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Abstract

We revisit an interesting example proposed by Maria Hoffmann-Ostenhof, the second author and Nikolai Nadirashvili of a bounded domain in $\mathbb{R}^2$ for which the second eigenvalue of the Dirichlet Laplacian has multiplicity 3. We also analyze carefully the first eigenvalues of the Laplacian in the case of the disk with two symmetric cracks placed on a smaller concentric disk in function of their size.

1 Introduction

The motivating problem is to analyze the multiplicity of the k-th eigenvalue of the Dirichlet problem in a domain Ω in $\mathbb{R}^2$. It is for example an old result of Cheng [3], that the multiplicity of the second eigenvalue is at most 3. In [13] an example with multiplicity 3 is given as a side product of the production of a counter example to the nodal line conjecture (see also [12], and the papers by Fournais [7] and Kennedy [16] who extend to higher dimensions these counter examples, introducing new methods). This example is based on the spectral analysis of the Laplacian in domains consisting of a disc in which we have introduced on an interior concentric circle suitable cracks.
We discuss the initial proof and complete it by one missing argument. For completion, we will also extend the validity of a theorem of Cheng to less regular domains. Although not needed for the positive results, we complete the paper

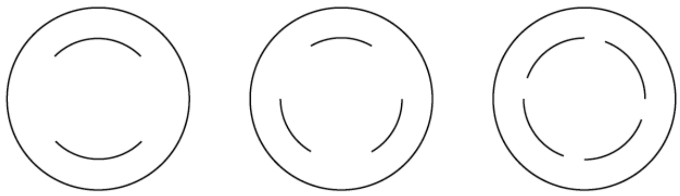

Figure 1: The domains with cracks for $N = 2$, $N = 3$ and $N = 4$.

with numerical results illustrating why some argument has to be modified and propose a fine theoretical analysis of the spectral problem when the cracks are closed.

2 Main statement

The starting point for the construction of counterexamples to the nodal line conjecture [12, 13] is the introduction of two concentric open discs B_{R_1} and B_{R_2} with $0 < R_1 < R_2$ and the corresponding annulus $M_{R_1,R_2} = B_{R_2} \setminus \bar{B}_{R_1}$. The authors choose R_1 and R_2 such that

$$\lambda_1(B_{R_1}) < \lambda_1(M_{R_1,R_2}) < \lambda_2(B_{R_1})\,, \tag{2.1}$$

where, for $\omega \subset \mathbb{R}^2$ bounded, $\lambda_j(\omega)$ denotes the j-th eigenvalue of the Dirichlet Laplacian H in ω.
We observe indeed that for fixed R_1, $\lambda_1(M_{R_1,R_2})$ tends to $+\infty$ as $R_2 \to R_1$ (from above) and tends to 0 as $R_2 \to +\infty$. Moreover $R_2 \mapsto \lambda_1(M_{R_1,R_2})$ is decreasing. Hence there is some interval $(a(R_1), b(R_1))$ with $a(R_1) > R_1$ such that (2.1) is satisfied if and only if $R_2 \in (a(R_1), b(R_1))$.
Then we introduce

$$D_{R_1,R_2} = B_{R_1} \cup M_{R_1,R_2}$$

and observe that

$$\begin{aligned}\lambda_1(D_{R_1,R_2}) &= \lambda_1(B_{R_1})\\ \lambda_2(D_{R_1,R_2}) &= \lambda_1(M_{R_1,R_2})\\ \lambda_3(D_{R_1,R_2}) &= \min(\lambda_2(B_{R_1}),\lambda_2(M_{R_1,R_2}))\,.\end{aligned} \tag{2.2}$$

If Condition (2.1) was important in the construction of the counter-example to the nodal line conjecture, the weaker assumption

$$\max(\lambda_1(B_{R_1}),\lambda_1(M_{R_1,R_2})) < \min(\lambda_2(B_{R_1}),\lambda_2(M_{R_1,R_2}))\,. \tag{2.3}$$

suffices for the multiplicity question. Under this condition, we have:

$$\begin{aligned}\lambda_1(D_{R_1,R_2}) &= \min(\lambda_1(B_{R_1}),\lambda_1(M_{R_1,R_2}))\\ \lambda_2(D_{R_1,R_2}) &= \max(\lambda_1(B_{R_1}),\lambda_1(M_{R_1,R_2}))\\ \lambda_3(D_{R_1,R_2}) &= \min(\lambda_2(B_{R_1}),\lambda_2(M_{R_1,R_2}))\,,\end{aligned} \tag{2.4}$$

and it is not excluded (we are in the non connected situation) to consider the case $\lambda_1(D_{R_1,R_2}) = \lambda_2(D_{R_1,R_2})$.

We now carve holes in ∂B_{R_1} such that D_{R_1,R_2} becomes a domain. For $N \in \mathbb{N}^* := \mathbb{N}\setminus\{0\}$ and $\epsilon \in [0,\frac{\pi}{N}]$, we introduce (see Figure 1 for $N = 2,3$)

$$\mathfrak{D}(N,\epsilon) = D_{R_1,R_2} \cup_{j=0}^{N-1} \{x\in\mathbb{R}^2\,,\ r = R_1\,,\ \theta \in (\frac{2\pi j}{N}-\epsilon,\frac{2\pi j}{N}+\epsilon)\}\,. \tag{2.5}$$

The theorem stated in [13] is the following:

Theorem 2.1. *Let $N \geq 3$, then there exists $\epsilon \in (0,\frac{\pi}{N})$ such that $\lambda_2(\mathfrak{D}(N,\epsilon))$ has multiplicity 3.*

We prove below that the theorem is correct. But the proof given in [13] works only for even integers $N \geq 4$ and in this case there is a need for additional arguments. So we improve in this paper the result in [13] by giving an example $\Omega := \mathfrak{D}(3,\epsilon)$ where the number of components of $\partial\Omega$ equals 4, hence $N = 3$.

