Transformed Lévy processes as state-dependent wear models

Ji Hwan Cha, Sophie Mercier

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TRANSFORMED LÉVY PROCESSES AS STATE-DEPENDENT WEAR MODELS

JI HWAN CHA,* *Ewha Womans University*

SOPHIE MERCIER,** *University of Pau & Pays Adour*

Abstract

Many wear processes used for modeling accumulative deterioration in a reliability context are non homogeneous Lévy processes and hence have independent increments, which may not be suitable in an application context. We here suggest to consider Lévy processes transformed by monotonous functions, which allow to overcome this restriction and provides a new state-dependent wear model. These transformed Lévy processes are first observed to remain tractable Markov processes. Some distributional properties are derived. The impact of the current state on the future increment level and on the overall accumulated level is investigated from a stochastic monotonicity point of view. Positive dependence properties and stochastic monotonicity of increments are also studied.

Keywords: Reliability; Deterioration model; Wear process; Stochastic order; Positive dependence; Non independent increments

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1. Introduction

Safety and dependability is a crucial issue in many industries, which has lead to the development of a huge literature devoted to the so-called reliability theory. In the

* Postal address: Department of Statistics, Ewha Womans University, Seoul, 120-750, Korea. Email address: jhcha@ewha.ac.kr

** Postal address: CNRS / Univ Pau & Pays Adour/ E2S UPPA, Laboratoire de Mathématiques et de leurs Applications de Pau - Fédération IPRA, UMR5142, 64000, Pau, France. Email address: sophie.mercier@univ-pau.fr

oldest literature, the lifetimes of industrial systems or components were usually directly modeled through random variables, see, e.g., [2] for a pioneer work on the subject. Based on the development of on-line monitoring which allows the effective measurement of a system deterioration, numerous papers nowadays model the degradation in itself, which often is considered to be accumulating over time. This is done through the use of continuous-time stochastic processes, which are usually assumed to be monotonous or with monotonous trend. Most common models include gamma processes [1, 8, 26], Wiener processes with trend [10, 29] and inverse gaussian processes [27, 30] (see also [15] for more references). All these models are (possibly non homogeneous) Lévy processes and hence have independent increments. However, in an application context, one could think that the current deterioration level of a system can have some influence on its future deterioration development. Typically, when the deterioration rate is increasing over time, one could expect that the more severe a system history is (and hence the higher the current deterioration level is), the higher the future deterioration rate is. Such a behavior cannot be modeled through processes with independent increments and some new "state-dependent wear models" (according to the vocabulary of [11]) need to be developed.

Some interesting attempts have been made in the previous literature for taking into account some stochastic dependence between the current state of a system and its future deterioration, such as [18, 28], where the deterioration process is constructed as the solution of a stochastic differential equation (see also [25]). However, these models do not seem to be very tractable, and up to our knowledge, generic tools still need to be developed for their practical use in an application context (such as estimation procedures). See however [11] for a practical use of such models in a specific setting. Another attempt has been made recently by Giorgio, Guida and Pulcini in a series of papers [12, 13, 14] where they suggest to consider gamma processes transformed by increasing functions. This provides a tractable Markovian state-dependent wear process. We here propose to use a similar procedure for general Lévy processes, which leads to a more generic state-dependent wear process that we call Transformed Lévy process. The new process includes, e.g., Transformed gamma processes in the sense of [12, 13, 14] but also classical geometric Brownian motion (see, e.g., [22]). As will be seen, a Transformed Lévy process remains a tractable Markov process, for which the

Markov kernel is easily obtained. This allows to derive the joint probability density function of successive observations of a deterioration path, from where a classical Maximum Likelihood Estimation procedure could easily be implemented (which is beyond the scope of the present paper). The model hence has a clear potential for practical use. For a better understanding of its modeling ability, we here focus on stochastic monotonicity/comparison results and on positive dependence properties.

The paper is organized as follows. The transformed Lévy process is defined in Section 2, and the first distributional properties are derived. Considering a system with deterioration level modeled by a transformed Lévy process, Section 3 is devoted to the study of the impact of the current state of the system on its future deterioration level, from a stochastic monotonicity point of view. Positive dependence properties are next developed in Section 4 and stochastic monotonicity of increments in Section 5. Conclusive remarks end the paper in Section 6.

2. Definition and the first properties

Throughout the paper, the term Lévy process stands for a possibly *non homogeneous* Lévy process. These processes are also called additive processes in the literature [23, page 3]. We recall that a process $(X_t)_{t\geq 0}$ is said to be a (non homogeneous) Lévy (or additive) process as soon as:

- $X_0 = 0$ almost surely (a.s.);
- $(X_t)_{t\geq 0}$ has independent increments;
- $(X_t)_{t\geq 0}$ is stochastically continuous;
- $(X_t)_{t\geq 0}$ has right-continuous paths with left-side limits, almost surely.

We refer to [23, page 3] for more details.

Definition 1. Let $(X_t)_{t\geq 0}$ be a Lévy process with range J where $J = \mathbb{R}$, $J = \mathbb{R}_+$, or $J = \mathbb{R}_-$ and let g be a (strictly) increasing differentiable function such that $g : I \subset \mathbb{R} \longrightarrow J$ with $g(I) = J$. A process $(Z_t)_{t\geq 0}$ is called a TRansformed (TR) Lévy process with baseline process $(X_t)_{t\geq 0}$ and state function g if $Z_t = g^{-1}(X_t)$ for all $t \geq 0$.

We first start with a well-known example.

Example 1. Let $(X_t)_{t\geq 0}$ be a time-scaled Wiener process with drift:

$$X_t = A(t) + \sigma\ W_{A(t)}, \forall t \geq 0,$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process such that $W_t \sim \mathcal{N}(0,t)$ and $A : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing such that $A(0) = 0$ and $\lim_{t\to+\infty} A(t) = +\infty$. Such a function is called a time-scaling function in the following. Then $(X_t)_{t\geq 0}$ is a (non homogeneous) Lévy process and each increment $X_t - X_s$ is normally distributed $\mathcal{N}\left(A(t) - A(s), \sigma^2 (A(t) - A(s))\right)$, where $0 \leq s < t$. Considering $z_0 > 0$ and $g(x) = \ln\left(\frac{x}{z_0}\right)$ for $x \in \mathbb{R}_+^*$, we have $g^{-1}(z) = z_0\ e^z$ and

$$Z_t = g^{-1}(X_t) = z_0\ e^{X_t} = z_0\ e^{A(t)+\sigma\ W_{A(t)}}$$

is a time-scaled geometric Brownian motion, which hence appears as a specific TR Lévy process.

Note that if $(X_t)_{t\geq 0}$ is a Lévy process, then $(-X_t)_{t\geq 0}$ also is a Lévy process so that $(g(-X_t))_{t\geq 0}$ with g increasing is a TR Lévy process. Then, any $(\overline{g}(X_t))_{t\geq 0}$ with $\overline{g}(x) = g(-x)$ decreasing is a TR Lévy process in the sense of the previous definition, which hence includes the case of any strictly monotonic function g.

In all the following, we assume that X_t admits a probability density function (pdf) with respect to Lebesgue measure denoted by f_{X_t}. The corresponding cumulative distribution function (cdf) and survival function are denoted by F_{X_t} and $\overline{F}_{X_t}$, respectively. Similar notations are used for other random variables, without any further notification.

We now come to the probabilistic structure of a TR Lévy process.

Proposition 1. *With the notations of Definition 1, a TR Lévy process is a Markov process with Markov transition kernel provided by:*

$$P(s,t;z,dx) = \mathbb{P}(Z_t \in dx | Z_s = z) = f_{Z_t|Z_s=z}(x)\ dx \tag{1}$$

with

$$f_{Z_t|Z_s=z}(x) = g'(x)\, f_{X_t-X_s}(g(x) - g(z)) \tag{2}$$

and

$$P(s,t;z,(x,\infty)) = \overline{F}_{Z_t|Z_s=z}(x) = \overline{F}_{X_t-X_s}(g(x) - g(z))$$

for all $0 \leq s < t$ *and all* $x, z \in I$.

Proof. Based on the fact that the baseline process $(X_t)_{t\geq 0}$ is a Markov process, it is clear that a TR Lévy process $\left(Z_t = g^{-1}(X_t)\right)_{t\geq 0}$ also is a Markov process. Also:

$$\begin{aligned} P(s,t;z,(x,\infty)) &= \mathbb{P}(Z_t > x | Z_s = z) \\ &= \mathbb{P}(X_t > g(x) | X_s = g(z)) \\ &= \mathbb{P}(X_t - X_s > g(x) - g(z) | X_s = g(z)) \\ &= \mathbb{P}(X_t - X_s > g(x) - g(z)) \\ &= \overline{F}_{X_t - X_s}(g(x) - g(z)) \end{aligned}$$

due to the independent increments of $(X_t)_{t\geq 0}$ for the previous to last line.

