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IN KOENIGS' FOOTSTEPS: DIAGONALIZATION OF COMPOSITION OPERATORS

W. ARENDT, B. CÉLARIÈS, AND I. CHALENDAR

ABSTRACT. Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic map with a fixed point $\alpha \in \mathbb{D}$ such that $0 \leq |\varphi'(\alpha)| < 1$. We show that the spectrum of the composition operator C_φ on the Fréchet space $\text{Hol}(\mathbb{D})$ is $\{0\} \cup \{\varphi'(\alpha)^n : n = 0, 1, \dots\}$ and its essential spectrum is reduced to $\{0\}$. This contrasts the situation where a restriction of C_φ to Banach spaces such as $H^2(\mathbb{D})$ is considered. Our proofs are based on explicit formulae for the spectral projections associated with the point spectrum found by Koenigs. Finally, as a byproduct, we obtain information on the spectrum for bounded composition operators induced by a Schröder symbol on arbitrary Banach spaces of holomorphic functions.

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1. INTRODUCTION

Let φ be a holomorphic self-map of the open unit disc \mathbb{D} and let $\text{Hol}(\mathbb{D})$ be the algebra of holomorphic functions on \mathbb{D} which is a Fréchet space endowed with the topology of uniform convergence on every compact subsets of \mathbb{D} .

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Denote by $\text{Aut}(\mathbb{D})$ the group of all automorphisms on \mathbb{D} . It is a well-known fact that such functions have the form $z \mapsto e^{i\theta} \frac{z-a}{1-\bar{a}z}$ where $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$.

The functional equation $f \circ \varphi = \lambda f$ where $\lambda \in \mathbb{C}$ is called the *homogeneous Schröder equation*.

For those φ which are not automorphisms of \mathbb{D} and which admit a fixed point $\alpha \in \mathbb{D}$, the solution was found by G. Koenigs in 1884. Note that a fixed point in \mathbb{D} is unique whenever it exists.

By \mathbb{N}_0 we denote the set of all nonnegative integers and let $\mathbb{N} = \mathbb{N}_0 \setminus \{0\} = \{1, 2, \dots\}$.

Theorem 1.1 (Koenigs' theorem). *Let φ be a holomorphic map on \mathbb{D} such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi \notin \text{Aut}(\mathbb{D})$ and assume that φ has a fixed point $\alpha \in \mathbb{D}$ with $\lambda_1 := \varphi'(\alpha)$. Then the following holds:*

- If $\lambda_1 = 0$ the equation $f \circ \varphi = \lambda f$ has a nontrivial solution $f \in \text{Hol}(\mathbb{D})$ if and only if $\lambda = 1$ and the constant functions are the only solutions.
- If $\lambda_1 \neq 0$, then:
 - (a) the equation $f \circ \varphi = \lambda f$ has a nontrivial solution $f \in \text{Hol}(\mathbb{D})$ if and only if $\lambda \in \{\lambda_1^n : n \in \mathbb{N}_0\}$;
 - (b) there exists a unique function $\kappa \in \text{Hol}(\mathbb{D})$ satisfying

$$\kappa \circ \varphi = \lambda_1 \kappa \text{ and } \kappa'(\alpha) = 1;$$

- (c) for $n \in \mathbb{N}_0$ and $f \in \text{Hol}(\mathbb{D})$, $f \circ \varphi = \lambda_1^n f$ if and only if $f = c\kappa^n$ for some $c \in \mathbb{C}$.

The case where $\varphi'(\alpha) \neq 0$ is the most interesting one. To be consistent with [23], we use the following terminology.

Definition 1.2. *A Schröder map is a holomorphic function φ satisfying the following conditions:*

$$\varphi(\mathbb{D}) \subset \mathbb{D}, \varphi \notin \text{Aut}(\mathbb{D}), \exists \alpha \in \mathbb{D} \text{ such that } \varphi(\alpha) = \alpha \text{ and } \varphi'(\alpha) \neq 0.$$

The function κ associated to a Schröder map in Theorem 1.1 is called the Koenigs' eigenfunction of φ .

As a consequence of the Schwarz lemma [19], a holomorphic self-map φ of \mathbb{D} with a fixed point $\alpha \in \mathbb{D}$ is a Schröder map if and only if $0 < |\varphi'(\alpha)| < 1$. Moreover, Koenigs' eigenfunction κ is then obtained as the limit of $\frac{\varphi_n}{\lambda_1^n}$ in $\text{Hol}(\mathbb{D})$ as $n \rightarrow \infty$, where $\varphi_n = \varphi \circ \dots \circ \varphi$.

The aim of this paper is to study the non homogeneous Schröder equation

$$(1) \quad f \circ \varphi - \lambda f = g$$

where $\lambda \in \mathbb{C}$ and $g \in \text{Hol}(\mathbb{D})$ are given and $f \in \text{Hol}(\mathbb{D})$ the solution.

As in Koenigs' work, we consider the case where $\varphi \notin \text{Aut}(\mathbb{D})$ and φ has a fixed point α in \mathbb{D} .

The study of the homogenous Schröder equation can be reformulated from an operator theory point of view in the following way: consider the composition operator $C_\varphi : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D})$ given by $C_\varphi(f) = f \circ \varphi$. We denote by $\sigma(C_\varphi)$ the spectrum and by $\sigma_p(C_\varphi)$ the point spectrum of C_φ . Thus (1) has a unique solution for all $g \in \text{Hol}(\mathbb{D})$ if and only if $\lambda \notin \sigma(T)$. Moreover, Koenigs' theorem implies that $\sigma_p(C_\varphi) = \{\lambda_n : n \in \mathbb{N}_0\}$.

Our main result consists in finding "the spectral projections" associated with $\lambda_n = \lambda_1^n$. The difficulty is that these spectral projections are not defined since a priori we do not know that the λ_n are isolated in the spectrum. We define projections P_n of rank 1 such that $P_n C_\varphi = C_\varphi P_n = \lambda_n P_n$. Using these "spectral" projections we then show that actually the spectrum of the composition operator C_φ on $\text{Hol}(\mathbb{D})$ is given by

$$\sigma(C_\varphi) = \{\lambda_n : n \in \mathbb{N}_0\} \cup \{0\}.$$

This looks very similar to spectral properties of compact operators. But we show that the operator C_φ is compact on $\text{Hol}(\mathbb{D})$ only in very special situations.

Nevertheless, our results show that the operator C_φ on $\text{Hol}(\mathbb{D})$ is always a Riesz operator; i.e. its essential spectrum is reduced to $\{0\}$. This contrasts the situation where a restriction $T = C_\varphi|_{H^2(\mathbb{D})}$ is considered. Indeed, in this case, the essential spectrum is a disc with $r_e(T) > 0$ in many cases. Actually, much is known on such restrictions to spaces such as $H^p(\mathbb{D})$, Bergman space, Dirichlet space and others. See the monographs [6] of Cowen and MacCluer and of Shapiro [21], as well as the articles [5, 10, 11, 14, 15, 16, 22, 24, 26] to name a few.

Our results on $\text{Hol}(\mathbb{D})$ allow us to prove some spectral properties of the restriction T of C_φ to some invariant Banach space $X \hookrightarrow \text{Hol}(\mathbb{D})$. For instance we will see that $0 \in \sigma(T)$ if and only if $\dim X = \infty$, and in this case we show that the essential spectrum $\sigma_e(T)$ is the connected component of 0 in $\sigma_e(T)$.

The paper is organized as follows. In Section 2 we characterize composition operators on $\text{Hol}(\mathbb{D})$ as the non-zero algebra homomorphisms. This is also an interesting example of automatic continuity. We also characterize when C_φ is compact as operators on $\text{Hol}(\mathbb{D})$ (which is much more restrictive than on $H^2(\mathbb{D})$, for example). Section 3 is devoted to the definition and investigation of the spectral projections. The main

theorem determining the spectrum of C_φ in $\text{Hol}(\mathbb{D})$ is established in Section 4. Finally, we deduce spectral properties of restrictions to arbitrary invariant Banach spaces in Section 5.

2. COMPOSITION OPERATORS ON $\text{Hol}(\mathbb{D})$

Let φ be a holomorphic self-map of \mathbb{D} . We define the composition operator C_φ on $\text{Hol}(\mathbb{D})$ by $C_\varphi(f) = f \circ \varphi$. Then C_φ is in $\mathcal{L}(\text{Hol}(\mathbb{D}))$ the algebra of linear and continuous operators on $\text{Hol}(\mathbb{D})$; indeed the linearity is trivial and the continuity follows from the definition of the topology of the Fréchet space (uniform convergence on compact subsets of \mathbb{D}) and the continuity of φ .

The next proposition is an algebraic characterization of composition operators. Note that $\text{Hol}(\mathbb{D})$ is an algebra. An algebra homomorphism $A : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D})$ is a linear map satisfying

$$A(f \cdot g) = A(f) \cdot A(g) \text{ for all } f, g \in \text{Hol}(\mathbb{D}).$$

Proposition 2.1. *Let $A : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D})$ be linear. The following assertions are equivalent.*

- (i) *There exists a holomorphic map $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ such that $A = C_\varphi$;*
- (ii) *A is an algebra homomorphism different from 0;*
- (iii) *A is continuous and $Ae_n = (Ae_1)^n$ for all $n \in \mathbb{N}_0$.*

Here we define $e_n \in \text{Hol}(\mathbb{D})$ by $e_n(z) = z^n$ for all $z \in \mathbb{D}$ and all $n \in \mathbb{N}_0$. For the proof we use the following well-known result.

