects, the birational geometry of cubic fourfolds is extremely hard to understand and many modern techniques (Hodge theory, derived categories, etc. - see *e.g.* [Kuz, Has 2, AT] for details) have been successfully deployed in order to have a deeper understanding. In any case, the rationality of the generic cubic fourfold is still an open problem. Also the finite dimensionality of the motive $h(X)$ of a cubic fourfold, as conjectured by several authors (see [Ki] , [An]), is known to hold only in some scattered cases.

In this paper we relate the finite dimensionality of $h(X)$ with the existence of an associated K3 surface and compare this condition with conjectures on the rationality of X.

We will denote by $\mathcal{M}_{rat}(\mathbf{C})$ the (covariant) category of Chow motives (with $\mathbf{Q}$-coefficients), whose objects are of the form (X, p, n), where X is a smooth projective variety over $\mathbf{C}$ of dimension d, p is an idempotent in the ring $A^d(X \times X) = CH^d(X \times X) \otimes \mathbf{Q}$ and $n \in \mathbf{Z}$. If X and Y are smooth projective varieties over $\mathbf{C}$, then the morphisms $\mathrm{Hom}_{\mathcal{M}_{rat}}(h(X), h(Y))$ of their motives $h(X)$ and $h(Y)$ are given by correspondences in the Chow groups $A^*(X \times Y) = CH^*(X \times Y) \otimes \mathbf{Q}$. More precisely, in our covariant setting, we have

$$\mathrm{Hom}_{\mathcal{M}_{rat}}(X, p, m), (Y, q, n)) = q \circ A_{d+m-n}(X \times Y) \circ p \subset A_{d+m-n}(X \times Y)$$

where X is irreducible of dimension d and $\circ$ means composition of correspondences (see [KMP, 7.1.1]). The category $\mathcal{M}_{rat}(\mathbf{C})$ is additive,pseudo-abelian, rigid and has a tensor structure (see [KMP]). The unit motive is $\mathbf{1} = (\mathrm{Spec}(\mathbf{C}, 1, 0)$: it is a

unit for the tensor structure. The *Lefschetz motive* $\mathbf{L}$ is defined via the motive of the projective line: $h(\mathbf{P}^1) = \mathbf{1} \oplus \mathbf{L}$ and there is an isomorphism $\mathbf{L} \simeq (\mathrm{Spec}(k), 1, 1)$ For every motive $M = (X, p, m)$ the Tate twist $M(r)$ is the motive $(X, p, m + r)$. Note that, with our covariant convention, $M(r) \simeq M \otimes \mathbf{L}^{\otimes r}$ for $r \geq 0$.
The Chow groups of a motive $(X, p, m) \in \mathcal{M}_{rat}(\mathbf{C})$ are defined as follows

$$A^i(X, p, m) = \mathrm{Hom}_{\mathcal{M}_{rat}}((X, p, m), \mathbf{L}^i) = p^* A^{i-m}(X)$$
$$A_i(X, p, m) = \mathrm{Hom}_{\mathcal{M}_{rat}}((\mathbf{L}^i, (X, p, m)) = p_* A_{i-m}(X).$$

A similar definition holds for the category $\mathcal{M}_{hom}(\mathbf{C})$ of homological motives, with respect to singular cohomology $H^*(X)$, where

$$H^i(X, p, m) = p^* H^{i-2m}(X) \ ; \ H_i(X, p, m) = p_* H_{i-2m}(X).$$

Let X be a smooth projective variety over $\mathbf{C}$. We say that its motive $h(X) \in \mathcal{M}_{rat}(\mathbf{C})$ has a *Chow-Künneth decomposition* (C-K for short) if there exist orthogonal projectors $\pi_i = \pi_i(X) \in \mathrm{Corr}_0(X, X) = A^d(X \times X)$, for $0 \leq i \leq 2d$, such that $cl^d(\pi_i)$ is the $(i, 2d - i)$-component of Δ_X in $H^{2d}(X \times X)$ and

$$[\Delta_X] = \sum_{0 \leq i \leq 2d} \pi_i.$$

This implies that in $\mathcal{M}_{\mathrm{rat}}$ the motive $h(X)$ decomposes as follows:

$$h(X) = \bigoplus_{0 \leq i \leq 2d} h_i(X),$$

where $h_i(X) = (X, \pi_i, 0)$. Moreover

$$H^*(h_i(X)) = H^i(X), \qquad H_*(h_i(X)) = H_i(X)$$

If we have $\pi_i = \pi^t_{2d-i}$ for all i, we say that the C-K decomposition is *self-dual*.

By the results in [KMP, 7.2.3] every smooth projective surface S has a *reduced C-K decomposition* $h(S) = \sum_{0 \leq 4} h_i(S)$ with

$$h_2(S) = h_2^{alg}(S) \oplus t_2(S) = (S, \pi_2^{alg}) \oplus (S, \pi_2^{tr}).$$

Here

$$\pi_2^{alg} = \sum_i \frac{[D_i] \times [D_i]}{< [D_i], [D_i] >}$$

where $[D_i]$ as an orthogonal basis of $NS(S)$ and $\pi_2^{tr} = \pi_2 - \pi_2^{alg}$. Then $h_2^{alg}(S) \simeq \mathbf{L}^{\rho(S)}$, where $\rho(S)$ is the rank of the Neron-Severi group. We also have

$$H^2(S) = H^2_{alg}(S) \oplus H^2_{tr}(S) = \pi_2^{alg} H^2(S) \oplus \pi_2^{tr} H^2(S)$$

The motive $t_2(S)$ is called the *transcendental motive* of S. It is a birational invariant and

$$H^*(t_2(S)) = H^2(t_2(S)) = T(S)_{\mathbf{Q}} \ ; \ A^2(t_2(S)) \simeq K(S),$$

where $T(S)$ is the transcendental lattice and $K(S)$ is the Albanese kernel, i.e the kernel of the map $A_0(S)_{hom} \to \mathrm{Alb}(S)$.

We recall the definition of finite dimensionality introduced by S.Kimura in [Ki]. Let $M = h(X) \in \mathcal{M}_{\mathrm{rat}}(\mathbf{C})$ and let Σ_n be the symmetric group of order n. Then we denote by $\wedge^n M$ the image of $M^{\otimes n}$ under the projector

$$(1/n!) \sum_{\sigma \in \Sigma_n} sgn(\sigma) \Gamma_\sigma$$

while S^nM is its image under the projector

$$(1/n!)\sum_{\sigma\in\Sigma_n}\Gamma_\sigma.$$

A motive M is said to be *evenly (oddly) finite-dimensional* if $\wedge^n M = 0$ ($S^nX = 0$) for some n. A motive M is finite-dimensional if it can be decomposed into a direct sum $M_+ \oplus M_-$ where M_+ is evenly finite-dimensional and M_- is oddly finite-dimensional. According to Kimura's conjecture in [Ki] all motives should be finite dimensional. The conjecture is known to hold for curves, rational surfaces, surfaces with $p_g(X) = 0$, which are not of general type, abelian varieties and some 3-folds. If $d = \dim X \leq 3$, then the finite dimensionality of $h(X)$ is a birational invariant (see [GG, Lemma 7.1]), the reason being that in order to make regular a birational map $X \to Y$ between smooth projective 3-folds one needs to blow up only points and curves, whose motives are finite dimensional.

If X is a complex Fano threefold, then $h(X)$ is finite dimensional and of abelian type, i.e. it lies in the subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by the motives of abelian varieties, see [GG, Thm. 5.1]. The proof is based on the fact that all the Chow groups $A_i(X)_{alg}$ of algebraically trivial cycles are representable. More generally, if $M \in \mathcal{M}_{rat}(\mathbf{C})$ is a motive such that $A_i(M)_{alg}$ is representable, for all $i \geq 0$, then M is finite dimensional of abelian type, see [Vial 2].
In particular, if X is a cubic threefold in $\mathbf{P}^4_{\mathbf{C}}$, then $h(X)$ has the following Chow-Künneth decomposition

$$h(X) = \mathbf{1} \oplus \mathbf{L} \oplus N \oplus \mathbf{L}^2 \oplus \mathbf{L}^3.$$

Here $N = h_1(J) \otimes \mathbf{L} = h_1(J)(1)$, with J an abelian variety, isogenous to the intermediate Jacobian $J^2(X)$. As proved by Clemens and Griffiths, X is not rational, because the principally polarized abelian variety $J^2(X)$ is not split by Jacobians of curves. Therefore in the case of a cubic threefold there is an "invariant" (up to the product with the Jacobian of a curve), the intermediate Jacobian, which controls the rationality of X and also determines the non-trivial part of its Chow motive.

Let X be a cubic fourfold in $\mathbf{P}^5_{\mathbf{C}}$. In Section 2, we show that the motive $h(X)$ has Chow-Künneth decomposition as follows

$$h(X) = \mathbf{1} \oplus \mathbf{L} \oplus (\mathbf{L}^2)^{\oplus\rho_2} \oplus t(X) \oplus \mathbf{L}^3 \oplus \mathbf{L}^4$$

where ρ_2 is the rank of $A^2(X)$ and all the summands of $h(X)$, but possibly $t(X)$, are finite dimensional, see (2.1). The motive $t(X)$ is the *transcendental motive* of X and the Chow group $A_1(X)_{hom} = A_1(X)_{alg} = A_1(t(X))$ is not representable. Let $F(X)$ be the Fano variety of lines contained in X. In fact, we show (see Prop. 2.8) that there exists a smooth projective surface S_l, the surface of lines in $F(X)$ meeting a general line $l \in X$, such that $t(X) \simeq t_2(S_l)(1)$.The surface S_l has $q(S_l) = 0$ and $p_g(S_l) > 0$. Therefore

$$A_0(S_l)_0 = A_0(t_2(S_l)) = A_1(X)_{hom}$$

and the group $A_0(S_l)_0$ of 0-cycles of degree 0 is not representable, by a famous result of Mumford. We also show (see Prop. 2.8 for details) that $A_1(X)_{hom} \simeq A_0(\Sigma_2)_0$, where Σ_2 is the surface of lines of the second kind in $F(X)$. Therefore the motive $h(X)$ is finite dimensional if either S_l or Σ_2 have a finite dimensional motive.

We relate then the transcendental motive of X to the motive of its Fano variety of lines $F(X)$ (F for short). For a smooth projective variety Y we denote by $\mathrm{Mot}(Y)$ the full pseudo-abelian tensor subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by $h(Y)$ and the Lefschetz motive $\mathbf{L}$.

Theorem 1.1. *Let $h(F)$ be the motive of $F(X)$, endowed with a Chow-Künneth decomposition, and let $h_2(F) \cong h_2^{alg} \oplus h_2^{tr}$ be the standard decomposition of $h_2(F)$. Then we have an isomorphism*

$$h_2^{tr}(F)(1) \cong h_4^{tr}(X) = t(X)$$

and therefore

$$\mathrm{Mot}(X) = \mathrm{Mot}(F(X)).$$

In some cases, we can say even more (see Sect. 3).

Theorem 1.2. *Suppose $F(X) \cong S^{[2]}$, with S a K3 surface. Then there is an isomorphism of motives*

$$t_2(S)(1) \cong t(X).$$

and hence

$$\mathrm{Mot}(X) = \mathrm{Mot}(F(X)) = \mathrm{Mot}(S)$$

In particular X has a finite dimensional motive if and only if the motive of S is finite dimensional, in which case the transcendental motives of X and S are both indecomposable. Note that, according to Kimura's conjecture and a conjecture by Y. André (see [An]), the motives of a cubic fourfold and of a K3 surface should be of abelian type.

The existence of a K3 surface whose transcendental motive is associated to $t(X)$ is also related to conjectures on the rationality of X, as follows. Let $\mathcal{C}_d$ be the Noether-Lefschetz divisor of special cubic fourfolds of discriminant d, as defined by B.Hassett [Has 1]. Recall that a cubic fourfold X is *special* if it contains a surface Z such that its cohomological class ζ in $H^4(X, \mathbf{Z})$ is not homologous to any multiple of γ^2, where γ is the hyperplane section. The discriminant d is defined as the discriminant of the intersection form on the sublattice of $H^4(X, \mathbf{Z})$ generated by ζ and the codimension 2 linear section. If d satisfies the following condition:

(**) d is not divisible by 4,9 or a prime $p \equiv 2(3)$

then X has an associated K3 surface, i.e the transcendental lattice $T(X)$ of X is Hodge isometric to the (twisted) transcendental lattice $T(S)$ of S. Also, according to a conjecture of Kuznetsov [Kuz] and results of Addington-Thomas [AT], a general member of $\mathcal{C}_d$ should be rational if (∗∗) holds.

In Section 4 we introduce the definition of associated motive for a cubic fourfold X. We say that the motive $h(X)$ is associated to the motive of a K3 surface S if there is an isomorphism between $t_2(S)(1)$ and $t(X)$, inducing an Hodge isometry between $T(S)_{\mathbf{Q}}(1)$ and $T(X)_{\mathbf{Q}}$. We relate this to conjectures about the rationality of X. Our main result in this direction is the following.

Proposition 1.3. *Let X be a general cubic fourfold in $\mathcal{C}_d$. Assuming Kimura's conjecture, the motive $h(X)$ is associated to the motive of a K3 surface, if d satisfies (∗∗). If d does not satisfy (∗∗), then there is no K3 surface S such that the motive $h(X)$ is associated to $h(S)$.*

In Section 5, by adapting some recent results by Vial [Vial 1] about motives of fibrations with rational fibers, we showcase classes of cubic fourfolds with an associated K3 surface - in a motivic sense. More precisely, thanks to [Has 1, Kuz, AHTV-A] we know that there are loci inside $\mathcal{C}_{18}$ and $\mathcal{C}_8$ where cubic fourfolds admit fibrations over $\mathbf{P}^2$ with rational fibers and a rational section. All those fourfolds are rational. For cubic fourfolds inside these loci we show that X has a finite dimensional motive if and only if $h(S)$ is finite dimensional, with S a K3 surface.

In section 6 we consider the case of a cubic fourfold with a symplectic automorphism σ and show that, if σ has order 3 and a fixed surface, then X is rational, its motive is of abelian type, and there is an associated K3 surface S such that $t_2(S) \simeq t_2(E_1 \times E_2)$ where the E_i are elliptic curves. If σ is a symplectic involution then $t(X) \simeq t_2(S)(1)$, where S is a K3 surface, a double cover of a cubic surface $C \subset \mathbf{P}^3$, ramified along a a degree 6 curve.

2. The motive of a cubic fourfold

In this section we give a Chow-Künneth decomposition of the motive $h(X)$ of a cubic fourfold and show that its transcendental part $h^{tr}_4(X) = t(X)$ is isomorphic to the (twisted) transcendental motive $h^{tr}_2(F(X))(1)$ coming from a suitable Chow-Künneth decomposition of the motive of the Fano variety of lines $F(X)$ (see Thm. 2.5). Note that, by a result of R.Laterveer [Lat 1], if $h(X)$ is finite dimensional then also $h(F(X))$ is finite dimensional. Then we show that

$$A_1(X)_{hom} = A_1(X)_{alg} \simeq A_1(t(X)) \simeq A_0(\Sigma_2)_0$$

where Σ_2 is the surface of lines of second type (*i.e.* such that $\mathcal{N}_{l/X} \simeq \mathcal{O}(1)^2 \oplus \mathcal{O}(-1)$) in $F(X)$. For a general X the surface Σ_2 is of general type with $p_g > 0$, see [Am], and hence the group $A_1(X)_{alg}$ is not representable.

