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# A new ordinary differential equation for the evaluation of the frequency-domain Green function

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## Abstract

Clément (2013) derived a second order ordinary differential equation (ODE) satisfied by the free-surface Green function in the frequency domain. Since then, similar ODEs for the gradient of the Green function have been developed. Unfortunately, all these ODEs degenerate at zero frequency. Therefore, it is not possible to initialize the numerical solution of these ODEs from this zero frequency. Alternative methods based on the shifting of the initial condition to frequencies strictly greater than zero have then been developed.

The present paper describes an alternative approach to address this issue. It involves a new function which is the solution of a modified ODE which can be solved from the zero frequency.

Finally, comparisons with evaluations of the Green function using the classical direct integration method are provided. They show that the new ODE can provide accurate estimates of the Green function.

*Keywords:* ordinary differential equation, Green function, frequency domain, singularity

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## 1. Introduction

Boundary element method (BEM)-based codes are widely used in the industry and in academia to investigate wave structure interactions effects on marine

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structures. BEM codes rely on the frequency domain linear free-surface potential flow theory which involves the free-surface Green function and its gradient. An important part of the numerical burden in BEM codes relates to the numerous numerical evaluations of the free-surface Green function. Indeed, they must be evaluated for a large set of geometrical configurations and for a wide range of frequencies to assess the structure response across all possible incident wave conditions.

Various analytical formulations and algorithms for the calculation of the free-surface Green function and its derivatives have been proposed. Pioneering work was performed by Noblesse [1, 2, 3] and Newman [4, 5, 6] using sub-domains methods with series expansions and polynomial approximations. Double Chebyshev polynomials approximations methods with special functions were proposed by Chen [7, 8] and Wang [9]. Other efficient methods include tabulated functions with Lagrange interpolations [10], approximation methods with coordinates-transformation [11, 12], eigen-function expansion [13], semi-analytical method based on a singularity subtraction technique [14], multipole expansion method [15]. Recently, Wu et al. [16, 17] proposed a global approximation based on Noblesse's method [2, 3]. It uses a simple approximation involving elementary functions for the local flow component. It does not required dividing the computational domain into multiple sub-domains. All those methods were recently reviewed in [18].

The analytical formulations typically include a source term, an image term, and a singular integral. The algorithms implemented in most common BEM softwares (WAMIT [19], HydroStar [20] or NEMOH [21] for example) are based on integral formulations.

A different approach was introduced by Clément [22] who showed that the frequency domain Green function is the solution of a second order ordinary differential equation (ODE) of the frequency variable. Thus, the Green function can be evaluated by integrating numerically this ODE, provided that initial conditions are available. Since then, similar ODEs for the gradient of the frequency domain Green function have been established (see [23] and [24]).

35 The remaining challenge for the evaluation of the frequency domain Green  
 function and its gradient using ODEs is the evaluation of the initial conditions.  
 Indeed, despite the values of the Green function and its first derivatives are  
 known for  $\omega = 0$ , the method cannot be started from that point because the  
 ODEs degenerate for  $\omega = 0$ . Thus, in [24], the ODEs are initialized at the non-  
 40 zero frequency  $\omega = 1$ . A power-series expansion method is used to solve the  
 ODEs for  $\omega < 1$  while a trigonometrically fitted block Numerov-type method is  
 used for  $\omega \geq 1$ . Note that it is not explained in [24] how the initial conditions  
 are obtained for  $\omega = 1$ . In [23], initial conditions for arbitrary values of  $\omega$  were  
 obtained using methods relying on the numerical approximations of the singular  
 45 integral.

In this paper, we present an alternative method which enables using  $\omega = 0$   
 for the initial conditions. The ODEs for the Green function and its derivatives  
 are recalled in section 2. The singular behaviour for  $\omega = 0$  is highlighted.  
 In section 3, the singularity is extracted and a modified ODE is introduced.  
 50 Numerical results and comparisons with direct numerical integration are shown  
 in section 4. Finally, a conclusion is given in section 5.

## 2. ODEs for the Green function and its derivatives

In this study, only the infinite water depth free-surface Green function is  
 considered. The coordinates and variables are depicted in Figure 1. The mean  
 55 free surface level is located at the plane  $z = 0$ . The vertical axis  $z$  points  
 upwards. The gravity constant is denoted  $g$ . The source point  $P(x_P, y_P, z_P)$   
 and the field point  $M(x_M, y_M, z_M)$  are both lying on or under the free surface  
 ( $z_P \leq 0, z_M \leq 0$ ). The image source point  $P'(x_P, y_P, -z_P)$  is the mirror of the  
 source point  $P$  with respect to the mean free surface. The horizontal distance  
 60 between the source point  $P$  and the field point  $M$  is denoted by  $r$ . The vertical  
 distance between the image source point  $P'$  and the field point  $M$  is  $-Z$ . The  
 distance between the source and field points is denoted by  $R$  and the distance  
 between the image source point and the field point is  $R_1 = \sqrt{r^2 + Z^2}$ . The