The result for $f_{Z_t|Z_s=z}(x)$ is next obtained through differentiation of $\mathbb{P}(Z_t > x|Z_s = z)$ with respect to x, which ends the proof. □

Remark 1. In [12], the authors consider a Transformed gamma process $(Z_t)_{t\geq 0}$ which they define as a Markov process such that the conditional survival function of $Z_t - Z_s$ given $Z_s = z$ is of the shape

$$\overline{F}_{Z_t - Z_s|Z_s=z}(x) = \overline{F}_{X_t - X_s}(g(z+x) - g(z))$$

for all $0 \leq s < t$ and all $x, z \geq 0$, where $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and the baseline process $(X_t)_{t\geq 0}$ is a gamma process. This means that their Transformed gamma process is defined as a Markov process with Markov transition kernel provided by (1). However, in order to get a consistent definition, it might have been necessary to show, as a preliminary step, that formula (1) actually gives raise to a Markov transition kernel, namely that

$$\int_{\{x\in I\}} P(s,t;z,dx) = 1 \tag{3}$$

and

$$\int_{\{x\in I\}} P(s,t;z,dx)\, P(t,u;x,dy) = P(s,u;z,dy) \tag{4}$$

for all $0 \leq s < t$ and all $z \in I$. In our definition, this point is useless to be checked because formula (1) is obtained by computing the kernel of a Markov process and as a consequence, formulas (3) and (4) are necessary true. As a by-product, this shows the coherency of the definition of a Transformed gamma process as proposed in [12]. Also, their definition leads to a similar notion to ours, in the specific context of gamma processes.

We now provide the conditional distribution of an increment of a TR Lévy process given its present state. The proof is a direct consequence of Proposition 1 and it is omitted.

Corollary 1. *For all $0 \leq s < t$ and $x, z \in I$, the conditional survival function of $Z_t - Z_s$ given $Z_s = z$ is*

$$\overline{F}_{Z_t - Z_s | Z_s = z}(x) = \overline{F}_{X_t - X_s}(g(z+x) - g(z)) \tag{5}$$

and the conditional pdf of $Z_t - Z_s$ given $Z_s = z$ is

$$f_{Z_t - Z_s | Z_s = z}(x) = g'(x+z) f_{X_t - X_s}(g(x+z) - g(z)). \tag{6}$$

In a general setting, the increment $Z_t - Z_s$ hence depends on its past through Z_s, and the increments of $(Z_t)_{t \geq 0}$ are *not* independent. However, it is easy to characterize the case where the increments are independent, as in the following corollary.

Corollary 2. *A TR Lévy process has independent increments if and only if g is of the shape $g(z) = az + b$, with $a > 0$ and $b \in \mathbb{R}$.*

Proof. Assuming $g(z) = az + b$, with $a > 0$, then $Z_t = g^{-1}(X_t) = \frac{X_t - b}{a}, \forall t \geq 0$ and $(Z_t)_{t \geq 0}$ clearly has independent increments.

Conversely, assume the process $(Z_t)_{t \geq 0}$ to have independent increments. Then $\overline{F}_{Z_t - Z_s | Z_s = z}(x)$ is independent on z for all $x \in I$ and all $0 \leq s < t$. Based on Corollary 1, this entails that $g(z+x) - g(z)$ is independent on z for all $x \in I$, which means that g' is a constant and provides the result. $\square$

Considering the fact that a Lévy process is assumed to start from 0 ($Z_0 = 0$), the only case for which a TR Lévy process is a Lévy process hence corresponds to a linear function $g(x) = ax$ with $a > 0$ (which entails $Z_t = \frac{X_t}{a}, \forall t \geq 0$).

Corollary 1 allows to easily write down the joint pdf of increments of a TR Lévy process on successive time intervals as in the following proposition, which could be used for the development of a likelihood estimation procedure in a parametric setting, based on successive observations of deterioration data. See, e.g. [12, 13] in the specific case of transformed gamma processes.

Proposition 2. *Let $0 < t_1 < \cdots < t_n$ and let $Z_{t_{i-1},t_i} = Z_{t_i} - Z_{t_{i-1}}$ for $i \in \{2, \ldots, n\}$. The pdf of $\left(Z_{t_1}, Z_{t_1,t_2}, \cdots, Z_{t_{n-1},t_n}\right)$ is equal to*

$$f_{\left(Z_{t_1},Z_{t_1,t_2},\cdots,Z_{t_{n-1},t_n}\right)}\left(z_1, \cdots, z_n\right) = g'\left(z_1\right) f_{X_{t_1}}\left(g\left(z_1\right) - g(0)\right) \times \prod_{i=1}^{n-1} g'\left(z_{1:i+1}\right) f_{X_{t_i,t_{i+1}}}\left(g\left(z_{1:i+1}\right) - g\left(z_{1:i}\right)\right)$$

where $z_{1:i} = \sum_{j=1}^{i} z_j$ for $1 \leq i \leq n$.

Proof. Using successive conditioning, we have:

$$\begin{aligned} f_{\left(Z_{t_1},Z_{t_1,t_2},\cdots,Z_{t_{n-1},t_n}\right)}\left(z_1, \cdots, z_n\right) &= f_{Z_{t_1}}\left(z_1\right) \times \prod_{i=1}^{n-1} f_{Z_{t_i,t_{i+1}} | \cap_{j=1}^{i}\{Z_{t_{j-1},t_j} = z_j\}}\left(z_{i+1}\right) \\ &= f_{Z_{t_1}}\left(z_1\right) \times \prod_{i=1}^{n-1} f_{Z_{t_i,t_{i+1}} | \cap_{j=1}^{i}\{Z_{t_j} = z_{1:j}\}}\left(z_{i+1}\right), \end{aligned}$$

where, in the first line, $t_0 = 0$. The Markov property now provides:

$$f_{\left(Z_{t_1},Z_{t_1,t_2},\cdots,Z_{t_{n-1},t_n}\right)}\left(z_1, \cdots, z_n\right) = f_{Z_{t_1}}\left(z_1\right) \times \prod_{i=1}^{n-1} f_{Z_{t_i,t_{i+1}} | Z_{t_i} = z_{1:i}}\left(z_{i+1}\right)$$

and the result follows from (2) and (6). $\square$

We end this section by noticing that given the present state, the future increment process still behaves according to a TR Lévy process. According the vocabulary of [6], this means that a TR Lévy process possesses the "restarting property".

Proposition 3. (Restarting property.) *For a fixed $s > 0$, let us set $Z_t^{(s)} = Z_{t+s} - Z_s$, for all $t \geq 0$. Then given $Z_s = x$, the process $\left(Z_t^{(s)}\right)_{t\geq 0}$ conditionally is a TR Lévy process with baseline Lévy process $X^{(s)} = \left(X_t^{(s)} = X_{t+s} - X_s\right)_{t\geq 0}$ and state function $g^{(x)} = g\left(x + \cdot\right) - g\left(x\right)$.*

Proof. Let the symbol $\stackrel{\mathcal{D}}{=}$ mean "is identically distributed as". Then:

$$\begin{aligned} \left[\left(Z_t^{(s)}\right)_{t\geq 0} | Z_s = x\right] &\stackrel{\mathcal{D}}{=} \left[\left(Z_{t+s} - Z_s\right)_{t\geq 0} | Z_s = x\right] \\ &\stackrel{\mathcal{D}}{=} \left[\left(g^{-1}\left(X_{t+s} - X_s + g\left(x\right)\right) - x\right)_{t\geq 0} | X_s = g\left(x\right)\right] \\ &\stackrel{\mathcal{D}}{=} \left(g^{-1}\left(X_{t+s} - X_s + g\left(x\right)\right) - x\right)_{t\geq 0} \end{aligned}$$

based on the independent increments of $(X_t)_{t\geq 0}$ for the last line. Hence, given $Z_s = x$, the process $\left(Z_t^{(s)}\right)_{t\geq 0}$ conditionally is a TR Lévy process with baseline Lévy process $X^{(s)} = \left(X_t^{(s)} = X_{t+s} - X_s\right)_{t\geq 0}$ and

$$\left(g^{(x)}\right)^{-1}(z) = g^{-1}(z + g(x)) - x,$$

or equivalently $g^{(x)}(y) = g(x+y) - g(x)$. □

Some of the results of the paper require the baseline Lévy process $(X_t)_{t\geq 0}$ to be non negative (or at least to keep a constant sign). We recall that this entails that $(X_t)_{t\geq 0}$ is non decreasing. In that case, $(Z_t)_{t\geq 0}$ is a non decreasing process with range $[g(0), g(\infty))$. For easiness, we will also assume that $g(0) = 0 (= Z_0)$. These assumptions will be referred to as "positive assumption" in the following. Note that dual results could be written under a similar "negative assumption".

3. Influence of the current state of a TR Lévy process on its future

In this section, we investigate the influence of the current state on an increment of the future deterioration process and on its overall cumulated level. We refer to [20] and [24] for the definition of the stochastic orders used in this section (usual stochastic order: $\prec_{st}$, hazard rate order: $\prec_{hr}$, reverse hazard rate order: $\prec_{rh}$, likelihood ratio order: $\prec_{lr}$), and to [19] for the aging properties (Increasing Hazard Rate: IHR, Decreasing Hazard Rate: DHR, Decreasing Reverse Hazard Rate: DRHR, Increasing Reverse Hazard Rate: IRHR).

