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# GLOBAL WELL-POSEDNESS OF THE AXISYMMETRIC NAVIER-STOKES EQUATIONS WITH MEASURE-VALUED INITIAL DATA.

GUILLAUME LÉVY<sup>1</sup>

## 1. INTRODUCTION

## 2. COLLECTION D'ESTIMATIONS CRITIQUES

There exists a constant  $C$  depending only on  $\|\mu\|_{TV}$  such that for any positive  $t$ ,

$$(1) \quad \|u(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{1}{2}}.$$

## 3. PROOFS ?

We begin with the global existence statement, which is rather simple to prove as most of the job has already been done in [1].

**Theorem 3.1** (Global existence). *Let  $\mu \in \mathcal{M}(\Omega)$ . There exists at least one global solution  $\omega : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  of the axisymmetric Navier-Stokes equations with initial data  $\mu$ , satisfying further the following assumptions.*

- *$L^1$  decay in time : the function  $t \mapsto \|\omega(t, \cdot)\|_{L^1(\Omega)}$  is decreasing and converges to  $\|\mu\|_{TV}$  as  $t$  goes to 0.*
- *Short time smoothness : for arbitrary small  $\delta > 0$ ,  $(t, x) \mapsto \omega(t, x)$  is smooth on  $[\delta, +\infty[ \times \Omega$ , with (critical estimates).*
- *Weak-\* continuity : as  $t \rightarrow 0$ , we have  $\omega(t, x)dx \rightharpoonup^* \mu$  in  $\mathcal{M}(\Omega)$ .*
- *(Maybe other properties here)*

*Proof.* We will simply smooth out the initial data, summon the result from [1] which provides us with estimates uniform in the regularization parameter, at which point we may pass to the limit.

Let  $S$  denote the linear semigroup associated to the cylindrical laplacian  $\delta_{\text{cyl}}$  and define for  $0 < \varepsilon < 1$  the regularization  $\mu_\varepsilon := S(\varepsilon)\mu$  of the initial data. Thanks to [1], we know that the axisymmetric Navier-Stokes equations with initial data  $\mu_\varepsilon$  possess a unique, global solution  $\omega_\varepsilon$ , satisfying estimates depending solely on  $\|\mu_\varepsilon\|_{L^1(\Omega)} \leq \|\mu\|_{TV}$ . More precisely, there exists a constant  $C$  depending only on  $\|\mu\|_{TV}$  such that the following inequalities hold.

(blabla smoothness en temps positif) (penser à des estimations avec  $r$  à une puissance négative, pour faire converger la condition au bord)

In particular, for any  $\delta > 0$ , the family  $(\omega_\varepsilon)_{0 < \varepsilon < 1}$  is locally relatively compact in, say,  $\mathcal{C}^2([\delta, +\infty[ \times \Omega)$ . Extracting a subsequence, we see that any accumulation point  $\omega$  of this family satisfies the axisymmetric Navier-Stokes equations on  $\mathbb{R}_+^* \times \Omega$ , with the same estimates (in particular, it is smooth in time and space). Moreover, thanks to Equation (estimée puissance négative), any such  $\omega$  satisfies the Dirichlet boundary condition.

As for the initial condition, we rely on the same technique as in [2], with minor modifications. Given a test function  $\varphi$  in  $\mathcal{C}^2(\Omega)$  with compact support, we define, for strictly positive values of both  $t$  and  $\varepsilon$ ,

$$I_\varphi(t, \varepsilon) := \int_\Omega \omega_\varepsilon(t, x)\varphi(x)dx.$$

We immediately notice that the estimates on  $\omega_\varepsilon$  yield the bound

$$\sup_{t>0} I_\varphi(t, \varepsilon) \leq \|\mu\|_{TV} \|\varphi\|_{L^\infty(\Omega)}.$$

Computing the time derivative of  $I_\varphi$  and using the equation on  $\omega$  to integrate by parts, we get

$$\partial_t I_\varphi(t, \varepsilon) = \int_{\Omega} \omega_\varepsilon(t, x) (u(t, x) \cdot \nabla \varphi(x) + \Delta_{cyl}^* \varphi(x)) dx,$$

where  $\Delta_{cyl}^*$  is the formal  $L^2$  adjoint of  $\Delta_{cyl}$ . Thanks to the bound (citer borne  $u$  ici), there exists a constant  $C$  depending only on  $\|\mu\|_{TV}$  such that

$$|\partial_t I_\varphi(t, \varepsilon)| \leq Ct^{-\frac{1}{2}}.$$

Consequently, for each  $\varphi$ , the family  $(I_\varphi(\cdot, \varepsilon))_{0 < \varepsilon < 1}$  is bounded in  $\mathcal{C}^{\frac{1}{2}}(\mathbb{R}_+)$ , hence locally relatively compact in  $\mathcal{C}^0(\mathbb{R}_+)$  by Ascoli's theorem. Finally, interchanging the two limits  $\varepsilon \rightarrow 0$  and  $t \rightarrow 0$  proves that for any  $\varphi$  in  $\mathcal{C}^2(\Omega)$  with compact support, we have, as  $t$  goes to 0,

$$\int_{\Omega} \omega(t, x) \varphi(x) dx \longrightarrow \int_{\Omega} \varphi(x) d\mu(x).$$