Lemma 2.2. *Let $L : \text{Hol}(\mathbb{D}) \rightarrow \mathbb{C}$ be a continuous algebra homomorphism, $L \neq 0$. Then there exists $z_0 \in \mathbb{D}$ such that $Lf = f(z_0)$ for all $f \in \text{Hol}(\mathbb{D})$.*

Proof. Since $Lf = L(f \cdot e_0) = L(f)L(e_0)$ for all $f \in \text{Hol}(\mathbb{D})$ and since $L \neq 0$, it follows that $L(e_0) = 1$. Set $z_0 := Le_1$. Then $z_0 \in \mathbb{D}$. Indeed, otherwise $g(z) = \frac{1}{z-z_0}$ defines a function $g \in \text{Hol}(\mathbb{D})$ such that $(e_1 - z_0e_0)g = e_0$. Hence

$$1 = Le_0 = (Le_1 - z_0Le_0)Lg = 0,$$

a contradiction. For $f \in \text{Hol}(\mathbb{D})$ such that $f(z_0) = 0$, we have $Lf = 0$. Indeed, since there exists $g \in \text{Hol}(\mathbb{D})$ such that $f = (e_1 - z_0e_0)g$, it follows that $L(f) = (L(e_1) - z_0L(e_0))L(g) = 0$.

For an arbitrary $f \in \text{Hol}(\mathbb{D})$, note that $h := f - f(z_0)e_0$ satisfies $h(z_0) = 0$. Hence $0 = L(h) = L(f) - f(z_0)$. \square

Remark 2.3. *We are grateful to H.G. Dales and J. Esterle for helping us with Lemma 2.2. For much more information about automatic continuity, we refer to the monograph of Dales [7] and the survey article of Esterle [9].*

Proof of Proposition 2.1. (ii) \Rightarrow (i): since $A \neq 0$, it follows as in Lemma 2.2 that $Ae_0 = e_0$. Let $z \in \mathbb{D}$. Then $L(f) := (Af)(z)$ is an algebra homomorphism and $L(e_0) = 1$. By Lemma 2.2, there exists $\varphi(z) \in \mathbb{D}$ such that $(Af)(z) = f(\varphi(z))$ for all $f \in \text{Hol}(\mathbb{D})$. In particular $\varphi = Ae_1 \in \text{Hol}(\mathbb{D})$.

(iii) \Rightarrow (ii): it follows from *(iii)* that $A(fg) = A(f)A(g)$ if f and g are polynomials. Since the set of polynomials is dense in $\text{Hol}(\mathbb{D})$ and since the multiplication is continuous, *(ii)* follows.

(i) \Rightarrow (iii) is trivial. □

For our purposes, the following corollary is useful.

Corollary 2.4. *Let $X = \text{Hol}(\mathbb{D})$ and φ a holomorphic self-map of \mathbb{D} . The following assertions are equivalent:*

- (i) C_φ is invertible in $\mathcal{L}(\text{Hol}(\mathbb{D}))$;
- (ii) φ is an automorphism of \mathbb{D} .

Proof. (ii) \Rightarrow (i) is clear since $C_{\varphi^{-1}}C_\varphi = C_\varphi C_{\varphi^{-1}} = \text{Id}$, where Id denotes the identity map on X .

(i) \Rightarrow (ii): let C_φ be invertible, $A = C_\varphi^{-1}$. Then A is an algebra homomorphism. By Proposition 2.1 there exists a holomorphic map $\psi : \mathbb{D} \rightarrow \mathbb{D}$ such that $A = C_\psi$. Then

$$e_1 = C_\varphi(C_\psi e_1) = \psi \circ \varphi \text{ and } e_1 = C_\psi(C_\varphi e_1) = \varphi \circ \psi.$$

Thus φ is an automorphism and $\psi = \varphi^{-1}$. □

Next we want to characterize those φ for which C_φ is compact on $\text{Hol}(\mathbb{D})$. The reason of this investigation is the following. One of our main points in the article is to show that the spectral properties of a composition operator C_φ for $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ with interior fixed point looks very much to what one knows from compact operators. However, as we will show now, for composition operators on $\text{Hol}(\mathbb{D})$, compactness is a very restrictive condition.

Recall that $\mathcal{V} \subset \text{Hol}(\mathbb{D})$ is a neighborhood of 0 if and only if there exist a compact subset $K \subset \mathbb{D}$ and $\varepsilon > 0$ such that

$$\mathcal{V}_{K,\varepsilon} := \{f \in \text{Hol}(\mathbb{D}) : |f(z)| < \varepsilon \text{ for all } z \in K\} \subset \mathcal{V}.$$

A linear mapping $T : X \rightarrow X$ where X is a Fréchet space, is called *compact* if there exists a neighborhood \mathcal{V} of 0 such that $T\mathcal{V}$ is relatively compact. Each compact linear mapping is continuous. We refer

to Kelley–Namioka [13] for these notions and properties of compact operators.

Theorem 2.5. *Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic. The following assertions are equivalent:*

- (i) C_φ is compact as operator from $\text{Hol}(\mathbb{D})$ to $\text{Hol}(\mathbb{D})$;
- (ii) $\sup_{z \in \mathbb{D}} |\varphi(z)| < 1$.

Proof. (i) \Rightarrow (ii): assume that $\varphi(\mathbb{D}) \not\subset r\overline{\mathbb{D}}$ for all $0 < r < 1$. Let \mathcal{V} be a neighborhood of 0. We show that $C_\varphi(\mathcal{V})$ is not relatively compact. There exists $0 < \varepsilon < 1$ and $0 < r_0 < 1$ such that

$$\mathcal{V}_0 := \{f \in \text{Hol}(\mathbb{D}) : |f(z)| < \varepsilon \forall z \in r_0\overline{\mathbb{D}}\} \subset \mathcal{V}.$$

Thus it is sufficient to show that $C_\varphi(\mathcal{V}_0)$ is not relatively compact. By our assumption there exists $w_0 \in \mathbb{D}$ such that $z_0 := \varphi(w_0) \notin r_0\overline{\mathbb{D}}$. Then there exist $r_0 < r_1 < 1$ and $\rho > 0$ such that $r_1\overline{\mathbb{D}} \cap D(z_0, \rho) = \emptyset$.

The set $K := r_0\overline{\mathbb{D}} \cup \{z_0\}$ is compact and $\mathbb{C} \setminus K$ is connected. Let $n \in \mathbb{N}_0$ and define h_n by

$$h_n(z) = 0 \text{ for } z \in r_1\overline{\mathbb{D}} \text{ and } h_n(z) = n + 1 \text{ for } z \in D(z_0, \rho).$$

Set $\Omega := r_1\overline{\mathbb{D}} \cup D(z_0, \rho)$. Then $K \subset \Omega$ and $h_n : \Omega \rightarrow \mathbb{C}$ is holomorphic. By Runge's theorem, there exists a polynomial $p_n : \mathbb{C} \rightarrow \mathbb{C}$ such that $|p_n(z) - h_n(z)| < \varepsilon$ for all $z \in K$. This implies that $p_n|_{\mathbb{D}} \in \mathcal{V}_0$ and $|p_n(z_0)| \geq n$. Since $|C_\varphi(p_n)(w_0)| = |p_n(z_0)| \geq n$, the sequence $(C_\varphi p_n)_{n \in \mathbb{N}_0}$ has no convergent subsequence.

(ii) \Rightarrow (i): Assume that $\sup_{z \in \mathbb{D}} |\varphi(z)| =: r_0 < 1$. The set

$$\mathcal{V} := \{f \in \text{Hol}(\mathbb{D}) : |f(z)| < 1 \text{ if } |z| \leq r_0\}$$

is a neighborhood of 0. Let $f \in \mathcal{V}$. Since $\varphi(\mathbb{D}) \subset r_0\overline{\mathbb{D}}$, one has $|f(\varphi(w))| < 1$ for all $w \in \mathbb{D}$. Now it follows from Montel's theorem that $C_\varphi \mathcal{V}$ is relatively compact in $\text{Hol}(\mathbb{D})$. \square

Remark 2.6. *The same characterization of compact composition operators is valid in some special Banach spaces of holomorphic functions, for example $X = H^\infty(\mathbb{D})$ [25]. However on $H^2(\mathbb{D})$, the class of mappings φ such that C_φ is compact is much larger [6].*

3. DIAGONALIZATION OF COMPOSITION OPERATORS

In this section we show that composition operators C_φ on $\text{Hol}(\mathbb{D})$ can be diagonalized if the symbol φ is a Schröder map.

For the following we fix the holomorphic function $\varphi : \mathbb{D} \rightarrow \mathbb{D}$, with interior fixed point $\alpha = \varphi(\alpha) \in \mathbb{D}$ and suppose that $\varphi \notin \text{Aut}(\mathbb{D})$, $\varphi'(\alpha) \neq 0$. We set $\lambda_n = \varphi'(\alpha)^n$ for $n \in \mathbb{N}_0$. Thus $\lambda_1 \in \mathbb{D}$ by the

Schwarz lemma and $|\lambda_n|$ tends to 0 as $n \rightarrow \infty$. The range of an operator T is denoted by $\text{rg } T$. We denote by κ Koenigs' eigenfunction associated with φ . The following properties of κ^n will be needed.