angle  $\theta$  is defined by  $\cos \theta = -Z/R_1$  and  $\sin \theta = r/R_1$ . The relations between  
65 the coordinates are given by:

$$\begin{aligned} r &= \sqrt{(x_M - x_P)^2 + (y_M - y_P)^2} \\ Z &= z_M + z_P \\ R &= \sqrt{(x_M - x_P)^2 + (y_M - y_P)^2 + (z_M - z_P)^2} \end{aligned} \quad (1)$$

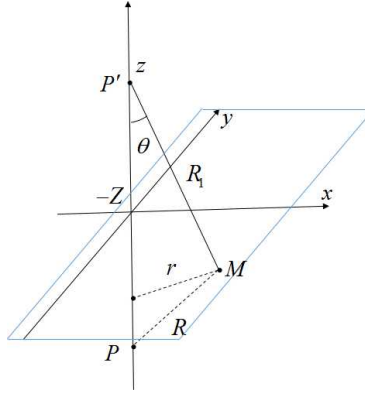


Figure 1: Definition of source point (P), field point (M) and other notations

### 2.1. ODEs with nondimensional original variables

The wave frequency is denoted  $\omega$  and the time dependent factor of the complex potential is  $e^{-i\omega t}$ . The frequency domain Green function  $G_\infty$  can be written:

$$-4\pi G_\infty(r, Z, \omega) = \frac{1}{R} - \frac{1}{R_1} + G(r, Z, \omega) \quad (2)$$

70 with

$$G(r, Z, \omega) = 2PV \int_0^\infty \frac{k}{k - k_0} e^{kZ} J_0(kr) dk + 2i\pi k_0 e^{k_0 Z} J_0(k_0 r) \quad (3)$$

where  $k_0 = \omega^2/g$  is the wave number and  $J_0(\cdot)$  is the Bessel function of the first kind and zero order.

Let us consider the nondimensional Green function  $\bar{G}_\infty = G_\infty L$ . Let:  $\bar{R} = R/L$ ,  $\bar{R}_1 = R_1/L$ ,  $\bar{r} = r/L$ ,  $\bar{Z} = Z/L$ ,  $\bar{k}_0 = k_0 L = \omega^2 L/g = \bar{\omega}^2$ ,  $\bar{k} = kL$ ,  $\bar{G} =$

75  $GL$  where  $L$  denotes a reference length. The non-dimensional Green function has the same form as the Green function with dimensional variables:

$$-4\pi\bar{G}_\infty(\bar{r}, \bar{Z}, \bar{\omega}) = \frac{1}{\bar{R}} - \frac{1}{\bar{R}_1} + \bar{G}(\bar{r}, \bar{Z}, \bar{\omega}) \quad (4)$$

with

$$\bar{G}(\bar{r}, \bar{Z}, \bar{\omega}) = 2PV \int_0^\infty \frac{\bar{k}}{\bar{k} - \bar{k}_0} e^{\bar{k}\bar{Z}} J_0(\bar{k}\bar{r}) d\bar{k} + 2i\pi\bar{k}_0 e^{\bar{k}_0\bar{Z}} J_0(\bar{k}_0\bar{r}) \quad (5)$$

The ODEs for the nondimensional Green function and its derivatives can be written [23]:

$$\frac{\bar{\omega}^2}{4}\bar{G}_{\bar{\omega}\bar{\omega}} - \bar{\omega}(\bar{\omega}^2\bar{Z} + \frac{3}{4})\bar{G}_{\bar{\omega}} + (\bar{\omega}^4(\bar{r}^2 + \bar{Z}^2) + \bar{\omega}^2\bar{Z} + 1)\bar{G} = \frac{2(1 + \bar{Z}\bar{\omega}^2)}{(\bar{r}^2 + \bar{Z}^2)^{1/2}} \quad (6)$$

$$\frac{\bar{\omega}^2}{4}\bar{G}_{\bar{r}\bar{\omega}\bar{\omega}} - \bar{\omega}(\bar{\omega}^2\bar{Z} + \frac{7}{4})\bar{G}_{\bar{r}\bar{\omega}} + (\bar{\omega}^4(\bar{r}^2 + \bar{Z}^2) + 3\bar{\omega}^2\bar{Z} + 3)\bar{G}_{\bar{r}} = \frac{-6\bar{r}(1 + \bar{Z}\bar{\omega}^2)}{(\bar{r}^2 + \bar{Z}^2)^{3/2}} \quad (7)$$