Arguing by density and using the uniform bound on  $\|\omega_\varepsilon(t)\|_{L^1(\Omega)}$  concludes the proof of the convergence  $\omega(t, x) dx \rightharpoonup^* \mu$  as  $t$  goes to 0.  $\square$

We now turn to a mass estimate near the symmetry axis, which is an improvement over the one in [2]. Given  $\mu$  in  $\mathcal{M}(\Omega)$  and  $R \geq 0$ , we define the near-axis mass function

$$M(\mu, R) := \|\mathbf{1}_{r \leq R} \mu\|_{TV}.$$

Of course, if  $\mu$  admits an integrable density  $\omega$ , we have

$$M(\mu, R) = \int_0^R \int_{\mathbb{R}} |\omega(r, z)| dz dr.$$

In this case, we will indifferently denote the above quantity by  $M(\omega, R)$  or  $M(\mu, R)$ . Following Lemma 5.1 in [1], we first assume that our initial data is positive, removing this assumption afterwards.

**Proposition 3.2.** *(Near the axis, positive case) Let  $\mu \in \mathcal{M}_+(\Omega)$  be a positive measure. There exists a constant  $C$  such that for any positive  $t$  and any  $\rho \geq C\sqrt{t}$ , we have*

$$(2) \quad M(\omega(t), \rho) \leq \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} M(\mu, y + 2\rho) dy.$$

*Proof.* Since the axisymmetric Navier-Stokes equations preserve the positivity of the solution, for any positive  $t$  and  $R$ , we have

$$M(\omega(t), R) = \int_0^R \int_{\mathbb{R}} \omega(t, r, z) dz dr.$$

In order to avoid technical singularities near the axis, where  $r = 0$ , we will rather work with the remainder function

$$f(t, R) := \int_R^\infty \int_{\mathbb{R}} \omega(t, r, z) dz dr.$$

From the estimates on  $\omega$ , it follows that  $f$  is smooth in all its variables. Computing its time derivative and using the equation on  $\omega$  gives, for positive  $t$  and  $R$ ,

$$\partial_t f(t, R) = \partial_R^2 f(t, R) + \frac{1}{R} \partial_R f(t, R) + \int_{\mathbb{R}} u(t, R, z) \omega(t, R, z) dz.$$

Thanks to the estimate (1), we have

$$\int_{\mathbb{R}} u(t, R, z) \omega(t, R, z) dz \geq -\frac{C}{\sqrt{t}} \int_{\mathbb{R}} \omega(t, R, z) dz = \frac{C}{\sqrt{t}} \partial_R f(t, R).$$

Let now  $g(t, R) := f(t, R + a(t))$  for some positive and decreasing function  $a$  to be chosen later. Computing the time derivative of  $g$  for positive  $t$  and  $R$ , we get

$$\partial_t g(t, R) \geq \partial_R^2 g(t, R) + \left( \frac{1}{R + a(t)} + \frac{C}{\sqrt{t}} + a'(t) \right) \partial_R f(t, R).$$

For some parameters  $\rho$  and  $t_1$  to be specified later, we choose, following [2],

$$a(t) = \rho + \frac{t_1 - t}{\rho} + 2C(\sqrt{t_1} - \sqrt{t}).$$

With this choice, for  $t$  in the time interval  $[0, t_1]$ , we have

$$a(t) \geq a(t_1) = \rho, \quad a'(t) + \frac{C}{\sqrt{t}} + \frac{1}{R + a(t)} \leq 0.$$

Hence, for  $t$  in  $[0, t_1]$  and any positive  $R$ , we have

$$\partial_t g(t, R) \geq \partial_R^2 g(t, R).$$

Since  $f$  is a decreasing function of its second argument, for any positive  $t$  (in particular, for  $t$  in  $[0, t_1]$ ), we have

$$\partial_R g(t, 0) = \partial_R f(t, R + a(0)) \leq 0.$$

By the parabolic maximum principle, for any  $t_0$  in  $]0, t_1[$ , for any  $t$  in  $[t_0, t_1]$  and any positive  $R$ ,  $g(t, R) \geq h(t, R)$  where  $h$  is the solution of

$$(3) \quad \begin{cases} \partial_t h(t, R) &= \partial_R^2 h(t, R), & t, R > 0, \\ \partial_R h(t, 0) &= 0, & t > 0, \\ h(t_0, R) &= g(t_0, R) = f(t_0, R + a(t_0)). \end{cases}$$