Lemma 3.1. *For all $n \in \mathbb{N}$, $(\kappa^n)^{(n)}(\alpha) = n!$ and $(\kappa^n)^{(l)}(\alpha) = 0$ for $l = 0, \dots, n-1$.*

Proof. Let $n \in \mathbb{N}$ and set $c := (\kappa^n)^{(n)}(\alpha)$. Then, for $z \rightarrow \alpha$,

$$(\kappa^n)^{(n)}(z) = c + o((z - \alpha)^n)$$

so, there exists $P \in \mathbb{C}[X]$ with $\text{deg}(P) < n$ such that

$$\kappa^n(z) = P(z - \alpha) + \frac{c}{n!}(z - \alpha)^n + o((z - \alpha)^n).$$

However, since $\kappa(\alpha) = 0$ and $\kappa'(\alpha) = 1$, we get

$$\begin{aligned} \kappa^n(z) &= [(z - \alpha) + o(z - \alpha)]^n \\ &= (z - \alpha)^n + o((z - \alpha)^n) \end{aligned}$$

Hence, $P = 0$, $c = n!$ and $(\kappa^n)^{(l)}(\alpha) = 0$ for $l = 0, \dots, n-1$. \square

In the following theorem we define inductively a series of rank-one projections which diagonalize the operator C_φ on $\text{Hol}(\mathbb{D})$.

Theorem 3.2. *Define iteratively rank-one projections $P_n \in \mathcal{L}(\text{Hol}(\mathbb{D}))$ by*

$$(2) \quad P_0 f = f(0)e_0 \text{ and for } n \in \mathbb{N}, P_n(f) = \frac{1}{n!}g^{(n)}(\alpha)\kappa^n,$$

where $g = f - \sum_{k=0}^{n-1} P_k f$. Then the following holds:

- (a) $P_n C_\varphi = C_\varphi P_n = \lambda_n P_n$.
- (b) $f^{(l)}(\alpha) = (\sum_{k=0}^n P_k f)^{(l)}(\alpha)$ for $l = 0, \dots, n$ and $f \in \text{Hol}(\mathbb{D})$.
- (c) There exist complex numbers $c_{n,m}$ ($n, m \in \mathbb{N}_0$) such that

$$P_n f = \left(\sum_{m=0}^n c_{n,m} f^{(m)}(\alpha) \right) \kappa^n.$$

- (d) $P_n P_m = \delta_{n,m} P_n$ for all $n, m \in \mathbb{N}_0$.

We deduce the following decomposition property from Theorem 3.2. Let

$$\text{Hol}_n(\alpha) := \{f \in \text{Hol}(\mathbb{D}) : f(\alpha) = f'(\alpha) = \dots = f^{(n)}(\alpha) = 0\},$$

and $Q_n = \sum_{k=0}^n P_k$, where P_k is given in Theorem 3.2.

Corollary 3.3. *The mappings Q_n are projections commuting with C_φ . Moreover $\{\kappa^l : l = 0, \dots, n\}$ is a basis of $\text{rg } Q_n$ and $\ker(Q_n) = \text{Hol}_n(\alpha)$. Thus we have the decomposition*

$$\text{Hol}(\mathbb{D}) = \text{Span}\{\kappa^m : m = 0, \dots, n\} \oplus \text{Hol}_n(\alpha)$$

into two subspaces which are invariant by C_φ .

As a consequence, $C_\varphi|_{\text{rg } Q_n}$ is a diagonal operator since $C_\varphi(\kappa^l) = \lambda_l \kappa^l$ for $l = 0, \dots, n$. Of course, by definition κ^0 is the constant function equal to 1.

Proof of Corollary 3.3. a) Let $g = \sum_{m=0}^n a_m \kappa^m \in \text{Hol}_n(\alpha)$ where $a_m \in \mathbb{C}$. Then by Lemma 3.1,

$$0 = g(\alpha) = a_0, 0 = g'(\alpha) = a_1, \dots, 0 = g^{(n)}(\alpha) = n! a_n.$$

This shows that the functions $\kappa^m, m = 0, \dots, n$ are linearly independent and that

$$\text{Span}\{\kappa^m : m = 0, \dots, n\} \cap \text{Hol}_n(\alpha) = \{0\}.$$

b) Let $f \in \text{Hol}(\mathbb{D})$. Then, by Theorem 3.2, $f - Q_n f \in \text{Hol}_n(\alpha)$. This shows that

$$\text{Hol}(\mathbb{D}) = \text{Span}\{\kappa^m : m = 0, \dots, n\} \oplus \text{Hol}_n(\alpha)$$

and that Q_n is the projection onto the first space along this decomposition. □

Proof of Theorem 3.2. We define P_n by the iteration equation (2).

At first we show (b) inductively. For $n = 0$ it is trivial. Let $n \geq 1$ and assume that (b) is true for $n - 1$. Let $f \in \text{Hol}(\mathbb{D})$ and $0 \leq l < n$. Since $\kappa(\alpha) = 0$, $(\kappa^n)^{(l)}(\alpha) = 0$ for $l < n$ (by Lemma 3.1), it follows that

$$\left(\sum_{k=0}^n P_k f \right)^{(l)}(\alpha) = \left(\sum_{k=0}^{n-1} P_k f \right)^{(l)}(\alpha) = f^{(l)}(\alpha),$$

by the inductive hypothesis. For $l = n$, we have

$$\begin{aligned} \left(\sum_{k=0}^n P_k f \right)^{(n)}(\alpha) &= \left(\sum_{k=0}^{n-1} P_k f \right)^{(n)}(\alpha) \\ &\quad + \frac{1}{n!} \left(f - \sum_{k=0}^{n-1} P_k f \right)^{(n)}(\alpha) (\kappa^n)^{(n)}(\alpha) \\ &= f^{(n)}(\alpha), \end{aligned}$$

since $(\kappa^n)^{(n)}(\alpha) = n!$ (see Lemma 3.1). Thus (b) is proved.

It is clear that $(P_n f) \circ \varphi = \lambda_n P_n f$ since $\kappa^n \circ \varphi = \lambda_n \kappa^n$. We show inductively that $P_n(f \circ \varphi) = \lambda_n P_n f$. For $n = 0$ this is trivial. Let $n \geq 1$ and assume now that $P_l(f \circ \varphi) = \lambda_l P_l f$ for all $l \leq n - 1$. Note that

$$P_n(f \circ \varphi) = \frac{1}{n!}(\tilde{g})^{(n)}(\alpha)\kappa^n,$$

where

$$\tilde{g} = f \circ \varphi - \sum_{k=0}^{n-1} P_k(f \circ \varphi) = g \circ \varphi$$

by the inductive hypothesis, where $g = f - \sum_{k=0}^{n-1} P_k f$. It follows that $P_n(f \circ \varphi) = \frac{1}{n!}(g \circ \varphi)^{(n)}(\alpha)\kappa^n$. Now let us introduce some more notations in order to compute $(g \circ \varphi)^{(n)}(\alpha)$. For $n \in \mathbb{N}$, let

$$J_n = \{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n \mid m_1 + 2m_2 + \dots + nm_n = n\}$$

and

$$K_n = J_n \setminus \{(n, 0, \dots, 0)\}$$

For $\mathbf{m} = (m_1, \dots, m_n) \in J_n$, set $|\mathbf{m}| = m_1 + \dots + m_n$ and note that, for $\mathbf{m} \in J_n$, $m \in K_n$ if and only if $|\mathbf{m}| < n$. For $\mathbf{m} \in J_n$, we also define the following coefficients

$$C_{\mathbf{m}}^n = \frac{n!}{m_1! m_2! \dots m_n!} \prod_{j=1}^n \left(\frac{\varphi^{(j)}(\alpha)}{j!} \right)^{m_j}$$

These coefficients are inspired by Faà di Bruno's Formula: indeed, if $g \in \text{Hol}(\mathbb{D})$, then, for every $n \in \mathbb{N}$,

$$(g \circ \varphi)^{(n)}(\alpha) = \sum_{\mathbf{m} \in J_n} C_{\mathbf{m}}^n g^{(|\mathbf{m}|)}(\alpha) = (\varphi'(\alpha))^n g^{(n)}(\alpha) + \sum_{\mathbf{m} \in K_n} C_{\mathbf{m}}^n g^{(|\mathbf{m}|)}(\alpha)$$

Since $g^{(|\mathbf{m}|)}(\alpha) = 0$ by (b), we get

$$(g \circ \varphi)^{(n)}(\alpha) = \varphi'(\alpha)^n g^{(n)}(\alpha) + \sum_{m \in K_n} C_m^n g^{(|\mathbf{m}|)}(\alpha) = \lambda_n g^{(n)}(\alpha).$$

Thus $P_n(f \circ \varphi) = \lambda_n P_n f$ for all $n \in \mathbb{N}_0$, which implies that $P_n C_\varphi = C_\varphi P_n = \lambda_n P_n$ for all $n \in \mathbb{N}_0$. Thus (a) is proved.

We show inductively that (c) holds for suitable coefficients. It is obvious

for $n = 0$ and assume that P_k has the property for all $k \leq n - 1$. Then

$$\begin{aligned} P_n f &= \frac{1}{n!} \left(f - \sum_{k=0}^{n-1} P_k f \right)^{(n)}(\alpha) \kappa^n \\ &= \frac{1}{n!} \left(f^{(n)}(\alpha) + \sum_{k=0}^{n-1} \left(\sum_{l=0}^k c_{k,l} f^{(l)}(\alpha) \right) (\kappa^k)^{(n)}(\alpha) \right) \kappa^n, \end{aligned}$$

which proves the claim for n .

In order to prove (d), note that by the properties defining the projections and proved previously, for all $k, l \in \mathbb{N}_0$, we have:

$$\lambda_k P_l \kappa^k = P_l (\kappa^k \circ \varphi) = \lambda_l P_l \kappa^k.$$

Since $\lambda_k \neq \lambda_l$ for $l \neq k$, it follows that $P_l \kappa^k = 0$. Hence $P_l P_k = 0$ for $k \neq l$.