$$\frac{\bar{\omega}^2}{4}\bar{G}_{\bar{Z}\bar{\omega}\bar{\omega}} - \bar{\omega}(\bar{\omega}^2\bar{Z} + \frac{7}{4})\bar{G}_{\bar{Z}\bar{\omega}} + (\bar{\omega}^4(\bar{r}^2 + \bar{Z}^2) + 3\bar{\omega}^2\bar{Z} + 4)\bar{G}_{\bar{Z}} = \frac{-4\bar{Z}(2 + \bar{Z}\bar{\omega}^2) + 2\bar{r}^2\bar{\omega}^2}{(\bar{r}^2 + \bar{Z}^2)^{3/2}} \quad (8)$$

80 For  $\bar{\omega} = 0$ , the values of the Green function and its first gradient can be obtained from equation (5). They are given by:

$$\bar{G} = \frac{2}{\bar{R}_1}, \bar{G}_{\bar{\omega}} = 0 \quad (9a)$$

$$\bar{G}_{\bar{r}} = \frac{-2\bar{r}}{\bar{R}_1^3}, \bar{G}_{\bar{r}\bar{\omega}} = 0 \quad (9b)$$

$$\bar{G}_{\bar{Z}} = \frac{-2\bar{Z}}{\bar{R}_1^3}, \bar{G}_{\bar{Z}\bar{\omega}} = 0 \quad (9c)$$

It can be observed that for  $\bar{\omega} = 0$ , equations (6),(7) and (8) degenerate into zero order equations. Thus, they cannot be used to obtain the second derivatives

85 of the Green function for that frequency. This singularity prevents using  $\bar{\omega} = 0$  as the initial condition for the ODEs.

The ODEs apply both to the real and imaginary part of the Green function. However, the imaginary part can be expressed analytically [18], [6], [2]. Thus, in the following, we shall focus only on the evaluation of the real part of the Green  
 90 function. Moreover, the  $\bar{\cdot}$  on top of the variables will be omitted as hereafter we only discuss the nondimensional Green function.

### 2.2. The singularity of the Green function for $\omega = 0$

The difficulty to use  $\omega = 0$  as the initial condition for the ODEs arises not only from the fact that they degenerate for that frequency but also because of  
 95 the singular behaviour of the Green function for  $\omega = 0$ .

Let us recall the modified Haskind representation of the Green function [25], [26]:

$$G(r, Z, \omega) = \frac{2}{R_1} - \pi\omega^2 e^{\omega^2 Z} [\mathbf{H}_0(\omega^2 r) + Y_0(\omega^2 r) + \frac{2}{\pi} \int_Z^0 \frac{e^{-\omega^2 t}}{\sqrt{t^2 + r^2}} dt] \quad (10)$$

where  $\mathbf{H}_0(\cdot)$  denotes the Struve function as defined by Abramowitz and Stegun [27]. By using a Taylor expansion, the Green function can be rewritten:

$$G(r, Z, \omega) = \frac{2}{R_1} - 2\omega^2 \ln(\omega^2) - 2\omega^2 \left( \gamma + \ln\left(\frac{r^2}{2(Z + R_1)}\right) \right) + O(\omega^4) \quad (11)$$

100 where  $\gamma$  is the Euler's constant.

Let us now differentiate twice this last equation to obtain the second derivative of the Green function as function of  $\omega$ . One can show that for  $\omega \rightarrow 0$ ,  $\partial^2 G / \partial \omega^2 \rightarrow -8 \ln \omega$ . Thus, the second derivative of the Green function tends to infinity for  $\omega = 0$ .

### 105 2.3. ODEs with nondimensional natural variables

In [23], ODEs as function of the natural variables  $(\mu, \varpi)$  were introduced. The natural variables are defined from the original variables through the change

of variables  $(r, Z, \omega) \leftrightarrow (R_1, \mu, \varpi)$ :

$$\begin{cases} r \\ Z \\ \omega \end{cases} \Rightarrow \begin{cases} R_1 &= (r^2 + Z^2)^{1/2} \\ \mu &= -Z/(r^2 + Z^2)^{1/2} = -Z/R_1 \\ \varpi &= \omega(r^2 + Z^2)^{1/4} = \omega\sqrt{R_1} \end{cases} \quad (12)$$

$$\begin{cases} R_1 \\ \mu \\ \varpi \end{cases} \Rightarrow \begin{cases} r &= R_1\sqrt{1 - \mu^2} \\ Z &= -\mu R_1 \\ \omega &= \varpi/\sqrt{R_1} \end{cases} \quad (13)$$

Note that as  $Z$  has values in  $[-\infty, 0]$  and  $r$  has values in  $[0, \infty]$ , the range of possible values for  $\mu$  is limited to  $[0, 1]$ .