In particular, taking  $R = 0$ , for any  $t$  in  $[t_0, t_1]$ , we have  $f(t, a(t)) \geq h(t, 0)$ . From the monotonicities of  $f$  and  $a$ , we get  $f(t, a(t)) \leq f(t, a(t_1)) = f(t, \rho)$ . On the other hand, the representation formula for  $h$  gives us

$$\begin{aligned} h(t, 0) &= \frac{1}{\sqrt{\pi(t-t_0)}} \int_0^\infty e^{-\frac{y^2}{4(t-t_0)}} h(t_0, y) dy \\ &= \frac{1}{\sqrt{\pi(t-t_0)}} \int_0^\infty e^{-\frac{y^2}{4(t-t_0)}} f(t_0, y + a(t_0)) dy. \end{aligned}$$

We will now choose  $t_1(\rho) \leq c\rho^2$ , with  $c$  chosen such that

$$c + 2C\sqrt{c} \leq \frac{1}{4}.$$

Hence, if  $t_0$  is sufficiently small, we have  $a(t_0) \leq \frac{3\rho}{2}$ . Under this condition, using again the monotonicity of  $f$ , for any  $t \in [t_0, t_1(\rho)]$  we have

$$h(t, 0) \geq \frac{1}{\sqrt{\pi(t-t_0)}} \int_0^\infty e^{-\frac{y^2}{4(t-t_0)}} f\left(t_0, y + \frac{3\rho}{2}\right) dy.$$

We now wish to take the limit  $t_0 \rightarrow 0$ . As we cannot hope  $f$  to be continuous up to the initial time, we will resort to a smooth cutoff function. Let  $\chi : \mathbb{R} \rightarrow [0, 1]$  be a smooth, positive and increasing function such that

$$\chi(r) = \begin{cases} 0 & \text{if } r \leq \frac{3\rho}{2}, \\ 1 & \text{if } r \geq 2\rho. \end{cases}$$

It is pretty clear that for any  $y$  and  $t$  in  $\mathbb{R}_+$ , we have

$$f\left(t, y + \frac{3\rho}{2}\right) \geq \int_\Omega \omega(t, r, z) \chi(r - y) dz dr.$$

By the dominated convergence theorem, we can now take the limit  $t_0 \rightarrow 0$  in order to get

$$(4) \quad f(t, \rho) \geq \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} \left( \int_\Omega \chi(r - y) d\mu(r, z) \right) dy.$$

This last integral over  $\Omega$  is clearly no less than  $\|\mu \mathbf{1}_{r \leq y + 2\rho}\|_{TV}$ , which yields

$$(5) \quad f(t, \rho) \geq \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} f(0, y + 2\rho) dy.$$

Reverting to the mass-function  $M$ , we have proven the existence of a small constant  $c$  such that for any positive  $t$  and any  $\rho \geq \sqrt{\frac{t}{c}}$ ,

$$(6) \quad M(\omega(t), \rho) \leq (\|\omega(t)\|_{L^1(\Omega)} - \|\mu\|_{TV}) + \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} M(\mu, y + 2\rho) dy.$$

The decay of the  $L^1$  norm of  $\omega$  with time allows us to close the proof.  $\square$

**Proposition 3.3** (Near the axis, general case). *Let  $\mu \in \mathcal{M}(\Omega)$  be a measure. There exists a constant  $C$  such that for any positive  $t$  and any  $\rho \geq C\sqrt{t}$ , we have*

$$(7) \quad M(\omega(t), \rho) \leq \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} M(\mu, y + 3\rho) dy.$$

*Proof.* We will mostly rely on the positive case, though with some necessary tweaks. Let us begin by picking some small time  $s$ . On the time interval  $[s, \infty[$ , we know that our mild solution  $\omega$  and its velocity field  $u$  are smooth. Hence, the linear system

$$\begin{cases} \partial_t a + \operatorname{div}_*(ua) - \Delta_{\text{cyl}} a = 0 \\ a(t, 0) = 0 \\ a(s) = a_0 \end{cases}$$

possesses a unique solution for any  $a$  in, say,  $L^1(\Omega)$ . Moreover, provided that  $a_0$  has a sign,  $a$  keeps the same sign for all subsequent times. We will choose the positive and negative parts of  $\omega(s)$  as initial data; the corresponding solutions will be denoted by  $\omega_\pm$ . Reproducing the proof in the positive case, we arrive at the following. There exists a constant  $C$  such that for any positive  $t$  and any  $\rho \geq C\sqrt{t}$ , we have

$$(8) \quad M(\omega_\pm(t), \rho) \leq \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} M(\omega_\pm(s), y + 2\rho) dy.$$