It remains to show that $P_n^2 = P_n$, which is equivalent to check that $P_n \kappa^n = \kappa^n$. We can show this easily and inductively since $P_k \kappa^n = 0$ for $k < n$ and $(\kappa^n)^{(n)}(\alpha) = n!$. \square

We can now give explicit expressions of P_n for $n = 0, 1, 2, 3$.

Corollary 3.4. *For all $f \in \text{Hol}(\mathbb{D})$, we have:*

$$\begin{aligned} P_0 f &= f(\alpha) 1 \\ P_1(f) &= f'(\alpha) \kappa \\ P_2(f) &= \frac{1}{2} \left(f''(\alpha) + \frac{\varphi''(\alpha)}{\lambda_2 - \lambda_1} \right) \kappa^2 \\ P_3(f) &= \frac{1}{3} \left(f'''(\alpha) + \frac{3\varphi''(\alpha)}{\lambda_2 - \lambda_1} f''(\alpha) + \left(\frac{\varphi'''(\alpha)}{\lambda_3 - \lambda_1} + \frac{3(\varphi''(\alpha))^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \right) f'(\alpha) \right) \kappa^3 \end{aligned}$$

A natural question concerns the density of $\text{Span}\{\kappa^n : n \in \mathbb{N}_0\}$ in $\text{Hol}(\mathbb{D})$. The following proposition gives the answer.

Proposition 3.5. *The space $\text{Span}\{\kappa^n : n \in \mathbb{N}_0\}$ is dense in the Fréchet space $\text{Hol}(\mathbb{D})$ if and only if φ is univalent.*

Proof. The function φ is univalent if and only if κ is univalent (see [23]). Thus univalence of φ is necessary for the density of $\text{Span}\{\kappa^n : n \in \mathbb{N}_0\}$. Conversely, assume that κ is univalent. Then $\Omega := \kappa(\mathbb{D})$ is a simply connected domain. It follows from Runge's theorem (see [18, Chap. 13 § 1 Section 2]) that the algebra $\mathcal{A}^{(\Omega)}$ of all polynomials on Ω is dense in $\text{Hol}(\Omega)$. Composition by κ shows that $\text{Span}\{\kappa^n : n \in \mathbb{N}_0\}$ is dense in $\text{Hol}(\mathbb{D})$. \square

We consider two illustrations.

Example 3.6. (a) *Consider the univalent Schröder symbol $\varphi(z) = \frac{z}{2-z}$. The Koenigs eigenfunction is $\kappa(z) = \frac{z}{1-z}$ and $\Omega = \kappa(\mathbb{D}) = \{z \in \mathbb{C} : \Re(z) > -1/2\}$.*

- (b) Let $\varphi(z) = z \frac{z+1/2}{1+z/2}$ which satisfies $\varphi(0) = 0$ and $\varphi'(0) = 1/2$. Since $\kappa \circ \varphi(z) = \kappa(z)/2$, it follows that $\kappa(0) = 0 = \kappa(-1/2)$, which obviously contradicts the density of $\text{Span}\{\kappa : n \in \mathbb{N}_0\}$ in the Fréchet space $\text{Hol}(\mathbb{D})$.

4. THE SPECTRUM OF COMPOSITION OPERATORS ON $\text{Hol}(\mathbb{D})$

In this section we determine the spectrum of C_φ on the Fréchet space $\text{Hol}(\mathbb{D})$. We suppose throughout that $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is a holomorphic map with an interior fixed point $\varphi(\alpha) = \alpha \in \mathbb{D}$ and that $0 < |\varphi'(\alpha)| < 1$. The case where $\varphi'(\alpha) = 0$ is treated at the very end of this section. We let $\lambda_n = \varphi'(\alpha)^n, n \in \mathbb{N}_0$. By $\sigma(C_\varphi)$ (resp. $\sigma_p(C_\varphi)$) we denote the *spectrum* (resp. *point spectrum*) of C_φ , that is the set $\{\lambda \in \mathbb{C} : \lambda \text{Id} - C_\varphi \text{ is not bijective}\}$ (resp. $\{\lambda \in \mathbb{C} : \lambda \text{Id} - C_\varphi \text{ is not injective}\}$). Note that for $\lambda \notin \sigma(C_\varphi)$, $(\lambda \text{Id} - C_\varphi)^{-1}$ is a continuous linear operator on $\text{Hol}(\mathbb{D})$ (by the closed graph theorem).

Since $\varphi \notin \text{Aut}(\mathbb{D})$, we already know that $0 \in \sigma(C_\varphi)$, by Corollary 2.4. Moreover, by Koenigs' theorem,

$$\sigma_p(C_\varphi) = \{\lambda_n : n \in \mathbb{N}_0\}.$$

Now we show that the entire spectrum $\sigma(C_\varphi)$ is equal to $\sigma_p(C_\varphi) \cup \{0\}$. This is surprising for several reasons. First of all, the operator C_φ is not compact in general (see Theorem 2.5). Nonetheless its spectral properties on $\text{Hol}(\mathbb{D})$ are exactly those of a compact operator (see [27] for the Riesz theory for compact operators on Fréchet spaces which is the same as for Banach spaces). The other surprise comes from the well developed spectral theory of $C_{\varphi|_X}$ for invariant Banach space $X \hookrightarrow \text{Hol}(\mathbb{D})$, which shows in particular that, on X , the spectrum is much larger in general (see Section 5).

Theorem 4.1. *One has*

$$\sigma(C_\varphi) = \{0\} \cup \{\varphi'(\alpha)^n : n \in \mathbb{N}_0\}.$$

In order to prove the surjectivity of $C_\varphi - \lambda \text{Id}$ on $\text{Hol}(\mathbb{D})$ for a complex number $\lambda \notin \{0\} \cup \{\varphi'(\alpha)^n : n \in \mathbb{N}_0\}$, we will use the following lemma.

Lemma 4.2. *Let $\psi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic, $\psi \notin \text{Aut}(\mathbb{D})$, such that $\psi(0) = 0$. Let $g \in \text{Hol}(\mathbb{D})$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Assume that there exist $0 < \varepsilon < 1$ and $f \in \text{Hol}(\varepsilon\mathbb{D})$ such that*

$$\lambda f - f \circ \psi = g \text{ on } \varepsilon\mathbb{D}.$$

Then f has an extension $\tilde{f} \in \text{Hol}(\mathbb{D})$ such that

$$\lambda \tilde{f} - \tilde{f} \circ \psi = g \text{ on } \mathbb{D}.$$

Proof. Let $\rho := \sup\{r \in [\varepsilon, 1] : f \text{ has an analytic extension on } r\mathbb{D}\}$. We show that $\rho = 1$. Assume that $\rho < 1$. Then there exists $\tilde{f} \in \text{Hol}(\rho\mathbb{D})$, a holomorphic extension of f , satisfying:

$$\lambda\tilde{f} - \tilde{f} \circ \psi = g \text{ on } \varepsilon\mathbb{D}.$$

Since both sides are holomorphic, by the uniqueness theorem, the identity remains true on $\rho\mathbb{D}$. Note that by the Schwarz lemma $\psi(r\mathbb{D}) \subset r\mathbb{D}$ for all $0 < r < 1$. It follows also from the Schwarz lemma that there exists $\rho < \rho' \leq 1$ such that $\psi(\rho'\mathbb{D}) \subset \rho\mathbb{D}$. Indeed, otherwise we find $(z_n)_n \in \mathbb{D}$, $|z_n| \downarrow \rho$ such that $|\psi(z_n)| > \rho$. Taking a subsequence we may assume that $z_n \rightarrow z$ and then $|z| = \rho$ and $|\psi(z)| \geq \rho$. This is not possible since φ is not an automorphism. Now, since

$$\lambda\tilde{f} = \tilde{f} \circ \psi + g \text{ on } \rho\mathbb{D},$$

and since $\psi(\rho'\mathbb{D}) \subset \rho\mathbb{D}$, it follows that f has a holomorphic extension to $\rho'\mathbb{D}$, a contradiction to the choice of ρ . \square

Proof of Theorem 4.1. First case: $\alpha = 0$. Let $\lambda \in \mathbb{C}$ and $\lambda \notin \{0\} \cup \{\lambda_n : n \in \mathbb{N}_0\}$. From Koenigs' theorem we know that $\lambda \text{Id} - C_\varphi$ is injective. Thus we only have to prove the surjectivity. Let $g \in \text{Hol}(\mathbb{D})$ and choose $n \in \mathbb{N}_0$ such that $|\lambda_{n+1}| < |\lambda|$. Since by Corollary 3.3, $\text{Hol}(\mathbb{D}) = \text{rg } Q_n \oplus \text{Hol}_n(0)$ we can write $g = g_1 + g_2$ where $g_1 \in \text{rg } Q_n$ and $g_2 \in \text{Hol}_n(0)$. Since $C_{\varphi|_{\text{rg } Q_n}}$ is a diagonal operator and $\lambda \notin \sigma(C_{\varphi|_{\text{rg } Q_n}})$, there exists $f_1 \in \text{rg } Q_n$ such that $\lambda f_1 - f_1 \circ \varphi = g_1$. Next we look at g_2 . Choose $|\lambda_1| < q < 1$ such that $q^{n+1} < |\lambda|$. Since $\lim_{z \rightarrow 0} \frac{\varphi(z)}{z} = \lambda_1$, there exists $0 < \varepsilon \leq 1$ such that $|\varphi(z)| \leq q|z|$ for $|z| < \varepsilon$. Consider the iterates $\varphi_k := \varphi \circ \dots \circ \varphi$ (k times) of φ . Then $|\varphi_k(z)| \leq q^k|z| \leq q^k\varepsilon$ for $|z| < \varepsilon$. Since $g_2 \in \text{Hol}_n(0)$, there exists $B \geq 0$ such that

$$|g_2(z)| \leq B|z|^{n+1} \text{ for } |z| < \varepsilon.$$

Hence for $k \in \mathbb{N}_0$, $|z| < \varepsilon$,

$$\begin{aligned} \left| \frac{g_2(\varphi_k(z))}{\lambda^{k+1}} \right| &\leq \frac{1}{|\lambda|} B \frac{|\varphi_k(z)|^{n+1}}{|\lambda|^k} \\ &\leq \frac{1}{|\lambda|} B \frac{q^{k(n+1)}}{|\lambda|^k} \varepsilon \\ &\leq \frac{B\varepsilon}{|\lambda|} \left(\frac{q^{n+1}}{|\lambda|} \right)^k. \end{aligned}$$