3.1. Influence of the current state on an increment of the wear process

Lemma 1. *Let $0 < s < t$.*

1. *Then $[Z_t | Z_s = z]$ increases in the usual stochastic ordering as z increases.*
2. *Assume g to be concave (resp. convex). Then, under the positive assumption, $[Z_t - Z_s | Z_s = z]$ increases (resp. decreases) in the usual stochastic ordering as z increases.*

Proof. The function

$$\overline{F}_{Z_t|Z_s=z}(x) = \overline{F}_{X_t - X_s}(g(x) - g(z))$$

increases with respect to z, which shows the first point.

For the second point, let us consider the case where g is concave. Let us observe that, for all $x \geq 0$,

$$\overline{F}_{Z_t - Z_s | Z_s = z}(x) = \overline{F}_{X_t - X_s}(g(z+x) - g(z))$$

increases with respect to z because $g(z+x) - g(z)$ decreases with respect to z, which shows the result. The convex case is similar and it is omitted. □

Remark 2. Based on the previous lemma, the future (cumulated) deterioration level will be all the higher as the current observation is high. However, the monotony of the future increment of deterioration with respect to the current observation depends on the concavity/convexity of the state function. Assume for instance that g is concave, or equivalently that g^{-1} is convex. Then the future increment of deterioration will be all the higher as the current observation is high. This seems coherent with the fact that $Z_t = g^{-1}(X_t)$ and that the rate of increasingness of the convex function g^{-1} is increasing.

When the increment $X_t - X_s$ has got some aging property, the previous result can be strengthened as shown in the next proposition.

Proposition 4. *Let $0 < s < t$. Assume the positive assumption to hold and $X_t - X_s$ to have an Increasing Hazard Rate (IHR). Then if g is concave (resp. convex), $[Z_t - Z_s | Z_s = z]$ increases (decreases) in the hazard rate (hr) ordering as z increases.*

Proof. We only look at the convex case as the concave case is similar.

Let $x, y \geq 0$ be fixed. The point is to show that

$$H(z) := \frac{\overline{F}_{Z_t - Z_s | Z_s = z}(x+y)}{\overline{F}_{Z_t - Z_s | Z_s = z}(x)} \tag{7}$$

decreases with respect to z. Let $z_1 \leq z_2$. We have

$$\begin{aligned} H(z_i) &= \frac{\overline{F}_{X_t - X_s}(g(z_i + x + y) - g(z_i))}{\overline{F}_{X_t - X_s}(g(z_i + x) - g(z_i))} \\ &= \frac{\overline{F}_{X_t - X_s}(u_i + v_i)}{\overline{F}_{X_t - X_s}(v_i)} \end{aligned}$$

with $u_i = g(z_i + x + y) - g(z_i + x) \geq 0$ and $v_i = g(z_i + x) - g(z_i)$ for $i = 1, 2$.

As $X_t - X_s$ is IHR , we know that

$$\frac{\overline{F}_{X_t - X_s}(u+v)}{\overline{F}_{X_t - X_s}(v)} \tag{8}$$

decreases with respect to v for any fixed $u \geq 0$.

Also, as g is convex, $z_1 \leq z_2$ and $x \geq 0$, we have $v_1 \leq v_2$ and hence:

$$H(z_1) = \frac{\overline{F}_{X_t - X_s}(u_1+v_1)}{\overline{F}_{X_t - X_s}(v_1)} \geq \frac{\overline{F}_{X_t - X_s}(u_1+v_2)}{\overline{F}_{X_t - X_s}(v_2)}.$$

Now, based again on the convexity of g, we have $u_1 \leq u_2$, from which we now derive:

$$H(z_1) \geq \frac{\overline{F}_{X_t - X_s}(u_2+v_2)}{\overline{F}_{X_t - X_s}(v_2)} = H(z_2),$$

which achieves the proof. □

In the following example, we explore the hazard rate monotony of $[Z_t - Z_s | Z_s = z]$ with respect to z, to see whether some dual results to Proposition 4 could be valid under the Decreasing Hazard Rate (DHR) assumption for $X_t - X_s$ instead of IHR.

Example 2. Let $A : \mathbb{R}_+ \to \mathbb{R}_+$ be a time-scaling function as defined in Example 1 and let $b > 0$. Let $(X_t)_{t\geq 0}$ be a non homogeneous gamma process with shape function $A(\cdot)$ and rate parameter b (denoted by $(X_t)_{t>0} \sim \mathcal{G}(A(\cdot), b)$). Then $(X_t)_{t\geq 0}$ is a Lévy process such that each increment $X_t - X_s$ (with $0 < s < t$) is gamma distributed $\mathcal{G}(A(t) - A(s), b)$, where the gamma distribution $\mathcal{G}(a, b)$ (with $a > 0, b > 0$) admits the following pdf

$$f(x) = \frac{1}{\Gamma(a)} b^a x^{a-1} \exp\{-bx\}, \ \forall x \geq 0.$$

The positive assumption holds and, if $A(t) - A(s) \geq (\leq) 1$, the random variable $X_t - X_s$ is IHR (DHR). Considering $A(t) = t^\beta$ ($\beta > 0$) and $b = 1$, the ratio $H(z)$ defined in (7) is plotted with respect to z in Figure 1 for $x = 0.5$, $y = 0.5$ with $s = 1$, $t = 1.25$, $\beta = 0.75$ for cases (a) and (c), with $g(x) = x\mathbf{1}_{\{x<2\}} + (1.5x-1)\mathbf{1}_{\{x\geq 2\}}$ for case (a) and $g(x) = x\mathbf{1}_{\{x<2\}} + (0.5x+1)\mathbf{1}_{\{x\geq 2\}}$ for case (c). We next take $s = 0.25$, $t = 1.75$, $\beta = 2$ for cases (b) and (d) with $g(x) = x^2$ for case (b) and $g(x) = x^{0.5}$ for case (d). This leads to $A(t) - A(s) \simeq 0.18$ (DHR) for cases (a) and (c), and to $A(t) - A(s) = 3$ (IHR) for cases (b) and (d). In Figure 1, we observe that $H(z)$ decreases with z for case (b), whereas it increases for case (d). This is coherent with what could be expected from Proposition 4 whenever g is convex (case (b)) or concave

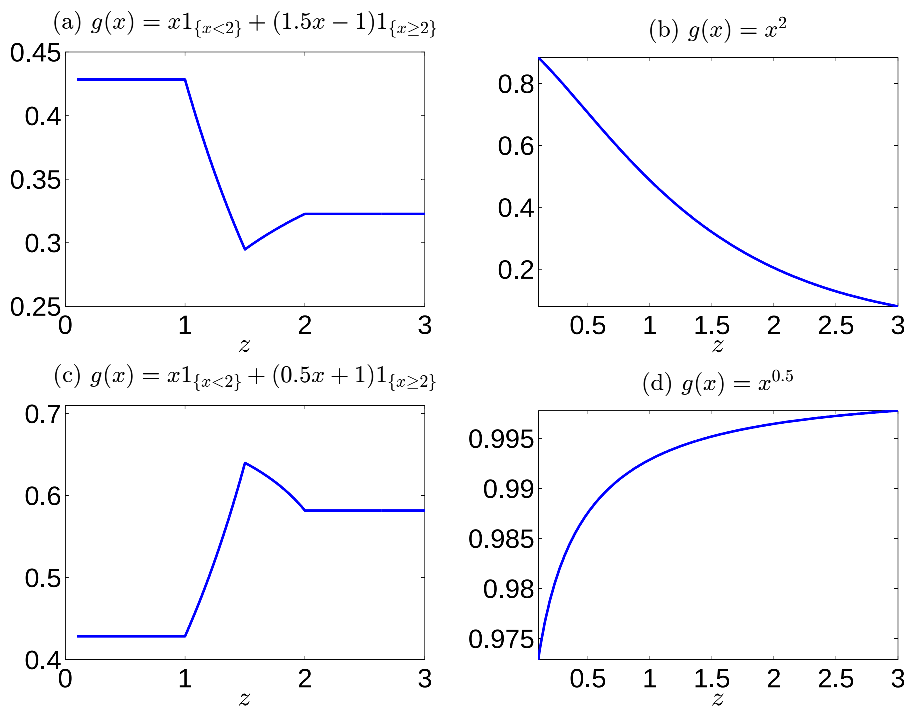

FIGURE 1: $H(z)$ with respect to z (when $X_t \sim \mathcal{G}(t^\beta, 1)$) for $x = 0.5$, $y = 0.5$ with $s = 1$, $t = 1.25$, $\beta = 0.75$ for cases (a) and (c), and $s = 0.25$, $t = 1.75$, $\beta = 2$ for cases (b) and (d) (and the function g is defined on each plot), Example 2.