Remark 2.2. *Theorem 2.1 leads to the following question:*
Is there a bounded domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ has strictly less than 4 components so that $\lambda_2(\Omega)$ has multiplicity 3? This is also a motivation for analyzing the cases $N = 1,2$. The natural conjecture (see Remark 4.3 for further discussion) would be that for simply connected domains Ω, $\lambda_2(\Omega)$ has at most multiplicity 2.

Remark 2.3. *For a specific choice of the pair (R_1,R_2) which will be introduced in Subsection 8.1, the numerics (see Figure 2) illustrates the statement of Theorem 2.1 when $N = 3$ and $N = 4$. Although the precision is not very good for ϵ close to 0 and $\frac{\pi}{3}$ (see Section 8), we can predict as $N = 3$ a second eigenvalue of multiplicity 3 for $\epsilon \sim 0.29$. A second crossing appears for $\epsilon \sim 0.96$ but corresponds to a third eigenvalue of multiplicity 3. The eigenvalues correspond (with the notation of Section 3) to $\ell = 0$ and to $\ell = 1$, the eigenvalues for $\ell = 1$ having multiplicity 2. When $N = 4$, we also see a first crossing for $\epsilon \sim 0.54$ where the multiplicity becomes 3, as the theory will show. The eigenvalues correspond (with the notation of Section 3) to $\ell = 0$, $\ell = 1$ and $\ell = 1$, the eigenvalues for $\ell = 1$ having multiplicity 2.The eigenvalues for $\ell = 0$ and 2 are simple for $\epsilon \in (0,\frac{\pi}{2})$.*

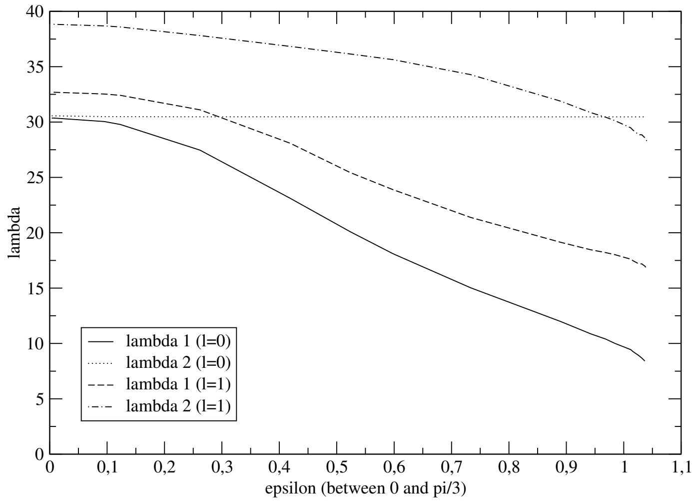

Figure 2: N=3. Six lowest eigenvalues of the Laplacian in $\mathfrak{D}(N,\epsilon))$ in function of $\epsilon \in (0, \frac{\pi}{3})$, with $R_1 = 0.4356$, $R_2 = 1$.

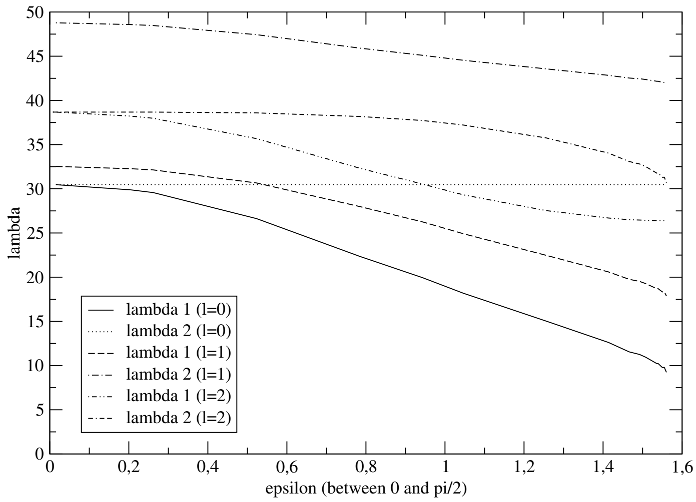

Figure 3: N=4. Eight lowest eigenvalues of the Laplacian in $\mathfrak{D}(N,\epsilon))$ in function of $\epsilon \in (0, \frac{\pi}{2})$, with $R_1 = 0.4356$, $R_2 = 1$.

We now explain what were the difficulties arising in the sketch of the proof given in [13].

The authors introduce a notion of symmetry or antisymmetry with respect to the inversion $x \mapsto -x$ but this does not work for odd N since $\mathfrak{D}(N,\epsilon)$ has no center of inversion. So the proof can only work for N even.

Considering N even ($N \geq 4$), the idea behind the proof in [13] is that there is a crossing for increasing ϵ between an eigenvalue associated with an antisymmetric eigenspace of multiplicity 2 and an eigenvalue associated with a symmetric eigenspace. With the considered antisymmetry proposed by the authors, it seems wrong that the multiplicity 2 results simply from the information that the eigenvalue corresponds to a non trivial antisymmetric eigenspace. We will give a theoretical analysis in Section 7 completed by a numerical study in Section 8 giving evidence that this guess is at least wrong in the simpler case $N = 2$ which is not considered in [13]. Hence one has also to change the argument for even N.