Since both  $\omega_\pm$  are positive functions and  $\omega(t) = \omega_+(t) - \omega_-(t)$  by the smoothness and  $u$  and the linear uniqueness it entails, we have  $|\omega(t)| \leq \omega_+(t) + \omega_-(t)$  for any  $t \geq s$ , with equality at  $t = s$ . In particular, there holds

$$M(\omega(t), \rho) \leq M(\omega_+(t), \rho) + M(\omega_-(t), \rho)$$

and

$$M(\omega(s), \rho) = M(\omega_+(s), \rho) + M(\omega_-(s), \rho).$$

Hence, summing the two inequalities gives

$$(9) \quad M(\omega(t), \rho) \leq \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} M(\omega(s), y + 2\rho) dy.$$

We will now resort to the same trick as above, using a smooth cutoff function  $\chi$ . Doing so, we finally arrive at the slightly weaker inequality

$$(10) \quad M(\omega(t), \rho) \leq \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} M(\mu, y + 3\rho) dy.$$

$\square$

Armed with this mass-spreading estimate in the general case, we are now ready to prove additional integrability on the velocity field for short times, under a weak decay assumption on  $M(\mu, \cdot)$ .

**Proposition 3.4.** *Let  $\mu \in \mathcal{M}(\Omega)$  be a measure. There exists a constant  $C > 0$  such that for any positive functions  $\alpha, \rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , the velocity field  $u$  associated to any solution  $\omega$  of the axisymmetric Navier-Stokes equations with initial data  $\mu$  satisfies*

$$(11) \quad \int_0^1 \left\| \frac{u_r(t)}{r} \right\|_{L^\infty(\Omega)} dt \leq C \int_0^1 \left( M(\mu, \alpha(t) + 3\rho(t)) + \|\mu\|_{TV}^{\frac{1}{3}} \frac{\sqrt{t}}{\alpha(t)} e^{-\frac{\alpha(t)^2}{4t}} \right) \frac{dt}{t} \\ + C \|\mu\|_{TV}^{\frac{1}{3}} \int_0^1 (t\rho(t))^{-\frac{2}{3}} dt.$$

*Proof.* As in the proof of Corollary 3.2 in [2], we will treat differently points close and far away from the axis. Let us denote by

$$\omega^-(t, r, z) := \omega(t, r, z)\mathbb{1}_{r \leq \rho}, \quad \omega^+(t, r, z) := \omega(t, r, z)\mathbb{1}_{r > \rho}.$$

The corresponding velocity fields obtained through the Biot-Savart law will be denoted by  $u^\pm(t, r, z)$ . The trick will be to choose  $\rho$  as a function of  $t$  in order to optimize the estimates. Using the estimate (Estimate here), we have

$$\left\| \frac{u_r^-(t)}{r} \right\|_{L^\infty(\Omega)} \lesssim \|\omega^-(t)\|_{L^1(\Omega)}^{\frac{1}{3}} \left\| \frac{\omega^-(t)}{r} \right\|_{L^\infty(\Omega)}^{\frac{2}{3}}$$

and

$$\left\| \frac{u_r^+(t)}{r} \right\|_{L^\infty(\Omega)} \lesssim \|\omega^+(t)\|_{L^1(\Omega)}^{\frac{1}{3}} \left\| \frac{\omega^+(t)}{r} \right\|_{L^\infty(\Omega)}^{\frac{2}{3}}.$$

Let us look at  $u_r^-$  first. Regarding the second term, Estimate (estimate here) gives us

$$\left\| \frac{\omega^-(t)}{r} \right\|_{L^\infty(\Omega)}^{\frac{2}{3}} \lesssim t^{-1}.$$

The first term is precisely  $M(\omega(t), \rho)^{\frac{1}{3}}$ . Using Proposition 3.3 and Jensen's inequality, we get

$$M(\omega(t), \rho)^{\frac{1}{3}} \lesssim \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y^2}{4t}} M(\mu, y + 3\rho)^{\frac{1}{3}} dy.$$