The set of ODEs as function of the natural variables are [23]:

$$\frac{1}{4}\varpi^2\tilde{G}_{\varpi\varpi} + \varpi(\mu\varpi^2 - \frac{3}{4})\tilde{G}_{\varpi} + (\varpi^4 - \mu\varpi^2 + 1)\tilde{G} = 1 - \varpi^2\mu \quad (14)$$

$$\frac{1}{4}\varpi^2\tilde{G}_{r\varpi\varpi} + \varpi(\mu\varpi^2 - \frac{7}{4})\tilde{G}_{r\varpi} + (\varpi^4 - 3\mu\varpi^2 + 3)\tilde{G}_r = -3\sqrt{1 - \mu^2}(1 - \mu\varpi^2) \quad (15)$$

$$\frac{1}{4}\varpi^2\tilde{G}_{Z\varpi\varpi} + \varpi(\mu\varpi^2 - \frac{7}{4})\tilde{G}_{Z\varpi} + (\varpi^4 - 3\mu\varpi^2 + 4)\tilde{G}_Z = 2\mu(2 - \varpi^2\mu) + (1 - \mu^2)\varpi^2 \quad (16)$$

with  $G(r, Z, \omega) = 2/R_1\tilde{G}(\mu, \varpi)$ .

The initial conditions for  $\varpi = 0$  as function of the natural variables can be obtained according to (9) are:

$$\tilde{G} = 1, \tilde{G}_{\varpi} = 0 \quad (17a)$$

$$\tilde{G}_r = -\frac{r}{R_1}, \tilde{G}_{r\varpi} = 0 \quad (17b)$$

$$\tilde{G}_Z = -\frac{Z}{R_1}, \tilde{G}_{Z\varpi} = 0 \quad (17c)$$



115 As for the case of the original variables, it can be observed that the ODEs degenerate for  $\varpi = 0$ . Thus, the ODEs as function of the natural variables cannot be integrated from  $\varpi = 0$  either.

#### 2.4. Relations between the Green function and its spatial derivatives

The vertical derivative of the Green function can be obtained directly from the Green function itself [18], [6], [2]. In the following, we show that the same 120 applies to the horizontal derivative.

Let us recall the nondimensional time-domain free-surface Green function,  $F_\infty(r, Z, t)$  [27], [28]:

$$-4\pi F_\infty(r, Z, t) = \delta(t) \left( \frac{1}{R} - \frac{1}{R_1} \right) + H(t)F(r, Z, t) \quad (18)$$

with

$$F(r, Z, t) = 2 \int_0^\infty J_0(Kr) e^{KZ} \sqrt{K} \sin(\sqrt{K}t) dK \quad (19)$$

125 where  $t$  is the time variable,  $\delta(\cdot)$  is the Dirac delta function and  $H(\cdot)$  is the Heaviside unit step function. The terms associated to the Dirac delta function are often referred to the impulsive part of the Green function while the term with the Heaviside unit step function is called the memory part.

$F$  is solution of the following partial differential equation :

$$\frac{\partial^2 F}{\partial t^2} + \frac{\partial F}{\partial Z} = 0 \quad Z \leq 0; \forall t \geq 0 \quad (20)$$

130 Using the change of variables (13) with  $\tau = t/\sqrt{R_1}$ , one can show:

$$2\tilde{F}_{\tau\tau} + \mu\tau\tilde{F}_\tau - 2(1 - \mu^2)\tilde{F}_\mu + 3\mu\tilde{F} = 0 \quad (21)$$

with  $F(r, Z; t) = 2R_1^{-3/2}\tilde{F}(\mu, \tau)$ .

Applying the Fourier transform as in [22], one can show that the Green function as function of the natural variables is a solution of the partial differential equation:

$$-\mu\varpi\tilde{G}_\varpi - (2\varpi^2 - 2\mu)\tilde{G} - 2(1 - \mu^2)\tilde{G}_\mu = 2\mu \quad (22)$$

135 Thus, the derivative of the Green function  $\tilde{G}_\mu$  can be obtained from the knowledge of  $\tilde{G}$  and  $\tilde{G}_\varpi$ .

From  $\tilde{G}_\mu$  and  $\tilde{G}_\varpi$ , the horizontal derivative of  $\tilde{G}$  can be obtained using:

$$\tilde{G}_r = \tilde{G}_\mu \frac{\partial \mu}{\partial r} + \tilde{G}_\varpi \frac{\partial \varpi}{\partial r} \quad (23)$$

Recalling that  $\tilde{G} = R_1/2G$ ,  $\tilde{G}_\varpi = \sqrt{R_1}/2G_\omega$ , the horizontal derivative of the Green function as function of the original variables can be obtained by using:

$$G_r = \frac{2}{R_1} \tilde{G}_r - \frac{r}{R_1^2} G \quad (24)$$

140 Thus, equations (23) and (24) provide a way to evaluate the horizontal derivative of the Green function from the Green function itself without having to solve an additional ODE.