3 Symmetry spaces

Before proving Theorem 2.1, we recall some basic representation theory. We consider a Hamiltonian which is the Dirichlet realization of the Laplacian in an open set Ω which is invariant by the action of the group G_N generated by the rotation g by $\frac{2\pi}{N}$. The Hilbert space is $\mathcal{H} := L^2(\Omega,\mathbb{R})$ but it is also convenient to work in $\mathcal{H}_{\mathbb{C}} := L^2(\Omega,\mathbb{C})$. In this case, it is natural to analyze the eigenspaces attached to the irreducible representations of the group G_N. This is standard, see for example [14] and references therein, but note that these authors work with a larger group of symmetry, i.e. the dihedral group $\mathbb{D}_{2N}$. Here we prefer to start with the smaller group G_N and it is important to note that we do not assume in our work that Ω is homeomorphic to a disk or to an annulus. The theory of this section will in particular apply for the family of open sets $\Omega = \mathfrak{D}(N,\epsilon)$ (which satisfy the $\mathbb{D}_{2N}$-symmetry). Hence in this case, Theorems 1.2 and 1.3 of [14] do not fully apply.
The theory is simpler for complex Hilbert spaces i.e. $\mathcal{H}_{\mathbb{C}} := L^2(\Omega,\mathbb{C})$, but the multiplicity property appears when considering operators on real Hilbert spaces, i.e $\mathcal{H} := L^2(\Omega,\mathbb{R})$. If we work in $\mathcal{H}_{\mathbb{C}}$, we introduce for $\ell = 0,\cdots,N-1$,

$$\mathcal{B}_\ell = \{w \in \mathcal{H}_{\mathbb{C}} \mid gw = e^{2\pi i\ell/N} w\}\,. \tag{3.1}$$

For $\ell = 0$, this corresponds to the invariant situation. Hence in the model above (where $\Omega = B_{R_2}$) u_0 and u_6 belong to $\mathcal{B}_0$. We also observe that the complex conjugation sends $\mathcal{B}_\ell$ onto $\mathcal{B}_{N-\ell}$. Hence, except in the cases $\ell = 0$ and $\ell = \frac{N}{2}$ the corresponding eigenspace are of even dimension.
The second case appears only if N is even.

For $2\ell \neq N$, one can alternately come back to real spaces by introducing for $0 < \ell < \frac{N}{2}$ ($\ell \in \mathbb{N}$)

$$\mathcal{C}_\ell = \mathcal{B}_\ell \oplus \mathcal{B}_{N-\ell} \tag{3.2}$$

and observing that $\mathcal{C}_\ell$ can be recognized as the complexification of the real space $\mathcal{A}_\ell$

$$\mathcal{A}_\ell = \{u \in \mathcal{H} \mid u - 2\cos(2\ell\pi/N)gu + g^2u = 0\} \tag{3.3}$$

such that

$$\mathcal{C}_\ell = \mathcal{A}_\ell \otimes \mathbb{C} \tag{3.4}$$

where (3.3) follows from an easy computation based on (3.1).
For $\ell = 0$ and $\ell = \frac{N}{2}$ (if N is even), we define $\mathcal{A}_\ell$ by

$$\mathcal{B}_\ell = \mathcal{A}_\ell \otimes \mathbb{C}\,. \tag{3.5}$$

Under the invariance condition on the domain, the Dirichlet Laplacian commutes with the natural action of g in L^2. Hence we get for $0 \leq \ell \leq N/2$ a family of well defined selfadjoint operators $H^{(\ell)}$ obtained by restriction of H to $\mathcal{A}_\ell$ (with domain $D(H) \cap \mathcal{A}_\ell$). Note that except for $\ell = 0$ and $\ell = \frac{N}{2}$ all the eigenspaces of $H^{(\ell)}$ have even multiplicity.

The other point is that Stollmann's theory [19] works for the spectrum of $H^{(\ell)}(\epsilon, N)$ associated with the Dirichlet realization $H(\epsilon, N)$ of the Laplacian in $\mathfrak{D}_{N,\epsilon}$. Hence we have continuity and monotonicity with respect to ϵ of the eigenvalues. Note also that

$$\sigma(H(\epsilon, N)) = \cup_{0 \leq \ell \leq \frac{N}{2}} \sigma(H^{(\ell)}(\epsilon, N))\,.$$

Remark 3.1. *When N is even, a particular role is played by $g^{\frac{N}{2}}$ which corresponds to the inversion considered in [13]. One can indeed decompose the Hilbert space $\mathcal{H}$ (or $\mathcal{H}_{\mathbb{C}}$) using the symmetry with respect to $g^{\frac{N}{2}}$ and get the decomposition*

$$\mathcal{H} = \mathcal{H}^{even} \oplus \mathcal{H}^{odd}\,, \tag{3.6}$$

and

$$H(\epsilon, N) = H^{even}(\epsilon, N) \oplus H^{odd}(\epsilon, N)\,. \tag{3.7}$$

One can compare this decomposition with the previous one. We observe that $\mathcal{A}_\ell$ belongs to $\mathcal{H}^{even}$ if ℓ is even and to $\mathcal{H}^{odd}$ if ℓ is odd.