Every cubic fourfold X is rationally connected and hence $CH_0(X) \simeq \mathbf{Z}$. Rational, algebraic and homological equivalences all coincide for cycles of codimension 2 on X. Hence the cycle map $CH^2(X) \to H^4(X, \mathbf{Z})$ is injective and $A^2(X) = CH^2(X) \otimes \mathbf{Q}$ is a vector subspace of dimension $\rho_2(X)$ of $H^4(X, \mathbf{Q})$. By the results in [TZ, Rk. 6.4] we have $A_1(X)_{hom} = A_1(X)_{alg}$. Moreover homological equivalence and numerical equivalence coincide for algebraic cycles on X, because the standard conjecture $D(X)$ holds true. Therefore $A_1(X)_{hom} = A_1(X)_{num}$.
A cubic fourfold X has no odd cohomology and $H^2(X, \mathbf{Q}) \simeq NS(X)_{\mathbf{Q}} \simeq A^1(X)$, because $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Let $\gamma \in A^1(X)$ be the class of a hyperplane section. Then $H^2(X, \mathbf{Q}) = A^1(X) \simeq \mathbf{Q}\gamma$ and $H^6(X, \mathbf{Q}) = \mathbf{Q}[\gamma^3/3]$. Here $< \gamma^2, \gamma^2 >= \gamma^4 = 3$, where $< , >$ is the intersection form on $H^4(X, \mathbf{Q})$.
Let $\pi_0 = [X \times P_0], \pi_8 = [P_0 \times X]$, where P_0 is a closed point and $\pi_2 = (1/3)(\gamma^3 \times \gamma)$, $\pi_6 = \pi_2^t = (1/3)(\gamma \times \gamma^3)$. Then

$$h(X) \simeq \mathbf{1} \oplus h_2(X) \oplus h_4(X) \oplus h_6(X) \oplus \mathbf{L}^4$$

where $\mathbf{1} \simeq (X, \pi_0)$, $\mathbf{L}^4 \simeq (X, \pi_8)$, $h_2(X) = (X, \pi_2)$, $h_6(X) = (X, \pi_6)$ and $h_4(X) = (X, \pi_4)$, with $\pi_4 = \Delta_X - \pi_0 - \pi_2 - \pi_6 - \pi_8$. The above decomposition of the motive $h(X)$ is in fact integral, because

$$\gamma^3 = 3|l|$$

for a line $l \in F(X)$, see [SV 2, Lemma A3].

Let ρ_2 be the dimension of $A^2(X)$, i.e. the rank of the algebraic part in $H^4(X)$. Choosing $2-cycles$ $\{D_1, D_2 \cdots, D_{\rho_2}\}$ and their Poincare' dual cycles $\{D'_1, D'_2 \cdots, D'_{\rho_2}\}$ we get a splitting

$$h_4(X) = h_4^{alg}(X) \oplus h_4^{tr}(X)$$

where $\pi_4^{alg} = \sum_{1 \leq i \leq \rho_2} [D_i, D'_i]$ and $h_4^{alg}(X) \simeq (\mathbf{L}^2)^{\rho_2}$. Therefore we get a *refined Chow-Künneth decomposition* of the motive $h(X)$

$$h(X) = \mathbf{1} \oplus \mathbf{L} \oplus (\mathbf{L}^2)^{\oplus \rho_2} \oplus t(X) \oplus \mathbf{L}^3 \oplus \mathbf{L}^4 \tag{2.1}$$

where $t(X) = h_4^{tr}(X)$ and $H^*(t(X)) = H^4(t(X)) = T(X)_{\mathbf{Q}}$ where $T(X)$ is the transcendental lattice. All the motives in (2.1), different from $t(X)$, are isomorphic to a multiple of $\mathbf{L}^i$, for some i. Therefore in the decomposition (2.1) all motives, but possibly $t(X)$, are finite dimensional. It follows that the motive $h(X)$ is finite dimensional if and only if $t(X)$ is evenly finite dimensional.

Lemma 2.2. *Let X be a cubic fourfold and let $t(X)$ be the transcendental motive in the Chow-Künneth decomposition (2.1). Then $A^i(t(X)) = 0$ for $i \neq 3$ and $A^3(t(X)) = A_1(X)_{hom}$*

Proof. The cubic fourfold X is rationally connected and hence $A^4(X) = A_0(X) = \mathbf{Q}$ that implies $A^4(t(X)) = 0$. Also from the Chow-Künneth decomposition in (2.1) we get $A^0(t(X)) = 0$ and $A^1(t(X)) = \pi_4^{tr} A^1(X) = 0$, because $A^1(h_2(X)) = \pi_2(A^1(X)) = A^1(X)$.

We first show that $A^2(t(X)) = 0$. Let $\alpha \in A^2(X)$, with $\alpha \neq 0$. Then α is not homologically trivial, because $A^2(X)_{hom} = 0$.

$$\pi_4^{tr}(\alpha) = \alpha - \pi_0(\alpha) - \pi_2(\alpha) - \pi_4^{alg}(\alpha) - \pi_6(\alpha) - \pi_8(\alpha),$$

where $\pi_0(\alpha) = \pi_8(\alpha) = 0$. We also have

$$\pi_2(\alpha) = (1/3)[\gamma^3 \times \gamma]_*(\alpha) = (1/3)(p_2)_*((\alpha \times X) \cdot [\gamma^3 \times \gamma])$$

where $p_2 : X \times X \to X$ and $\gamma^3 \in A^3(X)$. Therefore $\pi_2(\alpha) = 0$ in $A^2(X)$. Similarly

$$\pi_6(\alpha) = (1/3)[\gamma \times \gamma^3]_*(\alpha) = (1/3)(p_2)_*((\alpha \times X) \cdot [\gamma \times \gamma^3])$$

where $\alpha \cdot \gamma \in A_1(X)/A_1(X)_{hom} \simeq \mathbf{Q}[\gamma^3/3]$ and hence $\alpha \cdot \gamma = (a/3)[\gamma^3]$ with $a \in \mathbf{Q}$. Therefore $\pi_6(\alpha) = 0$ in $A^2(X)$. Let $\{D_1, \cdots, D_{\rho_2}\}$ be a $\mathbf{Q}$-basis for $A^2(X)$ and let $\alpha = \sum_{1 \leq i \leq \rho_2} m_i D_i$,with $m_i \in \mathbf{Q}$. Then $\pi_4^{alg}(\alpha) = \sum_{1 \leq i \leq \rho_2} \pi_{4,i}(\alpha) = \alpha$, because $(\pi_{4,i})_*(D_i) = D_i$. We get $\pi_4^{tr}(\alpha) = \alpha - \pi_4^{alg}(\alpha) = 0$ and hence

$$A^2(t(X)) = (\pi_4^{tr})_* A^2(X) = 0.$$

Therefore we are left to show that $A_1(t(X)) = A_1(X)_{hom}$. Let $\beta \in A_1(X) = A^3(X)$. From the Chow-Künneth decomposition in (2.1) we get

$$\pi_4^{tr}(\beta) = \beta - \pi_0(\beta) - \pi_2(\beta) - \pi_4^{alg}(\beta) - \pi_6(\beta) - \pi_8(\beta),$$

where $\pi_0(\beta) = \pi_4^{alg}(\beta) = \pi_8(\beta) = 0$. We also have $\pi_2(\beta) = 0$ because $\pi_2 = (1/3)(\gamma^3 \times \gamma)$. Therefore

$$\pi_4^{tr}(\beta) = \beta - \pi_6(\beta) = \beta - (1/3)(\gamma \times \gamma^3)_*(\beta) = \beta - (1/3)(\beta \cdot \gamma)\gamma^3 \in A^3(X)$$

and hence

$$(\pi_4^{tr}(\beta) \cdot \gamma) = (\beta \cdot \gamma) - (1/3)(\beta \cdot \gamma)(\gamma^3 \cdot \gamma) = 0$$

because $\gamma^4 = 3$. Since γ is a generator of $A^1(X)$ it follows that the cycle $\pi_4^{tr}(\beta)$ is numerically trivial. Therefore we get

$$A_1(t(X)) = \pi_4^{tr} A_1(X) = A_1(X)_{num} = A_1(X)_{hom}.$$

□

The following Lemma follows from the results in [Vial 1, Thm. 3.18] and [GG, Lemma 1].

Lemma 2.3. *Let $f : M \to N$ be a morphism of motives in $\mathcal{M}_{rat}(\mathbf{C})$ such that $f_* : A^i(M) \to A^i(N)$ is an isomorphism for all $i \geq 0$. Then f is an isomorphism.*

Proof. Let $M = (X, p, m)$ and $N = (Y, q, n)$ and let $k \subset \mathbf{C}$ be a field of definition of f, which is finitely generated . Then $\Omega = \mathbf{C}$ is a universal domain over k. By [Vial 1, Thm. 3.18] the map f has a right inverse, because the map $f_* : A^i(M) \to A^i(N)$ is surjective. Let $g : N \to M$ be such that $f \circ g = id_N$. Then g has an image T which is a direct factor of M and hence f induces an isomorphism of motives in $\mathcal{M}_{rat}(\mathbf{C})$

$$f : M \simeq N \oplus T$$

From the isomorphism $A^i(M) \simeq A^i(N)$, for all $i \geq 0$, we get $A^i(T) = 0$ and hence $T = 0$, by [GG, Lemma 1]. □

Let X be a cubic fourfold and let $F(X) = F$ be its Fano variety of lines , which is a smooth fourfold. Let

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ {\scriptstyle p}\big\downarrow & & \\ F & & \end{array} \tag{2.4}$$

be the incidence diagram, where $P \subset X \times F$ is the universal line over X. Let $p_*q^* : H^4(X, \mathbf{Z}) \to H^2(F, \mathbf{Z})$ be the Abel-Jacobi map. Let $\alpha_1, \cdots, \alpha_{23}$ be a basis of $H^4(X, \mathbf{Z})$ and let $\tilde{\alpha}_i = p_*q^*(\alpha_i)$. Then, by a result of Beauville-Donagi in [BD], $\tilde{\alpha}_i = p_*q^*(\alpha_i)$ form a basis of $H^2(F, \mathbf{Z})$. The lattice $H^2(F, \mathbf{Z})$ is endowed with the Beauville-Bogomolov bilinear form q_F, see [SV 2, Sect 19] . The Abel-Jacobi map induces an isomorphism between the primitive cohomology of $H^4(X, \mathbf{Z})_{prim}$ and the primitive cohomology $H^2(F, \mathbf{Z})_{prim}$. Here $H^2(F, \mathbf{Z})_{prim} =< g >^{\perp}$, with $g \in H^2(F, \mathbf{Z})$ the restriction to $F(X) \subset \mathrm{Gr}(2, 6)$ of the class on $\mathrm{Gr}(2, 6)$ defining the Plücker embedding. In particular $g = p_*q^*(\gamma^2)$. The Abel-Jacobi map induces an isomorphism between the Hodge structure of $H^4(X, \mathbf{C})_{prim}$ and the (shifted) Hodge structure of $H^2(F, \mathbf{C})_{prim}$.
The next result shows that the Abel-Jacobi map induces an isomorphism between $t(X)$ and the transcendental motive $h_2^{tr}(F)$ in a suitable Chow-Künneth decomposition for $h(F)$.

Theorem 2.5. *Let X be cubic fourfold and let $F(X)$ be its Fano variety of lines. Then there exists a Chow-Künneth decomposition*

$$h(F) = h_0(F) \oplus h_2(F) \oplus h_4(F) \oplus h_6(F) \oplus h_8(F)$$

with $h_2(F) \simeq h_2^{alg}(F) \oplus h_2^{tr}(F)$. The Abel-Jacobi map gives an isomorphism

$$h_2^{tr}(F)(1) \simeq h_4^{tr}(X) = t(X).$$

Proof. The hyperkähler manifold $F(X)$ is of $K3^2$-type, *i.e.* it is deformation equivalent to the Hilbert scheme of length-2 subschemes on a K3 surface. By the results in [SV 2, Sect. 19] there exists a cycle $L \in \mathrm{CH}^2(F \times F)$ whose cohomology class in $H^4(F \times F, \mathbf{Q})$ is the Beauville-Bogomolov class $\mathcal{B}$, i.e. the class corresponding to q_F^{-1}. Let us set $l := (i_\Delta)^* L \in CH^2(F)$, where $i_\Delta : F \to F \times F$ is the diagonal embedding. By [SV 2, Thm. 2] the Chow groups of the variety F have a *Fourier decomposition*. In particular the group $A^4(F) = A_0(F)$ has a canonical decomposition

$$A^4(F) = A^4(F)_0 \oplus A^4(F)_2 \oplus A^4(F)_4$$

with $A^4(F)_0 =< l^2 >$, $A^4(F)_2 = l \cdot L_* A^4(F)$ and $A^4(F)_4 = L_* A^4(F) \cdot L_* A^4(F)$. Here $< l^2 >= \mathbf{Q} c_F$, with c_F a special degree 1 cycle coming from a surface $W \subset F$ such that any two points on W are rationally equivalent on F, see [SV 2, Lemma A.3].

