



**HAL**  
open science

# Topological derivatives of leading-and second-order homogenized coefficients in bi-periodic media

Rémi Cornaggia, Bojan B Guzina, Marc Bonnet

► **To cite this version:**

Rémi Cornaggia, Bojan B Guzina, Marc Bonnet. Topological derivatives of leading-and second-order homogenized coefficients in bi-periodic media. WAVES 2017 - 13th International Conference on Mathematical and Numerical Aspects of Wave Propagation, May 2017, Minneapolis, United States. hal-02065548

**HAL Id: hal-02065548**

**<https://hal.science/hal-02065548v1>**

Submitted on 12 Mar 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Topological derivatives of leading- and second-order homogenized coefficients in bi-periodic media

Marc Bonnet<sup>1</sup>, Rémi Cornaggia<sup>2,\*</sup>, Bojan Guzina<sup>3</sup>

<sup>1</sup>Poems (UMR 7231 CNRS-INRIA-ENSTA), Palaiseau, France

<sup>2</sup>IRMAR, Université Rennes 1, Rennes, France

<sup>3</sup>CEGE, University of Minnesota, Minneapolis, USA

\*Email: remi.cornaggia@univ-rennes1.fr

**Abstract** We derive the topological derivatives of the homogenized coefficients associated to a periodic material, with respect of the small size of a penetrable inhomogeneity introduced in the unit cell that defines such material. In the context of antiplane elasticity, this work extends existing results to (i) time-harmonic wave equation and (ii) second-order homogenized coefficients, whose contribution reflects the dispersive behavior of the material.

**Keywords:** homogenization, topological derivatives.

**Introduction** Consider an elastic material occupying a 2D domain and characterized by periodic shear modulus  $\mu$  and density  $\rho$ . The unit cell  $Y$  has characteristic length  $\ell$ . Under time-harmonic conditions, the antiplane displacement  $u$  satisfies the wave equation:

$$\nabla \cdot (\mu \nabla u) + \omega^2 \rho u = 0$$

For long-wavelength configurations (i.e.  $\ell \ll \lambda$ ), two-scale periodic homogenization of this equation in terms of  $\varepsilon = \ell/\lambda$  [4] leads to the equation satisfied by the mean field  $U$ :

$$\begin{aligned} \mu^0 : \nabla^2 U + \omega^2 \rho^0 U \\ = -\varepsilon^2 [\mu^2 :: \nabla^4 U + \omega^2 \rho^2 : \nabla^2 U] + O(\varepsilon^4), \end{aligned}$$

where the leading-order and second-order homogenized coefficients  $(\mu^0, \rho^0, \mu^2, \rho^2)$  are constant tensors and  $\nabla^k U$  stands for the  $k$ -th gradient of  $U$ .

This study considers a periodic perturbation of this material, whereby a penetrable inhomogeneity  $B_a$ , of size  $a$  and shape  $\mathcal{B}$ , characterized by contrasts  $(\Delta\mu, \Delta\rho)$  is introduced at point  $z \in Y$  (Fig. 1). Then, the leading-order expansion coefficients of  $(\mu^0, \rho^0, \mu^2, \rho^2)$  w.r.t.  $a$ , namely their topological derivatives, are computed, as in [3] for in-plane elastostatics.

**Leading-order coefficients** Let  $\langle \cdot \rangle = \frac{1}{|Y|} \int_Y \cdot$  denote an average on the unit cell. The homogenized density  $\rho^0$  is defined by  $\rho^0 = \langle \rho \rangle$ , so that

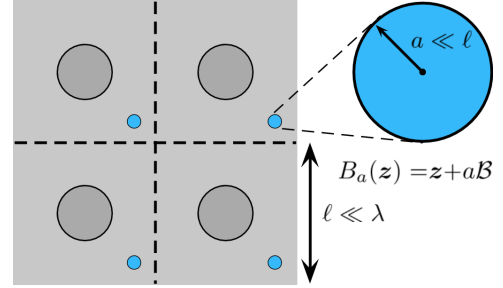


Figure 1: Perturbed periodic material

the perturbed coefficient  $\rho_a^0$  and the topological derivative  $\mathcal{D}\rho^0$  are exactly given by:

$$\rho_a^0 = \rho^0 + a^2 |Y|^{-1} \mathcal{D}\rho^0; \quad \mathcal{D}\rho^0 = |\mathcal{B}| \Delta\rho.$$