(case (d)), under the IHR assumption for $X_t - X_s$. Now, when $X_t - X_s$ is DHR, it can be seen from cases (a) and (c) in Figure 1 that the ratio $H(z)$ is not monotonous with respect to z, neither for a convex function g (case (a)) nor for a concave function g (case (c)). Hence, it seems that nothing more can be said than the results of Proposition 4 in a general setting.

We next look at a similar example to the previous one, now considering an inverse Gaussian process instead of a gamma process. Note that the inverse Gaussian distribution is known not to have a monotonic hazard rate in a general setting (that is, it is neither IHR nor DHR, see [7]) so that the conclusions of Proposition 4 do not apply in this case.

Example 3. Let $A : \mathbb{R}_+ \to \mathbb{R}_+$ be a a time-scaling function and let $b > 0$. Let $(X_t)_{t\geq 0}$ be a non homogeneous inverse Gaussian process with mean function $A(t)$ and

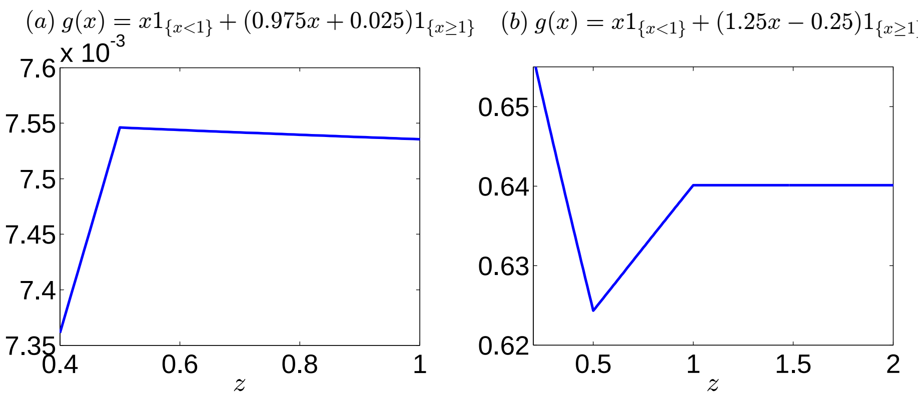

FIGURE 2: $H(z)$ for $x = 0.5$, $y = 0.5$ with respect to z (when $X_t \sim \mathcal{IG}\left(t^\beta, 1\right)$) with $s = 1$, $t = 1.25$, $\beta = 0.95$ for case (a), and with $s = 0.5$, $t = 1.5$, $\beta = 2$ for case (b) (and the function g is defined on each plot), Example 3.

rate parameter b (denoted by $(X_t)_{t>0} \sim \mathcal{IG}(A(\cdot), b)$). Then $(X_t)_{t\geq 0}$ is a Lévy process such that each increment $X_t - X_s$ (with $0 < s < t$) is inverse Gaussian distributed $\mathcal{IG}(A(t) - A(s), b)$, where the inverse Gaussian distribution $\mathcal{IG}(a, b)$ (with $a > 0, b > 0$) admits the following pdf

$$f(x) = \sqrt{\frac{b}{2\pi}} x^{-\frac{3}{2}} \exp\{-\frac{b(x-a)^2}{2a^2 x}\}, \ \forall x > 0$$

and the positive assumption holds. Considering $A(t) = t^\beta$ and $b = 1$, the ratio $H(z)$ defined in (7) is plotted with respect to z in Figure 2 for $x = 0.5$, $y = 0.5$ with $s = 1$, $t = 1.25$, $\beta = 0.95$ (which leads to $A(t) - A(s) \simeq 0.24$) and $g(x) = x\mathbf{1}_{\{x<1\}} + (0.975x + 0.025)\mathbf{1}_{\{x\geq 1\}}$ for case (a), and with $s = 0.5$, $t = 1.5$, $\beta = 2$ (which leads to $A(t) - A(s) = 2$) and $g(x) = x\mathbf{1}_{\{x<1\}} + (1.25x - 0.25)\mathbf{1}_{\{x\geq 1\}}$ for case (b). In Figure 2, we observe that $H(z)$ is not monotonic with respect to z neither when g is concave (case (a)) nor convex (case (b)). It is easy to check that in those two cases, $X_t - X_s$ is not IHR (nor DHR). The IHR assumption hence appears as necessary to derive the results in Proposition 4.

We next look at aging properties (IHR/DHR) of an increment of a TR Lévy process.

Proposition 5. *Let* $0 < s < t$*.*

1. *Assume that* $X_t - X_s$ *has Increasing Hazard Rate (IHR) and that* g *is convex.*

Then $[Z_t - Z_s | Z_s = z]$ *is IHR.*

2. *Assume that* $X_t - X_s$ *has Decreasing Hazard Rate (DHR) and that* g *is concave. Then* $[Z_t - Z_s | Z_s = z]$ *is DHR.*

Proof. We only look at the first point as the second one is similar. Let $y \geq 0$ and z be fixed. We have to show that

$$G(x) := \frac{\overline{F}_{Z_t - Z_s | Z_s = z}(x + y)}{\overline{F}_{Z_t - Z_s | Z_s = z}(x)} \tag{9}$$

decreases with respect to x. This can proved in a similar way as for the proof of Proposition 4 and it is omitted. □

Remark 3. Note that contrary to Proposition 4, the results in Proposition 5 do not require the positive assumption to hold. However, it is well-known that DHR distributions have a bounded support from below (see, e.g., [2]), so that the second point is useless in case of a baseline Lévy process with $\mathbb{R}$ as support (such as a Wiener process).

The first point of Proposition 5 is now illustrated considering a Wiener process as baseline Lévy process.

Example 4. Let $(X_t)_{t \geq 0}$ be a time-scaled Wiener process with drift as defined in Example 1, where $X_t \sim \mathcal{N}\left(t^\beta, \sigma\, t^\beta\right)$ with $\sigma = 1$ and $\beta = 0.75$. The ratio $G(x)$ defined in (9) is plotted with respect to x in Figure 3 for $y = 0.5$, $z = 0.5$, $s = 1$ and $t = 1.25$ with $g(x) = (x-1)^3 + 1$ for case (a), and $g(x) = x^{1.5}$ for case (b). Recall from [3] that any normal random variable is IHR so that $X_t - X_s$ is IHR. In case (b), we can see that $G(x)$ is decreasing with respect to x. This is coherent with Proposition 5 based on the fact that g is convex. In case (a), the function g is neither convex nor concave and nothing can be said from Proposition 5. It can be observed that indeed, G is not monotonic. This shows that the convexity/concavity assumption is required in Proposition 5 in order to derive the conditional IHR/DHR property of an increment.

Remark 4. One may also be interested in the unconditional aging property of $Z_t - Z_s$. Actually, $Z_t - Z_s$ can be regarded as the mixture of $[Z_t - Z_s | Z_s = z]$ with respect to the mixture distribution of Z_s. It is well known that the mixture of DHR r.v.s is DHR. Thus, from the second point of Proposition 5, we can conclude that $Z_t - Z_s$ is DHR

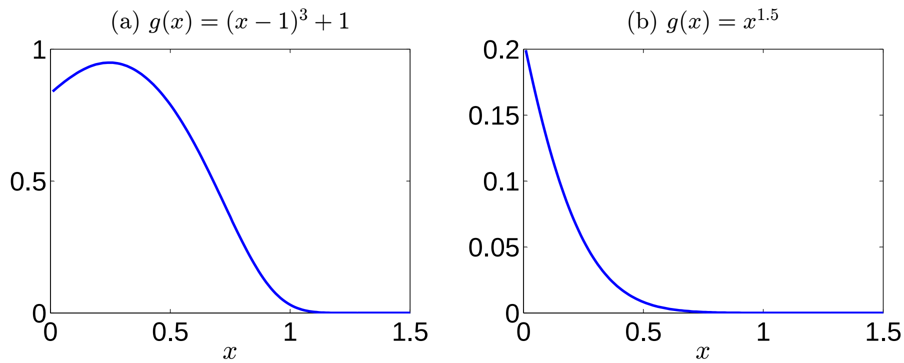

FIGURE 3: $G(x)$ with respect to x (when $X_t \sim \mathcal{N}\left(t^{0.75}, t^{0.75}\right)$) for $y = 0.5$, $z = 0.5$, $s = 1$ and $t = 1.25$ with $g(x) = (x-1)^3 + 1$ for case (a), and $g(x) = x^{1.5}$ for case (b), Example 3.

as soon as $X_t - X_s$ is DHR and g is concave. However, the mixture of IHR r.v.s is not necessarily IHR and nothing can be said on the IHR property of $Z_t - Z_s$ in general.

3.2. Influence of the current state on the future deterioration level

Proposition 6. *Let $0 < s < t$. If $X_t - X_s$ is IHR (DHR), then $[Z_t | Z_s = z]$ increases (decreases) in the hazard rate (hr) ordering as z increases, whatever g is.*

Proof. We only look at the IHR case as the DHR case is similar.