### 3. A new ODE free of the singularity at the origin

#### 3.1. Derivation

145 The ODEs (6, 7, 8, 14, 15, 16) can be written in a general form as follows:

$$x^2 y'' + (c_{21}x^2 + c_{01})xy' + (c_{40}x^4 + c_{20}x^2 + c_{00})y = c_0 + c_2x^2 \quad (25)$$

where  $c_{21}, c_{01}, c_{40}, c_{20}, c_{00}, c_0, c_2$  are parameters independent on the variable  $x$ ,  $x \geq 0$ . Note that  $x$  is the frequency variable, which is either  $\omega$  or  $\varpi$  depending on the considered ODE.  $y$  is the non-dimensional Green function or one of its derivatives as function of the non-dimensional or natural variables. The parameters are given for each equation in AppendixA.

The initial conditions are:

$$y(0) = c_0/c_{00} \quad (26a)$$

$$y'(0) = 0 \quad (26b)$$

According to (11), the asymptotic form for  $y$  when  $x$  is close to 0 is:

$$y(x) = c_0/c_{00} + a_1x^2 \log x + b_1x^2 + o(x^2) \quad (27)$$

with

$$a_1 = -4, b_1 = -2(\gamma + \log(r^2/(2(Z + R_1))))), \quad \text{for } G \quad (28a)$$

$$a_1 = 0, b_1 = -4\left(\frac{1}{r} - r/(2R_1(Z + R_1))\right), \quad \text{for } G_r \quad (28b)$$

$$a_1 = -2, b_1 = \log 2 - \gamma - \log(1 + \mu), \quad \text{for } \tilde{G} \quad (28c)$$

$$a_1 = 0, b_1 = \mu\sqrt{1 - \mu^2}/(R_1(1 + \mu)), \quad \text{for } \tilde{G}_r \quad (28d)$$

The asymptotic form for the first derivative of  $y$  is:

$$y'(x) = a_1x(2\log x + 1) + 2b_1x + o(x) \quad (29)$$

155 Let us define the new function  $z(x)$  as the difference of the original function  $y$  minus the two leading terms in (27):

$$z(x) = y(x) - c_0/c_{00} - a_1x^2 \log x \quad (30)$$

Thus,  $y(x)$  and its derivative are given by:

$$y(x) = z + c_0/c_{00} + a_1x^2 \log x \quad (31a)$$

$$y'(x) = z' + 2a_1x \log x + a_1x \quad (31b)$$

By introducing (31) into (25), one can show that  $z(x)$  is the solution of a the following ODE:

$$x^2z'' + c(x)xz' + d(x)z = x^2f(x) \quad (32)$$

with

$$c(x) = c_{21}x^2 + c_{01} \quad (33a)$$

$$d(x) = c_{40}x^4 + c_{20}x^2 + c_{00} \quad (33b)$$

$$f(x) = -c_{40}a_1x^4 \log x + [-(2c_{21} + c_{20})a_1 \log x - c_{21}a_1 - c_{40}c_0/c_{00}]x^2 \quad (33c)$$

$$+ [-(2 + 2c_{01} + c_{00})a_1 \log x - (3 + c_{01})a_1 + c_2 - c_{20}c_0/c_{00}]$$

For the case  $y = \tilde{G}$ , the coefficients involved in the last bracket of  $f(x)$  are  $c_{01} = -3$ ,  $a_1 = -2$ ,  $c_2 = -4\mu$ ,  $c_{20} = -4\mu$ ,  $c_0 = 4$  and  $c_{00} = 4$  according to (14). Thus,  $2 + 2c_{01} + c_{00} = 0$  and  $-(3 + c_{01})a_1 + c_2 - c_{20}c_0/c_{00} = 0$ . It can be shown that it is also the case for  $y = G$ ,  $y = G_r$  and  $\tilde{G}_r$  using (6), (7) and (15).  $f(x)$  can then be rewritten:

$$f(x) = -c_{40}a_1x^4 \log x + [-(2c_{21} + c_{20})a_1 \log x - c_{21}a_1 - c_{40}c_0/c_{00}]x^2 \quad (34)$$

For  $x \rightarrow 0$ ,  $c(x) \rightarrow c_{01}$  and  $d(x) \rightarrow c_{00}$  according to equation (33) while  $f(x) \rightarrow 0$  according to equation (34).