4 Upper bound: the regularity assumptions in Cheng's statement revisited

In [3], S.Y. Cheng proved that the multiplicity of the second eigenvalue is at most 3. Cheng's proof is actually using a regularity assumption which is not satisfied by $D(N, \epsilon)$. This domain has indeed cracks and we need a description of the nodal line structure near corners or cracks. But we will explain how to complete the proof in this case. We recall that for an eigenfunction u the nodal set $N(u)$ of u is defined by

$$\mathcal{N}(u) := \overline{\{x \in \Omega\,, |\, u(x) = 0\}}\,.$$

For other reasons (this was used in the context of spectral minimal partitions) this analysis was needed and treated in the paper of Helffer, Hoffmann-Ostenhof, and Terracini [8] (Theorem 2.6). With this complementary analysis near the cracks, we can follow the main steps of the proof given in the first part of [11] (Theorem B). This proof includes an extended version of Euler's Polyhedral formula (Proposition 2.8 in [11] with a stronger regularity assumption).

Proposition 4.1. *Let Ω be a $C^{1,+}$-domain*[1] *with possibly corners of opening*[2] *$\alpha\pi$ for $0 < \alpha \leq 2$. If u is an eigenfunction of the Dirichlet Laplacian in Ω, $\mathcal{N}$ denotes the nodal set of u and $\mu(\mathcal{N})$ denotes the cardinality of the components of $\Omega \setminus \mathcal{N}$, i.e. the number of nodal domains, then*

$$\mu(\mathcal{N}) \geq \sum_{x \in \mathcal{N} \cap \Omega} (\nu(x) - 1) + 2\,, \tag{4.1}$$

where $\nu(x)$ is the multiplicity of the critical point $x \in \mathcal{N}$ (i.e. the number of lines crossing at x).

For a second eigenfunction $\mu(\mathcal{N}) = 2$, and the upper bound of the multiplicity by 3 comes by contradiction. Assuming that the multiplicity of the second eigenvalue is ≥ 4, one can, for any $x \in \Omega$, construct some u in the second eigenspace such that $\nu(x) \geq 2$. This gives the contradiction with (4.1). Hence we have

Proposition 4.2. *Let Ω be a $C^{1,+}$-domain with possibly corners of opening $\alpha\pi$ for $0 < \alpha \leq 2$. Then the multiplicity of the second eigenvalue of the Dirichlet Laplacian in Ω is not larger than* 3.

Remark 4.3. *An upper bound of the multiplicity by* 2 *is obtained by C.S. Lin when Ω is convex (see [17]). As observed at the end of Section 2 in [18], Lin's theorem can be extended to the case of a simply connected domain for which the nodal line conjecture holds. If the multiplicity of the second eigenvalue is larger than 2, one can indeed find in the associated spectral space an eigenfunction whose nodal set contains a point in the boundary where two half lines hit the boundary. This will contradict either the nodal line conjecture or Courant's theorem. See also [13] for some sufficient conditions on domains for the nodal line conjecture to hold.*

5 Proof of Theorem 2.1

We first observe that for the disk of radius R we have

$$\lambda_1(B_R) < \lambda_2(B_R) = \lambda_3(B_R) < \lambda_4(B_R) = \lambda_5(B_R) < \lambda_6(B_R)\,. \tag{5.1}$$

The eigenfunctions u_1 and u_6 are radial. We will use this property with $R = R_2$.

Proposition 5.1. *For $N \geq 3$, there exists $\epsilon \in (0, \frac{\pi}{N})$ such that $\lambda_2(H(\epsilon, N))$ belongs to $\sigma(H^{(\ell)}(\epsilon, N))$ for some $0 < \ell < \frac{N}{2}$ AND to $\sigma(H^{(\ell)}(\epsilon, N))$ for $\ell = 0$ or (in the case N even) $\frac{N}{2}$. In particular, the multiplicity of λ_2 for this value of ϵ is exactly* 3.

Proof.
Note that the condition $N \geq 3$ implies the existence of at least one $\ell \in (0, \frac{N}{2})$. We now proceed by contradiction. Suppose the contrary. By continuity of the second eigenvalue, we should have

[1] $C^{1,+}$ means $C^{1,\epsilon}$ for some $\epsilon > 0$.
[2] $\alpha = 2$ corresponds to the crack case.

- either $\lambda_2(H(\epsilon,N))$ belongs to $\cup_{0<\ell<\frac{N}{2}}\sigma(H^{(\ell)}(\epsilon,N)))$ and not to $\sigma(H^{(0)}(\epsilon,N))\cup\sigma(H^{(N/2)}(\epsilon,N))$ for any ϵ,
- or $\lambda_2(H(\epsilon,N))$ belongs to $\sigma(H^{(0)}(\epsilon,N))\cup\sigma(H^{(N/2)}(\epsilon,N))$ and not to $\cup_{0<\ell<\frac{N}{2}}\sigma(H^{(\ell)}(\epsilon,N)))$ for any ϵ.

But, as we shall see below, the analysis for $\epsilon>0$ small enough shows that we should be in the first case and the analysis for ϵ close to $\frac{\pi}{N}$ that we should be in the second case. Hence a contradiction.

The analysis for $\epsilon>0$ very small is by perturbation a consequence of the analysis of $\epsilon=0$. Here we see from (2.2) that $\lambda_2(D_{R_1,R_2})$ is simple and belongs to $\sigma(H^{(0)}(0,N))$.

Remark 5.2. *If we only have* (2.3)*, we observe that the two first eigenvalues belong to* $\sigma(H^{(0)}(0,N))$ *and the argument is unchanged.*

The analysis for ϵ close to $\frac{\pi}{N}$ is by perturbation a consequence of the analysis of $\epsilon=\frac{\pi}{N}$. More details (which are not necessary for the argument) will be given in Section 7. Here we see from (5.1) that $\lambda_2(B_{R_2})$ has multiplicity two corresponding to $\sigma(H^{(1)}(\frac{\pi}{N},N))$.