Now, given some arbitrary positive real number  $\alpha$ , we split the integral into two parts, depending on whether the integrand is greater than or smaller than  $\alpha$ . We have

$$M(\omega(t), \rho)^{\frac{1}{3}} \lesssim M(\mu, \alpha + 3\rho) + \|\mu\|_{TV}^{\frac{1}{3}} \frac{\sqrt{t}}{\alpha} e^{-\frac{\alpha^2}{4t}}.$$

where we used the bound

$$\int_X^\infty e^{-u^2} du \leq \frac{1}{X} e^{-X^2},$$

valid for any  $X > 0$ . We now turn to the second half of the velocity field. Thanks to the  $L^1$  decay, we obviously have

$$\|\omega^+(t)\|_{L^1(\Omega)}^{\frac{1}{3}} \leq \|\mu\|_{TV}^{\frac{1}{3}}$$

for any positive  $t$ . On the other hand, from the definition of  $\omega^+$  and estimate (estimate here), we have

$$\left\| \frac{\omega^+(t)}{r} \right\|_{L^\infty(\Omega)}^{\frac{2}{3}} \lesssim (\rho t)^{-\frac{2}{3}}.$$

Gathering all the pieces, we got the desired estimate.  $\square$

The following corollary is now an obvious consequence of the preceding Propositions.

**Corollary 3.5.** *Let  $\mu \in \mathcal{M}(\Omega)$  be a measure. Assume that its near-axis mass function  $M$  decays fast enough near 0 so that there exists increasing functions  $\alpha, \rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  for which the three following integrals are finite.*

$$\int_0^1 (t\rho(t))^{-\frac{2}{3}} dt; \quad \int_0^1 \frac{1}{\sqrt{t}\alpha(t)} e^{-\frac{\alpha(t)^2}{4t}} dt; \quad \int_0^1 \frac{M(\mu, \alpha(t) + 3\rho(t))^{\frac{1}{3}}}{t} dt.$$

*Then, for any mild solution  $\omega$  of the axisymmetric Navier-Stokes equations, its velocity field  $u$  satisfies*

$$\int_0^1 \left\| \frac{u_r(t)}{r} \right\|_{L^\infty(\Omega)} dt < \infty.$$

**Remark.** For the first two integrals to be finite, we need  $\sqrt{t}$  to be negligible in front of both  $\alpha(t)$  and  $\rho(t)$ . In particular, the third integral can be finite only if

$$\int_0^1 \frac{M(\mu, \sqrt{t})^{\frac{1}{3}}}{t} dt < \infty.$$

Changing variables, this last condition is equivalent to

$$\int_0^1 \frac{M(\mu, t)^{\frac{1}{3}}}{t} dt < \infty.$$

This decay condition seems indepassable if one needs the velocity field integrability for short times.

**Remark.** Having a  $M(\mu, \cdot)$  which decays like a power function near the origin, no matter how small the exponent, is enough to ensure that Corollary 3.5 holds. In particular, if  $M(\mu, \cdot)$  vanishes identically near the origin as it is in the case of one or several Dirac masses in  $\Omega$ , all the integrability conditions are easily seen to be satisfied.

Mettre ici un blabla explicatif

**Proposition 3.6.** Let  $T$  be an arbitrary time. Let  $U, V : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be continuous vector fields satisfying the following assumptions :

- (1)  $\operatorname{div} U \equiv 0$ ;
- (2)  $K_1 := \sup_{0 < t < T} t^{\frac{1}{2}} \|U(t)\|_{L^\infty(\mathbb{R}^3)} < \infty$ ;
- (3)  $K_2 := \int_0^T \|V(t)\|_{L^\infty(\mathbb{R}^3)} dt < \infty$ .

Consider the equation

$$\partial_t f + U \cdot \nabla f + V f - \Delta f = 0.$$

There exists a unique fundamental solution  $\tilde{\Phi}$  to the above equation. Moreover,  $\tilde{\Phi}$  is Hölder-continuous in space and time on  $[0, T] \times \mathbb{R}^3$  and satisfies the pointwise bounds

$$0 < \tilde{\Phi}(t, x; s, y) \leq \frac{C}{(t-s)^{\frac{3}{2}}} \exp\left(-\frac{|x-y|^2}{4(t-s)} + K_1 \frac{|x-y|}{\sqrt{t-s}} + K_2\right).$$

*Proof.* Déjà fait ailleurs, flemme de recopier. □

**Lemma 3.7.** The fundamental solution  $\tilde{\Phi}$  is contravariant with respect to axial rotations, in the following sense. For any rotation matrix  $R$  whose invariant axis is the symmetry axis  $\{r = 0\}$ , we have

$$(12) \quad \tilde{\Phi}(t, x; 0, Ry) = \tilde{\Phi}(t, R^{-1}x; 0, y).$$

*Proof.* Par unicité de la solution fondamentale (les données initiales sont égales avec cette convention et les champs  $U, V$  sont invariants). □