The Fourier decomposition of the Chow groups $A^*(F)$ is compatible with a Chow-Künneth decomposition of the motive $h(F)$ given by projectors

$$\{\pi_0(F), \pi_2(F), \pi_4(F), \pi_6(F), \pi_8(F)\},$$

as in [SV 1, Thm. 8.4]. Here $\pi_2(F) = \pi_2^{alg} \oplus \pi_2^{tr}, \pi_6(F) = \pi_6^{alg} \oplus \pi_6^{tr}$ and

$$\pi_4 = \Delta_F - (\pi_0 - \pi_2 - \pi_6 - \pi_8)$$

We have $h(F) = M \oplus N$ where

$$N = (F, \pi_0) \oplus (F, \pi_2^{alg}) \oplus (F, \pi_4^{alg}) \oplus (F, \pi_6^{alg}) \oplus (F, \pi_8)$$

and N is isomorphic to a direct sum of $\mathbf{L}^i$, for $i \geq 0$. We also have $A^*(F)_{hom} = A^*(M)$. Let us set $h_2(F) = h_2^{alg}(F) \oplus h_2^{tr}(F)$, where $h_2^{tr}(F) = (F, \pi_2^{tr}(F))$, with $\pi_2^{tr}(F) \in \mathrm{End}_{\mathcal{M}_{rat}} M$ and $H^*(h_2^{tr}(F)) = H^2_{tr}(F)$. Then

$$A^2(F) = \mathrm{Im}(\pi_4)_* \oplus \mathrm{Im}(\pi_2)_* = \mathrm{Im}(\pi_4)_* \oplus \mathrm{Im}(\pi_2^{tr})_*$$

because $\pi_2^{alg}(F)$ acts as 0 on $A^2(F)$.
Let us denote $\mathcal{A} = I_* A^4(F) \subset A^2(F)$, with I the incidence correspondence, i.e. $I = (p \times p)_*(q \times q)^* \Delta_X$. The group $\mathcal{A}_{hom}$ is generated by the classes $[S_{l_1}] - [S_{l_2}]$ where, for a line l on X, S_l denotes the surface in $F(X)$ of all lines meeting l, see [SV 2, Thm. 21.9]. By [SV 2, Prop. 21.10] the group $\mathcal{A}_{hom}$ coincides with the subgroup $A^2(F)_2$ in the Fourier decomposition $A^2(F) = A^2(F)_0 \oplus A^2(F)_2$.
The Abel-Jacobi map $q_* p^* : A^i(F) \to A^{i-1}(X)$ induces a surjective map $\Psi_0 : A^4(F) \to A^3(X) = A_1(X)$, where $A_1(X)$ is generated by the classes of lines, see [TZ]. The map induced by Ψ_0 on the subgroup $A^4(F)_{hom}$ has a kernel isomorphic to $F^4 A^4(F) = \mathcal{A}_{hom} \otimes \mathcal{A}_{hom}$, see [SV 2, Thm 20.2], where $F^4 A^4(F) = \mathrm{Ker}\{I_* : A^4(F) \to A^2(F)\}$. The maps I_* and Ψ_0 yield two exact sequences

$$0 \longrightarrow F^4 A^4(F) \longrightarrow A^4(F)_{hom} \xrightarrow{I_*} \mathcal{A}_{hom} \longrightarrow 0$$

$$0 \longrightarrow F^4 A^4(F) \longrightarrow A^4(F)_{hom} \xrightarrow{\Psi_0} A_1(X)_{hom} \longrightarrow 0$$

where $(A^4(F))_{hom} = (A^4(F)_2)_{hom} \oplus (A^4(F)_4)_{hom}$, with $(A^4(F)_2)_{hom} \simeq \mathcal{A}_{hom}$ and $(A^4(F)_4)_{hom} \simeq \mathcal{A}_{hom} \cdot \mathcal{A}_{hom}$. Therefore we get the following isomorphisms

$$\mathcal{A}_{hom} \simeq A^4(F)_2 \simeq A_1(X)_{hom};$$
$$\mathcal{A}_{hom} \simeq A^2(F)_2 \simeq A_1(X)_{hom}.$$

By [SV 1, Proposition 7.7] we also have

$$A^2(F)_{hom} \simeq \mathcal{A}_{hom} \Longleftrightarrow \mathrm{Im}(\pi_2)_* = A^2(F)_{hom}.$$

Therefore $A^2(F)_{hom} = \mathrm{Im}(\pi_2^{tr})$ and we get an isomorphism

$$A^2(h_2^{tr}(F)) \simeq A_1(X)_{hom}.$$

The universal line P, viewed as a correspondence in $A_5(F \times X)$, yields a map in $\mathcal{M}_{rat}(\mathbf{C})$

$$P_* : h(F)(1) \to h(X).$$

Therefore, by composing with the projection $h(X) \to t(X)$ and the inclusion $h_2^{tr}(F)(1) \subset h(F)(1)$, the correspondence P yields a map of motives

$$\bar{P}_* : h_2^{tr}(F)(1) \to t(X)$$

The above map induces a map of Chow groups

$$A^i(h_2^{tr}(F)(1) \to A^i(t(X))$$

that is an isomorphism for all $i \geq 0$ because

$$A^3(h_2^{tr}(F)(1) = A^2(h_2^{tr}(F)) \simeq A_1(X)_{hom} = A^3(t(X))$$

and $A^i(h_2^{tr}(F)) = A^i(t(X)) = 0$ for $i \neq 3$. By Lemma 2.3 we get $h_2^{tr}(F)(1) \simeq t(X)$. □

Remark 2.6. If the motive $h(F(X))$ is finite dimensional then, by Theorem 2.5, also $t(X)$ is finite dimensional and hence $h(X)$ is finite dimensional. Conversely if $h(X)$ is finite dimensional then, by [Lat 1], also $h(F(X))$ is finite dimensional .

Let X be a cubic fourfold and let $l \in F(X)$ be a general line. There exists a unique plane $P_l \subset \mathbf{P}^5$ containing l and which is everywhere tangent to X along l. Then

$$P_l \cdot X = 2[l] + [l_0]$$

A line l on X is said to be of the second type if

$$\mathcal{N}_{l/X} \simeq \mathcal{O}(1)^2 \oplus \mathcal{O}(-1).$$

In this case there is a linear $\mathbf{P}^3_l$ containing the line l and which is tangent to X along l and hence a family of planes $\{\Pi_t / t \in \mathbf{P}^1\}$ containing l such that each Π_t is tangent to X along l. Let Σ_2 be the surface of the lines of second type in $F = F(X)$. For a general X the surface Σ_2 is smooth and is the indeterminacy locus of the rational map, defined by C.Voisin

$$\phi : F \dashrightarrow F.$$

The map ϕ sends a general line $l \subset X$ to its residual line with respect to the unique plane $\mathbf{P}^2 \subset \mathbf{P}^5$ tangent to X along l. The class of Σ_2 in the Chow group $A^2(F)$ is given by $5(g^2 - c)$, where $g = \Phi(\gamma^2) \in A^1(F)$ and $c = [\Sigma_Y]$, with $\Sigma_Y = F(Y)$ for a smooth hyperplane section Y of X. Here $\Phi = p_* q^* : A^i(X) \to A^{i-1}(F)$ is the Abel-Jacobi map and $\Psi : A^i(F) \to A^{i-1}(X)$ is the cylinder homomorphism. We have $\Psi(g^2) = 21\gamma$, with $g^2 \in A^2(F)$ and $A^1(X) \simeq \mathbf{Q}\gamma$, see [SV 2, A.4].

Let $S_l \subset F$ be the surface of lines meeting a general line l. Then S_l is a smooth surface and there is a natural involution $\sigma : S_l \to S_l$. If $[l'] \in S_l$ is a point different from $[l]$ then $\sigma([l'])$ is the residue line of $l \cup l'$, while $\sigma([l]) = [l_0]$. The involution σ has 16 isolated fixed points and the quotient $Y_l = S_l/\sigma$ is a quintic surface in $\mathbf{P}^3$ with 16 ordinary double points, see [Shen, Remark 4.4]. Also, by [Shen, Lemma

4.5] $q(S_l) = 0$. Let $\tilde{X}$ be the blow-up of X along l. Then the projection from the line l defines a conic bundle $\pi : \tilde{X} \to \mathbf{P}^3$. The surface S_l parametrizes lines in the singular fibers, the discriminant divisor $D \subset \mathbf{P}^3$ is the quintic surface Y_l and the induced map $S_l \to D$ is the double cover $f_l : S_l \to Y_l$ associated to the involution σ. The map $f_l : S_l \to Y_l$ induces a commutative diagram

$$\begin{array}{ccc} \tilde{S}_l & \xrightarrow{g_l} & S_l \\ {\scriptstyle \bar{f}_l}\downarrow & & \downarrow{\scriptstyle f_l} \\ \tilde{Y}_l & \longrightarrow & Y_l \end{array}$$

where $\tilde{S}_l$ is the blow-up of the set of isolated fixed points of σ and $\tilde{Y}_l$ is a desingularization of Y_l. Since $t_2(-)$ is a birational invariant for smooth projective surfaces the above diagram yields a map

$$\theta : t_2(\tilde{S}_l) = t_2(S_l) \to t_2(\tilde{Y}_l)$$

which is a projection onto a direct summand. Since $q(S_l) = 0$ the motive $t_2(S_l)$ splits as follows, see [Ped, Prop. 1],

$$t_2(S_l) \simeq t_2(S_l)^+ \oplus t_2(S_l)^-,$$

where $t_2(S_l)^+$ and $t_2(S_l)^-$ are the direct summand of $t_2(S_l)$ on which the involution σ acts as $+1$ and -1 respectively . We also have

$$A^2(t_2(Y_l)) = A^2(t_2(S_l))^+ = A_0(S_l)_0^+ \ ; \quad A^2(t_2(S_l))^- = A_0(S_l)_0^-;$$

where $A_0(S_l)_0 = A_0(S_l)_0^+ \oplus A_0(S_l)_0^-$. Here $A_0(S_l)_0$ is the group of 0-cycles of degree 0 (with $\mathbf{Q}$-coefficients) and $A_0(S_l)_0^+$ is the subgroup fixed by σ.
The surface S_l, for a general line l, has a class $1/3(g^2 - c)$ in $A^2(F)$. Therefore $[\Sigma_2] = 15[S_l]$ in $A^2(F)$. For a general fourfold X any Lagrangian surface in F has a cohomology class proportional to c. Therefore both the surfaces Σ_2 and S_l are not Lagrangian and hence $p_g(\Sigma_2) > 0$ and $p_g(S_l) > 0$.

Lemma 2.7. *Let S_l be the surface of lines meeting a general line $l \subset X$. Then $A_0(S_l)_0^+ = 0$ and hence $t_2(Y_l) = 0$, $t_2(S_l) = t_2(S_l)^-$.*

Proof. The group $A_0(S_l)_0$ is generated by classes α of the form $[l_1] - [l_2]$ where $l_i \in S_l$. If $l_i \neq l$, for $i = 1, 2$, then $l_i \cap l \neq \emptyset$ and there are planes $P_1, P_2 \subset \mathbf{P}^5$ such that

$$P_1 \cdot X = [l] + [l_1] + \sigma([l_1]) = \gamma^3 \ ; \ P_2 \cdot X = [l] + [l_2] + \sigma([l_2]) = \gamma^3,$$

where $\gamma \in A^1(X)$ is the class of a hyperplane section. Therefore $[l_1] - [l_2] = \sigma([l_2]) - \sigma([l_1])$ and hence $[l_1] - [l_2] \in A_0(S_l)_0^-$. Similarly if $\alpha = [l] - [l_1]$, then there exists a plane P_l such that $P_l \cdot X = 2[l] + \sigma([l]) = \gamma^3$ and hence $[l] + [l_1] + \sigma([l_1]) = 2[l] + \sigma([l])$. Therefore we have

$$[l] - [l_1] = \sigma([l_1]) - \sigma([l]) \in A_0(S_l)_0^-.$$

This in turn implies that $A_0(S_l)_0 = A_0(S_l)_0^-$ and $A_0(S_l)_0^+ = A^2(t_2(Y_l)) = 0$. Since $A^i(t_2(Y_l)) = 0$, for $i \neq 2$, we get $t_2(Y_l) = 0$. □

The following result shows the relation between the transcendental motive $t(X)$ and the transcendental motives of the surfaces Σ_2 and S_l.

Proposition 2.8. *Let X be a cubic fourfold and let S_l be the surface of lines meeting a general line $l \subset X$. Then*

(i) $t(X)$ is isomorphic to $t_2(\Sigma_2)(1) \oplus h_3(\Sigma_2)(1)$, where

$$h(\Sigma_2) \simeq \mathbf{1} \oplus h_1(\Sigma_2) \oplus \mathbf{L}^{\oplus \rho(\Sigma_2)} \oplus t_2(\Sigma_2) \oplus h_3(\Sigma_2) \oplus \mathbf{L}^2$$

is a reduced Chow-Künneth decomposition.

(ii) $t(X) \simeq t_2(S_l)(1)$

(iii) $A_0(S_l)_0 \simeq A_0(\Sigma_2)_0 \simeq A_1(X)_{hom}$.

Proof. **(i)** Let $Z = \Sigma_2$ and let

$$\begin{array}{ccc} P_Z & \xrightarrow{q_Z} & X \\ {\scriptstyle p_Z}\downarrow & & \\ Z & & \end{array}$$

be the incidence diagram in (2.4) restricted to $Z \subset F(X)$, where P_Z is the universal line over Z. Then, as a correspondence in $A_3(Z \times X)$, P_Z gives a map of motives

$$(P_Z)_* : h(Z)(1) \to h(X)$$

that composed with the inclusion $t_2(Z)(1) \oplus h_3(Z)(1)$, coming from the C-K decomposition of $h(Z)$, and the projection $h(X) \to t(X)$, gives a map

$$(P_Z)_* : t_2(Z)(1) \oplus h_3(Z)(1) \to t(X)$$

We claim that the above map induces an isomorphism on all Chow groups.
For a general X the surface $Z = \Sigma_2$ is smooth, see [Am], and, by [SV 2, Prop. 19.5]

$$A^4(F)_0 \oplus A^4(F)_2 = \mathrm{Im}\{\phi_* : A_0(Z) \to A^4(F)\}.$$

We get

$$(A^4(F)_2) = \mathrm{Im}\{\phi_* : A_0(Z)_0 \to A^4(F)_{hom}\},$$

because $(A^4(F)_0)_{hom} = 0$. The map $\phi_* : A_0(Z)_0 \to A^4(F)_{hom}$ is injective because $\phi_*(z - z') = -2(z - z')$, by [SV 2, Lemma 18.3]. Therefore $A_0(Z)_0 \simeq A^4(F)_2$ and hence

$$A_0(Z)_0 \simeq A^4(F)_2 \simeq A_1(X)_{hom} \tag{2.9}$$

By [KMP, Prop. 7.2.3] we obtain

$$A^3(h(Z)(1)) = A^3(t_2(Z)(1) \oplus h_3(Z)(1)) = A^2(t_2(Z) \oplus h_3(Z)) = T(Z) \oplus \mathrm{Alb}(Z) \simeq A_0(Z)_0,$$

where $T(Z)$ is the Albanese kernel. We also have

$$A^3(h(X)) = A_1(h(X)) = A_1(X)_{hom} \oplus \mathbf{Q}[\gamma^3/3],$$

where $A_1(X)_{hom} = A^3(t(X))$. Since $A^i(t(X)) = 0$ for $i \neq 3$ and $A^i(t_2(Z)(1) \oplus h_3(Z)1)) = 0$ for $i \neq 3$, by (2.9) we get an isomorphism $A^i(t_2(Z)(1) \oplus h_3(Z)1)) \simeq A^i(t(X))$, for all $i \geq 0$.

By Lemma 2.3 $(P_Z)_*$ is an isomorphism of motives.