So we have proven that for this value of ϵ the multiplicity is at least three, hence equals three by the extension of Cheng's statement [3] proven in the previous section. □

Comparison with the former proof proposed in [13]
When $N/2$ is even, we deduce from Remark 3.1 that

$$\sigma(H^{(odd)})\subset\cup_{0<\ell<\frac{N}{2}}\sigma(H^{(\ell)}(\epsilon,N)))\text{ and }\sigma(H^{(1)}(\epsilon,N))\subset\sigma(H^{(odd)}),$$

with equality for $N=4$.
From these two properties which imply that the eigenvalues in $\sigma(H^{(odd)})$ have even multiplicity we can rewrite the previous proof in the way presented in [13]:

Proposition 5.3. *For* $N\geq 4$ *and* $N/2$ *even, there exists* $\epsilon\in(0,\frac{\pi}{N})$ *such that* $\lambda_2(H(\epsilon,N))$ *belongs to* $\sigma(H^{even}(\epsilon,N))$ *AND to* $\sigma(H^{odd}(\epsilon,N))$*. In addition, the multiplicity of* λ_2 *for this value of* ϵ *is exactly* 3.

For $N/2$ odd integer some extra argument is necessary to exclude that an eigenvalue in $\sigma(H^{(N/2)}(\epsilon,N))$ (which belongs to $\sigma(H^{(odd)})$) becomes a second eigenvalue. More precisely we should prove that $\lambda_1(H^{(N/2)}(\epsilon,N))>\lambda_1(H^{(1)}(\epsilon,N))$ for any $\epsilon>0$. Here we have to use the additional dihedral invariance and use the arguments in [14]. The inequality follows from comparing the nodal sets of corresponding eigenfunctions (see (3.3) and (3.4) in [14] after having verified that the proof does not use the assumption that Ω is homeomorphic to a disk or an annulus). Hence, we have completed the proof sketched in [13] but the new proof looks more natural.

6 Further discussion for the case $N = 2$

In the previous sections, we have excluded the case $N = 2$ because we were unable to prove that the eigenspaces of $\mathcal{H}^{(N/2)}$ have even dimension and there were no more spaces $\mathcal{A}_\ell$ with $0 < \ell < \frac{N}{2}$ to play with. We now assume $N = 2$ and consider $\mathfrak{D}(2, \epsilon)$. Note that this time it will be quite important to have not only the dihedral symmetry but also the property that the cracks are on a circle.
As in [14] (see (1.16) and (1.17) there), we will use the decomposition of L^2:

$$\mathcal{H} := L^2(\mathfrak{D}(2, \epsilon)) = \mathcal{A}_0 \oplus \mathcal{A}_1^a \oplus \mathcal{A}_1^s\,.$$

Here

$$\begin{array}{ll} \mathcal{A}_1^s & = \{u \in \mathcal{H}\,,\, gu = -u\,,\, Tu = u\}\,, \\ \mathcal{A}_1^a & = \{u \in \mathcal{H}\,,\, gu = -u\,,\, Tu = -u\}\,, \end{array}$$

where $Tu(x_1, x_2) = u(x_1, -x_2)$.
We also observe that g is for $N = 2$ the inversion.
We similarly define the operators $H^{(1,a)}(\epsilon, 2)$ and $H^{(1,s)}(\epsilon, 2)$. The question is then to compare the spectra of these two operators and more specifically the first eigenvalue.
If we observe what is imposed by the symmetry or the antisymmetry with respect to $\{x_1 = 0\}$ or $\{x_2 = 0\}$ we can replace $\mathfrak{D}(2, \epsilon)$ by

$$\widehat{\mathfrak{D}}(2, \epsilon) := \mathfrak{D}(2, \epsilon) \cap \{x_1 > 0, x_2 > 0\}\,.$$

The problem corresponding to $H^{(1,s)}(\epsilon, 2)$ is the problem where we assume on $\partial\widehat{\mathfrak{D}}(2, \epsilon) \cap \{x_2 = 0\}$ the Neumann condition and on $\partial\widehat{\mathfrak{D}}(2, \epsilon) \cap \{x_1 = 0\}$ the Dirichlet condition, keeping the Dirichlet condition on the other parts of the boundary.

The problem corresponding to $H^{(1,a)}(\epsilon, 2)$ is the problem where we assume on $\partial\widehat{\mathfrak{D}}(2, \epsilon) \cap \{x_2 = 0\}$ the Dirichlet condition and on $\partial\widehat{\mathfrak{D}}(2, \epsilon) \cap \{x_1 = 0\}$ the Neumann condition, keeping the Dirichlet condition on the other parts of the boundary.
For $\epsilon = 0$ and $\epsilon = \frac{\pi}{2}$ the two operators are isospectral. Hence the question is:
Are the ground state energies of the two problems the same or are they different for $\epsilon \neq 0, \frac{\pi}{2}$?
We will show in Section 7 that for a given pair $(R_1, , R_2)$ with $R_1 < R_2$ this can only be true, in any closed subinterval of $(0, \frac{\pi}{2}]$, for a finite number of different values of ϵ's. Moreover, for a specific natural pair (R_1, R_2) we can give in Section 8 the following numerically assisted answer:
The ground state energies of $H^{(1,a)}(\epsilon, 2)$ and $H^{(1,s)}(\epsilon, 2)$ are equal if and only if $\epsilon = 0$ or $\frac{\pi}{2}$.

7 Theoretical asymptotics in domains with cracks

In this section, we analyze theoretically the behavior of the eigenvalue as ϵ tends to $\frac{\pi}{2}$. This improves the general results based on [19] and explains why we have to modify the sketch of [13] for the proof of Theorem 2.1.