Since $\frac{q^{n+1}}{|\lambda|} < 1$, the series $f_0(z) := \sum_{k=0}^{\infty} \frac{g_2(\varphi_k(z))}{\lambda^{k+1}}$ converges uniformly on $\varepsilon\mathbb{D}$ and defines a function $f_0 \in \text{Hol}(\varepsilon\mathbb{D})$. Moreover, since $\varphi(\varepsilon\mathbb{D}) \subset \varepsilon\mathbb{D}$,

$$\begin{aligned} f_0(\varphi(z)) &= \sum_{k=0}^{\infty} \frac{g_2(\varphi_{k+1}(z))}{\lambda^{k+1}} \\ &= \lambda \sum_{k=1}^{\infty} \frac{g_2(\varphi_k(z))}{\lambda^{k+1}} \\ &= \lambda f_0(z) - g_2(z) \end{aligned}$$

on $\varepsilon\mathbb{D}$. It follows from Lemma 4.2 that f_0 has a holomorphic extension $f \in \text{Hol}(\mathbb{D})$ satisfying $\lambda f - f \circ \varphi = g_2$. This shows that $\lambda \notin \sigma(C_\varphi)$ in the case $\alpha = 0$.

Second case: $\alpha \in \mathbb{D}$, $\alpha \neq 0$. Consider the Möbius transform $\psi_\alpha : \mathbb{D} \rightarrow \mathbb{D}$ defined by $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ and note that $\psi_\alpha(0) = \alpha$ and $\psi_\alpha = \psi_\alpha^{-1}$. Then $\tilde{\varphi} := \psi_\alpha \circ \varphi \circ \psi_\alpha$ maps \mathbb{D} into \mathbb{D} and satisfies $\tilde{\varphi}(0) = 0$. Since

$$C_{\tilde{\varphi}} = C_{\psi_\alpha} C_\varphi C_{\psi_\alpha} = C_{\psi_\alpha} C_\varphi C_{\psi_\alpha}^{-1},$$

the operators $C_{\tilde{\varphi}}$ and C_φ are similar. From the first case we deduce that

$$\sigma(C_\varphi) \setminus \{0\} = \sigma(C_{\tilde{\varphi}}) \setminus \{0\} = \sigma_p(C_{\tilde{\varphi}}) \setminus \{0\} = \sigma_p(C_\varphi) \setminus \{0\} = \{\lambda_n : n \in \mathbb{N}_0\}.$$

□

For later purposes we extract the following lemma from the previous proof.

Lemma 4.3. *Let $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ such that $|\lambda| > |\lambda_{n+1}|$. Then for each $h \in \text{Hol}_n(\alpha)$, there exists a unique $f \in \text{Hol}_n(\alpha)$ solving the inhomogeneous Schröder equation $\lambda f - f \circ \varphi = h$.*

Proof. Since $\kappa^k \notin \text{Hol}_n(\alpha)$ for $k \in \{0, 1, \dots, n\}$, uniqueness follows from Koenigs' theorem. In order to prove existence, we only have to show that there exists $f \in \text{Hol}(\mathbb{D})$ such that $\lambda f - f \circ \varphi = g$. Then, since $Q_n g = 0$, $f_1 = f - Q_n f \in \ker Q_n = \text{Hol}_n(\alpha)$ satisfies $\lambda f_1 - f_1 \circ \varphi = g$ as well.

In the case $\alpha = 0$ the existence of f follows from the proof of Theorem 4.1. So let $\alpha \neq 0$. Consider the Möbius transform ψ_α defined in the proof of Theorem 4.1 and let $h = g \circ \psi_\alpha$. Then $h \in \text{Hol}_n(0)$. Indeed, $h(0) = g(\alpha) = 0$. Moreover, for $l \in \{1, \dots, n\}$, using Faà di

Bruno's formula (see the proof of Theorem 3.2 for notations), we get:

$$\begin{aligned} h^{(l)}(0) &= (g \circ \psi)^{(l)}(\alpha) \\ &= \psi'_\alpha(\alpha)^l g^{(l)}(\alpha) + \sum_{m \in K_n} C_m^m g_2^{(l|m)}(\alpha) \\ &= 0. \end{aligned}$$

Consider $\tilde{\varphi} = \psi_\alpha \circ \varphi \circ \psi_\alpha$. Since $\tilde{\varphi}(0) = 0$ we can apply the first case and find $\tilde{f} \in \text{Hol}(\mathbb{D})$ such that $\lambda \tilde{f} - \tilde{f} \circ \tilde{\varphi} = h$. Then $f := \tilde{f} \circ \psi_\alpha \in \text{Hol}(\mathbb{D})$ and

$$g = h \circ \psi_\alpha = \lambda f - \tilde{f} \circ \tilde{\varphi} \circ \psi_\alpha = \lambda f - f \circ \psi_\alpha \circ \tilde{\varphi} \circ \psi_\alpha = \lambda f - f \circ \varphi.$$

□

Our next aim is to show that each P_n is the spectral projection associated with λ_n for each $n \in \mathbb{N}$. We use the following definition.

Definition 4.4. *Let Y be a Fréchet space and $S : Y \rightarrow Y$ linear and continuous.*

- (1) *A number $\lambda \in \sigma(T)$ is called a Riesz point if λ is isolated and if there exists a decomposition $Y = Y_1 \oplus Y_2$ in closed subspaces which are invariant by S such that:*

$$\dim Y_1 < \infty, \sigma(S|_{Y_1}) = \{\lambda\} \text{ and } (\lambda \text{Id} - S)|_{Y_2} \text{ is invertible.}$$

It is not difficult to see that this decomposition is unique. The projection $P : Y \rightarrow Y$ onto Y_1 along this decomposition is called the spectral projection associated with the Riesz point λ .

- (2) $\sigma_e(T) := \{\lambda \in \sigma(T) : \lambda \text{ is not a Riesz point}\}$ is the essential spectrum and $r_e(T) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$ is the essential spectral radius.
- (3) T is a Riesz operator if $r_e(T) = 0$.

Remark 4.5. (1) *If X is a Banach space, the existence of the decomposition as in (1) of Definition 4.4 for $\lambda \in \sigma(T)$ implies that λ is an isolated point since the set of all invertible operators is open in $\mathcal{L}(X)$. This last property is no longer true if X is a Fréchet space (see Example 4.9).*

- (2) *If X is a Banach space, then an isolated point $\lambda \in \sigma(T)$ is a Riesz point if and only if λ is a pole of the resolvent whose residuum P has finite rank. In that case P is the spectral projection. Note that*

$$P = \frac{1}{2i\pi} \int_{|\mu-\lambda|=\varepsilon} R(\mu, T) d\mu.$$

- (3) See [8], in particular [8, Theorem 3.19] for other equivalent definitions of Riesz operators on Banach spaces.

Let P_n be the rank-one projections defined in Theorem 3.2 where $n \in \mathbb{N}_0$.

Theorem 4.6. *The operator C_φ on $\text{Hol}(\mathbb{D})$ is a Riesz operator. Moreover, for each $n \in \mathbb{N}_0$, the spectral projection associated with λ_n is P_n .*

Proof. Let $n \in \mathbb{N}_0$. We show that λ_n is a Riesz point with spectral projection P_n . Since $P_n C_\varphi = C_\varphi P_n = \lambda_n P_n$ and $\text{rg } P_n = \mathbb{C}\kappa^n$, it follows that the decomposition

$$\text{Hol}(\mathbb{D}) = \mathbb{C}\kappa^n \oplus \ker P_n$$

is invariant under C_φ . Moreover, $\sigma(C_{\varphi|_{\mathbb{C}\kappa^n}}) = \{\lambda_n\}$. Thus it suffices to show that $(\lambda_n \text{Id} - C_\varphi)|_{\ker P_n}$ is bijective. Since $\kappa^n \notin \ker P_n$ injectivity follows from Koenigs' theorem. In order to prove surjectivity, let $g \in \text{Hol}(\mathbb{D})$. Then, by Corollary 3.3, $g = g_1 + g_2$ where $g_1 \in \text{Span}\{\kappa^m : m = 0, \dots, n\} =: Z$, $g_2 \in \text{Hol}_n(\alpha)$. Since $P_n g_1 = P_n g - P_n g_2 = 0$ and since $C_{\varphi|_Z}$ is diagonal, there exists $f_1 \in Z$ such that $\lambda f_1 - f_1 \circ \varphi = g_1$. Note that $|\lambda_n| > |\lambda_{n+1}|$. Thus it follows from Lemma 4.3 that there exists $f_2 \in \text{Hol}(\mathbb{D})$ such that $\lambda f_2 - f_2 \circ \varphi = g_2$. Therefore $f := f_1 + f_2$ solves $\lambda f - f \circ \varphi = g$. This shows that λ_n is a Riesz point and P_n is the associated spectral projection. \square

We want to prove that a version of the formula in (2), Remark 4.5, remains true for the operator C_φ on $\text{Hol}(\mathbb{D})$.