(ii) Let l be general line and let S_l be the surface of lines meeting l. Let $\mathcal{C}_l$ be the total space of lines meeting l and let

$$\begin{array}{ccc} \mathcal{C}_l & \xrightarrow{q_l} & X \\ {\scriptstyle p_l}\big\downarrow & & \\ S_l \subset F & & \end{array}$$

be the incidence diagram. Then the Abel-Jacobi map Φ and the cylinder homomorphism Ψ induce homomorphisms

$$\Phi_l : A_i(X) \to A_{i-1}(S_l) \ ; \ \Psi_l : A_i(S_l) \to A_{i+1}(X)$$

By [Shen, Thm. 4.7] the composition $\Phi_l \circ \Psi_l$ equals $\sigma - id$ and $\Psi_l \circ \Phi_l = -2$. The Abel-Jacobi map Φ_l induces an isomorphism between $A_1(X)$ and $\Pr(A_0(S_l)_0, \sigma)$, where $\Pr(A_0(S_l)_0, \sigma) = A_0(S_l)_0^- = A^3(t_2(S_l)^-(1))$ [Shen, Def. 3.6]. By Lemma 2.7 we get $A_0(S_l)_0^- = A_0(S_l)_0$. Therefore the map

$$\Phi_l : A_1(X) \simeq \mathbf{Q}[\gamma^3/3] \oplus A_1(X)_{hom} \to A_0(S_l)_0$$

yields an isomorphism between $A_1(X)_{hom}$ and $A_0(S_l)_0 = A_0(S_l)_{hom}$.
In the incidence diagram $\mathcal{C}_l$ is a $\mathbf{P}^1$-bundle over S_l and hence $h(\mathcal{C}_l) \simeq h(S_l) \oplus h(S_l)(1)$. Therefore we get a map of motives

$$g : t_2(S_l)(1) \to t(X)$$

which induces a map of Chow groups $(g)_* : A^i(t_2(S_l)(1)) \to A^i(t(X))$. We have $A^3(t_2(S_l)(1)) = A_0(S_l)_0 = A_0(S_l)_0^-$ and $A^i(t_2(S_l)(1)) = 0$ for $i \neq 3$. Also $A^3(t(X)) = A_1(X)_{hom}$ and $A^i(t(X)) = 0$ for $i \neq 3$. Therefore g induces a map

$$g^- : t_2(S_l)(1) \to t(X)$$

such that $g_* : A^i(t_2(S_l)(1)) \to A^i(t(X))$ is an isomorphism for all $i \geq 0$ and hence g^- is an isomorphism in $\mathcal{M}_{rat}(\mathbf{C})$.

(iii) follows immediately from the isomorphisms

$$A_1(X)_{hom} \simeq A_0(S_l)_0 \simeq A_0(\Sigma_2)_0.$$

□

Remark 2.10. (1) The isomorphism in (ii) answers a question raised by M.Shen in a private communication. For a smooth projective surface S, with $q(S) = 0$ and $p_g(S) > 0$, equipped with an involution σ, we can define the *Prym motive* $\mathrm{Prym}(S, \sigma)$ to be the motive

$$Prym(S, \sigma) = t_2(S)^-$$

where, as in [Ped, Prop 1], $t_2(S)^-$ is the direct summand of $t_2(S)$ where the involution σ acts as -1. The action of σ on $t_2(S)$ is defined via the homomorphism

$$\Psi_S : A^2(S \times S) \to \mathrm{End}_{\mathcal{M}_{rat}}(t_2(S))$$

which sends the correspondence $\Gamma_\sigma \in A^2(S \times S)$ to $\pi_2^{tr} \circ \Gamma_\sigma \circ \pi_2^{tr}$. Here $t_2(S) = (S, \pi_2^{tr})$ and hence the projector π_2^{tr} corresponds to the identity in $\mathrm{End}_{\mathcal{M}_{rat}}(t_2(S))$.

(2) If $l \in X$ is a general line then the blow-up $\tilde{X}$ of X along l is a conic bundle $\pi : \tilde{X} \to \mathbf{P}^3$ and $Y_l = S_l/\sigma$ is the discriminant divisor. Therefore (ii) may be viewed as a generalization of a result appearing in [N-S] for a conic bundle $f : X \to \mathbf{P}^2$. In [N-S] it is proved that the transcendental part of $h(X)$ is given by the Prym motive

$\mathrm{Prym}(\tilde{C}/C)$, where the curve C is the discriminant of f and $\tilde{C} \to C$ is a double cover.

Remark 2.11. In the case $p_g(S) = 1$ (e.g. S a K3 surface) then $t_2(S)$ is indecomposable if $h(S)$ is finite dimensional. Therefore the Prym motive of S is either 0 or it coincides with $t_2(S)$. If S is a K3 surface and σ is a symplectic involution then $t_2(S) \simeq t_2(S/\sigma)$ and hence σ acts as the identity on $t_2(S)$, i.e $\mathrm{Prym}(S, \sigma) = 0$. If σ is non-symplectic then the quotient surface S/σ is either an Enriques surface or a rational surface. In any case $t_2(S/\sigma) = 0$ and $\Psi_S(\Gamma_\sigma) = -id_{t_2(S)}$. Therefore $\mathrm{Prym}(S, \sigma) = t_2(S)$. By Lemma 2.7 the same result holds for $t_2(S_l)$.

3. Special cubic fourfolds

In this section we prove (see Thm.3.2) that, if $F(X)$ is isomorphic to $S^{[2]}$, with S a K3 surface, then $t(X)$ is isomorphic to $t_2(S)(1)$. Therefore $h(X)$ is finite dimensional if and only if $h(S)$ is finite dimensional.
Recall that a cubic fourfold X is *special* if it contains a surface Z such that its cohomological class ζ in $H^4(X, \mathbf{Z})$ is not homologous to any multiple of γ^2. Therefore $\rho_2(X) > 1$. The discriminant d is defined as the discriminant of the intersection form $<,>_D$ on the sublattice D of $H^4(X, \mathbf{Z})$ generated by ζ and γ^2. B.Hassett in [Has 1] proved that special cubic fourfolds of discriminant d form an irreducible divisor $\mathcal{C}_d$ in the moduli space $\mathcal{C}$ of cubic fourfolds if and only if $d > 0$ and $d \equiv 0, 2(6)$.

Definition 3.1. Let X be a special cubic fourfold and let D be the sublattice of $H^4(X, \mathbf{Z})$ generated by ζ and γ^2. A polarized K3 surface S is *associated* to X if there is an isomorphism of lattices $K \simeq H^2(S, \mathbf{Z})_{prim}(-1)$, where $K = D^\perp$ and $H^2(S, \mathbf{Z})_{prim}$ denotes primitive cohomology with respect to a polarization $l \in H^2(S, \mathbf{Z})$.

If X is a generic special cubic fourfold with discriminant of the form $d = 2(n^2+n+1)$, where n is an integer ≥ 2, then the Fano variety of X is isomorphic to $S^{[2]}$, with S a K3 surface associated to X. Special cubic fourfolds of discriminant $d > 6$ have associated K3 surface S if and only if d is not divisible by 4 or 9 or any odd prime $p \equiv 2(3)$. In this case the transcendental lattice $T(X)$ is Hodge isometric to $T(S)(-1)$, see [Add].
In the case $d = 14$ the special surface is a smooth quartic rational normal scroll. By the results in [BD] and in [BRS] all the fourfolds X in $\mathcal{C}_{14}$ are rational (see also [ABBV] for details on the derived categories approach). Moreover if $X \in (\mathcal{C}_{14} - \mathcal{C}_8)$, then $F(X) \simeq S^{[2]}$, where S is the K3 surface of degree 14 and genus 8, parametrizing smooth quartic rational normal scrolls contained in X.

More generally, suppose that X is special and $F(X) \simeq S^{[2]}$, with S a K3 surface. Then the homomorphism $H^2(S, \mathbf{Q}) \to H^2(F, \mathbf{Q})$ induces an orthogonal direct sum decomposition with respect to the Beauville-Bogomolov form

$$H^2(F, \mathbf{Q}) \simeq H^2(S, \mathbf{Q}) \oplus \mathbf{Q}\delta,$$

with $q_F(\delta, \delta) = -2$ and q_F restricted to $H^2(S, \mathbf{Q})$ is the intersection form, see [SV 2, Rmk. 10.1]. Therefore

$$H^4_{tr}(X, \mathbf{Q}) \simeq H^2_{tr}(S, \mathbf{Q})$$

where $\dim H^4_{tr}(X, \mathbf{Q}) = 23 - \rho_2(X)$. Here $\rho_2(X) \geq 2$ and hence we get

$$\dim H^2_{tr}(F, \mathbf{Q}) = \dim H^2_{tr}(S, \mathbf{Q}) = 22 - \rho(S) = 23 - \rho_2(X) \leq 21,$$

where $\rho(S)$ is the rank of $NS(S)$.

Theorem 3.2. *Let X be a cubic fourfold and let $F = F(X)$ be the Fano variety of lines. Suppose that $F \simeq S^{[2]}$, with S a K3 surface. Let p and q be the morphisms in the incidence diagram (2.4). Then q induces a map of motives $\bar{q} : t_2(S)(1) \to t(X)$ in $\mathcal{M}_{rat}(\mathbf{C})$ which is an isomorphism.*

Proof. In (2.4) the universal line P is seen as a correspondence in $A_5(F \times X)$ and gives a map

$$P_* : h(F)(1) \to h(X)$$

By the results in [deC-M, Thm. 6.2.1] $h(S)$ is a direct summand of $h(S^{[2]}) = h(F)$. Therefore we get a map

$$h(S)(1) \longrightarrow h(F)(1) \xrightarrow{P_*} h(X)$$

Let

$$h(S) \simeq \mathbf{1} \oplus \mathbf{L}^{\oplus \rho(S)} \oplus t_2(S) \oplus \mathbf{L}^2$$

be a refined Chow-Künneth decomposition, as in [KMP, Sect. 7.2.2]. By composing with the inclusion $t_2(S)(1) \to h(S)(1)$ and the surjection $h(X) \to t(X)$ we get a map of motives in $\mathcal{M}_{\rm rat}(\mathbf{C})$,

$$\bar{P} : t_2(S)(1) \to t(X)$$

For two distinct points $x, y \in S$ let us denote by $[x, y] \in F = S^{[2]}$ the point of F that corresponds to the subscheme $x \cup y \subset S$. If $x = y$ then $[x, x]$ denotes the element in $A^4(F)$ represented by any point corresponding to a non reduced subscheme of length 2 on S supported on x. With these notations the special degree 1 cycle $c_F \in A^4(F)$ (see [SV 2, Lemma A.3]), given by any point on a rational surface $W \subset F$, is represented by the point $[c_S, c_S] \in F$, where c_S is the Beauville-Voisin cycle in $A_0(S)$ such that $c_2(S) = 24c_S$. We also have (see [SV 2, Prop. 15.6])

$$(A^4(F)_2)_{hom} =< [c_S, x] - [c_S, y] > .$$

We claim that the map $\phi : A_0(S) \to A_0(S^{[2]}) = A^4(F)$ sending $[x]$ to $[c_S, x]$ is injective and hence

$$A_0(S)_0 \simeq (A^4(F)_2)_{hom}.$$

The variety $S^{[2]}$ is the blow-up of the symmetric product $S^{(2)}$ along the diagonal $\Delta \cong S$. Let $\tilde{S}$ be the inverse image of Δ in $S^{[2]}$. Then $\tilde{S}$ is the image of the closed embedding $s \to [c_S, s]$. By a result proved in [Ba, Thm. 2.1] the induced map of 0-cycles $A_0(\tilde{S}) \to A_0(S^{[2]})$ is injective. Therefore the map ϕ is injective.
From the isomorphism $A_0(S)_0 \simeq (A^4(F)_2)_{hom}$ we get

$$A^3(t_2(S)(1) = A^2(t_2(S)) = A_0(S)_0 \simeq A_1(X)_{hom} \simeq A^3(t(X))$$

Since $A^i(t_2(S)(1)) = A^i(t(X)) = 0$ for $i \neq 3$ the map $\bar{P} : t_2(S)(1) \to t(X)$ gives an isomorphism on all Chow groups .Therefore $t_2(S)(1) \simeq t(X)$ □

Remark 3.3. Let X be a cubic fourfold such that there exist K3 surfaces S_1 and S_2 and isomorphisms $r_1 : F(X) \to S_1^{[2]}$ and $r_2 : F(X) \to S_2^{[2]}$ with $r_1^*\delta_1 \neq r_2^*\delta_2$, as in [Has 1, Def. 6.2.1], where $H^2(F, \mathbf{Q}) \simeq H^2(S_1, \mathbf{Q}) \oplus \mathbf{Q}\delta_1 \simeq H^2(S_2, \mathbf{Q}) \oplus \mathbf{Q}\delta_2$. Then, by Thm. 3.2, we get $t_2(S_1) \simeq t_2(S_2)$, and hence the motives $h(S_1)$ and $h(S_2)$ are isomorphic.

Corollary 3.4. *Let X be a cubic fourfold and let $F = F(X)$ be the Fano variety of lines. Suppose that $F \simeq S^{[2]}$, with S a K3 surface. Then $h(X)$ is finite dimensional if and only if $h(S)$ is finite dimensional in which case the motive $t(X)$ is indecomposable.*

Proof. If $h(X)$ is finite dimensional then also $t(X)$ is finite dimensional and hence, by Theorem 3.2, $t_2(S)$ is finite dimensional. Therefore $h(S)$ is finite dimensional. Conversely, if $h(S)$ is finite dimensional then also $t_2(S)$ and $t(X)$ are finite dimensional, by Theorem 3.2. From the Chow-Künneth decomposition in (2.1) we get that $h(X)$ is finite dimensional. If $h(S)$ is finite dimensional then the motive $t_2(S)$ is indecomposable, see [Vois 1, Cor 3.10], and hence also $t(X)$ is indecomposable. □

Remark 3.5. If the motive $h(X)$ of a cubic fourfold is finite dimensional then the transcendental part $t(X)$ of $h(X)$ is, up to isomorphisms in $\mathcal{M}_{rat}(\mathbf{C})$, independent of the Chow-Künneth decomposition $h(X) = \sum_i h_i(X)$ in (2.1). If $h(X) = \sum_i \tilde{h}_i(X)$ is another Chow-Künneth decomposition, with $\tilde{h}_i(X) = (X, \tilde{\pi}_i)$, then, by [KMP, Thm. 7.6.9], there is an isomorphism $\tilde{h}_i(X) \simeq h_i(X)$ and $\tilde{\pi}_i = (1+Z) \circ \pi_i \circ (1+Z)^{-1}$, where $Z \in A^4(X \times X)_{hom}$ is a nilpotent correspondence. In particular

$$\tilde{\pi}_4 = (1+Z) \circ \pi_4 \circ (1+Z)^{-1} = (1+Z) \circ (\pi_4^{alg} + \pi_4^{tr}) \circ (1+Z)^{-1}$$

and hence $\tilde{h}_4(X)$ contains as a direct summand a submotive $\tilde{t}(X) = (X, (1+Z) \circ \pi_4^{tr} \circ (1+Z)^{-1})$ isomorphic to $t(X)$.
However, differently from the case of the transcendental motive $t_2(S)$ of a surface S, the motive $t(X)$ is not a birational invariant. In fact $t(X) \neq 0$ for a rational cubic fourfold X such that $F(X) \simeq S^{[2]}$, with S a K3 surface, while $\mathbf{P}^4_{\mathbf{C}}$ has no transcendental motive.

According to Cor. 3.4 if X is a special cubic fourfold with $F(X) \simeq S^{[2]}$, and $h(X)$ is finite dimensional, then $t(X)$ is indecomposable. The following proposition shows that, if X is not special and $h(X)$ is finite dimensional, then $t(X)$ is indecomposable.