Moreover, it can be noted that for  $x \rightarrow 0$ :

$$z(x) = b_1x^2 + o(x^2) \quad (35a)$$

$$z'(x) = 2b_1x + o(x) \quad (35b)$$

according to the asymptotic expression (30) of  $y(x)$ . Thus,  $z(x) \rightarrow 0$  and  $z'(x) \rightarrow 0$  for  $x \rightarrow 0$ .

In comparison with the second-order ODE (25) for  $y(x)$ , the second order ODE (32) for  $z(x)$  is of the same form on the left hand side. However, it has a non-homogeneous term on the right hand side whose leading order is  $x^4 \log x$  for  $x \rightarrow 0$ . Therefore, in contrast to  $y''(x)$ ,  $z''(x)$  tends to a finite value for  $x \rightarrow 0$  (that is 0 according to equation (34)). The ODE for  $z$  equation (32) is free of the singularity for  $x \rightarrow 0$ .

### 3.2. Practical representation of the new ODE for its numerical integration

The direct application of a numerical integration scheme (e.g. Runge-kutta (RK4)) for the ODE (32) remains difficult in practice because of the coefficient  $x^2$  associated with  $z''$ . Hereafter, we present a method to deal with this issue.

Let us assume that  $x \neq 0$ . The ODE (32) can be re-written as:

$$z'' + c(x)z'/x + d(x)z/x^2 = f(x) \quad (36)$$

Let us define the new variables  $(u, v) = (z', z/x)$ . Their derivatives are:  
 $(u', v') = (z'', u/x - v/x)$ .

By introducing the new variables in the second-order ODE (36), one can obtain the system of differential equations:

$$\mathbf{Y}' = \mathbf{F}(x, \mathbf{Y}/x) \quad (37)$$

185 with  $\mathbf{Y}$  and  $\mathbf{F}$  defined by

$$\mathbf{Y} = \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \text{and} \quad \mathbf{F}(x, \mathbf{Y}/x) = \begin{Bmatrix} f(x) \\ 0 \end{Bmatrix} - \begin{bmatrix} c(x) & d(x) \\ -1 & 1 \end{bmatrix} \mathbf{Y}/x \quad (38)$$

The initial values for  $x \rightarrow 0$  are

$$\begin{aligned} \mathbf{Y}_0 &= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ \hat{\mathbf{Y}}_0 &= \frac{\mathbf{Y}}{x} \Big|_{x \rightarrow 0} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} b_1 \\ \mathbf{F}_0 &= \begin{Bmatrix} -(c_{00} + 2c_{01}) \\ 1 \end{Bmatrix} b_1 \end{aligned} \quad (39)$$

in which we used (35) to obtain  $\hat{\mathbf{Y}}_0$  and (33) for  $c(0) = c_{01}$ ,  $d(0) = c_{00}$  and  $f(0) = 0$  to obtain  $\mathbf{F}_0$  defined by (38).

Both the initial values of  $\hat{\mathbf{Y}}_0$  and  $\mathbf{F}_0$  being finite, usual numerical integration  
 190 schemes can be used to integrate (37) in order to obtain values of the Green  
 function from the initial conditions at  $x = \omega = \varpi = 0$ . The coefficients in  
 equations (38) and (39) and relation between  $y(x)$  and  $z(x)$  for  $G$ ,  $G_r$ ,  $\tilde{G}$ ,  $\tilde{G}_r$   
 are given in AppendixA.

#### 4. Results

195 In this section, the results of the evaluation of the Green function and its  
 horizontal derivative by using the ODE are compared to the results obtained  
 using the direct integration method as described in [23].

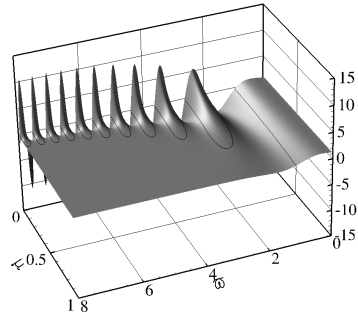
The Green function as function of the original and natural variables are evaluated by solving the system (37) by using the Runge-Kutta 4 method with  
200 a constant step size  $\Delta\varpi = 0.001$ .

The initial conditions are for  $\varpi = 0$  from equation (39). The results are shown in Figure 2 for  $\varpi \in (0, 8]$ ,  $\mu \in [0, 1)$ . The error is defined as the difference between the values of the Green function obtained by integrating the ODE and the direct integration method with 8 decimal accuracy [18]. One can see  
205 that a 6 decimals (6D) accuracy is obtained. Thus, the method using the ODE is as accurate as the most accurate of the other numerical methods for evaluating the Green function [18], that are based on the approximations of integral formulations.