At first we deduce from [27, Lemma 3.2] that the following holds.

Lemma 4.7. *Let $z \in \mathbb{D}$, $f \in \text{Hol}(\mathbb{D})$. The functions*

$$\lambda \mapsto ((\lambda \text{Id} - C_\varphi)^{-1} f)(z) : \mathbb{C} \setminus \sigma(C_\varphi) \rightarrow \mathbb{C}$$

is holomorphic.

This can also be seen directly from our proof of Theorem 4.1.

The following contour formula for P_n will be useful in Section 5.

Lemma 4.8. *Let $n \in \mathbb{N}$, $\varepsilon > 0$ such that $\lambda_k \notin D(\lambda_n, 2\varepsilon)$ for all $k \in \mathbb{N} \setminus \{n\}$. Then, for all $f \in \text{Hol}(\mathbb{D})$*

$$\frac{1}{2i\pi} \int_{|\lambda - \lambda_n| = \varepsilon} ((\lambda \text{Id} - C_\varphi)^{-1} f)(z) d\lambda = (P_n f)(z).$$

Proof. Write $f = (\text{Id} - P_n)f + P_n f$. The function

$$\lambda \mapsto ((\lambda \text{Id} - C_\varphi)^{-1} (\text{Id} - P_n)f)(z)$$

is holomorphic on $D(\lambda_n, 2\varepsilon)$ whereas $(\lambda \text{Id} - C_\varphi)^{-1} P_n f = \frac{1}{\lambda - \lambda_n} P_n f$. From this the claim follows. \square

The spectrum in a Fréchet space may be neither closed nor bounded. Indeed, here is an example of a composition operator on $\text{Hol}(\mathbb{D})$.

Example 4.9. *Let $r \in (0, 1)$ and the automorphism*

$$\psi(z) = \frac{z + r}{1 + rz}.$$

By Corollary 2.4, $0 \notin \sigma(C_\psi)$ but $\sigma_p(C_\psi) = \mathbb{C} \setminus \{0\}$ since for all $\lambda \in \mathbb{C} \setminus \{0\}$ we have

$$g_\lambda \circ \psi = \left(\frac{1+r}{1-r} \right)^\lambda g_\lambda \text{ where } g_\lambda(z) = \left(\frac{1+z}{1-z} \right)^\lambda.$$

So, for $\mu = se^{i\theta}$ with $s > 0$ and $\theta \in \mathbb{R}$, $g_\lambda \circ \psi = \mu g_\lambda$ when

$$\Re(\lambda) = \frac{\ln s}{\ln((1+r)/(1-r))} \text{ and } \text{Im}(\lambda) = \frac{\theta + 2k\pi}{\ln((1+r)/(1-r))}, k \in \mathbb{Z}.$$

For this reason one defines a larger spectrum, the Waelbroeck spectrum $\sigma_w(T)$ in the following way (see [27]).

Let $T \in \mathcal{L}(\text{Hol}(\mathbb{D}))$. Then

$$\mathbb{C} \setminus \sigma_w(T) := \{ \lambda \in \mathbb{C} \setminus \sigma(T) : \text{there exists a neighborhood } \mathcal{V} \text{ of } \lambda \text{ such that the family } ((\lambda \text{Id} - T)^{-1})_{\lambda \in \mathcal{V}} \text{ is bounded} \}.$$

Here a subset A of $\text{Hol}(\mathbb{D})$ is called bounded if

$$\sup_{f \in A} \sup_{z \in K} |f(z)| < \infty$$

for all compact subsets K of \mathbb{D} .

From the proof of Theorem 4.1 one sees that, in our case, $\sigma(C_\varphi) = \sigma_w(C_\varphi)$. Now we can use Fréchet theory (see Theorem 3.11 in [27]). It tells us in particular that for the isolated point $\lambda_n \in \sigma_w(C_\varphi)$, there exists a unique projection R_n commuting with C_φ such that

$$\sigma(C_{\varphi|_{\text{rg } R_n}}) = \{\lambda_n\} \text{ and } \sigma(C_{\varphi|_{\ker R_n}}) = \sigma(C_\varphi) \setminus \{\lambda_n\}.$$

It is clear that $R_n = P_n$. Anyhow, we needed to define them differently since a priori it is not clear at all that λ_n is an isolated point.

Finally, we determine the spectrum of composition operators in the case where the symbol is not Schröder but has an interior fixed point.

Theorem 4.10. *Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic, $\alpha \in \mathbb{D}$ such that $\varphi(\alpha) = \alpha$. Assume that $\varphi'(\alpha) = 0$. Then*

$$\sigma(C_\varphi) = \sigma_w(C_\varphi) = \{0, 1\}.$$

Proof. Since $\varphi'(\alpha) = 0$, $\varphi \notin \text{Aut}(\mathbb{D})$ and thus, $0 \in \sigma(C_\varphi)$ by Corollary 2.4. Since the constant functions are in the kernel of $C_\varphi - \text{Id}$, $1 \in \sigma_p(C_\varphi) \subset \sigma(C_\varphi)$. By Theorem 1.1, for $\lambda \notin \{0, 1\}$, $C_\varphi - \lambda \text{Id}$ is injective. It remains to check that it is also surjective.

First case: $\alpha = 0$. Then $\varphi_n(z) = z^{2^n} \tau_n(z)$ where τ_n is a holomorphic self-map of the unit disc ($\tau_n(\mathbb{D}) \subset \mathbb{D}$ follows from the Schwarz lemma). Let $g \in \text{Hol}(\mathbb{D})$, $g(z) = g(0) + g_1(z)$ where $g_1 \in \text{Hol}(\mathbb{D})$ and $g_1(z) = zg_2(z)$ with $g_2 \in \text{Hol}(\mathbb{D})$. Note that $(\lambda \text{Id} - C_\varphi)(\frac{g(0)}{\lambda-1} \mathbf{1}) = g(0) \mathbf{1}$. Moreover, the series

$$f(z) := \frac{1}{\lambda} \sum_{n \geq 0} \frac{g_1(\varphi_n(z))}{\lambda^n} = \frac{1}{\lambda} \sum_{n \geq 0} \frac{z^{2^n} \tau_n(z) g_2(\varphi_n(z))}{\lambda^n}$$

uniformly converges on every compact $K \subset \mathbb{D} \cap \{|z| < \sqrt{|\lambda|}\}$ (since $|z^{2^n}| \leq |z^{2^n}|$ for $z \in \mathbb{D}$). Note also that $\lambda f - f \circ \varphi = g_1$ on such K . The surjectivity of $\lambda \text{Id} - C_\varphi$ follows from Lemma 4.2.

Second case: $\alpha \neq 0$. We proceed as in the proof of Theorem 4.1. \square

5. SPECTRAL PROPERTIES ON ARBITRARY BANACH SPACES

In this section we study spectral properties of composition operators on arbitrary Banach spaces which are continuously injected in $\text{Hol}(\mathbb{D})$.

Throughout this section we assume that $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic, $\varphi \notin \text{Aut}(\mathbb{D})$ and that there exists $\alpha \in \mathbb{D}$ such that $\varphi(\alpha) = \alpha$ and $\varphi'(\alpha) \neq 0$; i.e. φ is a Schröder function. By κ we denote Koenigs' eigenfunction. Let X be a Banach space such that $X \hookrightarrow \text{Hol}(\mathbb{D})$ (which means that X is a subspace of $\text{Hol}(\mathbb{D})$ and the injection is continuous; see [1] for equivalent formulations). Assume that $C_\varphi X \subset X$ and define $T : X \rightarrow X$ by $T = C_{\varphi|_X}$. Then $T \in \mathcal{L}(X)$ by the closed graph theorem.

As before we will consider the spectral projections P_n on $\text{Hol}(\mathbb{D})$ and let $\lambda_n = \varphi'(\alpha)^n$, $n \in \mathbb{N}_0$. By Theorem 3.2, $P_n f = \langle \Psi_n, f \rangle \kappa^n$, where Ψ_n is a functional given by

$$\langle \Psi_n, f \rangle = \sum_{m=0}^n c_{nm} f^{(m)}(\alpha).$$

It follows from Theorem 3.2 (a) that,

$$\langle C_\varphi f, \Psi_n \rangle = \lambda_n \langle f, \Psi_n \rangle,$$

for all $f \in \text{Hol}(\mathbb{D})$. This implies that $T' \Psi_n|_X = \lambda_n \Psi_n|_X$. Thus, if $\Psi_n|_X \neq 0$, then $\lambda_n \in \sigma_p(T') \subset \sigma(T)$. We note this as a first result.

Proposition 5.1. *Let $n \in \mathbb{N}_0$. Assume that $\Psi_{n|X} \neq 0$. Then*

$$\lambda_n \in \sigma_p(T') \subset \sigma(T).$$

The following corollary concerns all the classical Banach spaces X of holomorphic functions on the unit disc.