7.1 Preliminaries

We now fix $N = 2$ and consider $0 < R_1 < R_2$. Motivated by the previous question, we analyze the different spectral problems according to the symmetries. This leads us to consider on the quarter of a disk ($0 < \theta < \frac{\pi}{2}$) four different models. On the exterior circle and on the cracks, we always assume the Dirichlet condition and then, according to the boundary conditions retained for $\theta = 0$ and $\theta = \pi/2$, we consider four test cases :

- Case NND (homogeneous Neumann boundary conditions for $\theta = 0$ and $\theta = \pi/2$).
- Case DDD (homogeneous Dirichlet boundary conditions for $\theta = 0$ and $\theta = \pi/2$).
- Case NDD (homogeneous Neumann boundary conditions for $\theta = 0$ and homogeneous Dirichlet boundary conditions for $\theta = \pi/2$).
- Case DND (homogeneous Dirichlet boundary conditions for $\theta = 0$ and homogeneous Neumann boundary conditions for $\theta = \pi/2$).

This is immediately related to the problem on the cracked disk by using the symmetries with respect to the two axes. The symmetry properties lead either to Dirichlet or Neumann.

7.2 The cases NND and DND

We use the notation

$$\begin{array}{ll} B^+_{R_2} & := B_{R_2} \cap \{x_2 > 0\}\,; \\ x_\pm & := (0, \pm R_1)\,; \\ \delta & := \frac{\pi}{2} - \epsilon\,; \\ K_\delta & := \{x \in \mathbb{R}^2\,;\, r = R_1, \theta \in [-\pi/2 - \delta, -\pi/2 + \delta] \cap [\pi/2 - \delta, \pi/2 + \delta]\}\,; \\ K^+_\delta & := K_\delta \cap \{x_2 > 0\}\,; \\ K^-_\delta & := K_\delta \cap \{x_2 < 0\}\,. \end{array}$$

By the symmetry arguments of Section 6,

$$\lambda_1^{NND}(\widehat{\mathfrak{D}}(2,\epsilon)) = \lambda_1(B_{R_2} \setminus K_\delta);$$
$$\lambda_1^{DND}(\widehat{\mathfrak{D}}(2,\epsilon)) = \lambda_1(B^+_{R_2} \setminus K^+_\delta).$$

The family of compact sets $(K_\delta)_{\delta>0}$ concentrates to the set $\{x_+, x_-\}$, in the sense that K_δ is contained in any open neighborhood of $\{x_+, x_-\}$ for δ small enough. Reference [1] provides two-term asymptotic expansions in this situation.

A direct application of Theorem 1.7 in [1] gives

$$\lambda_1(B^+_{R_2} \setminus K^+_\delta) = \lambda_1(B^+_{R_2}) + u(x_+)^2 \frac{2\pi}{\left|\log(\mathrm{diam}(K^+_\delta)\right|} + o\left(\frac{1}{\left|\log(\mathrm{diam}(K^+_\delta)\right|}\right),$$

where $\mathrm{diam}(K^+_\delta)$ is the diameter of K^+_δ and u an eigenfunction associated with $\lambda_1(B^+_{R_2})$, normalized in $L^2(B^+_{R_2})$. Using $\mathrm{diam}(K^+_\delta) = 2R_1 \sin(\delta)$ and the normalized eigenfunction given by Proposition 1.2.14 in [9] we find, after simplification

$$\lambda_1^{DND}(\widehat{\mathfrak{D}}(2,\epsilon)) = j_{1,1}^2 + \\ + \frac{8}{R_2^2}\left(\frac{J_1(j_{1,1}R_1/R_2)}{J_1'(j_{1,1})}\right)^2 \frac{1}{|\log(\pi/2-\epsilon)|} + o\left(\frac{1}{|\log(\pi/2-\epsilon)|}\right), \quad (7.1)$$

where $j_{\ell,k}$ is the k-th zero of the Bessel function J_ℓ corresponding to the integer $\ell \in \mathbb{N}$ (see Subsection 8.2 for more details and numerical values).

We obtain a similar expansion for the other eigenvalue, starting from Theorem 1.4 in [1], which gives us

$$\lambda_1(B_{R_2} \setminus K_\delta) = \lambda_1(B_{R_2}) + \mathrm{Cap}_{B_{R_2}}(K_\delta, u) + o\left(\mathrm{Cap}_{B_{R_2}}(K_\delta, u)\right).$$

In this formula, u is an eigenfunction associated with $\lambda_1(B_{R_2})$ and normalized in $L^2(B_{R_2})$, and $\mathrm{Cap}_{B_{R_2}}(K_\delta, u)$ is defined by Equation (6) in [1]. Since u is radially symmetric, $u(x_+) = u(x_-)$. We then observe that the proof of Proposition 1.5 in [1] can be adapted to give

$$\mathrm{Cap}_{B_{R_2}}(K_\delta, u) = u(x_\pm)^2 \mathrm{Cap}_{B_{R_2}}(K_\delta) + o\left(\mathrm{Cap}_{B_{R_2}}(K_\delta)\right),$$

where $\mathrm{Cap}_{B_{R_2}}(K_\delta)$ is the classical (condenser) capacity of K_δ relative to B_{R_2}. Since $K_\delta = K_\delta^+ \cup K_\delta^-$, and since K_δ^+ and K_δ^- concentrate to x_+ and x_- respectively, we have

$$\mathrm{Cap}_{B_{R_2}}(K_\delta) \sim \mathrm{Cap}_{B_{R_2}}(K_\delta^+) + \mathrm{Cap}_{B_{R_2}}(K_\delta^-)$$

as $\delta \to 0$. This last fact seems to be well known (see [6], page 178), but we give a proof in Appendix A for completeness. Finally, Proposition 1.6 in [1] gives an asymptotic expansion for $\mathrm{Cap}_{B_{R_2}}(K_\delta^\pm)$. Gathering these estimates, we find

$$\lambda_1^{NND}(\widehat{\mathfrak{D}}(2,\epsilon)) = j_{0,1}^2 + \\ + \frac{4}{R_2^2}\left(\frac{J_0(j_{0,1}R_1/R_2)}{J_0'(j_{0,1})}\right)^2 \frac{1}{|\log(\pi/2-\epsilon)|} + o\left(\frac{1}{|\log(\pi/2-\epsilon)|}\right). \quad (7.2)$$