Proposition 3.6. *Let X be a very general cubic fourfold, i.e. $\rho_2(X) = 1$. If $h(X)$ is finite dimensional the transcendental motive $t(X)$ is indecomposable.*

Proof. Let us define the primitive motive $h(X)_{prim} = (X, \pi_{prim}, 0)$ as in [Ki, Sect. 8.4], where

$$\pi_{prim} = \Delta_X - (1/3) \sum_{0 \leq i \leq 4} (\gamma^{4-i} \times \gamma^i).$$

and

$$H^*(h(X)_{prim}) = H^4(X, \mathbf{Q})_{prim} = P(X)_{\mathbf{Q}}$$

If X is very general then $\rho_2(X) = 1$ and $A^2(X)$ is generated by the class γ^2. Therefore in the Chow-Künneth decomposition of $h(X)$ in (2.1) we have $h(X)_{prim} = h_4^{tr}(X) = t(X)$ and

$$h_4(X) = h_4^{alg}(X) + h_4^{tr}(X) \simeq \mathbf{L} \oplus h(X)_{prim}.$$

If X is very general, then $\mathrm{End}_{HS}(H^4(X, \mathbf{Q})_{prim}) = \mathbf{Q}[id]$, see [Vois 2, Lemma 5.1]. Let $\mathcal{M}_{hom}(\mathbf{C})$ be the category of homological motives and let $\widetilde{\mathcal{M}}_{hom}(\mathbf{C})$ be the subcategory generated by the motives of all smooth projective varieties V such

that the Künneth components of the diagonal in $H^*(V \times V)$ are algebraic. The Hodge realization functor

$$H_{Hodge} : \mathcal{M}_{rat}(\mathbf{C}) \to HS_{\mathbf{Q}}$$

to the Tannakian category of $\mathbf{Q}$-Hodge structures induces a faithful functor $\widetilde{\mathcal{M}}_{hom}(\mathbf{C}) \to HS_{\mathbf{Q}}$. Let us denote $\bar{h}(X) := h^{hom}(X) \in \widetilde{\mathcal{M}}_{hom}(\mathbf{C})$: then $\mathrm{End}_{\mathcal{M}_{hom}}(\bar{h}(X)_{prim}) \simeq \mathbf{Q}[id]$ and hence

$$\mathrm{End}_{\mathcal{M}_{hom}}(\bar{h}_4^{tr}((X)) \simeq \mathrm{End}_{\mathcal{M}_{hom}}(\bar{h}(X)_{prim}) \simeq \mathbf{Q}[id]$$

If $h(X)$ is finite dimensional then the indecomposability of $\mathrm{End}_{\mathcal{M}_{hom}}(\bar{h}_4^{tr}(X))$ in $\mathcal{M}_{hom}(\mathbf{C})$ implies the indecomposability in $\mathcal{M}_{rat}(\mathbf{C})$. Therefore

$$\mathrm{End}_{\mathcal{M}_{rat}}(t(X)) \simeq \mathrm{End}_{\mathcal{M}_{rat}}(h(X)_{prim}) \simeq \mathbf{Q}[id]$$

and the transcendental motive of X is indecomposable.

□

4. A motivic conjecture

Let X be a cubic fourfold.

Definition 4.1. We will say that the motive $h(X)$ is associated to the motive of a K3 surface S if there is an isomorphism between $t_2(S)(1)$ and $t(X)$, inducing an Hodge isometry between $T(S)_{\mathbf{Q}}(1)$ and $T(X)_{\mathbf{Q}}$.

In this section we relate this isomorphism with a conjecture by Kuznetsov on the rationality of X. Indeed, it was conjectured in [Kuz] that a cubic fourfold is rational if and only if there exists a semi-orthogonal decomposition of the derived category $\mathbf{D}^b(X)$ of bounded complexes of coherent sheaves

$$\mathbf{D}^b(X) =< \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) >,$$

such that $\mathcal{A}_X$ is equivalent to the category $\mathbf{D}^b(S)$ where S is a K3 surface. If X has an associated K3 surface S , in the sense of Kuznetsov, then the motive $h(S)$ is uniquely determined, up to isomorphisms, by X. Let $\mathbf{D}^b(S_1)$ and $\mathbf{D}^b(S_2)$ be equivalent. It was conjectured by Orlov that this implies that the motives $h(S_1)$ and $h(S_2)$ are isomorphic. The conjecture has been proved in [DelP-P] in the case $h(S_1)$ (and hence also $h(S_2)$) is finite dimensional and recently extended by D.Huybrechts in [Huy 2] to all K3 surfaces over an algebraically closed field.
Let us denote by $\mathcal{C}$ the moduli space of smooth cubic fourfolds. As it is customary, we will denote by $\mathcal{C}_d \subset \mathcal{C}$ the irreducible divisors that parametrize special cubic fourfolds with an intersection lattice whose determinant is d. Let X be a general cubic fourfold inside $\mathcal{C}_d$, where d satisfies the following condition

(**) d is not divisible by 4,9 or a prime $p \equiv 2(3)$.

Hassett [Has 1] has shown that $X \in \mathcal{C}_d$ has an associated K3 surface, in the sense of Defn. 3.1, if and only if satisfies (**). Then Addington and Thomas in [AT] proved that a general such X has an associated K3 surface in the sense of Kuznetsov. Therefore, for a general cubic fourfold, Kuznetsov conjecture is equivalent to the following conjecture, that has been certainly around for a while.

Conjecture 4.2. *A cubic fourfold $X \subset \mathbf{P}^5$ is rational if and only if it is contained in $\mathcal{C}_d$, with d satisifying (**).*

Proposition 4.3. *Let X be a general cubic fourfold in $\mathcal{C}_d$. Assuming Kimura's conjecture, the motive $h(X)$ is associated to the motive of a K3 surface, if d satisfies (**). If d does not satisfy (**) there is no K3 surface S such that the motive $h(X)$ is associated to $h(S)$*

Proof. If d satisfies (**) then by Theorem 1.2 in [AT] there exists a polarized K3 surface S of degree d and a correspondence $\Gamma \in A^3(S \times X)$ which induces an Hodge isometry between the (shifted) primitive cohomology of S and the lattice $< \gamma^2, T >^{\perp}$ inside $H^4(X, \mathbf{Z})$. Here the class of T is not homologous to γ^2. Let $PHS_{\mathbf{Q}}$ be the semisimple abelian category of polarized Hodge structures. Then Γ induces an isomorphism between the polarized Hodge structures $T(S)_{\mathbf{Q}}(1)$ and $T(X)_{\mathbf{Q}}$ in $PHS_{\mathbf{Q}}$, where $T(S)$ and $T(X)$ are the transcendental lattices of S and X respectively. Let $\mathcal{M}^B_{hom}(\mathbf{C})$ be the subcategory of $\mathcal{M}_{hom}(\mathbf{C})$ generated by the homological motives $h_{hom}(X)$ of smooth complex projective varieties X satisfying the standard conjecture $B(X)$, that is satisfied by curves, surfaces, abelian varieties and is stable under products and hyperplane sections. Since $B(X)$ implies the standard conjecture $D(X)$, for smooth varieties over $\mathbf{C}$, the category $\mathcal{M}^B_{hom}$ is contained in the category $\mathcal{M}_{num}(\mathbf{C})$ of numerical motives and hence it is semisimple. The Hodge realization functor

$$H_{Hodge} : \mathcal{M}_{rat}(\mathbf{C}) \to PHS_{\mathbf{Q}},$$

factors trough $\mathcal{M}^B_{hom}(\mathbf{C})$ and the induced functor $H_{Hodge} : \mathcal{M}^B_{hom}(\mathcal{C}) \to PHS_{\mathbf{Q}}$ is faithful and exact. Both the K3 surface S and the cubic fourfold X satisfy $B(X)$ and hence M_{hom} and N_{hom} belong to $\mathcal{M}^B_{hom}(\mathbf{C})$, where M_{hom} and N_{hom} are the images of $t_2(S)(1)$ and $t(X)$ in $\mathcal{M}_{hom}(\mathbf{C})$, respectively. Then M_{hom} and N_{hom} have isomorphic images in $PHS_{\mathbf{Q}}$ because the polarized Hodge structures $T(S)_{\mathbf{Q}}(1)$ and $T(X)_{\mathbf{Q}}$ are isomorphic. Therefore the correspondence Γ induces an isomorphism between M_{hom} and N_{hom} in $\mathcal{M}^B_{hom}(\mathbf{C})$. By Kimura's conjecture on the finite dimensionality of motives the functor $F : \mathcal{M}_{rat}(\mathbf{C}) \to \mathcal{M}^B_{hom}(\mathbf{C})$ is conservative, i.e. it preserves isomorphisms, see [AK, Thm. 8.2.4]. Therefore the correspondence Γ gives an isomorphism between $t_2(S)(1)$ and $t(X)$ in $\mathcal{M}_{rat}(\mathbf{C})$ that induces an Hodge isometry between $T(S)_{\mathbf{Q}}(1)$ and $T(X)_{\mathbf{Q}}$.

Suppose now that d does not satisfy (**). Let Γ be a correspondence in $A^3(S \times X)$, where S is a K3 surface. Then Γ is a Hodge cycle in $H^6(S \times X, \mathbf{Q})$ such that $\Gamma_* : T(S)_{\mathbf{Q}}(1) \to T(X)_{\mathbf{Q}}$ is not an Hodge isometry, because, by the results in [AT, Theorem 1.3] this would imply that $X \in \mathcal{C}_d$ for some d satisfying the condition (**). Therefore there is no isomorphism between $t_2(S)(1)$ and $t(X)$ inducing a Hodge isometry. □

Proposition 4.3, and Conjecture 4.2 suggest the following motivic conjecture

Conjecture 4.4. *A general cubic fourfold in $\mathcal{C}_d$ is rational if and only if its motive is associated to the motive of a K3 surface.*

5. Cubic fourfolds fibered over a plane

Let X be a cubic fourfold containing a surface T, with $t_2(T) = 0$ and such that the blow-up $\tilde{X}$ of X at T admits a fibration $\pi : \tilde{X} \to \mathbf{P}^2$ whose fibers are rational. Examples of such X are general elements of the divisor $\mathcal{C}_8$, in which case T is a plane, π is a quadric bundle, and general $X \in \mathcal{C}_{18}$, where T is an elliptic ruled

surface, and the fibers of π are del Pezzo surfaces of degree 6. According to [Vial 1, (2)], the motive $h(\tilde{X})$ splits as follows

$$h(\tilde{X}) \simeq h(\mathbf{P}^2) \oplus h(\mathbf{P}^2)(1) \oplus h(\mathbf{P}^2)(2) \oplus h(Z)(1) \tag{5.1}$$

where Z is a smooth projective surface. Moreover $h(\tilde{X}) \simeq h(X) \oplus h(T)(1)$, where the motive of T has no transcendental part. Therefore the transcendental part $t(X)$ is isomorphic to $t(\tilde{X})$ and hence $h(X)$ is finite dimensional if and only if the motive $h(Z)$ is finite dimensional.
In order to identify the surfaces Z appearing in (5.1) we will use the following proposition, that comes from the results in [Vial 1, Prop. 6.7].

Proposition 5.2. *Let X be a cubic fourfold containing a surface T with $t_2(T) = 0$ and such that the blow-up $\pi : \tilde{X}$ of X at T admits a fibration $f : \tilde{X} \to \mathbf{P}^2$. Let C be the discriminant curve of the fibration f and let $B^o = \mathbf{P}^2 - C$. Assume that*

(1) For all $t \in B^o$, the fibers $\tilde{X}_t$ are smooth rational surfaces.

(2)For all $P \in C$ the map $A_1(\tilde{X}_P) \to A_1(\tilde{X})_{hom}$ is the 0-map.

Then there are a finite number of smooth surfaces $\tilde{B}_i$, for $i = 1 \cdots n$, with surjective and generically finite maps $r_i : \tilde{B}_i \to \mathbf{P}^2$, such that the transcendental motive $t(X)$ is isomorphic to a direct summand of $h(\tilde{B})$, where $\tilde{B} = \coprod_i \tilde{B}_i$.

Proof. Let $t \in B^o$ and let $f_t : \tilde{X}_t \to \tilde{X}$ be the inclusion. The induced map on Chow groups $(f_t)_* : A_1(\tilde{X}_t) \to A_1(\tilde{X})$ fits into the following diagram

$$\begin{array}{ccc} A_1(\tilde{X}_t) & \xrightarrow{(f_t)_*} & A_1(\tilde{X}) \\ \simeq\downarrow & & {\scriptstyle cl}\downarrow \\ H^2(\tilde{X}_t) & \xrightarrow{j_*} & H^6(\tilde{X}, \mathbf{Q}) \end{array}$$

Here $H^6(X, \mathbf{Q}) \simeq \mathbf{Q}[\gamma^3/3]$, with $\gamma \in A^1(X)$ a hyperplane section and therefore $H^6(\tilde{X}, \mathbf{Q}) \simeq \mathbf{Q}[\gamma^3/3] \oplus \mathbf{Q}$, where the second summand is generated by the class of any exceptional curve contained in the exceptional divisor E, over a point $P \in T$. Therefore the image of $(f_t)_*$ lies in $A_1(\tilde{X})_{hom}$. In the exact sequence of Chow groups, associated to the blow-up $\pi : \tilde{X} \to X$

$$0 \to A_1(T) \to A_1(X) \oplus A_1(E) \to A_1(\tilde{X}) \to 0$$

We have $A_1(T) \subset H^2(T, \mathbf{Q})$, because $p_g(T) = 0$ and hence $A_1(T)_{hom} = 0$. Also $A_1(\tilde{X})_{hom} \simeq A_1(X)_{hom}$, because the transcendental part $t(X)$ is isomorphic to $t(\tilde{X})$. Therefore $A_1(E)_{hom} = 0$ and hence every class in $A_1(\tilde{X})_{hom}$ comes from the class of a line in $A_0(\tilde{F})$ lying in a fiber $\tilde{X}_t$. Here $\tilde{F} = F(\tilde{X})$ is the Fano variety of lines of $\tilde{X}$ and $A_0(\tilde{F}) \simeq A_0(F)$, because $\tilde{F}$ is birational to F. It follows that the map

$$\bigoplus_{t \in \mathbf{P}^2} A_1(\tilde{X}_t) \to A_1(\tilde{X})_{hom} = A_1(X)_{hom} \tag{5.3}$$

is surjective.
Let $\mathcal{H} = \mathrm{Hilb}_1(\tilde{X}/\mathbf{P}^2)$ be the relative Hilbert scheme whose fibers parametrize curves on the fibers of f. Let

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{q} & \tilde{X} \\ {\scriptstyle p}\downarrow & & \\ \mathcal{H} & & \\ \downarrow & & \\ \mathbf{P}^2 & & \end{array}$$

be the incidence diagram, where $\mathcal{C}$ is the universal family over $\mathcal{H}$, i.e. $\mathcal{C} = \{(C,x)/x \in \tilde{X}\} \subset \mathcal{H} \times \tilde{X}$. Then the map

$$p^*q_* : A_0(\mathcal{H}) \to A_1(\tilde{X})_{hom} \simeq A_1(X)_{hom}$$

factors trough $A_0(\mathcal{H}) \to A_1(\tilde{X}_t)$ and $f_t : A_1(\tilde{X}_t) \to A_1(\tilde{X})_{hom} \simeq A_1(X)_{hom}$, for every fiber $\tilde{X}_t$. By [Vial 1, Lemma 6.6] there is finite set $\mathcal{E} = \{\mathcal{H}_1, \cdots, \mathcal{H}_n\}$ of irreducible components of $\mathrm{Hilb}_1(\tilde{X}/\mathbf{P}^2)$, such that they obey the following technical condition:

$\forall t \in B^o$, *the set* $\{cl(q_*[p^{-1}(u)]/u \in \mathcal{H}_i, t = \pi(u)\}$ *span* $H^2(\tilde{X}_t, \mathbf{Q})$. $\quad (*)$.