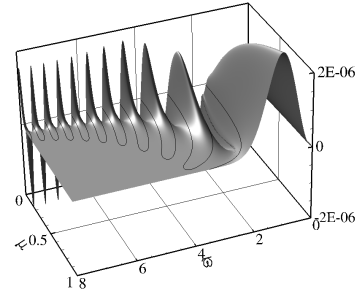
The horizontal derivative of the Green function is obtained from the Green  
210 function itself using (23). The values and errors are shown in Figure 3 for  $\varpi \in (0, 8]$ , and  $\mu \in [0, 1)$ . A satisfactory absolute accuracy of 4 decimals is obtained.

Results for the Green function as function of the original variables and its horizontal derivative are shown in Figure 4 and 5. A 6D accuracy is obtained.

215 Therefore, the ODE-based method can predict the Green function as accurately (6D accuracy) as existing alternative methods. Nevertheless, it can be noted that 4D accuracy is expected to be sufficient for practical applications according to [16, 17, 29]

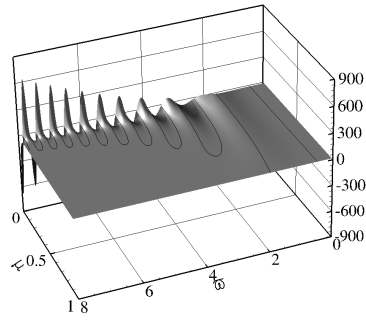


(a)  $\tilde{G}$

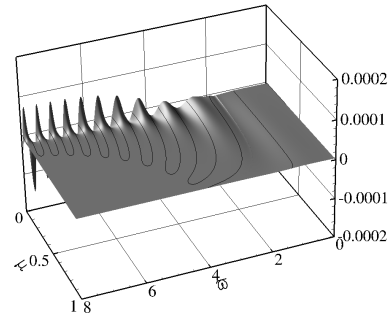


(b)  $\tilde{G}error$

Figure 2: Results for the evaluation of  $\tilde{G}$  with the ODE and error with the direct integration method. The ODE is solved using the RK4 method with  $\Delta\varpi = 0.001$   $\varpi \in (0, 8]$ , and  $\mu \in [0, 1)$

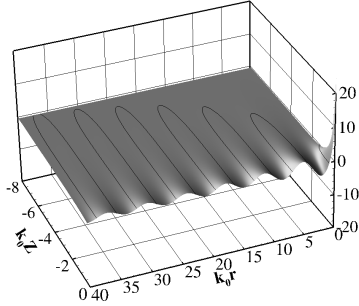


(a)  $\tilde{G}_r$

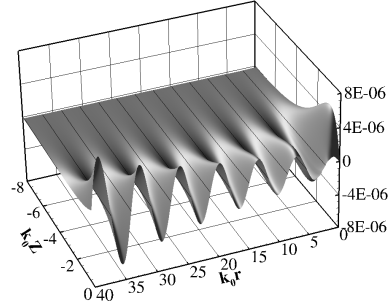


(b)  $\tilde{G}_rerror$

Figure 3: Results for the evaluation of  $\tilde{G}_r$  and error with the direct integration method. The ODE is solved using the RK4 method with  $\Delta\varpi = 0.001$   $\varpi \in (0, 8]$ , and  $\mu \in [0, 1)$

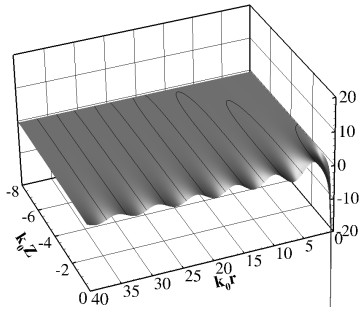


(a)  $G$

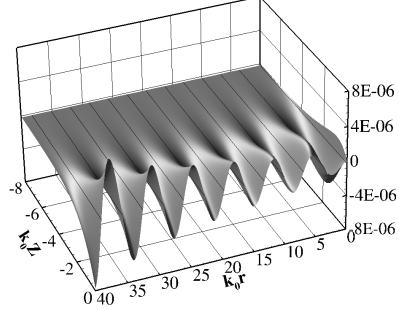


(b)  $Gerror$

Figure 4: Results for the evaluation of  $G$  and error with the direct integration method. The ODE is solved using the RK4 method with  $\Delta\omega = 0.001$ .



(a)  $G_r$



(b)  $G_rerror$

Figure 5: Results for the evaluation of  $G_r$  and error with the direct integration method. The ODE is solved using the RK4 method with  $\Delta\omega = 0.001$ .