7.3 Analysis of the cases NDD and DDD

In these cases, the results in [1] give an estimate of the eigenvalue variation but no explicit first term for the expansion. However, they strongly suggest the form of this term, which we present as a conjecture in each case. By the symmetry arguments of Section 6, we have, for ϵ close to $\pi/2$,

$$\lambda_1^{DDD}(\widehat{\mathfrak{D}}(2,\epsilon)) = \lambda_2(B_{R_2}^+ \setminus K_\delta^+).$$

We further note that $\lambda_2(B_{R_2}^+)$ $(= j_{2,1}^2)$ is simple and that an associated eigenfunction u, normalized in $L^2(B_{R_2}^+)$, is given by

$$u(r\cos\theta, r\sin\theta) = \frac{2}{\sqrt{\pi}} \frac{1}{R_2 |J_2'(j_{2,1})|} J_2\left(\frac{j_{2,1} r}{R_2}\right) \sin(2\theta).$$

In particular, it follows that

$$\partial_{x_1} u(x_+) = \frac{1}{R_1} \frac{4}{\sqrt{\pi}} \frac{1}{R_2 |J_2'(j_{2,1})|} J_2\left(\frac{j_{2,1} R_1}{R_2}\right),$$

so that,

$$u(x_+ + \rho(\cos t, \sin t)) = \frac{4}{\sqrt{\pi}} \frac{1}{R_1 R_2 |J_2'(j_{2,1})|} J_2\left(\frac{j_{2,1} R_1}{R_2}\right) \rho \cos t + \mathcal{O}\left(\rho^2\right).$$

Theorem 1.4 in [1] gives us

$$\lambda_1(B_{R_2}^+ \setminus K_\delta^+) = \lambda_1(B_{R_2}^+) + \mathrm{Cap}_{B_{R_2}^+}(K_\delta^+, u) + o\left(\mathrm{Cap}_{B_{R_2}^+}(K_\delta^+, u)\right). \tag{7.3}$$

Proposition 7.1. *There exists $C > 0$ such that*

$$j_{2,1}^2 \leq \lambda_1^{DDD}(\widehat{\mathfrak{D}}(2, \epsilon)) \leq j_{2,1}^2 + C\left(\left(\frac{\pi}{2} - \epsilon\right)^2\right). \tag{7.4}$$

Proof. By monotonicity of the Dirichlet eigenvalues with respect to the domain, we immediately have

$$j_{2,1}^2 = \lambda_1^{DDD}(\widehat{\mathfrak{D}}(2, \pi/2)) \leq \lambda_1^{DDD}(\widehat{\mathfrak{D}}(2, \epsilon)).$$

On the other hand, since x_+ is a (regular) point in the nodal set of u, we have $\mathrm{Cap}_{B_{R_2}^+}(K_\delta^+, u) = O\left(\delta^2\right)$ as $\delta \to 0$, according to Lemma 2.2 in [1]. The upper bound follows from Equation (7.3). $\square$

Conjecture 7.2. *As $\delta \to 0^+$,*

$$\mathrm{Cap}_{B_{R_2}^+}(K_\delta^+, u) \sim \mathrm{Cap}_{B_{R_2}^+}(s_\delta, u),$$

where s_δ is the segment $[-\delta R_1, \delta R_1] \times \{R_1\}$.

The first term of the asymptotics of $\mathrm{Cap}_{B_{R_2}^+}(s_\delta, u)$ is given by Theorem 1.13 in [1], so that Conjecture 7.2 would imply that as $\epsilon \to \pi/2^-$,

$$\lambda_1^{DDD}(\widehat{\mathfrak{D}}(2, \epsilon)) = j_{2,1}^2 + \\ + \frac{16}{R_2^2}\left(\frac{J_2(j_{2,1} R_1/R_2)}{J_2'(j_{2,1})}\right)^2 \left(\frac{\pi}{2} - \epsilon\right)^2 + o\left(\left(\frac{\pi}{2} - \epsilon\right)^2\right). \tag{7.5}$$

Using again Section 6, we have

$$\lambda_1^{NDD}(\widehat{\mathfrak{D}}(2, \epsilon)) = \lambda_2(B_{R_2} \setminus K_\delta) = \mu_2(B_{R_2}^+ \setminus K_\delta^+),$$

where $(\mu_j)_{j \geq 1}$ denotes the eigenvalues of the Laplacian in $B_{R_2}^+$, with a Dirichlet condition on the semi-circle $\{|x| = R_2\} \cap x_2 > 0$ and a Neumann condition on the diameter $[-R_2, R_2] \times \{0\}$. We note that $\mu_2(B_{R_2}^+)$ is simple. The eigenvalue $\lambda_2(B_{R_2})$ is double when the Laplacian is understood as acting on $L^2(B_{R_2})$, but simple if we restrict the Laplacian to $\mathcal{A}_1^s$ (as defined in Section 6). We denote by $(\lambda_j^s)_{j\geq 1}$ the spectrum of $H^{(1,s)}(\epsilon, 2)$ (by a slight abuse of notation we do not specify the value of ϵ). We remark that

$$\mu_2(B_{R_2}^+) = \lambda_2(B_{R_2}) = \lambda_1^s(B_{R_2}).$$

An eigenfunction associated with $\lambda_1^s(B_{R_2})$ and normalized in $\mathcal{A}_1^s$ is given by

$$u(r\cos\theta, r\sin\theta) = \sqrt{\frac{2}{\pi}} \frac{1}{R_2|J_1'(j_{1,1})|} J_1\left(\frac{j_{1,1} r}{R_2}\right)\cos\theta\,.$$