Corollary 5.2. *Assume that $e_n \in X$ for all $n \in \mathbb{N}_0$. Then $\lambda_n \in \sigma(T') \subset \sigma(T)$ for all $n \in \mathbb{N}_0$.*

Proof. We know that $\Psi_n \neq 0$ for all $n \in \mathbb{N}_0$. Since the polynomials are dense in $\text{Hol}(\mathbb{D})$, it follows that $\Psi_{n|X} \neq 0$. \square

It follows from the decomposition result, Corollary 3.3, that the ψ_n separate points in $\text{Hol}(\mathbb{D})$, i.e. for $f \in \text{Hol}(\mathbb{D})$, $\langle \Psi_n, f \rangle = 0$ for all $n \in \mathbb{N}_0$ implies $f = 0$.

Corollary 5.3. *If $X \neq \{0\}$, then $r(T) > 0$, where $r(T)$ is the spectral radius of T .*

Proof. Since the Ψ_n , $n \in \mathbb{N}_0$, separate the functions of $\text{Hol}(\mathbb{D})$, there exists $n \in \mathbb{N}_0$ such that $\Psi_{n|X} \neq 0$. Hence $\lambda_n \in \sigma(T)$ by Proposition 5.1. \square

We need the following characterization of the finite dimension also for further arguments.

Proposition 5.4. *The following assertions are equivalent:*

- (i) $0 \notin \sigma(T)$;
- (ii) for only finitely many $n \in \mathbb{N}_0$ one has $\Psi_{n|X} \neq 0$;
- (iii) $\exists J \subset \mathbb{N}_0$ finite such that $X = \text{Span}\{\kappa^l : l \in J\}$;
- (iv) $\dim X < \infty$.

Proof. (i) \Rightarrow (ii): Since $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, this follows from Proposition 5.1.

(ii) \Rightarrow (iii): Let $J := \{n \in \mathbb{N}_0 : \Psi_{n|X} \neq 0\}$. It follows from Corollary 3.3 that the Ψ_n , $n \in \mathbb{N}_0$, separate $\text{Hol}(\mathbb{D})$. Thus the mapping

$$f \mapsto (\langle \Psi_n, f \rangle)_{n \in J} : X \rightarrow \mathbb{C}^d,$$

with $d = |J|$ is injective and linear. It follows that $\dim X \leq d$. It follows from Proposition 5.1 that $\{\lambda_n : n \in J\} \subset \sigma_p(T)$. Since all λ_n are different, it follows that $\dim X \geq d$. We have shown that $\dim X = d$ and $\sigma_p(T) = \{\lambda_n : n \in J\}$. Now it follows from Koenigs' theorem that $X = \text{Span}\{\kappa^l : l \in J\}$.

(iii) \Rightarrow (iv) is trivial.

(iv) \Rightarrow (i): Since $\dim X < \infty$, by Koenigs' theorem,

$$\sigma(T) = \sigma_p(T) \subset \{\lambda_n : n \in \mathbb{N}_0\} \subset \mathbb{C} \setminus \{0\}.$$

□

We note the following corollary which will be useful later.

Corollary 5.5. *Assume that $\dim X = \infty$. Then there exist infinitely many $n \in \mathbb{N}_0$ such that $\lambda_n \in \sigma(T)$.*

This follows from Proposition 5.1 and Proposition 5.4.

Next we show that each isolated point in the spectrum of T is necessarily a simple pole, and thus equal to some λ_n .

Recall that if μ is an isolated point of the spectrum, for the resolvent $R(\lambda, T)$ we have the Laurent development

$$R(\lambda, T) = \sum_{k=-\infty}^{\infty} A_k(\lambda - \mu)^k,$$

which is valid for $0 < |\lambda - \mu| < \delta$ and $\delta = \text{dist}(\mu, \sigma(T) \setminus \{\mu\})$. Here $A_k \in \mathcal{L}(X)$ are the coefficients and $A_{-1} = P$ is the spectral projection associated with μ . Thus the spectral projection is equal to the residuum.

One says that $\mu \in \sigma(T)$ is a *simple pole* if it is isolated in $\sigma(T)$ and if $\dim(\text{rg } P) = 1$. This implies that $A_k = 0$ for $k \leq -2$. Moreover $\text{rg } P = \ker(\mu \text{Id} - T)$ and $PT = TP = \mu P$.

Theorem 5.6. *Let $\mu \in \mathbb{C} \setminus \{0\}$ be an isolated point of $\sigma(T)$. Then there exists $n \in \mathbb{N}_0$ such that $\mu = \lambda_n$ and μ is a simple pole. Moreover $P_n X \subset X$ and $P_{n|X}$ is the spectral projection associated with μ . Here P_n is the projection from Theorem 3.2.*

Proof. Assume that $\mu \notin \{\lambda_n : n \in \mathbb{N}_0\}$, $\lambda_n = \varphi'(\alpha)^n$. Let $\varepsilon > 0$ such that $D(\mu, 2\varepsilon) \cap \{\lambda_n : n \in \mathbb{N}_0\} = \emptyset$. Denote by

$$P = \frac{1}{2i\pi} \int_{|\lambda - \mu| = \varepsilon} (\lambda \text{Id} - T)^{-1} d\lambda$$

the spectral projection associated with μ . Let $f \in X, z \in \mathbb{D}$. Since for $|\lambda - \mu| = \varepsilon, \lambda \in \rho(C_\varphi) \cap \rho(T)$, one has:

$$((\lambda \text{Id} - T)^{-1} f)(z) = ((\lambda \text{Id} - C_\varphi)^{-1} f)(z),$$

it follows from Lemma 4.7 and Cauchy's theorem that $(Pf)(z) = 0$. Since $f \in X, z \in \mathbb{D}$ are arbitrary, it follows that $P = 0$, a contradiction.

Thus $\mu = \lambda_{n_0}$ for some $n_0 \in \mathbb{N}$. Let $\varepsilon > 0$ such that $\lambda_n \notin D(\lambda_{n_0}, 2\varepsilon)$ for all $n \neq n_0$. Let

$$P = \frac{1}{2i\pi} \int_{|\lambda - \lambda_{n_0}| = \varepsilon} (\lambda \text{Id} - T)^{-1} d\lambda$$

be the spectral projection. It follows from Lemma 4.8 that $P = P_{n_0}$. Thus P has rank 1 and this means by definition that $\mu = \lambda_{n_0}$ is a simple pole. \square

Remark 5.7. *In [1] it was proved that each pole is necessarily simple. Now we know more: each isolated point in the spectrum is a simple pole.*

We note the following consequence of Theorem 5.6

Corollary 5.8. *If the spectrum of $T = C_{\varphi|X}$ is countable, then $\sigma(T) \subset \{\lambda_n : n \in \mathbb{N}_0\} \cup \{0\}$.*

Proof. Let \mathcal{U} be an open neighborhood of $\{\lambda_n : n \in \mathbb{N}_0\} \cup \{0\}$ and $K := \sigma(T) \setminus \mathcal{U}$. Then K is compact and countable. If $K \neq \emptyset$, then, by Baire's theorem, K has an isolated point. This is impossible by Theorem 5.6. Thus $K = \emptyset$. Since \mathcal{U} is arbitrary, the claim follows. \square

Our next aim is to describe the connected component of 0 in $\sigma(T)$. Assume that $X \hookrightarrow \text{Hol}(\mathbb{D})$ is invariant under C_φ and let $T = C_{\varphi|X}$, as before, where $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is the given Schröder map. Let us assume that $\dim X = \infty$. Then $0 \in \sigma(T)$ and the set

$$J := \{n \in \mathbb{N}_0 : \Psi_{n|X} \neq 0\}$$

is infinite by Proposition 5.4

We let $\lambda_n = \varphi'(\alpha)^n$ where α is the interior fixed point of φ . By Proposition 5.1, $\lambda_n \in \sigma(T)$ for $n \in J$. Moreover, let

$$J_0 := \{n \in \mathbb{N}_0 : \lambda_n \text{ is an isolated point of } \sigma(T)\}.$$

We know that for $n \in J_0$, $\kappa^n \in X$ and $T\kappa^n = \lambda_n\kappa^n$.

Our main result in this section is the following quite precise description of the spectrum of T .

Theorem 5.9. *Assume that $\dim X = \infty$. Denote by $\sigma_0(T)$ the connected component of 0 in $\sigma(T)$. Then*

$$\sigma(T) = \sigma_0(T) \cup \{\lambda_n : n \in J_0\}.$$

In particular, $\sigma_0(T)$ is the essential spectrum of T .

Of course it may happen that $J_0 = \emptyset$. This is the case if and only if $\sigma(T)$ is connected.

For the proof, we need the following.