Let x and y be fixed ($y \geq 0$). The point is to show that

$$J(z) := \frac{\overline{F}_{Z_t|Z_s=z}(x+y)}{\overline{F}_{Z_t|Z_s=z}(x)} \tag{10}$$

increases with respect to z, which can be shown similarly as in the proof of Proposition 4. □

Proposition 7. *Let $0 < s < t$. If $X_t - X_s$ is Decreasing/Increasing Reverse Hazard Rate (DRHR/IRHR), then $[Z_t | Z_s = z]$ decreases/increases in the reverse hazard rate (rh) ordering as z increases, whatever g is.*

Proof. We only look at the DRHR case as the IRHR case is similar.

Let x and y be fixed ($y \geq 0$). The point is to show that

$$K(z) := \frac{F_{Z_t|Z_s=z}(x+y)}{F_{Z_t|Z_s=z}(x)} \tag{11}$$

decreases with respect to z. Let $z_1 \leq z_2$. We have

$$K(z_i) = \frac{F_{X_t-X_s}(g(x+y) - g(z_i))}{F_{X_t-X_s}(g(x) - g(z_i))} = \frac{F_{X_t-X_s}(u+v_i)}{\overline{F}_{X_t-X_s}(v_i)}$$

with $u = g(x+y) - g(x) \geq 0$ and $v_i = g(x) - g(z_i)$ for $i = 1, 2$.

As g is increasing, we have $v_1 \geq v_2$. Assuming $X_t - X_s$ to be DRHR, we get:

$$K(z_1) = \frac{F_{X_t - X_s}(u + v_1)}{F_{X_t - X_s}(v_1)} \leq \frac{F_{X_t - X_s}(u + v_2)}{F_{X_t - X_s}(v_2)} = K(z_2),$$

which achieves the proof. □

Proposition 8. *Let $0 < s < t$. If the pdf of $X_t - X_s$ is log-concave (log-convex), then $[Z_t | Z_s = z]$ increases (decreases) in the likelihood ratio (lr) ordering as z increases, whatever g is.*

Proof. We only look at the log-concave case as the log-convex case is similar. Let $z_1 \leq z_2$. We have to show that

$$L(x) := \frac{f_{Z_t|Z_s = z_2}(x)}{f_{Z_t|Z_s = z_1}(x)} = \frac{f_{X_t - X_s}(g(x) - g(z_2))}{f_{X_t - X_s}(g(x) - g(z_1))} \tag{12}$$

increases with respect to x. Based on the log-concavity of $f_{X_t - X_s}$, we know that

$$\frac{f_{X_t - X_s}(u + h)}{f_{X_t - X_s}(v + h)} \tag{13}$$

increases with respect to h whenever $u \leq v$. Considering $h = g(x) - g(z_2)$, $u = 0 \leq v = g(z_2) - g(z_1)$, we derive that $L(x)$ increases with respect to h and hence with respect to x, which completes the proof. □

4. Positive dependence properties

We now come to positive (negative) dependence properties and we first look at the dependence properties between the increments of a transformed Lévy process.

Proposition 9. *Under the positive assumption:*

1. *Assume g to be concave. Then, for all $t_0 = 0 < t_1 < \cdots < t_n$, the random vector $\left(Z_{t_1}, Z_{t_2} - Z_{t_1}, \cdots, Z_{t_n} - Z_{t_{n-1}}\right)$ is Conditionally Increasing in Sequence (CIS), namely:*

$$\begin{aligned}&\left[Z_{t_i} - Z_{t_{i-1}} | Z_{t_1} = z_1, Z_{t_2} - Z_{t_1} = z_2 \cdots, Z_{t_{i-1}} - Z_{t_{i-2}} = z_{i-1}\right] \\ &\prec_{st} \left[Z_{t_i} - Z_{t_{i-1}} | Z_{t_1} = z'_1, Z_{t_2} - Z_{t_1} = z'_2 \cdots, Z_{t_{i-1}} - Z_{t_{i-2}} = z'_{i-1}\right]\end{aligned} \tag{14}$$

for all $i \in \{2, \cdots, n\}$ and all $z_j \leq z'_j$, $j \in \{1, \cdots, i-1\}$, see, e.g., [9, Def 5.3.22.].

2. *Assume g to be convex. Then, for all $t_0 = 0 < t_1 < \cdots < t_n$, the random vector $\left(Z_{t_1}, Z_{t_2} - Z_{t_1}, \cdots, Z_{t_n} - Z_{t_{n-1}}\right)$ is Conditionally Decreasing in Sequence (CDS), namely the inequality in* (14) *is reversed.*

Proof. We only look at the concave case, as the convex case is similar. The point is to show (14).

Based on the Markov property, we know that

$$\begin{aligned}&\left[Z_{t_i} - Z_{t_{i-1}} | Z_{t_1} = z_1, Z_{t_2} - Z_{t_1} = z_2 \cdots, Z_{t_{i-1}} - Z_{t_{i-2}} = z_{i-1}\right]\\ &\overset{\mathcal{D}}{=} \left[Z_{t_i} - Z_{t_{i-1}} | Z_{t_{i-1}} = z_{1:i-1}\right]\end{aligned}$$

with

$$z_{1:i-1} = \sum_{j=1}^{i-1} z_j.$$

As $z_j \leq z'_j$, $j \in \{1, \cdots, i-1\}$, we also have $z_{1:i-1} \leq z'_{1:i-1}$ (similar notation).

Due to the second point of Lemma 1, we derive that

$$\left[Z_{t_i} - Z_{t_{i-1}} | Z_{t_{i-1}} = z_{1:i-1}\right] \prec_{st} \left[Z_{t_i} - Z_{t_{i-1}} | Z_{t_{i-1}} = z'_{1:i-1}\right].$$

Then $\left(Z_{t_1}, Z_{t_2} - Z_{t_1}, \cdots, Z_{t_n} - Z_{t_{n-1}}\right)$ is CIS. □

Remark 5. As CIS is closed under marginalization (consequence of [20, Thm 3.10.19.]), we can derive that when g is concave, the random vector $\left(Z_{t_1}, Z_{t_3} - Z_{t_2}, \cdots, Z_{t_{2n+1}} - Z_{t_{2n}}\right)$ is CIS. This shows that when g is concave, the CIS property is still true for non overlapping but non necessarily consecutive intervals. However, up to our knowledge, CDS is not known to be closed under marginalization. The question hence remains open whether CDS property is true for increments of a TR Lévy process over non overlapping intervals.

Remark 6. Let us recall that a random vector $\mathbf{V} = (V_1, V_2, \cdots, V_n)$ is said to be (positively) associated if

$$\mathbb{E}[h(\mathbf{V})w(\mathbf{V})] \geq \mathbb{E}[h(\mathbf{V})]\,\mathbb{E}[w(\mathbf{V})],$$

for all non decreasing functions h and w such that $\mathbb{E}[h(\mathbf{V})]$, $\mathbb{E}[w(\mathbf{V})]$ and $\mathbb{E}[h(\mathbf{V})w(\mathbf{V})]$ exist, see, e.g., [20]. Furthermore, a random vector $\mathbf{V} = (V_1, V_2, \cdots, V_n)$ is said to be

Positive Upper (Lower) Orthant Dependent (PUOD/PLOD) if

$$\mathbb{P}\left[V_i > (\leq) v_i,\ i = 1, 2, \cdots, n\right] \geq \prod_{i=1}^{n} \mathbb{P}\left[V_i > (\leq) v_i\right], \tag{15}$$

for all $v_i,\ i = 1, 2, \cdots, n$. If the inequality "$\geq$" in (15) is reversed, the random vector $\mathbf{V} = (V_1, V_2, \cdots, V_n)$ is said to be Negative Upper (Lower) Orthant Dependent (NUOD/NLOD). We recall that CIS implies association, which itself implies PUOD/PLOD properties, see, e.g., [9, Property 7.2.11]. The previous result hence shows that when g is concave, the random vector $(Z_{t_1}, Z_{t_2} - Z_{t_1}, \cdots, Z_{t_n} - Z_{t_{n-1}})$ is associated, and consequently both PUOD and PLOD. As for negative dependence properties, it is known from [21] that CDS implies Negative Lower Orthant Dependence (NLOD) property. Then, if g is convex, the random vector $\left(Z_{t_1}, Z_{t_2} - Z_{t_1}, \cdots, Z_{t_n} - Z_{t_{n-1}}\right)$ is NLOD. However, up to our knowledge, it seems that nothing else can be derived from the CDS property.

We now come to positive (negative) dependence properties for successive overall deterioration levels in a TR Lévy process.

Proposition 10. *For all $0 < t_1 < \cdots < t_n$, the random vector $(Z_{t_1}, Z_{t_2}, \cdots, Z_{t_n})$ is Conditionally Increasing in Sequence (CIS), and hence associated and PUOD/PLOD, whatever g is.*

Proof. The point is to show that

$$\begin{aligned} &\left[Z_{t_i} | Z_{t_1} = z_1, Z_{t_2} = z_2 \cdots, Z_{t_{i-1}} = z_{i-1}\right] \\ &\prec_{st} \left[Z_{t_i} | Z_{t_1} = z'_1, Z_{t_2} = z'_2 \cdots, Z_{t_{i-1}} = z'_{i-1}\right] \end{aligned}$$

for all $i \in \{2, \cdots, n\}$ and all $z_j \leq z'_j,\ j \in \{1, \cdots, i-1\}$, where we set $t_0 = 0$ and $Z_{t_0} = 0$.