We will be more interested in the averaged version of  $\tilde{\Phi}$ , denoted by  $\Phi$ , obtained by integrating it in the  $\theta$  variable. The bounds for the original  $\tilde{\Phi}$  translates, for any  $\eta > 0$  into

$$(13) \quad 0 < \Phi(t, x; 0, y) \leq \frac{C_\eta}{t} \exp\left(-(1-\eta) \frac{|x-y|^2}{4t}\right).$$

An immediate consequence of the above proposition is a decomposition of  $\omega$  into simpler pieces. Indeed, for  $t > 0$  and  $x \in \Omega$ , we have

$$\omega(t, x) = \sum_i \alpha_i \omega_i(t, x) + \omega_d(t, x),$$

where

$$\omega_i(t, x) := \alpha_i \Phi(t, x; 0, x_i)$$

is the contribution from the vortex filament starting at  $x_i$  and

$$\omega_d(t, x) := \int_{\Omega} \Phi(t, x; 0, y) d\mu_d(y)$$

is the diffuse contribution, coming from the continuous (or non atomic) part  $\mu_d$  of  $\mu$ .

The bound (13) translates into

$$(14) \quad |\omega_i(t, x)| \leq \frac{C_\eta |\alpha_i|}{t} e^{-\frac{1-\eta}{4t} |x-x_i|^2}$$

and

$$(15) \quad |\omega_d(t, x)| \leq \frac{C_\eta |\alpha_i|}{t} \int_{\Omega} e^{-\frac{1-\eta}{4t} |x-y|^2} d|\mu_d|(y).$$

Our goal now is to gain more information on the Gaussian behavior of  $\Phi$  for small times. While this was performed originally in (GS16) and (avecYanlin) for the atomic part of the initial measure, at least at the level of the full solution, we still need to prove this statement for the diffuse part. We begin with a few easy lemmas relative to rescaling operators which we will frequently use.

**Definition.** Let  $y$  be a point in  $\Omega$  and  $\lambda \in \mathbb{R}_+^*$ . Define the rescaling operators around  $y$  at the scale  $\lambda$  acting on  $f \in L^1(\Omega)$  by

$$(R_{y,\lambda}^{ve} f)(x) = \sqrt{\lambda} f(y + \sqrt{\lambda} x),$$

$$(R_{y,\lambda}^{vo} f)(x) = \lambda f(y + \sqrt{\lambda} x).$$

**Remark.** If  $\omega$  and  $u$  are two functions such that for some constant  $C$  and any  $x$  in  $\Omega$

$$|u(x)| \leq C \int_{\Omega} \frac{|\omega(z)|}{|x-z|} dz,$$

then for any  $y$  in  $\Omega$  and  $\lambda > 0$ ,

$$|R_{y,\lambda}^{ve} u(x)| \leq C \int_{\Omega_\lambda} \frac{|R_{y,\lambda}^{vo} \omega(z)|}{|x-z|} dz,$$

where  $\Omega_\lambda := \frac{\Omega-y}{\sqrt{\lambda}}$ .

**Lemma 3.8.** The following local uniform convergences hold. For any  $y$  in  $\Omega$ ,

$$R_{y,t}^{ve} u_d(t) \rightarrow 0 \text{ and } R_{y,t}^{vo} \omega_d(t) \rightarrow 0.$$

Moreover, for any index  $j$  such that  $y \neq x_j$ ,

$$R_{y,t}^{ve} u_j(t) \rightarrow 0 \text{ and } R_{y,t}^{vo} \omega_j(t) \rightarrow 0.$$

*Proof.* Let us first assume that there exists an index  $j$  such that  $y \neq x_j$ . Then, owing to the bound (14), we have

$$|R_{y,t}^{vo} \omega_j(t)(x)| \leq C_\eta |\alpha_j| e^{-\frac{1-\eta}{4t} |y-x_i+\sqrt{t}x|^2}.$$

Hence, locally uniformly in  $x$  and for  $t$  small enough,

$$|R_{y,t}^{vo} \omega_j(t)(x)| \leq C_\eta |\alpha_j| e^{-\frac{1-\eta}{8t} |y-x_i|^2}.$$

Owing to the above remark, it is straightforward to show that also  $R_{y,t}^{ve} u_j(t)$  goes to 0 locally uniformly in  $x$  as  $t \rightarrow 0$ .