## 5. Conclusion

220 In this paper, we present a new ordinary differential equation for the evaluation of the frequency domain Green function. In contrast to original Clément's ODE, this modified ODE allows for the zero frequency to be used for the initial conditions. Comparisons with the direct integration method show that 6D accurate estimates of the Green function can be obtained with this ODE. This  
 225 method may be used for calculating the Green function and its gradient. It will



be investigated in future work whether it is more efficient than conventional methods for the calculation of hydrodynamic coefficients in BEM codes.

## 6. Acknowledgements

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### 230 AppendixA. The coefficients of ODEs for $G$ , $G_r$ , $\tilde{G}$ , $\tilde{G}_r$

For sake of clarity, the parameters  $c_{21}$ ,  $c_{01}$ ,  $c_{40}$ ,  $c_{20}$ ,  $c_{00}$ ,  $c_0$ ,  $c_2$  for  $G$ ,  $G_r$ ,  $\tilde{G}$ ,  $\tilde{G}_r$  are listed in following. The  $c(x)$ ,  $d(x)$ ,  $f(x)$  and  $-(c_{00} + 2c_{01})$  in equations (38), (39) are also given and  $y(x)$  is expressed by  $z(x)$ .

- $G$

235 According to equation (6),

$$\begin{aligned} c_{21} &= -4Z; c_{01} = -3; \\ c_{40} &= 4(r^2 + Z^2); c_{20} = 4Z; c_{00} = 4; \\ c_0 &= 8/R_1; c_2 = 8Z/R_1; \end{aligned} \tag{A.1}$$

Thus:

$$c(x) = -4Zx^2 - 3 \tag{A.2a}$$

$$d(x) = 4(r^2 + Z^2)x^4 + 4Zx^2 + 4 \tag{A.2b}$$

$$f(x) = -4x^2(R_1^2 a_1 x^2 \log x - Za_1 \log x - Za_1 + 2R_1) \tag{A.2c}$$

$$-(c_{00} + 2c_{01}) = 2 \tag{A.2d}$$

$$y(x) = z(x) + 2/R_1 - 4x^2 \log x \tag{A.2e}$$

- $G_r$

According to equation (7),

$$\begin{aligned} c_{21} &= -4Z; c_{01} = -7; \\ c_{40} &= 4(r^2 + Z^2); c_{20} = 12Z; c_{00} = 12; \\ c_0 &= -24r/R_1^3; c_2 = -24rZ/R_1^3; \end{aligned} \tag{A.3}$$

Thus:

$$c(x) = -4Zx^2 - 7 \quad (\text{A.4a})$$

$$d(x) = 4(r^2 + Z^2)x^4 + 12Zx^2 + 12 \quad (\text{A.4b})$$

$$f(x) = 8rx^2/R_1 \quad (\text{A.4c})$$

$$-(c_{00} + 2c_{01}) = 2 \quad (\text{A.4d})$$

$$y(x) = z(x) - 2r/R_1^3 \quad (\text{A.4e})$$

•  $\tilde{G}$

According to equation (14),

$$\begin{aligned} c_{21} &= 4\mu; c_{01} = -3; \\ c_{40} &= 4; c_{20} = -4\mu; c_{00} = 4; \\ c_0 &= 4; c_2 = -4\mu; \end{aligned} \quad (\text{A.5})$$

Thus:

$$c(x) = 4\mu x^2 - 3 \quad (\text{A.6a})$$

$$d(x) = 4x^4 - 4\mu x^2 + 4 \quad (\text{A.6b})$$

$$f(x) = -4x^2(a_1 x^2 \log x + \mu a_1 \log x + \mu a_1 + 1) \quad (\text{A.6c})$$

$$-(c_{00} + 2c_{01}) = 2 \quad (\text{A.6d})$$

$$y(x) = z(x) + 1 - 2x^2 \log x \quad (\text{A.6e})$$

240

•  $\tilde{G}_r$

According to equation (15),

$$\begin{aligned} c_{21} &= 4\mu; c_{01} = -7; \\ c_{40} &= 4; c_{20} = -12\mu; c_{00} = 12; \\ c_0 &= -12\sqrt{1 - \mu^2}; c_2 = 12\mu\sqrt{1 - \mu^2}; \end{aligned} \quad (\text{A.7})$$

Thus:

$$c(x) = 4\mu x^2 - 7 \quad (\text{A.8a})$$

$$d(x) = 4x^4 - 12\mu x^2 + 12 \quad (\text{A.8b})$$

$$f(x) = 4\sqrt{1 - \mu^2} \quad (\text{A.8c})$$

$$-(c_{00} + 2c_{01}) = 2 \quad (\text{A.8d})$$

$$y(x) = z(x) - \sqrt{1 - \mu^2} \quad (\text{A.8e})$$

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