The homogenized shear modulus  $\mu^0$  is defined by  $\mu^0 = \langle \mu(\mathbf{I} + \nabla \mathbf{P}) \rangle^S$ , where  $\mathbf{I}$  is the identity tensor, the first *cell function*  $\mathbf{P}$  [4] is the  $Y$ -periodic and zero-mean vector-valued solution of:

$$\nabla \cdot (\mu(\mathbf{I} + \nabla \mathbf{P})) = \mathbf{0} \quad (1)$$

and the superscript  $\cdot^S$  means symmetrization w.r.t. all index permutations. Consequently,  $\mu_a^0$  is computed as:

$$\mu_a^0 = \mu^0 + \langle \mu \nabla \mathbf{p}_a \rangle^S + \langle \chi_{B_a} \Delta\mu(\mathbf{I} + \nabla \mathbf{P}_a) \rangle^S$$

where  $\mathbf{p}_a := \mathbf{P}_a - \mathbf{P}$  is the perturbation of  $\mathbf{P}$ . The analysis of this perturbation is done by reformulating problem (1) and its perturbed counterpart using domain integral equations [2]. With the help of the adjoint state method, it leads to the following leading-order expansion:

$$\mu_a^0 = \mu^0 + a^2 |Y|^{-1} \mathcal{D}\mu^0(z) + o(a^2 |Y|^{-1}), \quad (2)$$

with the topological derivative  $\mathcal{D}\mu^0$  given by:

$$\mathcal{D}\mu^0(z) = [(\mathbf{I} + \nabla \mathbf{P}) \cdot \mathbf{A} \cdot (\mathbf{I} + \nabla \mathbf{P})^T](z)$$

and  $\mathbf{A}(z) = \mathbf{A}(\mathcal{B}, \mu(z), \Delta\mu)$  is the polarization tensor [1] associated to shape  $\mathcal{B}$  and moduli  $\mu(z)$  and  $\mu(z) + \Delta\mu$ . Under notational adjustments, this result is similar to [3]. For homogeneous background materials, in which case  $\mathbf{P} = \mathbf{0}$ , it reduces to  $\mathcal{D}\mu^0 = \mathbf{A}$  as shown by [1].

**Second-order coefficients** The second-order homogenized density is defined by  $\boldsymbol{\rho}^2 = \langle \rho \mathbf{Q} \rangle^S$ , where the *second cell function*  $\mathbf{Q}$  is the  $Y$ -periodic, zero-mean, tensor-valued solution of:

$$\begin{aligned} \nabla \cdot (\mu(\mathbf{P} \otimes \mathbf{I} + \nabla \mathbf{Q})) \\ = -\mu(\mathbf{I} + \nabla \mathbf{P}) + (\rho/\rho^0)\boldsymbol{\mu}^0 \end{aligned} \quad (3)$$

Relying on the same integral equation framework, and with careful analysis of the influence of the source terms involving  $\mathbf{P}_a$  when addressing the perturbed cell function  $\mathbf{Q}_a$ , we show that  $\boldsymbol{\rho}_a^2$  has an expansion of the same form as (2), with its topological derivative  $\mathcal{D}\boldsymbol{\rho}^2$  given by:

$$\begin{aligned} \mathcal{D}\boldsymbol{\rho}^2(\mathbf{z}) = & \left[ (\mathbf{I} + \nabla \mathbf{P}) \cdot \mathbf{A} \cdot (\beta \mathbf{I} + \nabla \hat{\mathbf{X}}[\beta])^T \right. \\ & - (\mathbf{P} \otimes \mathbf{I} + \nabla \mathbf{Q}) \cdot \mathbf{A} \cdot \nabla \beta \\ & - (\mathcal{D}\boldsymbol{\mu}^0 - (\mathcal{D}\rho^0/\rho^0)\boldsymbol{\mu}^0) \langle \rho(\beta/\rho^0) \rangle \\ & \left. - \mathcal{D}\rho^0 ((\beta/\rho^0)\boldsymbol{\mu}^0 - \mathbf{Q}) \right]^S(\mathbf{z}). \end{aligned} \quad (4)$$

The above expression features (i) various combinations of the previously computed cell solutions and topological derivatives and (ii) two new *adjoint fields*  $\beta$  and  $\hat{\mathbf{X}}[\beta]$  defined as the ( $Y$ -periodic, zero-mean) solutions of:

$$\begin{aligned} \nabla \cdot (\mu \nabla \beta) &= -(\rho - \rho^0) \\ \text{and } \nabla \cdot (\mu(\beta \mathbf{I} + \nabla \hat{\mathbf{X}}[\beta])) &= -\mu \nabla \beta. \end{aligned}$$

In particular, all the fields involved in (4) solve problems posed on the *unperturbed* cell.