Let us now turn to the diffuse parts of the solution. Beginning with the vorticity and using the Gaussian bound, we have

$$|R_{y,t}^{vo} \omega_d(t)(x)| \leq C \int_{\mathbb{R}^2} e^{-\frac{1-\eta}{4t} |y-y'+\sqrt{t}x|^2} d|\mu_d|(y').$$

Choosing some arbitrarily small parameter  $\varepsilon$ , we split the integral according to whether  $|y-y'| \leq \varepsilon$  or  $|y-y'| > \varepsilon$ . Since  $\mu_d$  is by definition atom-free, the first integral goes to 0 as  $\varepsilon$  goes to 0,



uniformly in  $t$ ; the other one goes to 0 as  $t \rightarrow 0$  for any  $\varepsilon$ . Repeating the same argument as for  $u_j$  above shows that  $R_{y,t}^{ve} u_d(t)$  goes to 0 as  $t \rightarrow 0$  locally uniformly.  $\square$

**Lemma 3.9.** *Let  $(t_m)_{m \in \mathbb{N}}$  be a sequence converging to 0. Define sequences of rescaled functions  $(u^{(m)})_{m \in \mathbb{N}}$  and  $(\Phi^{(m)})_{m \in \mathbb{N}}$  by*

$$(16) \quad \begin{cases} u_y^{(m)}(s, z) := \sqrt{t_m} u(t_m s, y + \sqrt{t_m} z), & s > 0, z \in \Omega \\ \Phi_y^{(m)}(s, z) := t_m \Phi(t_m s, y + \sqrt{t_m} z; 0, y). \end{cases}$$

Then, any pair of limit points  $(\bar{u}_y, \bar{\Phi}_y)$  of these sequences as  $m \rightarrow \infty$  satisfy

$$(17) \quad \begin{cases} \partial_s \bar{\Phi}_y + \bar{u}_y \cdot \nabla \bar{\Phi}_y - \Delta \bar{\Phi}_y = 0, \\ \bar{\Phi}_y(0) = \delta_0. \end{cases}$$

*Proof.* We recall that the pair  $(u, \tilde{\Phi})$  satisfies the equation

$$\partial_t \tilde{\Phi} + u \cdot \nabla \tilde{\Phi} - \Delta \tilde{\Phi} = \tilde{\Phi} \frac{u_r}{r}, \quad t > 0, x \in \mathbb{R}^3.$$

Integrating on the circle orthogonal to the  $z$ -axis going through  $y$  and applying the rescalings, the equation for the pair  $(u_y^{(m)}, \Phi_y^{(m)})$  reads

$$\partial_s \Phi_y^{(m)} + \operatorname{div}_*(u_y^{(m)} \Phi_y^{(m)}) - \Delta \Phi_y^{(m)} = \sqrt{t_m} \Phi_y^{(m)} \frac{u_{y,r}^{(m)}}{R_m},$$

where  $R_m^2 := (y_1 + \sqrt{t_m} z_1)^2 + (y_2 + \sqrt{t_m} z_2)^2$ . (préservation des bornes, point d'accumulation) Hence, the right-hand side converges locally uniformly to 0 as  $m \rightarrow \infty$ . Taking the limit  $m \rightarrow \infty$  proves that any accumulation point  $\bar{\Phi}$  indeed satisfies the transport-diffusion equation. Since the equations on the rescalings provide us with uniform bounds in  $L^1 H^{-3}$  for the derivative in time, the initial condition is what we want (réécrire ce passage).  $\square$

**Proposition 3.10.** *With the same notations as in the above lemma, for any  $y \in \Omega$ , we have*

$$\bar{\Phi}_y(s, z) = \frac{1}{4\pi s} e^{-\frac{|z|^2}{4s}}, \quad s > 0, z \in \mathbb{R}^2.$$

*Proof.* We split the proof into two cases, depending on whether  $y$  is an atom of  $\mu$ .

**Case 1 :**  $y$  is not an atom of  $\mu$ .

In this case, Lemma 3.8 entails the nullity of  $\bar{u}$  and hence,  $\bar{\Phi}$  is the fundamental solution of the two-dimensional heat equation. The conclusion is thus immediate.

**Case 2 :** there exists an index  $i$  such that  $y = x_i$ .

Owing again to Lemma 3.8,  $u^{(m)} - u_i^{(m)}$  goes to 0 locally uniformly as  $m \rightarrow \infty$  (et idem avec  $\Phi$ ).

Et alors, on vérifie qu'on se ramène au non linéaire 2D et on invoque Proposition 1.3 in (Gallay-Wayne).  $\square$

**Remark.** *Étape suivante : donner un taux de convergence du reste en temps petit. Espoir : que ça soit un minimum uniforme. Est-ce qu'on travaillerait pas encore un peu sur  $\Phi$ , plutôt que la solution  $\omega$  ? Ça a l'air plus efficace, ça évite les blowups qui mènent juste à zéro.*

*C'est encore à vérifier, mais il est possible qu'on puisse reproduire la preuve de Gallay-Sverak sur la solution fondamentale en tout point. C'est peut-être l'idée qui va débloquent toute la fin (aka l'unicité), à voir.*

Let  $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a smooth, decreasing function such that

$$\chi(x) = 1 \text{ if } x \geq \frac{1}{2}, \quad \chi(x) = 0 \text{ for } x \leq \frac{1}{4}.$$

For  $y \in \Omega$  and  $X \in \mathbb{R}^2$ , let

$$f_{0,y}(t, X) := G(X) \chi \left( \frac{r(y + \sqrt{t} X)}{r(y)} \right).$$

The function  $f_{0,y}$  is tailored to satisfy the Dirichlet condition on  $\partial\Omega_t$  for all positive  $t$ .