Based on the Markov property, we know that:

$$\left[Z_{t_i} | Z_{t_1} = z_1, Z_{t_2} = z_2 \cdots, Z_{t_{i-1}} = z_{i-1}\right] \stackrel{\mathcal{D}}{=} \left[Z_{t_i} | Z_{t_{i-1}} = z_{i-1}\right]$$

which stochastically increases with respect to z_{i-1}, based on Lemma 1. This achieves the proof. $\square$

Before going to the last positive (negative) dependence result, let us recall that a function $f : \mathbb{R}^n \longmapsto \mathbb{R}_+$ is said to be Multivariate Totally Positive of order 2 (MTP2) as soon as

$$f(\mathbf{x})f(\mathbf{y}) \leq f(\mathbf{x} \vee \mathbf{y})f(\mathbf{x} \wedge \mathbf{y}), \forall \mathbf{x},\, \mathbf{y} \in \mathbb{R}^n$$

where $\vee$ and $\wedge$ are the max and min component-wise operations, respectively. The function f is said to be Multivariate Reverse Rule of order 2 (MRR2) when the previous inequality is reversed, see [16, 17] for more details on these notions.

Proposition 11. *Let $t_0 = 0 < t_1 < \cdots < t_n$. If the pdf of $X_{t_i} - X_{t_{i-1}}$ is log-concave (log-convex) for each $i \in \{1, \cdots, n\}$, then $(Z_{t_1}, Z_{t_2}, \cdots, Z_{t_n})$ is MTP2 (MRR2), whatever g is.*

Proof. We only look at the log-concave case, as the log-convex case is similar.

We have:

$$f_{\left(Z_{t_1}, Z_{t_2}, \cdots, Z_{t_n}\right)}\left(z_1, \cdots, z_n\right) = f_{Z_{t_1}}\left(z_1\right) \prod_{i=2}^{n-1} f_{Z_{t_i}|Z_{t_{i-1}}=z_{i-1}}\left(z_i\right).$$

Based on Proposition 8, we know that $\left[Z_{t_i}|Z_{t_{i-1}} = x_{i-1}\right] \prec_{lr} \left[Z_{t_i}|Z_{t_{i-1}} = z_{i-1}\right]$ for all $x_{i-1} \leq z_{i-1}$ so that

$$f_{Z_{t_i}|Z_{t_{i-1}}=z_{i-1}}\left(x_i\right) f_{Z_{t_i}|Z_{t_{i-1}}=x_{i-1}}\left(z_i\right) \leq f_{Z_{t_i}|Z_{t_{i-1}}=z_{i-1}}\left(z_i\right) f_{Z_{t_i}|Z_{t_{i-1}}=x_{i-1}}\left(x_i\right)$$

for all $x_{i-1} \leq z_{i-1}$ and $x_i \leq z_i$. This shows that $f_{Z_{t_i}|Z_{t_{i-1}}=z_{i-1}}\left(z_i\right)$ is TP2 in (z_{i-1}, z_i) and hence MTP2 as a function of $(z_1, \cdots, z_n)$. Also $f_{Z_{t_1}}\left(z_1\right)$ is MTP2 as it is a univariate function of $(z_1, \cdots, z_n)$. As a product of MTP2 functions is MTP2, $f_{\left(Z_{t_1}, Z_{t_2}, \cdots, Z_{t_n}\right)}\left(z_1, \cdots, z_n\right)$ hence is MTP2. $\square$

Under log-concavity assumption on the increments of the baseline Lévy process $(X_t)_{t\geq 0}$, the successive deterioration levels of the TR Lévy process $(Z_t)_{t\geq 0}$ hence fulfills the MTP2 property, so that they are strongly positively dependent. (We recall that, among others, the MTP2 property implies CIS). However, when the increments of the baseline Lévy process are log-convex, the successive deterioration levels of the TR Lévy process fulfill the MRR2 property, which is a negative dependence property. This was not necessarily expected, as from Proposition 10, this vector also exhibits CIS, which is a positive dependence property. These results are illustrated in the following example.

Example 5. Let $g(x) = x^{1.5}$ and $X_t \sim \mathcal{G}(t, 1)$. Then $X_t - X_s \sim \mathcal{G}(t-s, 1)$ for $0 < s < t$ and if $t - s < (>)1$, the pdf of $X_t - X_s$ is log-convex (log-concave), see, e.g., [5]. Two cases are considered: $t_1 = 0.25 < t_2 = 1.2, x_2 = 1.5 > y_2 = 1 > x_1 = 0.25$ and $t_1 = 1.35 < t_2 = 2.75, x_2 = 3 > y_2 = 2 > x_1 = 1$, which leads to log-convex and log-concave pdf for both X_{t_1} and $X_{t_2} - X_{t_1}$ in the first and second cases, respectively. The function

$$d_{(x_1,x_2,y_2)}(y_1) = f_{(Z_{t_1},Z_{t_2})}(x_1,x_2) f_{(Z_{t_1},Z_{t_2})}(y_1,y_2) - f_{(Z_{t_1},Z_{t_2})}(x_1,y_2) f_{(Z_{t_1},Z_{t_2})}(y_1,x_2)$$

is plotted for $y_1 \in [x_1, y_2]$ in Figures 4 (a) and (c), whereas $\bar{F}_{Z_{t_2}|Z_{t_1}=y_1}(y_2)$ is plotted as a function of y_1 in Figures 4 (b) and (d) for $y_2 = 1$ and $y_2 = 2$, respectively. Figures 4 (a) and (b) correspond to the first case, and (Z_{t_1}, Z_{t_2}) is observed to be both MRR2 and CIS. Figures 4 (c) and (d) correspond to the second case, and (Z_{t_1}, Z_{t_2}) is observed to be both MTP2 and CIS. All figures hence are in coherence with what was expected from the previous results.

5. Stochastic monotonicity of increments

Proposition 12. *Assume the positive assumption to hold and let $h > 0$ be fixed. Assume further that g is convex (concave) and that $X_{t+h} - X_t$ decreases (increases) in the sense of the usual stochastic order with respect to t. Then $Z_{t+h} - Z_t$ decreases (increases) in the sense of the usual stochastic order with respect to t (with $h > 0$ fixed).*

Proof. We only consider convex case, as the concave one is similar.

Let $h > 0$ and $x \geq 0$ be fixed. For $t \geq 0$, we have

$$\mathbb{P}(Z_{t+h} - Z_t > x) = \mathbb{E}(\varphi_t(Z_t))$$

with

$$\begin{aligned}\varphi_t(z) &= \mathbb{P}(Z_{t+h} - Z_t > x | Z_t = z) \\ &= \overline{F}_{X_{t+h}-X_t}(g(z+x) - g(z)).\end{aligned}$$

Now let $t_1 < t_2$. Let us note that, under the positive assumption, then $X_{t_1} \leq X_{t_2}$ and hence $X_{t_1} \prec_{st} X_{t_2}$. Also $X_{t_1+h} - X_{t_1} \succ_{st} X_{t_2+h} - X_{t_2}$ by assumption. Then

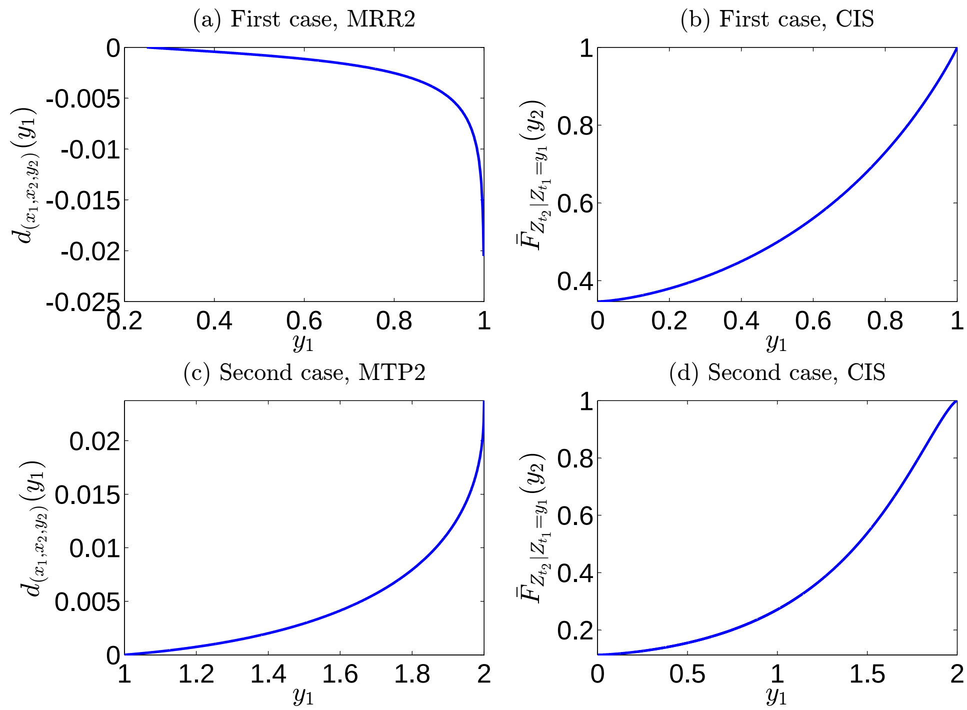

FIGURE 4: $d_{(x_1,x_2,y_2)}(y_1)$ and $\bar{F}_{Z_{t_2}|Z_{t_1}=y_1}(y_2)$ as a function of y_1 with (x_1, x_2, y_2) fixed, Example 5