Let $f_i : \tilde{\mathcal{H}}_i \to \mathcal{H}_i$ be a resolution. By [Vial 1, Prop. 6.7], for all i there is a smooth surface $\tilde{B}_i \to \tilde{\mathcal{H}}_i$, such that, for every $i \in (1, \cdots, n)$ the following map is surjective and generically finite

$$r_i : \tilde{B}_i \to \tilde{\mathcal{H}}_i \xrightarrow{f_i} \mathcal{H}_i \to \mathbf{P}^2$$

Moreover , for every point $P \in \mathbf{P}^2$, $r^{-1}(P)$ contains at least a point in every connected component of the fiber of $\mathcal{H}_i$ over P. Let $\tilde{B} = \coprod_{1 \le i \le n} \tilde{B}_i$ be the disjoint union of the surfaces $\tilde{B}_i$. Again by [Vial 1, Prop. 6.7], there is a correspondence $\Gamma \in A^3(\tilde{B} \times \tilde{X})$ such that $\Gamma = \oplus_i \Gamma_i$ where $\Gamma_i \in A^3(\tilde{B}_i \times \tilde{X})$ and $\tilde{\Gamma}_i$ is the class of the image of $\mathcal{C}_i$ inside $\tilde{B}_i \times \tilde{X}$ in the incidence diagram

$$\begin{array}{ccc} \mathcal{C}_i & \xrightarrow{q_i} & \tilde{X} \\ {\scriptstyle p_i}\downarrow & & \\ \tilde{B}_i & & \end{array} \tag{5.4}$$

Then , by [Vial 1, Prop. 6.7],

$$\mathrm{Im}(\bigoplus_{t \in B^o} A_1(\tilde{X}_t) \to A_1(\tilde{X})_{hom}) \subseteq \mathrm{Im}(\Gamma_* : A_0(\tilde{B})_0 \to A_1(\tilde{X})_{hom}) \tag{5.5}$$

From the assumption in (2) it follows that the map $\bigoplus_{P \in C} A_1(\tilde{X}_P) \to A_1(\tilde{X})_{hom}$ is the 0-map and hence the map $\mathrm{Im}(\bigoplus_{t \in B^o} A_1(\tilde{X}_t) \to A_1(\tilde{X})_{hom})$, having the same image as the map in (5.3), is surjective. Therefore the map

$$A_0(\tilde{B})_0 \to A_1(\tilde{X})_{hom}$$

is surjective. The correspondence Γ induces a map of motives $h(\tilde{B})(1) \to h(\tilde{X})$. Let $h(\tilde{B}) = \sum_{0 \le i \le 4} h_i(\tilde{B})$ be a reduced Chow -Künneth decomposition, with $h_2(\tilde{B}) =$

$h_2^{alg}(\tilde{B}) \oplus t_2(\tilde{B})$. We get a map

$$\Gamma_* : t_2(\tilde{B})(1) \oplus h_3(\tilde{B})(1) \to t(\tilde{X}) \simeq t(X)$$

which induces a map of Chow groups

$$A^3(t_2(\tilde{B})(1) \oplus h_3(\tilde{B})(1)) \to A^3(t(\tilde{X})) = A_1(\tilde{X})_{hom},$$

that is surjective because

$$A^3(t_2(\tilde{B})(1) \oplus h_3(\tilde{B})(1)) = A^2(t_2(\tilde{B}) \oplus h_3(\tilde{B})) \simeq A_0(\tilde{B})_0.$$

Since $A^i(t_2(\tilde{B})(1) \oplus h_3(\tilde{B})(1)) = 0$ and $A^i(t(X)) = 0$, for $i \neq 3$, the map Γ_* induces a surjective map on all Chow groups. Therefore the motive $t(X)$ is isomorphic to a direct summand of the motive

$$t_2(\tilde{B})(1) \oplus h_3(\tilde{B})(1) = \bigoplus_i (t_2(\tilde{B}_i)(1) \oplus h_3(\tilde{B}_i)(1)$$

□

5.1. **Cubic fourfolds containing a plane.** Let $X \subset \mathbf{P}^5$ be a cubic fourfold containing a plane P. Call $\tilde{X}$ the blow-up of X along P and $\pi : \tilde{X} \to \mathbf{P}^2$ the morphism that resolves the projection off P. The morphism π is a fibration in quadric surfaces, whose fibers degenerate along a plane sextic C, which is smooth in the general case. The double cover $S \to \mathbf{P}^2$ ramified along C is a K3 surface. We will assume that the quadric bundle has simple degeneration and a rational section and hence X is rational. The first is an open condition inside the divisor $\mathcal{C}_8$ whereas the second is verified on a countable infinity of codimension two loci inside $\mathcal{C}_8$ [Has 1, Kuz].
Recall that the relative Hilbert scheme of lines $\mathcal{H}(0,1)$ of the morphism π is an étale projective bundle over S. To such an object one can associate a Brauer class $\beta \in Br(S)$. If the quadric bundle has a section then β is trivial and the projective bundle is the projectivized space $\mathbf{P}(\mathcal{E})$ of a vector bundle $\mathcal{E}$.

$$\begin{array}{c} \mathcal{H}(0,1) \\ {\scriptstyle \mathbf{P}^1}\big\downarrow \\ S \\ \big\downarrow \\ \mathbf{P}^2 \end{array}$$

Proposition 5.6. *Under the above hypotheses on the quadric bundle $\pi : \tilde{X} \to \mathbf{P}^2$, the transcendental motive $t(X)$ is isomorphic to the motive $t_2(S)(1)$. Therefore If the motive of S is finite dimensional then also $h(X)$ is finite dimensional .*

Proof. The incidence diagram in (2.4) yields

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{H}} & \xrightarrow{q} & X \\ \downarrow & & \\ \mathcal{H}(0,1) & & \\ {\scriptstyle p}\downarrow & & \\ S & & \end{array}$$

where $\mathcal{C}_{\mathcal{H}}$ is the universal line over $\mathcal{H}(0,1)$. Since $\mathcal{H}(0,1)$ is the projectivized space $\mathbf{P}(\mathcal{E})$ of a vector bundle $\mathcal{E}$, there is an isomorphism of motives $h(\mathcal{H}(0,1)) \simeq h(S) \oplus h(S)(1)$ and hence from the above diagram we get a map of motives $h(S)(1) \to h(X)$ which induces

$$\bar{q} : t_2(S)(1) \to t(X)$$

By the results in [S-Y-Z, Theorem 3.6], the diagram above induces an isomorphism between the Chow groups $A_0(S)_0$ and $A_0(F)_2$, where $F = F(X)$ is the Fano variety of lines. Since $A_0(F)_2 \simeq A_1(X)_{hom}$ it follows that $\bar{q}$ induces an isomorphism between $A^3(t_2(S)(1) = A_0(S)_0$ and $A^3(t(X)) = A_1(X)_{hom}$. Since the other Chow groups of $t_2(S)(1)$ and $t(X)$ vanish we get that $\bar{q}$ is an isomorphism of motives. □

Remark 5.7. Note that the same result can be obtained by applying Proposition 5.2. First of all, since $\tilde{X}$ has a rational section, each quadric has a rational point t and we can isolate one line for each ruling by imposing the condition that it passes through t (one line inside the single ruling if the quadric is singular). This gives a rational section $\pi : S \to \mathcal{H}(0,1)$. Since both S and $\mathcal{H}(0,1)$ are smooth we can resolve π to a regular section of the (pull-back of) the projective bundle to the blow-up $\tilde{S}$ of S over a finite number of points. Call B the image of this regular section. We remark that $t_2(\tilde{S}) = t_2(S) = t_2(B)$. In order to show that $\mathcal{H}(0,1)$ is the only component that we need to apply Prop. 5.2, we need to check that the technical condition $(*)$ holds true for this Hilbert scheme. This is not hard to show, since the $H^2(\tilde{X}_t, \mathbf{Q})$ is generated by the classes of any line of the two rulings of the quadric. In fact the two connected components of $\mathcal{H}(0,1)$ over a point $p \in \mathbf{P}^2$ not lying on the discriminant parametrize the lines in each ruling.

Let $r : S \to \mathbf{P}^2$ be the double cover map. Then, also the condition (2) in proposition 5.2 is satisfied because the curve $D := r^{-1}(C) \subset S$ is a constant cycle curve, see [Huy 1, 7.1]. Let $j : D \to S$ be the inclusion. Since D is a constant cycle curve, the map $j_* : A_0(D)_0 \to A_0(S)_0$ is the 0-map. Then the map $\bigoplus_{P \in C} A_1(\tilde{X}_P) \to A_1(\tilde{X})$ vanishes, because the map $A_0(D)_0 \to A_1(X)_{hom}$, coming from the diagram in (5.4), when restricted to $\mathcal{C}/D \to D \subset S$, factors through j_*. Finally the injectivity of the map $A_0(S)_0 \to A_1(X)_{hom} \simeq A_0(F)_2$, as proved in [S-Y-Z, Theorem 3.6], gives an isomorphism $t_2(S)(1) \simeq t(X)$.

5.2. **Cubic fourfolds fibered in del Pezzo sextics.** Let X be a generic fourfold in $\mathcal{C}_{18}$. The fourfold X contains an elliptic ruled surface T of degree 6 such that the linear system of quadrics in $\mathbf{P}^5$ containing T is two dimensional. Let once again $r : \tilde{X} \to X$ be the blow-up of X at T and $\pi : \tilde{X} \to \mathbf{P}^2$ the (resolution of the) map induced by the linear system of quadrics containing T. The generic fiber of π is a del Pezzo surface of degree 6. The generic del Pezzo fibration π obtained from a cubic fourfold in $\mathcal{C}_{18}$ is a *good del Pezzo fibration* in the sense of [AHTV-A,

Def. 11]. The discriminant curve D of π has two irreducible components, a smooth sextic C and a sextic $\tilde{C}$ with 9 cusps. As in the previous case the double cover $S \to \mathbf{P}^2$ branched on C is a smooth K3 surface of degree 2. The goal of this section is to show that, if the del Pezzo fibration has a rational section (and hence X is rational) then the motive $t(X)$ is a direct summand of $t_2(S)(1)$. Therefore, if S has a finite dimensional motive, there is an isomorphism $t_2(S)(1) \simeq t(X)$ and also $h(X)$ is finite dimensional. Exactly as in the case of $\mathcal{C}_8$ the existence of the section is verified on a countable infinity of codimension two loci in the moduli space, clearly all contained in $\mathcal{C}_{18}$. The main difference with the $\mathcal{C}_8$ case is that here the Picard rank of the generic fiber is higher, so we will need to consider surfaces inside two different Hilbert schemes of curves in order to obey the technical condition $(*)$ and hence to apply the constructions of Prop. 5.2.

Associated to the good del Pezzo fibration $\pi : \tilde{X} \to \mathbf{P}^2$ there is a non-singular degree 3 cover $f : Z \to \mathbf{P}^2$ branched along a cuspidal sextic $\tilde{C}$ (see [AHTV-A]) where Z is a non singular surface. Let $\mathcal{H}(0,2) \to \mathbf{P}^2$ be the relative Hilbert scheme of connected genus 0 curves of anti canonical degree 2 on the fibers. The Stein factorization yields an étale $\mathbf{P}^1$-bundle $\mathcal{H}(0,2) \to Z$. It is easy to see that, on every fiber, the $\mathbf{P}^1$-bundle is given by the strict transform of the lines through each of the 3 blown-up points $P_1, P_2, P_3 \in \mathbf{P}^2$ of the corresponding del Pezzo of degree 6. This gives a diagram analogous to the one in the previous section

$$\begin{array}{c} \mathcal{H}(0,2) \\ {\scriptstyle \mathbf{P}^1}\big\downarrow \\ Z \\ \big\downarrow \\ \mathbf{P}^2 \end{array} .$$

Proposition 5.8. *The triple cover Z is an elliptic ruled surface and hence $t_2(Z) = 0$ and $A_0(Z)_0 \simeq \mathrm{Alb}\, Z \simeq \mathrm{Jac}E$, with E an elliptic curve.*

Proof. Let $\tilde{C} \subset \mathbf{P}^2$ be the ramification locus of the triple cover $f : Z \to \mathbf{P}^2$. As it has been observed in [AHTV-A], for a generic cubic $X \in \mathcal{C}_{18}$, $\tilde{C}$ is a cuspidal degree 6 curve with 9 cusps. It is well known [Mir] that such a triple cover is completely determined by the Tschirnhausen rank two vector bundle on $\mathbf{P}^2$ and a section of (a twist of) the relative $\mathcal{O}(3)$ on the associated projectivized $\mathbf{P}^1$-bundle. Let us denote V the Tschirnhausen module. From Prop. 4.7 of [Mir] we see that D belongs to the linear system $|-2c_1(V)|$, hence $c_1(V) = \mathcal{O}_{\mathbf{P}^2}(-3)$. Then, by [Mir, Lemma 10.1], the number of cusps is exactly $3c_2$, this means that $c_2(V) = 3$. With these data in mind we can use [Mir, Prop. 10.3] to compute the invariants of Z and get

$$\chi = 0, \quad K^2 = 0, \quad e(Z) = 0.$$

Now, by [Shi, Cor 2.3] we see that that $V \cong \Omega_{\mathbb{P}^2}$, hence by [Mir, Cor 10.6] we have $p_g(Z) = 0$, and $q(Z) = 1$. This easily implies that the surface Z is again an elliptic ruled surface. Note that such a triple plane being an elliptic ruled surface was first observed by Du Val in [DV] by different methods. Since $p_g(Z) = 0$ and Z is not of general type we get $t_2(Z) = 0$. The rest follows from the isomorphism $A_0(Z)_0 \simeq \mathrm{Jac}Z$.