## APPENDIX A. ON REFAIT L'ESTIMATION DE MASSE

On travaille pour l'instant au-delà de  $s > 0$ , comme ça tout le monde est lisse et intégrable comme on veut. La  $\Phi$  est solution de

$$(18) \quad \begin{cases} \partial_t \Phi(t, x; s, y) + \operatorname{div}_*(u(t, x)\Phi(t, x; s, y)) - \Delta_{\text{cyl}}\Phi(t, x; s, y) = 0, \\ \Phi(s, \cdot; s, y) = \delta_y. \end{cases}$$

On a des estimations sur  $u$  : on sait que  $\|u\|_{L^\infty} \lesssim t^{-\frac{1}{2}}$ .

**Proposition A.1.** *On a une estimation gaussienne de la masse de  $\Phi(t, x; s, y)$  pour  $t \leq cr(y)^2$  avec  $c$  petite constante et  $r(x) \leq \frac{1}{2}r(y)$ .*

*Proof.* Il suffit de recopier la preuve de Gallay-Sverak, qui est complètement linéaire.  $\square$

**Corollary A.2.** *Soit  $y$  tel que  $r(y) > C\sqrt{t}$ . Alors*

$$\left\| \frac{BS(\Phi(t, \cdot; s, y))}{r} \right\|_{L^\infty} \lesssim (\text{relou})$$

*Proof.* On divise comme d'habitude  $\Phi$  en deux bouts, selon que  $r(z) > \frac{1}{2}r(y)$  ou  $r(z) \leq \frac{1}{2}r(y)$  et on les appelle  $\Phi^\pm(t, z; s, y)$ . Pour  $\Phi^-$  :

$$\|\Phi^-(t, \cdot; s, y)\|_{L^1}^{\frac{1}{3}} \lesssim e^{-X^2}/X \text{ avec } X = \frac{r(y)}{\sqrt{t}}$$

$$\|\Phi^-(t, z; s, y)/z\|_{L^\infty}^{\frac{2}{3}} \lesssim t^{-1}? \text{ à vérifier}$$

Pour  $\Phi^+$  :

$$\|\Phi^+(t, \cdot; s, y)\|_{L^1}^{\frac{1}{3}} \lesssim 1$$

$$\|\Phi^+(t, z; s, y)/z\|_{L^\infty}^{\frac{2}{3}} \lesssim r(y)^{-\frac{2}{3}} \|\Phi^+(t, z; s, y)\|_{L^\infty}^{\frac{2}{3}} \lesssim (tr(y))^{-\frac{2}{3}}? \text{ à vérifier aussi.}$$

$\square$

**Proposition A.3.** *Borne finale sur le champ : soit  $C$  une constante assez grande (définie plus haut).*

$$\|V(t)\|_{L^\infty} \lesssim \int_{r(y) > C\sqrt{t}} \left( \frac{1}{t} e^{-\frac{cr(y)^2}{t}} \left( \frac{r(y)}{\sqrt{t}} \right)^{-1} + \frac{1}{(tr(y))^{\frac{2}{3}}} \right) d|\mu|(y) + \int_{r(y) \leq C\sqrt{t}} \frac{C}{t} d|\mu|(y).$$

**Lemma A.4.** *L'intégrale de la partie  $r(y) > C\sqrt{t}$  est  $\lesssim \|\mu\|_{TV}$ .*

*Proof.* Calculs pas compliqués, on permute les intégrales, on intègre en temps et on fait un petit changement de variable, c'est cool.  $\square$

**Proposition A.5.** *Condition finale :  $\int_0^1 \frac{M(t)}{t} dt < \infty$  est suffisant (et en fait plus ou moins CNS avec cette méthode en tout cas) pour avoir l'intégrabilité du champ de vitesses.*

## APPENDIX B. RÉÉCRIRE LÉGÈREMENT LA CONVERGENCE RESCALÉE

Avec

$$u = \int_{\Omega} BS(\Phi_y) d\mu(y),$$

$$u^{(m)} \rightarrow \bar{u} = \mu\{y\} BS(\bar{\Phi}).$$

Comme ça c'est plus joli, c'est une écriture unifiée.

## REFERENCES

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