$\varphi_{t_1} \geq \varphi_{t_2}$ and

$$\mathbb{P}\left(Z_{t_1+h} - Z_{t_1} > x\right) = \mathbb{E}\left(\varphi_{t_1}\left(Z_{t_1}\right)\right) \geq \mathbb{E}\left(\varphi_{t_2}\left(Z_{t_1}\right)\right).$$

As $\varphi_{t_2}(z)$ decreases with respect to z (because g is convex and $x \geq 0$) and as $Z_{t_1} = g^{-1}(X_{t_1}) \prec_{st} Z_{t_2} = g^{-1}(X_{t_2})$ (because $X_{t_1} \prec_{st} X_{t_2}$ and g^{-1} increases), we derive that $\mathbb{E}\left(\varphi_{t_2}\left(Z_{t_1}\right)\right) \geq \mathbb{E}\left(\varphi_{t_2}\left(Z_{t_2}\right)\right)$ and consequently

$$\mathbb{E}\left(\varphi_{t_1}\left(Z_{t_1}\right)\right) \geq \mathbb{E}\left(\varphi_{t_2}\left(Z_{t_2}\right)\right),$$

which achieves the proof.

□

Example 6. Let $(X_t)_{t>0} \sim \mathcal{IG}(A(\cdot), 1)$. Considering the expression of the failure rate of an IG distribution given in [7, top of page 463], it is easy to check that $\mathcal{IG}(a, b)$

increases in the hazard rate ordering when a increases with b fixed. This entails that, if $A(\cdot)$ is convex (concave), then $X_{t+h} - X_t$ increases (decreases) in the hazard rate ordering and hence in the usual stochastic ordering [20]. We take $A(t) = t^{\beta}$, $g(x) = t^{\gamma}$, $h = 0.5$, $x = 0.5$ and the survival function $\overline{F}_{Z_{t+h}-Z_t}(x)$ is plotted with respect to t in Figure 5 for $\beta = 0.5$ and $\gamma = 2$ in case (a), $\beta = \gamma = 2$ in case (b), $\beta = 0.65$ and $\gamma = 0.5$ in case (c) and $\beta = 1.25$ and $\gamma = 0.5$ in case (d). Case (a) (case (d)) corresponds to the case where $A(\cdot)$ is concave (convex) and $g(\cdot)$ is convex (concave). The results are coherent with what was expected from Proposition 12. Cases (b) and (c) correspond to the cases where $A(\cdot)$ and $g(\cdot)$ are both convex and both concave, respectively. As can be seen on the two plots, $X_{t+h} - X_t$ is not stochastically monotonous with respect to t and hence, it seems that nothing more can be said than the results of Proposition 12 in a general setting.

6. Stochastic comparison of two transformed Lévy processes

We here consider stochastic comparison of two transformed Lévy processes keeping the same baseline process or the same state function for both processes, which allows a better understanding of the influence of each item (baseline process/state function) on the behavior of the resulting TR Lévy process.

6.1. Common baseline process with different state functions

Proposition 13. *Consider two processes $(Z_{1t})_{t\geq 0}$ and $(Z_{2t})_{t\geq 0}$ having a common baseline process $(X_t)_{t\geq 0}$ and corresponding state functions $g_1(x)$ and $g_2(x)$, respectively. Assume the positive assumption to hold for both processes.*

1. *If one among $g_i(x)$, $i = 1, 2$ is concave and $g_1(z+x) - g_1(z) \leq g_2(z+x) - g_2(z)$, for all $z \geq 0$, $x > 0$, then $Z_{1t} - Z_{1s} \succ_{st} Z_{2t} - Z_{2s}$, for any $0 \leq s < t$.*

2. *If one among $g_i(x)$, $i = 1, 2$, is convex and $g_1(z+x) - g_1(z) \geq g_2(z+x) - g_2(z)$, for all $z \geq 0$, $x > 0$, then $Z_{1t} - Z_{1s} \prec_{st} Z_{2t} - Z_{2s}$, for any $0 \leq s < t$.*

Proof. We only deal with the first point, as the second one is similar.

As a first step, assume g_1 to be concave. Then, for any fixed $x \geq 0$,

$$\overline{F}_{Z_{1t}-Z_{1s}|Z_{1s}=z}(x) = \overline{F}_{X_t-X_s}(g_1(z+x) - g_1(z))$$

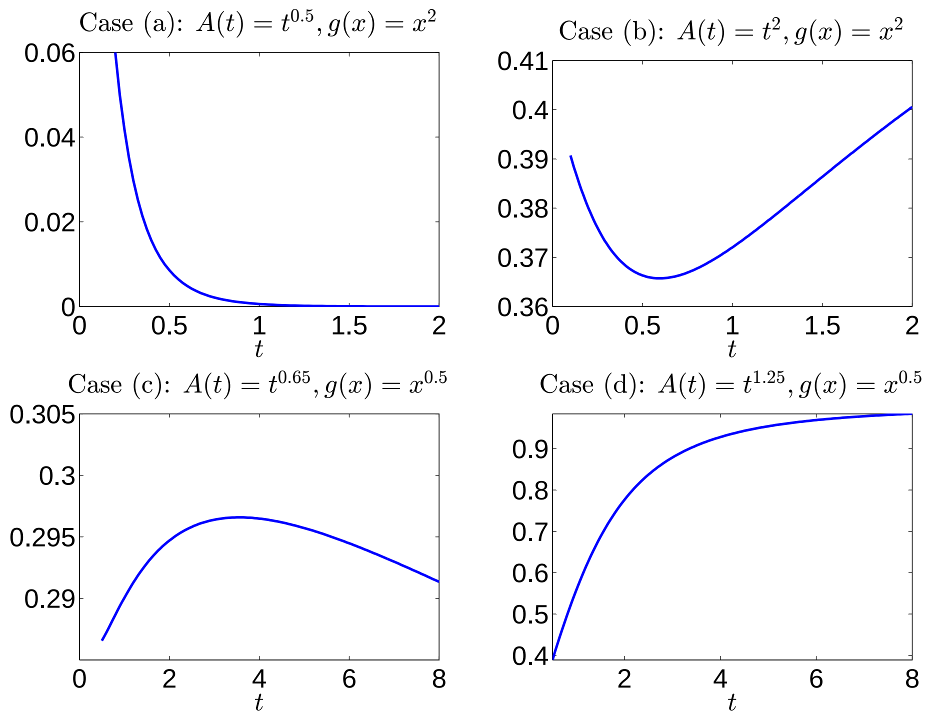

FIGURE 5: $\overline{F}_{Z_{t+0.5}-Z_t}(0.5)$ with respect to t (when $X_t \sim \mathcal{IG}\left(t^\beta, 1\right)$) for $g(x) = x^\gamma$ with $\beta = 0.5$ and $\gamma = 2$ (case (a)), $\beta = \gamma = 2$ (case (b)), $\beta = 0.65$ and $\gamma = 0.5$ (case (c)) and $\beta = 1.25$ and $\gamma = 0.5$ (case (d)), Example 6

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is increasing in z. Furthermore, under the positive assumption, $g_1(0) = g_2(0) = 0$, which implies $\overline{F}_{Z_{1s}}(x) = \overline{F}_{X_s}(g_1(x)) \geq \overline{F}_{Z_{2s}}(x) = \overline{F}_{X_s}(g_2(x))$, $i = 1, 2$, and $Z_{1s} \succ_{st} Z_{2s}$. Therefore,

$$\begin{aligned}\overline{F}_{Z_{1t}-Z_{1s}}(x) &= \mathbb{E}[\overline{F}_{X_t-X_s}(g_1(Z_{1s}+x) - g_1(Z_{1s}))] \\ &\geq \mathbb{E}[\overline{F}_{X_t-X_s}(g_1(Z_{2s}+x) - g_1(Z_{2s}))].\end{aligned}$$

Now, as $g_1(z+x) - g_1(z) \leq g_2(z+x) - g_2(z)$ by assumption, we get:

$$\begin{aligned}\overline{F}_{Z_{1t}-Z_{1s}}(x) &\geq \mathbb{E}[\overline{F}_{X_t-X_s}(g_2(Z_{2s}+x) - g_2(Z_{2s}))] \\ &= \overline{F}_{Z_{2t}-Z_{2s}}(x).\end{aligned}$$