□

Lemma 5.9. *If the del Pezzo fibration $\pi : \tilde{X} \to \mathbf{P}^2$ has a rational section, then the projective bundle $\mathcal{H}(0,2) \to Z$ has a rational section as well.*

Proof. Remark that by [AHTV-A, Prop. 8] we have that $\tilde{X}$ is rational over $\mathbf{P}^2$ (and hence over $\mathbf{C}$). Hence we can assume that generically the image of the section is not contained in the exceptional divisor. Then, since by construction the conics are the proper transforms of lines through one of the 3 blown-up points, there exist only one conic in each of the three pencils passing through the section. □

The same way as we did in Rmk. 5.1, we can resolve the rational section constructed here above to a regular section f defined on $\tilde{Z}$, the blow up of Z in a finite number of points. The projective bundle $\mathcal{H}$ pulls back to a projective bundle $\tilde{\mathcal{H}}$ on $\tilde{Z}$:

$$\begin{array}{ccc} \tilde{\mathcal{H}} & \longrightarrow & \mathcal{H} \\ \downarrow & & \downarrow \\ \tilde{Z} & \longrightarrow & Z, \end{array}$$

and $f : \tilde{Z} \to \tilde{\mathcal{H}}$ is a regular section of $\tilde{\mathcal{H}}$. We will call B_1 the image of this section in $\mathcal{H}$ via the birational map $\tilde{\mathcal{H}} \to \mathcal{H}$ and we have $t_2(\tilde{Z}) = t_2(Z) = t_2(B_1)$.

Remark also that, as it was already remarked in [AHTV-A], this implies that the Brauer class of the $\mathbf{P}^1$ bundle $\mathcal{H}(0,2) \to Z$ is trivial whenever $\tilde{X}$ has a rational section over $\mathbf{P}^2$.

As we have already anticipated, in this case, considering just one Hilbert scheme will not be enough in order to apply Prop. 5.2, since the fibers of π have higher Picard rank. Hence we need to consider also $\mathcal{H}(0,3)$, the relative Hilbert scheme of curves of genus zero and canonical degree 3 inside the fibers. There are two 2-dimensional families of such curves on a del Pezzo sextic. One is given by the strict transforms of the lines in $\mathbf{P}^2$ that do not pass through any of the three base points. The second is given by conics passing through the three base points. We will call the former cubic curves of first type and the latter cubic curves of second type. The Stein factorization of the natural projection $p : \mathcal{H}(0,3) \to \mathbf{P}^2$ reflects this difference and displays $\mathcal{H}(0,3)$ as an étale $\mathbf{P}^2$-bundle over a smooth degree two K3 surface S [AHTV-A]. It is straightforward to see that one $\mathbf{P}^2$ parametrizes the curves of first type and the other those of second type.

Lemma 5.10. *If the del Pezzo fibration $\pi : \tilde{X} \to \mathbf{P}^2$ has a rational section, then the projective bundle $\mathcal{H}(0,3) \to S$ has a rational section as well.*

Proof. As we did before, we can assume that the rational section of π generically does not intersect the exceptional divisors. If π has a rational section then $\tilde{X}$ is rational over $\mathbf{P}^2$, hence we can choose a second rational section with the same features of the first one. There exists only one (proper transform of a) line through the two sections inside each fiber of π, and also only one (proper transform of a) conic through the 3 base points and the two sections. This means that we also have a rational section of the projective bundle $\mathcal{H}(0,3) = \mathcal{H} \to S$, its Brauer class is trivial and it is hence a projectivized vector bundle. □

As we did before one can resolve the rational section to a regular section defined on $\tilde{S}$, the blow up in a finite number of points of S, of the pull-back $\tilde{\mathcal{H}}$ in the diagram, where $\mathcal{H} = \mathcal{H}(0,3)$

$$\begin{array}{ccc} \tilde{\mathcal{H}} & \longrightarrow & \mathcal{H} \\ \downarrow & & \downarrow \\ \tilde{S} & \longrightarrow & S. \end{array}$$

Let $f : \tilde{S} \to \mathcal{H}$ be the composite section map. Then the image B_2 of f is a surface which has a surjective and generically finite map of degree 2 onto $\mathbf{P}^2$.

We also have $t_2(S) = t_2(\tilde{S}) = t_2(B_2)$, because $t_2(-)$ is a birational invariant. In order to apply Prop. 5.2 to the fibration $\tilde{X} \to \mathbf{P}^2$ we prove the following lemma.

Lemma 5.11. *Let $\pi : \tilde{X} \to \mathbf{P}^2$ a del Pezzo fibration with a rational section and let $D \subset \mathbf{P}^2$ be the discriminant curve. Then the two surfaces B_1 and B_2 in $\mathcal{H}(0,2) = \mathcal{H}_1$ and $\mathcal{H}(0,3) = \mathcal{H}_2$ obey the technical condition $(*)$, i.e. $\forall t \in B^o = \mathbf{P}^2 - D$, the set $\{cl(q_*[p^{-1}(u)]/u \in \mathcal{H}_i, t = \pi(u)\}$ span $H^2(\tilde{X}_t, \mathbf{Q})$.*

Proof. Fix a point $p \in \mathbf{P}^2$, such that the fiber $\tilde{X}_p$ over p is a smooth del Pezzo sextic. Its Picard rank is 4 and the generators are the proper transform of a line and the three exceptional divisors. Let us denote H, E_1, E_2 and E_3 these divisor classes. Then, the fiber of $B_2 \subset \mathcal{H}(0,3)$ over p contains at least a curve from the linear system $|H|$ and a curve from the linear system $|2H - E_1 - E_2 - E_3|$. On the other hand, the fiber of $B_1 \subset \mathcal{H}(0,2)$ over p contain at least 3 curves from the linear systems $|H - E_1|$, $|H - E_2|$ and $|H - E_3|$. It is straightforward to see that linear combinations of these 5 divisor classes generate the whole $H^2(\tilde{X}_p, \mathbf{Q})$. □

Theorem 5.12. *If the del Pezzo fibration $\pi : \tilde{X} \to \mathbf{P}^2$ has a rational section then $t(X)$ is isomorphic to a direct summand of $t_2(S)(1)$, where S a K3 surface. If $h(S)$ is finite dimensional then $t(X) \simeq t_2(S)(1)$ and hence also $h(X)$ is finite dimensional.*

Proof. From Lemma 5.11 it follows that we can use the formula in (5.5), where $\tilde{B} = B_1 \oplus B_2 = \tilde{Z} \oplus \tilde{S}$ and $r_i : B_i \to \mathcal{H}_i \to \mathbf{P}^2$. Let $\Gamma \in A^3(\tilde{B} \times \tilde{X})$, with $\Gamma = \Gamma_1 \oplus \Gamma_2$. Here $\Gamma_1 \in A^3(\tilde{Z} \times \tilde{X})$ and $\Gamma_2 \in A^3(\tilde{S} \times \tilde{X})$. The correspondence Γ induces a map of motives

$$\Gamma_* : h(\tilde{Z})(1) \oplus h(\tilde{S})(1) \to h(\tilde{X}).$$

The surfaces $\tilde{Z}$ and $\tilde{S}$ have reduced Chow-Künneth decompositions with $t_2(\tilde{Z}) = t_2(Z) = 0$, $h_3(\tilde{S}) = h_3(S) = 0$ and $t_2(\tilde{S}) = t_2(S)$. Therefore Γ_* gives a map

$$\Gamma_* : h_3(Z)(1) \oplus t_2(S)(1) \to t(X) \tag{5.13}$$

From (5.5) we also get

$$\mathrm{Im}(\bigoplus_{t \in \mathbf{P}^2 - D} A_1(\tilde{X}_t) \to A_1(\tilde{X})_{hom}) \subseteq \mathrm{Im}(\Gamma_* : A_0(\tilde{Z})_0 \oplus A_0(\tilde{S})_0 \to A_1(\tilde{X})_{hom}). \tag{5.14}$$

where $\mathbf{P}^2 - D = \mathbf{P}^2 - (\tilde{C} \cup C) = (\mathbf{P}^2 - \tilde{C}) \cap (\mathbf{P}^2 - C)$. We also have $A_0(\tilde{Z})_0 = A_0(Z)_0$ and $A_0(\tilde{S})_0 = A_0(S)_0$.
The discriminant curve D has two irreducible components, the cuspidal sextic $\tilde{C}$ and the smooth sextic C. We first show, using the same argument as in Remark 5.1, that

$$\mathrm{Im}(\bigoplus_{P \in C} A_1(\tilde{X}_P) \to A_1(\tilde{X})_{hom}) = 0.$$

The K3 surface S is a double cover of $\mathbf{P}^2$ ramified along C. Therefore the curve $R = r_2^{-1}(C)$, where $r_2 : S \to \mathbf{P}^2$, is a constant cycle curve in S. Let $j : R \to S$; then the map of Chow groups : $A_0(R)_0 \to A_0(S)_0$ vanishes. Therefore the map $\bigoplus_{P \in C} A_1(\tilde{X}_P) \to A_1(\tilde{X})_{hom}$ is 0, because the map $j_* : A_0(R)_0 \to A_1(\tilde{X})_{hom}$, coming from the diagram in (5.4), when restricted to $\mathcal{C}/R \to D \subset S$, factors through j_*. From the above equality we get

$$\mathrm{Im}(\bigoplus_{t \in \mathbf{P}^2} A_1(\tilde{X}_t) \to A_1(\tilde{X})_{hom}) = \mathrm{Im}(\bigoplus_{t \in \mathbf{P}^2 - C} A_1(\tilde{X}_t) \to A_1(\tilde{X})_{hom}) \tag{5.15}$$

By (5.3) the left hand side in (5.15) equals $A_1(\tilde{X})_{hom}$ and hence $\mathrm{Im}(\bigoplus_{t \in \mathbf{P}^2 - C} A_1(\tilde{X}_t) \to A_1(\tilde{X})_{hom}) = A_1(\tilde{X})_{hom}$.
The diagram in (5.4) gives

$$\begin{array}{ccc} (\mathcal{C}_1)|_{\bar{C}} & \xrightarrow{q_1'} & \tilde{X} \\ {\scriptstyle p_1'}\downarrow & & \\ \bar{C} & & \\ {\scriptstyle r_1}\downarrow & & \\ \tilde{C} & & \end{array} \tag{5.16}$$

where $\bar{C}$ is a desingularization of $r_1^{-1}(\tilde{C}) \subset \tilde{Z}$ and $p_1' : (\mathcal{C}_1)|_{\bar{C}} \to \bar{C}$ is the pull-back of $\mathcal{C}_1$ in the diagram (5.4), along $\bar{C} \to r^{-1}(\tilde{C}) \to \tilde{Z}$. The curve $\bar{C}$ is smooth of genus 1 and hence it is an elliptic curve birational (and hence isomorphic) to a curve E, such that Z is birational to the product $\mathbf{P}^1 \times E$. Let $\Gamma_1' \in A_2(\bar{C} \times \tilde{X})$ be the correspondence induced by Γ_1 . Then Γ_1' is the class of the image of $(\mathcal{C}_1)|_{\bar{C}}$ inside $\bar{C} \times \tilde{X}$ and, from diagram (5.16), we get

$$(\Gamma_1')_* = (q_1')_*(p_1')^* : A_0(\bar{C})_0 \to A_1((\mathcal{C}_1)|_{\bar{C}}) \to A_1(\tilde{X})_{hom},$$

where $A_0(\bar{C})_0 = A_0(\tilde{C})_0$. Since $\tilde{Z}$ is (birational to) an elliptic ruled surface we have

$$A_0(\tilde{C})_0 = A_0(\bar{C})_0 \simeq \mathrm{Jac}E \simeq \mathrm{Alb}(\tilde{Z}) \simeq A_0(\tilde{Z})_0 = A_0(Z)_0.$$

Therefore the map $A_0(\bar{C})_0 \to A_1(\tilde{X})_{hom}$ factors through the isomorphism $A_0(\bar{C})_0 \simeq A_0(\tilde{Z})_0$ and hence

$$\mathrm{Im}((\Gamma_1)_* : A_0(Z)_0 \to A_1(\tilde{X})_{hom}) = \mathrm{Im}((\Gamma_1')_* : A_0(\tilde{C})_0 \to A_1(\tilde{X})_{hom})$$

Let us now prove the inclusion

$$\mathrm{Im}(\bigoplus_{P\in\tilde{C}} A_1(\tilde{X}_P)\to A_1(\tilde{X})_{hom})\subseteq \mathrm{Im}((\Gamma_1')_*: A_0(\tilde{C})_0\to A_1(\tilde{X})_{hom}),$$

In order to show this, we will need to apply [Vial 1, Prop. 6.7] and hence we have to check the technical condition $(*)$, for all the fibers of the Del Pezzo fibration over the desingularized curve $\tilde{C}$. That is: for each *degenerate* Del Pezzo sextic $\tilde{X}_p$ over $p\in\tilde{C}$ we need to check that linear combinations of the 5 divisor classes $H, 2H-E_1-E_2-E_3, H-E_1, H-E_2, H-E_3$ given in Lemma 5.11 (1) generate $H^2(\tilde{X}_p,\mathbb{Q})$. These degenerate Del Pezzo are described in Sect. 5 of [AHTV-A] and they are all obtained by blowing up three (non generic) points on $\mathbf{P}^2$ and then blowing down some rational curves. It is straightforward to see that one can obtain H, E_1, E_2, E_3 form the five classes above. Then we observe that, in all the possible degenerate cases, $H^2(\tilde{X}_p,\mathbb{Q})$ will be a quotient of the $\mathbb{Q}^{\oplus 4}$ generated by the 4 classes above. Hence the technical condition $(*)$ is verified by the two Hilbert schemes $\mathcal{H}(0,2)$ and $\mathcal{H}(0,3)$.

The map in (5.3) is surjective, the map $A_1(\tilde{X}_P)\to A_1(\tilde{X})_{hom}$ vanishes, for all $P\in C\subset\mathbf{P}^2$ and $\mathbf{P}^2=(\mathbf{P}^2-D)\cup(C\cup\tilde{C})$. Hence, the equality in (5.14) and the inclusion $\mathrm{Im}(\bigoplus_{P\in\tilde{C}} A_1(\tilde{X}_P)\to A_1(\tilde{X})_{hom})\subseteq \mathrm{Im}((\Gamma_1)_*: A_0(Z)_0\to A_1(\tilde{X})_{hom})$ show that every class $\alpha\in A_1(\tilde{X})_{hom}$ belongs either to the image of $(\Gamma_1)_*: A_0(Z)_0\to A_1(\tilde{X})_{hom}$ or to the image of $\Gamma_*: A_0(Z)_0\oplus A_0(S)_0\to A_1(\tilde{X})_{hom}$. The map $(\Gamma_1)_*$ cannot be surjective, because otherwise we would get a surjective map

$$A_0(Z)_0\simeq \mathrm{Jac}E\to A_1(\tilde{X})_{hom}=A_1(X)_{hom}$$

while the group $A_1(X)_{hom}$ is not representable. Therefore the map $\Gamma_*: A_0(Z)_0\oplus A_0(S)_0\to A_1(\tilde{X})_{hom}=A_1(X)_{hom}$ is surjective and hence the map of motives in (5.13) induces a surjective map of Chow groups

$$A^3(h_3(Z)(1))\oplus A^3(t_2(S)(1))=A_0(Z)_0\oplus A_0(S)_0\to A^3(t(X)=A_1(X)_{hom}.$$

Since the other Chow groups, for $i\neq 3$, vanish on both sides, the transcendental motive $t(X)$ is isomorphic to a direct summand of $h_3(Z)(1)\oplus t_2(S)(1)$. We also have $h_3(Z)\simeq h_1(E)(1)$ with E an elliptic curve, because Z is an elliptic ruled surface . Therefore $t(X)$ is isomorphic to a direct summand of $h_1(E)(2)\oplus t_2(S)(1)$. However the same argument as before implies that the motive $t(X)$ cannot be isomorphic to a direct summand of $h_1(E)(2)$. Therefore $t(X)$ is isomorphic to a direct summand of $t_2(S)(1)$. If $h(S)$ is finite dimensional then the motive $t_2(S)$ is indecomposable and hence $t(X)\simeq t_2(S)(1)$.