Lemma 5.10. *Let σ_1 be an open and closed subset of $\sigma(T)$. If $0 \notin \sigma_1$, then there exists a finite set $J_1 \subset J_0$ such that*

$$\sigma_1 = \{\lambda_n : n \in J_1\}.$$

Proof. Assume that $0 \notin \sigma_1$. Let Γ be a rectifiable Jordan curve such that $\sigma_1 \subset \text{int } \Gamma$ and $\sigma(T) \setminus \sigma_1 \subset \text{ext } \Gamma$. We can choose Γ such that $\lambda_n \notin \Gamma$ for all $n \in \mathbb{N}_0$, and $0 \in \text{ext } \Gamma$. Consequently $J_2 := \{n \in \mathbb{N}_0 : \lambda_n \in \text{int } \Gamma\}$ is finite. Denote by

$$P = \frac{1}{2i\pi} \int_{\Gamma} R(\lambda, T) d\lambda$$

the spectral projection with respect to σ_1 . Let $Y := PX$. Then $TY \subset Y$ and $\sigma(T|_Y) = \sigma_1$. Let $f \in X, z \in \mathbb{D}$. Then

$$(R(\lambda, T)f)(z) = ((\lambda \text{Id} - C_{\varphi})^{-1}f)(z)$$

for all $\lambda \in \Gamma$. Now we choose $\varepsilon > 0$ small enough and deduce from Lemma 4.7 and 4.8 that

$$\begin{aligned} (Pf)(z) &= \frac{1}{2i\pi} \int_{\Gamma} (R(\lambda, T)f)(z) d\lambda \\ &= \frac{1}{2i\pi} \int_{\Gamma} ((\lambda \text{Id} - C_{\varphi})^{-1}f)(z) d\lambda \\ &= \sum_{k \in J_2} \frac{1}{2i\pi} \int_{|\lambda - \lambda_k| = \varepsilon} ((\lambda \text{Id} - C_{\varphi})^{-1}f)(z) d\lambda \\ &= \sum_{k \in J_2} \langle \Psi_k, f \rangle \kappa^k(z) \\ &= \sum_{k \in J_1} \langle \Psi_k, f \rangle \kappa^k(z), \end{aligned}$$

where $J_1 := J_2 \cap J$.

Since $T'\Psi_k|_X = \lambda_k \Psi_k|_X$ for all $k \in J_1$, and since the λ_k are all different, it follows that the $\Psi_k, k \in J_1$, are linearly independent. Consequently we find $f_l \in X$ such that $\langle \Psi_k, f_l \rangle = \delta_{kl}$ for $k, l \in J_1$. It follows that the $\kappa^k, k \in J_1$, form a basis of Y consisting of eigenvectors of $T, T\kappa^k = \lambda_k \kappa^k, k \in J_1$. Thus

$$\sigma_1 = \sigma(T|_Y) = \{\lambda_k : k \in J_1\}.$$

□

Proof of Theorem 5.9. Let $K = \sigma(T) \setminus \{\lambda_n : n \in J_0\}$. Then K is compact and $0 \in K$. It suffices to show that K is connected.

Let $\sigma_1 \subset K$ be open and closed. We have to show that $\sigma_1 = \emptyset$ or $\sigma_1 = K$.

First case: $0 \notin \sigma_1$.

We show that σ_1 is open in $\sigma(T)$. Let $z_0 \in \sigma_1$. Then $z_0 \neq 0$ and $z_0 \neq \lambda_n$ for all $n \in J_0$. Thus there exists $\varepsilon > 0$ such that $z \neq \lambda_n$

for all $n \in J_0$ if $|z - z_0| < \varepsilon$ and, if $z \in K$, then $z \in \sigma_1$. Hence also $D(z_0, \varepsilon) \cap \sigma(T) \subset \sigma_1$. This proves that σ_1 is open in $\sigma(T)$. It is trivially closed. By Lemma 5.10 there exists a finite set $J_1 \subset J_0$ such that $\sigma_1 \subset \{\lambda_n : n \in J_1\}$. Since $\sigma_1 \subset K$, it follows that $\sigma_1 = \emptyset$.

Second case: $0 \in \sigma_1$.

Then $K \setminus \sigma_1 = \emptyset$ by the first case. Thus $\sigma_1 = K$.

By Corollary 5.5, the point 0 is not isolated in $\sigma(T)$. Thus $0 \in \sigma_e(T)$. It follows that $\sigma_0(T) \subset \sigma_e(T)$. Since (by Theorem 5.6) $\sigma(T) \setminus \sigma_0(T)$ consists of Riesz points, we conclude that $\sigma_0(T) = \sigma_e(T)$. \square

Of course, it can happen that $\sigma_0(T) = \{0\}$. Here is a situation where it is bigger.

Corollary 5.11. *Let $n_0 \in \mathbb{N}_0$. Assume that $\lambda_{n_0} \in \sigma(T)$ but $\kappa^{n_0} \notin X$. Then $\lambda_{n_0} \in \sigma_0(T)$. In particular $r_e(T) \geq |\lambda_{n_0}|$.*

Proof. It follows from Theorem 5.6 that λ_{n_0} is not an isolated point of $\sigma(T)$. Thus $\lambda_{n_0} \notin \{\lambda_n : n \in J_0\}$. Hence $\lambda_{n_0} \in \sigma_0(T)$. \square

We cannot expect in the general situation we are considering here to prove more precise results on the geometry of $\sigma_0(T)$. However, for several concrete Banach spaces X , it is known that $\sigma_0(T)$ is a disc. Moreover one can estimate the essential radius $r_e(T) := \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$ by knowing whether $\kappa^n \in X$ or $\kappa^n \notin X$. We explain this in the following examples.

Example 5.12. (1) *Let $X = H^2(\mathbb{D})$ and let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be a Schröder map with fixed point $\alpha \in \mathbb{D}$ and Koenigs eigenfunction κ . Then $C_\varphi(X) \subset X$. Let $T = C_{\varphi|_X}$. Then it is known that $\sigma_0(T)$ ($= \sigma_e(T)$) is a disc (see [6, Theorem 7.30]). A question which has been investigated is for which n the eigenfunction κ^n of C_φ lies in X , which means, for which n actually $\lambda_n = \varphi'(\alpha)^n \in \sigma_p(T)$. This is related to the essential spectral radius $r_e(T)$.*

- (a) *One has $\sigma_0(T) = \{0\}$ if and only if $\kappa^n \in X$ for all $n \in \mathbb{N}_0$ (see [23, Section 6]). To say that $\sigma_0(T) = \{0\}$ is the same as saying that T is a Riesz operator. We had seen in Section 4 that C_φ is always a Riesz operator on $\text{Hol}(\mathbb{D})$. So the situation is very different if we restrict C_φ to a Banach space.*
- (b) *Let $n \in \mathbb{N}$. Then $\kappa^n \in X$ if and only if $|\lambda_n| > r_e(T)$.*

Proof. If $|\lambda_n| > r_e(T)$, then $\lambda_n \in \sigma_p(T)$. Thus $\kappa^n \in X$ by Koenigs' theorem (Theorem 1.1). The converse implication follows from deep results by Poggi-Corradini [17] and

Bourdon–Shapiro [3]. We follow the survey article [23] by Shapiro. Assume that $\kappa^n \in X$. Then $\kappa \in H^{2n}(\mathbb{D})$. Using the notation of [23, Section 7], this implies that $h(\kappa) \geq 2n$. By [23, (12) in Section 6], one has $r_e(T) = |\lambda_1|^{h(\kappa)/2}$. Since $|\lambda_1| < 1$ this implies $r_e(T) \leq |\lambda_1|^n = |\lambda_n|$. We need the strict inequality. Assume that $|\lambda_n| = r_e(T)$. Then $h(\kappa) = 2n$. By the "critical exponent result" in [23, Section 8], this implies that $\kappa^n \in X$. thus $|\lambda_n| > r_e(T)$. \square

(c) *The result of (b) can be reformulated by saying that there are no "hidden eigenvalues" besides possibly λ_0 . More precisely, if λ_n is an eigenvalue, then $\lambda_n \notin \sigma_e(T)$. For $\lambda_0 = 1$ the situation is different. In Example 3.6 an inner function φ is defined which is Schröder and 0 at 0. Thus $T = C_{\varphi|_{H^2}}$ is isometric and non invertible (see [2, Section 1] for further informations on isometric composition operators on various Banach spaces). Hence $\sigma(T) = \overline{\mathbb{D}} = \sigma_e(T)$, and thus the eigenvalue $\lambda_0 = 1$ is in the essential spectrum.*

(2) *Also on some weighted Hardy spaces the essential spectrum is a disc. However it can happen that $\lambda_{n+1} < r_e(T) < \lambda_n$ for some n . In fact Hurst [10] considered Schröder symbols φ which are linear fractional maps with a fixed point $\alpha \in \mathbb{D}$ such that $\varphi'(\alpha) \in (0, 1)$ and a second fixed point of modulus one. The Banach spaces on which the composition operators are defined are the weighted Hardy spaces*

$$H^2(\beta) := \left\{ f(z) = \sum_{n \geq 0} a_n z^n : \|f\|^2 = \sum_{n \geq 0} |a_n|^2 \beta(n)^2 < \infty \right\}$$

where $\beta(n) = (n+1)^a$, $a < 0$. Recall that for $a = -1/2$, the Banach space is the classical Bergman space \mathcal{B} . In this case the spectrum is

$$\{\lambda \in \mathbb{C} : |\lambda| \leq \varphi'(\alpha)^{(2|a|+1)/2}\} \cup \{\varphi'(\alpha)^n : n \in \mathbb{N}_0\},$$

and $\kappa^p \in H^2(\beta)$ if and only if $p < |a| + 1/2$. For $a = -1$, it follows that $\lambda_2 < r_e(T) < \lambda_1$, whereas, for $a = -1/2$, we get $r_e(T) = \lambda_1$.

Remark 5.13. *By Proposition 5.1 we have seen that the composition operators associated with a Schröder symbol φ are not quasinilpotent on a large class of Banach spaces of holomorphic functions. Note that if φ has a fixed point $\alpha \in \mathbb{D}$ such that $\varphi'(\alpha) = 0$, the description of the spectrum of $T := C_{\varphi|_X}$ may be very different. For example, if $\varphi(z) = z^2$*

and $X = zH^2(\mathbb{D})$, then the spectrum of T is the closed unit disc (T is a non-invertible isometry) whereas for $X = z\mathcal{B}$, T is quasinilpotent (see [2, Theorem 2.9]).

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