As a second step, assume g_2 to be concave. Using similar arguments, we have:

$$\begin{aligned}\overline{F}_{Z_{1t}-Z_{1s}}(x) &= \mathbb{E}[\overline{F}_{X_t-X_s}(g_1(Z_{1s}+x)-g_1(Z_{1s}))] \\ &\geq \mathbb{E}[\overline{F}_{X_t-X_s}(g_2(Z_{1s}+x)-g_2(Z_{1s}))] \\ &\geq \mathbb{E}[\overline{F}_{X_t-X_s}(g_2(Z_{2s}+x)-g_2(Z_{2s}))] \\ &= \overline{F}_{Z_{2t}-Z_{2s}}(x).\end{aligned}$$

□

Proposition 14. *Consider two processes $(Z_{1t})_{t\geq 0}$ and $(Z_{2t})_{t\geq 0}$ having a common baseline process $(X_t)_{t\geq 0}$ and corresponding state functions $g_1(x)$ and $g_2(x)$, respectively. Assume the positive assumption to hold for both processes, X_t to be Increasing Hazard Rate (IHR) and $g_2 - g_1$ to be non decreasing. Then $Z_{1t} \succ_{hr} Z_{2t}$, for any $t > 0$.*

Proof. Let λ_{X_t} stand for the hazard rate of X_t. As $g_1(0) = g_2(0)$ under the positive assumption, observe that

$$\begin{aligned} r\,(x) &= \frac{\overline{F}_{Z_{1t}}(x)}{\overline{F}_{Z_{2t}}(x)} = \frac{\overline{F}_{X_t}(g_1(x))}{\overline{F}_{X_t}(g_2(x))} = \frac{\exp\left\{-\int_0^{g_1(x)}\lambda_{X_t}(u)du\right\}}{\exp\left\{-\int_0^{g_2(x)}\lambda_{X_t}(u)du\right\}} \\ &= \exp\left\{\int_0^{g_2(x)-g_1(x)}\lambda_{X_t}(v+g_1\,(x))dv\right\}.\end{aligned}$$

As $g_2 - g_1$ is non decreasing with $g_1(0) = g_2(0) = 0$, then $g_2(x) - g_1\,(x) \geq 0$, $\forall x \geq 0$. As g_1 and λ_{X_t} also are non decreasing (because X_t is IFR), we derive that $r\,(x)$ is non decreasing, which achieves the proof. □

6.2. Common state function with different baseline processes

Proposition 15. *Consider two processes $(Z_{1t})_{t\geq 0}$ and $(Z_{2t})_{t\geq 0}$ having a common state function $g(x)$ and corresponding baseline processes $(X_{1t})_{t\geq 0}$ and $(X_{2t})_{t\geq 0}$, respectively. Assume the positive assumption to hold for both processes and the common state function g to be concave. Let $0 \leq s < t$. Then, if $X_{1t} - X_{1s} \prec_{st} X_{2t} - X_{2s}$ and $X_{1s} \prec_{st} X_{2s}$, we have $Z_{1t} - Z_{1s} \prec_{st} Z_{2t} - Z_{2s}$.*

Proof. Observe that, as $X_{1t} - X_{1s} \prec_{st} X_{2t} - X_{2s}$:

$$\begin{aligned}\overline{F}_{Z_{1t}-Z_{1s}|Z_{1s}=z}(x) &= \overline{F}_{X_{1t}-X_{1s}}(x)(g(z+x)-g(z)) \\ &\leq \overline{F}_{X_{2t}-X_{2s}}(x)(g(z+x)-g(z)) \\ &= \overline{F}_{Z_{2t}-Z_{2s}|Z_{2s}=z}(x),\end{aligned}$$

for all $x, z \geq 0$. Furthermore, as $X_{1s} \prec_{st} X_{2s}$ and g^{-1} increases, $Z_{1s} = g^{-1}(X_1(s)) \prec_{st} Z_{2s} = g^{-1}(X_2(s))$. Also, as g is concave, $\overline{F}_{Z_{1t}-Z_{1s}|Z_{1s}=z}(x)$ is increasing in z for all $x \geq 0$. Thus

$$\begin{aligned}\overline{F}_{Z_{1t}-Z_{1s}}(x) &= \int_0^\infty \overline{F}_{Z_{1t}-Z_{1s}|Z_{1s}=z}(x) f_{Z_{1s}}(z)dz \\ &\leq \int_0^\infty \overline{F}_{Z_{1t}-Z_{1s}|Z_{1s}=z}(x) f_{Z_{2s}}(z)dz \\ &\leq \int_0^\infty \overline{F}_{Z_{2t}-Z_{2s}|Z_{2s}=z}(x) f_{Z_{2s}}(z)dz \\ &= \overline{F}_{Z_{2t}-Z_{2s}}(x).\end{aligned}$$

□

Note that similar results would not be valid for a convex function g. This is illustrated in Example 7.

Example 7. Let $X_{it} \sim \mathcal{G}\left((t+1)^{\beta_i} - 1, 1\right), i = 1, 2$ with $\beta_1 = 1, \beta_2 = 2$. Then, it can be checked that $X_{1s} \prec_{st} X_{2s}$ and $X_{1t} - X_{1s} \prec_{st} X_{2t} - X_{2s}$ for all $0 \leq s \leq t$. The survival functions of $Z_{it} - Z_{is}$, $i = 1, 2$ are plotted with respect to x in Figure 6 for $s = 4.5, t = 4.53, g(x) = x^2$ in case (a) (left) and for $s = 3, t = 3.03, g(x) = x^{0.8}$ in case (b) (right). It can be seen that, as expected from Proposition 15, when g is concave (case (b)), we have $Z_{1t} - Z_{1s} \prec_{st} Z_{2t} - Z_{2s}$. However, in the convex case (case (a)), $Z_{1t} - Z_{1s}$ and $Z_{2t} - Z_{2s}$ are not comparable with respect to the usual stochastic order.

7. Conclusion and perspectives

A new class of state-dependent wear models has been proposed in this paper, which includes the transformed gamma process proposed in [12] and the classical geometric Brownian motion. Transformed Lévy process allows to overcome the independent

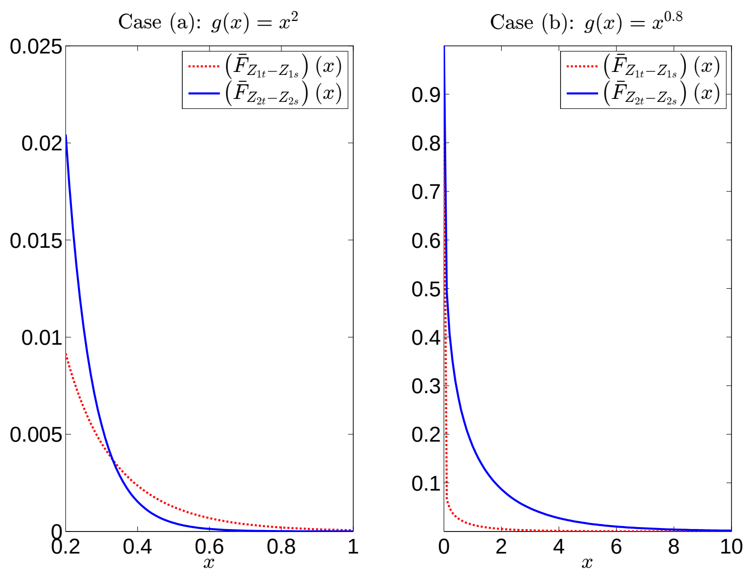

FIGURE 6: Survival functions of $Z_{it} - Z_{is}$, $i = 1, 2$ with respect to x, Example 7

increments property of standard Lévy processes, and hence enlarge their modeling ability. They however remain tractable Markov processes. Several results provide some insight in the influence of the current state of a TR Lévy process on its future, which typically differs according to the (log-)concavity/convexity property of the state function. Some positive (negative) dependence properties have also been highlighted for the increments of deterioration and for the overall deterioration levels of a TR Lévy process, which here again are highly dependent on the (log-)concavity/convexity property of the state function. In case of a (log-)concave state function, we observe strong positive dependence properties (such as MTP2). This seems to be in coherence with wear phenomena where the rate of deterioration increases over time. When the state function exhibits a (log-)convex property, we observed that some positive and negative dependence properties can hold on the same process (see the end of Section 5). Also, there remain open questions about negative dependence properties (see, e.g., Remarks 5 and 6). This is coherent with the previous literature where it has already been observed that negative dependence properties are much more involved than positive dependence ones, see, e.g., [4], or [17], where the authors exhibit a 3-

dimensional MRR2 vector with a 2-dimensional MTP2 vector as margin.

Clearly, there remains many work to do on the new Transformed Lévy process. For instance, even if a first study can be found in [12] in a specific parametric setting, generic estimation procedures still require to be developed.

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