□

Remark 5.17. In [AHTV-A, Remark 17 (c)], it is conjectured that the surface Z identifies with the elliptic ruled surface $T\subset X$. If this is the case then from the isomorphism $A_0(Z)_0\simeq A_0(T)_0$ one gets that the map $A_0(Z)_0\to A_1(\tilde{X})_{hom}$ vanishes, because every class in $A_1(\tilde{X})_{hom}$ comes from a class in a fiber $\tilde{X}_t$, with $t\in\mathbf{P}^2$. Therefore the image of $A_0(Z)_0\oplus A_0(S)_0\to A_1(\tilde{X})_{hom}$ equals the image of $A_0(S)_0=A_0(\tilde{S})_0$. Then, by applying the same argument as in the proof of [S-Y-Z, Theorem 3.6], it is easy to show that the map $A_0(\tilde{S})_0\to A_1(\tilde{X})_{hom}$ is also injective.

The map $A_0(\tilde{S})_0 \to A_1(\tilde{X})_{hom}$ comes from the incidence diagram

$$\begin{array}{ccc} \mathcal{C}_2 & \xrightarrow{q_2} & \tilde{X} \\ {\scriptstyle p_2}\downarrow & & \\ \mathcal{H}_2 & & \\ \downarrow & & \\ \mathbf{P}^2 & & \end{array}$$

where $\tilde{S} \subset \mathcal{H}_2$. If $(q_2)_*(p_2)^*(\alpha) = 0$ in $A_1(\tilde{X})_{hom}$, with $\alpha \in A_0(\tilde{S})_0 = A_0(S)_0$ then $\sigma_*(\alpha) = 0$, with σ the involution on S coming from the double cover $S \to \mathbf{P}^2$.Therefore $\alpha = 0$. From the isomorphism $A_0(S)_0 \simeq A_1(X)_{hom}$ we get

$$t_2(S)(1) \simeq t(X)$$

6. Cubic fourfolds with a symplectic automorphism

Let $X \subset \mathbf{P}^5$ be a cubic fourfold and let σ be a symplectic automorphism of X inducing a polarized automorphism σ_F on $F(X)$, i.e such that σ_F preserves the Plücker polarization on $F(X)$ (see [LFu 2, Sect 1]). Then σ comes from an automorphism of finite order on $\mathbf{P}^5$. According to [LFu 1], σ_F acts as the identity on $A_1(X)_{hom}$ and on $A^2(F)_{hom}$. Assume that σ_F has prime order and fixes a codimension 2 subvariety Z in $F(X)$. Then, by [LFu 2], σ_F is either an automorphism of order 3 or a symplectic involution. Here we show that, in the first case, the fourfold X is rational, its motive $h(X)$ is finite dimensional and of abelian type and there exists a K3 surface S such that $t(X) \simeq t_2(S)(1)$. In the second case X is a conic bundle over $\mathbf{P}^3$ with reducible discriminant, that is the union of a cubic and a (singular) quadric surface. Moreover, there is a K3 surface S which parametrizes the lines on $F(X)$ fixed by σ_F, which is a double cover of the cubic surface in $\mathbf{P}^3$ ramified along a smooth sextic (in fact the intersection of the two components). In Prop. 6.4 we show that $t(X) \simeq t_2(S)(1)$. Let us start with the order 3 case.

(1) By the results in [LFu 2] if σ is an automorphism of order 3 on $\mathbf{P}^5$ acting on X, such that the fixed locus in $F(X)$ is a surface, then σ is the automorphism defined by

$$[x_0, x_1, x_2, x_3, x_4, x_5] \to [x_0, x_1, x_2, \omega x_3, \omega x_4, \omega x_5],$$

with ω a primitive third root of 1, and the the cubic fourfold X has an equation of the form

$$f(x_0, x_1, x_2) + g(x_3, x_4, x_5) = 0$$

where f and g are homogeneous of degree 3. The fixed locus in $\mathbf{P}^5$ is given by two disjoint planes $W_1 = (x_3 = x_4 = x_5 = 0)$ and $W_2 = (x_0 = x_1 = x_2 = 0)$. The fixed locus of σ_F in $F(X)$ is the abelian surface $E_1 \times E_2$, where E_i is an elliptic curve. The surface $E_1 \times E_2$ parametrizes all lines joining a point Q_1 on the plane W_1 satisfying the equation $f = 0$ and a point Q_2 on the plane W_2 satisfying the equation $g = 0$.
Let $Z \subset \mathbf{P}^3$ and $T \subset \mathbf{P}^3$ be the cubic surfaces defined by $f(x_0, x_1, x_2) - t^3 = 0$ and $g(x_3, x_4, x_5) - t^3 = 0$, respectively. By [CT, Prop. 1.2] there is a rational map $\mathbf{P}^3 \times \mathbf{P}^3 \to \mathbf{P}^5$ which induces a rational dominant map $\psi : Z \times T \to X$ and whose

locus of indeterminacy is $E_1 \times E_2$. Let Y be the blow-up of $S \times T$ at $C \times D$. By Manin's formula there is an isomorphism

$$h(Y) \simeq h(Z \times T) \oplus h(E_1 \times E_2)(1).$$

Hence the motive of the fourfold $Z \times T$ is finite dimensional and has no transcendental part, since both the surfaces Z and T are rational. Therefore the motive $h(Y)$ is finite dimensional and of abelian type because its transcendental part coincides with the transcendental motive $t_2(E_1 \times E_2)(1)$, which lies in the subcategory of $\mathcal{M}_{rat}$ generated by the motives of curves. The map ψ induces a finite morphism $\tilde{\psi} : Y \to X$. It follows that also $h(X)$ is finite dimensional and of abelian type. Moreover the transcendental motive $t(X)$ is isomorphic to the transcendental motive of Y and hence

$$t(X) \simeq t_2(E_1 \times E_2)(1)$$

The fourfold X contains two skew planes P_1 and P_2 (see [CT, Rmk. 2.4]) and therefore is rational. Let

$$\rho : P_1 \times P_2 \to X$$

be the birational map, as defined [Has 1], whose locus of indeterminacy is the K3 surface S parametrizing the lines $l(p_1, p_2)$ joining two points $p_1 \in P_1$ and $p_2 \in P_2$, which are contained in X. Let $\bar{\rho} : \tilde{Y} \to X$ be the dominant morphism, induced by ρ, where $\tilde{Y}$ is the blow-up of $P_1 \times P_2$ at S. The motive of $\tilde{Y}$ splits as

$$h(\tilde{Y}) \simeq h(P_1 \times P_2) \oplus h(S)(1)$$

and hence its transcendental part is isomorphic to $t_2(S)(1)$. The finite morphism $\bar{\rho}$ induces a map $h(\tilde{Y}) \to h(X)$ and an isomorphism between the transcendental motive $t(X)$ and $t_2(S)(1)$. Therefore $t_2(S)$ is isomorphic to $t_2(E_1 \times E_2)$ and hence is of abelian type.

(2) Let σ be the involution on $\mathbf{P}^5$ defined by

$$[x_0, x_1, x_2, x_3, x_4, x_5] \to [x_0, x_1, x_2, x_3, -x_4, -x_5]$$

A cubic fourfold X fixed by σ has an equation of the form

$$C(x_0, x_1, x_2, x_3) + x_4^2 L_1 + x_5^2 L_2 + x_4 x_5 L_3 = 0 \tag{6.1}$$

where C has degree 3 and L_1, L_2, L_3 are linear forms in x_0, x_1, x_2, x_3. Camere shows that this is the unique automorphism of $\mathbf{P}^5$ inducing a symplectic involution on $F(X)$ [Ca, Sect. 7]. The locus of fixed points of σ on $\mathbf{P}^5$ is the disjoint union of a $\mathbf{P}^3$ defined by $x_4 = x_5 = 0$ and the line r joining the base points P_4 and P_5. The line r is contained in X and the fixed locus on $\mathbf{P}^3$ is the cubic surface $C = 0$. The symplectic involution σ_F on $F(X)$ has 28 isolated points, i.e. the line r and the 27 lines on the cubic surface, plus a K3 surface S, consisting of the lines joining a fixed point Q_1 on $\mathbf{P}^3$ and a point Q_2 on r (see again [Ca]). Let us now project with center the line r and let $\tilde{X}$ denote the blow-up of X along r. The projection resolves into a morphism $\delta : \tilde{X} \to \mathbf{P}^3$, which is well-known to be a conic bundle with quintic degeneration locus D.

Lemma 6.2. *The quintic hypersurface $D \subset \mathbf{P}^3$ has a cubic and a quadric irreducible components. For appropriate choices of the L_i and of C the sextic intersection curve is smooth and parametrizes rank one conics. For general choices of the L_i the quadric has rank 3.*

Proof. Let $p := [a : b : c : d] \in \mathbf{P}^3$, in order to study the conic over p we need to study the intersection of X with the plane $\mathbf{P}^2_p := \langle p, P_4, P_5 \rangle \subset \mathbf{P}^5$, where $\langle \cdot \rangle$ denotes as usual the linear span, and P_4, P_5 are the base points on r. Hence we substitute inside equation 6.1 the values $\lambda[0:0:0:0:1:0] + \mu[0:0:0:0:0:1] + \gamma[a:b:c:d:0:0]$, with $\lambda, \mu, \gamma \in \mathbb{C}$. Recall that in this plane the equation of r is $\gamma = 0$. Then, dividing by γ the cubic equation in γ, λ and μ we obtain

$$\gamma^2 C(a,b,c,d) + \lambda^2 L_1(a,b,c,d) + \mu^2 L_2(a,b,c,d) + \lambda\mu L_3(a,b,c,d). \tag{6.3}$$

This is the conic obtained from the symmetric matrix

$$\begin{pmatrix} C(a,b,c,d) & 0 & 0 \\ 0 & L_1 & \frac{1}{2}L_3 \\ 0 & \frac{1}{2}L_3 & L_2 \end{pmatrix}.$$

Hence one easily sees that the equation of D is $C \cdot (L_1 L_2 - \frac{1}{4}L_3)$. The mere equation $L_1 L_2 - \frac{1}{4}L_3$ shows that the quadric has at most rank 3 and that for general L_i this is the case. Let us denote by Q the quadric surface. Suppose now $L_3 = x_0 - x_1 - x_2$, $L_1 = (t - z)$, $L_2 = (t + z)$ and C is the Fermat cubic. Then the quadric has equation $-(x - y - z)^2 + t^2 - z^2$ and rank 3. A quick Macaulay2 [Mac2] routine shows that the intersection with the Fermat cubic is a smooth sextic curve Y, and from the matrix representing the conic one sees that the sextic curves parametrizes conics of rank 1. □

The surface S is a double cover of the cubic surface $C = 0$ ramified along the degree 6 curve Y. It is straightforward to see that S parametrizes irreducible (linear) components of degenerate conics, that are fixed by the involution. If one takes the double cover $W \stackrel{2:1}{\to} Q$, ramified along Y, this parametrizes the irreducible components of degenerate conics that are not fixed by the involution (except for double lines, parametrized by Y). It is a classical construction that double covers W of quadric cones, ramified along a smooth genus 4 sextic are del Pezzo surfaces of degree 1, and the double cover is induced by the linear system $|-2K_W|$, where K_W is the canonical bundle. We observe that by Kodaira vanishing it is easy to see that $q(W) = 0$. The Abel-Jacobi map induces an isomorphism

$$H^{3,1}(X) \simeq H^{2,0}(F(X)) \simeq H^{2,0}(S)$$

and hence $H^2_{tr}(F, \mathbf{Q}) \simeq H^2_{tr}(S, \mathbf{Q})$. By [Lat 2, Thm. 3.1] there is a correspondence $\Gamma \in A^3(S \times X)$ inducing a surjective homomorphism

$$A^2(S)_0 \to A_1(X)_{hom}$$

Let $h(S) = \mathbf{1} \oplus h^{alg}(2) \oplus t_2(S) \oplus \mathbf{L}^2$ be a Chow-Künneth decomposition and let $\Gamma_* : t_2(S)(1) \to t(X)$ be the map of motives induced by Γ. We have

$$A^3(t(X)) = A_1(X)_{hom} \ ; \ A^3(t_2(S)(1)) = A^2(t_2(S)) = A^2(S)_0$$

and $A^i(t(X)) = A^i(t_2(S)(1)) = 0$ for $i \neq 3$. Therefore Γ induces a surjective map on all Chow groups and hence $t(X)$ is a direct summand of $t_2(S)(1)$. The following result shows that Γ_* is in fact an isomorphism.

Proposition 6.4. *Let S_r be the surface of lines meeting the fixed line r. Then the following isomorphisms hold in $\mathcal{M}_{rat}(\mathbf{C})$*

$$t_2(S_r)^- = t_2(S_r) \simeq t_2(S) \ ; \ t_2(S)(1) \simeq t(X)$$

.

Proof. By the same argument as in the proof of 2.8 the conic bundle $\delta : \tilde{X} \to \mathbf{P}^3$ is obtained by the blow-up of the line r and the surface S_r parametrizes lines in the singular fibers of δ. Let $\tilde{D}$ be the desingularization of the quintic surface $D \subset \mathbf{P}^3$ and let $\tilde{S}_r$ be the blow up of S_r in the finite set of points fixed by σ. In the commutative diagram

$$\begin{array}{ccc} \tilde{S}_r & \longrightarrow & S_r \\ \downarrow & & \downarrow \\ \tilde{D} & \longrightarrow & D \end{array}$$

the surface $\tilde{D}$ is rational and hence $t_2(\tilde{D}) = 0$. Also $q(S_r) = 0$. Therefore, by [Ped, Prop 1 (iv)], $A_0(S_r)_0^+ = 0$ and hence

$$A_0(S_r)_0 = A_0(S_r)_0^-$$

The surface S parametrizes lines in the singular fibers of δ fixed by σ_F and W parametrizes singular fibers which are not fixed by σ_F (except for the double lines , parametrized by the sextic Y). The surface W being rational we have $A_0(W)_0 = 0$ and hence the inclusion $S \subset S_r$ induces an equality

$$A_0(S)_0 = A_0(S_r)_0 = A_0(S_r)_0^-$$

From Lemma 6.4 and Prop. 2.8, (ii) there is an isomorphism $t_2(S_r)^-(1) \simeq t(X)$. Therefore the map $\Gamma_* : t_2(S)(1) \to t(X)$ is an isomorphism, because it induces an isomorphisms on all Chow groups. □

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