The second-order homogenized shear modulus is defined by  $\boldsymbol{\mu}^2 = \langle \mu(\mathbf{Q} \otimes \mathbf{I} + \nabla \mathbf{R}) \rangle^S$  in terms of  $\mathbf{Q}$  and a third cell function  $\mathbf{R}$ . Once again, an analysis of the problems satisfied by  $\mathbf{R}$  and  $\mathbf{R}_a$  is conducted. As a result,  $\boldsymbol{\mu}_a^2$  is found to have an expansion similar to (2), and its topological derivative  $\mathcal{D}\boldsymbol{\mu}^2$  (not shown here for brevity) is expressed in terms of the cell solutions  $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$  and the previously determined topological derivatives  $(\mathcal{D}\rho^0, \mathcal{D}\boldsymbol{\mu}^0, \mathcal{D}\boldsymbol{\rho}^2)$ .

**Perspectives.** The obtained expansions of the homogenized coefficients are useful on their own right, e.g. for computing quickly an approximation of the properties of a perturbed periodic material for several trial inhomogeneity locations  $\mathbf{z}$  without solving the new cell problems. As an example, an approximation of  $\boldsymbol{\mu}_a^0$  is obtained by neglecting the remainder in (2), as illustrated on Fig. 2 for a chessboard-like cell.

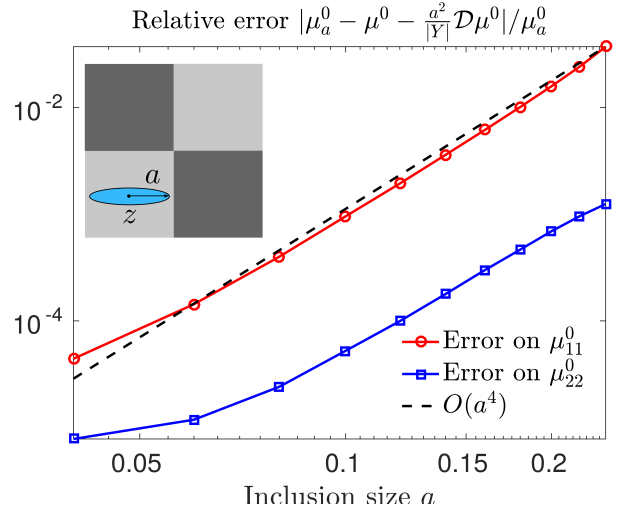


Figure 2: Relative error on shear modulus  $\boldsymbol{\mu}_a^0$  approximated by expansion (2) for an ellipsoidal inhomogeneity of semi-axes  $(a, 0.2a)$  placed at  $\mathbf{z} = (0.25, 0.25)$  in a chessboard-like cell  $Y = [0, 1]^2$ . In this case, since the medium is locally homogeneous around  $\mathbf{z}$ , the remainder can be shown to be in  $O(a^4)$  as observed.

However, as already intended in [3], the main usefulness of such expansions occurs for optimizing a periodic structure towards some desirable property. Since they address the time-harmonic case and the second-order homogenized coefficients, our results should notably allow to tune the dispersive properties of the homogenized material, in particular the so-called *band-gaps* (forbidden frequencies for which no wave propagates through the structure).

## References

- [1] H. Ammari, H. Kang, and K. Touibi. Boundary layer techniques for deriving the effective properties of composite materials. *Asymptot. Anal.*, 41:119–140, 2005.
- [2] M. Bonnet. Higher-order topological sensitivity for 2-d potential problems. application to fast identification of inclusions. *Int. J. Solids Struct.*, 46:2275–2292, 2009.
- [3] S. M. Giusti, A. A. Novotny, and E. A. de Souza Neto. Sensitivity of the macroscopic response of elastic microstructures to the insertion of inclusions. *Proc. Roy. Soc. London. Series A.*, 2010.
- [4] A. Wautier and B. B. Guzina. On the second-order homogenization of wave motion in periodic media and the sound of a chessboard. *J. Mech. Phys. Solids*, 78:382–404, 2015.