An eigenfunction associated with $\mu_2(B_{R_2}^+)$ and normalized in $L^2(B_{R_2}^+)$ is given by $\sqrt{2}u_{|B_{R_2}^+}$.

Proposition 7.3. *There exists $C > 0$ such that*

$$j_{1,1}^2 \leq \lambda_1^{NDD}(\widehat{\mathfrak{D}}(2,\epsilon)) \leq j_{1,1}^2 + C\left(\frac{\pi}{2} - \epsilon\right)^2. \tag{7.6}$$

Proof. We use $\lambda_1^{NDD}(\widehat{\mathfrak{D}}(2,\epsilon)) = \lambda_2(B_{R_2} \setminus K_\delta)$. To find an upper bound, we would like to apply Theorem 1.4 in [1] to get

$$\lambda_2(B_{R_2} \setminus K_\delta) = \lambda_2(B_{R_2}) + \mathrm{Cap}_{B_{R_2}}(K_\delta, u) + o\Big(\mathrm{Cap}_{B_{R_2}}(K_\delta, u)\Big).$$

Unfortunately, this cannot be done directly, since $\lambda_2(B_{R_2})$ is a double eigenvalue. But the result is easily obtained by repeating the proof in Appendix A of [1] for the Laplacian acting on the symmetric space $\mathcal{A}_1^s$. A straightforward adaptation of the proof of Lemma 2.2 in [1], taking into account the fact that K_δ concentrates to two points, gives $\mathrm{Cap}_{B_{R_2}}(K_\delta, u) = O\left(\delta^2\right)$ as $\delta \to 0$. □

Let us assume again that Conjecture 7.2 holds. Let us also assume that Theorem 1.4 in [1] holds for the eigenvalue problems with mixed boundary conditions which define $(\mu_j)_{j\geq 1}$. We obtain, as $\delta \to 0^+$,

$$\mu_2(B_{R_2}^+ \setminus K_\delta^+) = \mu_2(B_{R_2}^+) + 2\pi|\partial_{x_1} u(x_+)|^2 R_1^2\delta^2 + o\left(\delta^2\right).$$

Alternatively, we could work in the symmetric space $\mathcal{A}_1^s$. Repeating the proof in Appendix A of [1] in this space and assuming that the u-capacity defined in [1] is asymptotically additive for small distant sets, we obtain

$$\lambda_1^s(B_{R_2} \setminus K_\delta) = \lambda_1^s(B_{R_2}) + \pi|\partial_{x_1} u(x_+)|^2 R_1^2\delta^2 + \pi|\partial_{x_1} u(x_-)|^2 R_1^2\delta^2 + o\left(\delta^2\right).$$

Both methods would give

$$\lambda_1^{NDD}(\widehat{\mathfrak{D}}(2,\epsilon)) = j_{1,1}^2 + \\ + \frac{4}{R_2^2}\left(\frac{J_1(j_{1,1}R_1/R_2)}{J_1'(j_{1,1})}\right)^2 \left(\frac{\pi}{2} - \epsilon\right)^2 + o\left(\left(\frac{\pi}{2} - \epsilon\right)^2\right). \tag{7.7}$$

7.4 Comparison

As a consequence of Propositions 7.1 and 7.3 and using also the analyticity with respect to ϵ, we obtain

Proposition 7.4. *There exists $\epsilon_0 \in (0, \frac{\pi}{2})$ such that for $\epsilon \in [\epsilon_0, \frac{\pi}{2})$ we have*

$$\lambda_1^{NDD}(\widehat{\mathfrak{D}}(2,\epsilon)) < \lambda_1^{DND}(\widehat{\mathfrak{D}}(2,\epsilon))\,.$$

Moreover $\delta(\epsilon) := \lambda_1^{NDD}(\widehat{\mathfrak{D}}(2,\epsilon)) - \lambda_1^{DND}(\widehat{\mathfrak{D}}(2,\epsilon))$ can at most vanish in $(0, \frac{\pi}{2})$ on a sequence of ϵ's with no accumulation point except possibly at 0.

A more accurate analysis as $\epsilon \to 0$ would be useful for excluding the possibility of a sequence of zeros of δ tending to 0. We will see in the next section that numerics strongly suggests that $\delta(\epsilon)$ is negative in $(0, \frac{\pi}{2})$. The argument used in Proposition 7.4 is general and not related to $N = 2$.

8 Some illustrating numerics

8.1 Preliminaries

In this section, we complete the theoretical study of the previous section by using numerics. With the discussion around (2.4) in mind, a particular choice for the pair (R_1, R_2) is to start from B_{R_2} with $R_2 = 1$, and then to take as $0 < R_1 < 1$ the radius of the circle on which the second radial eigenfunction (which is associated with the sixth eigenvalue) vanishes. In this case, we have $\lambda_1(B_{R_1}) = \lambda_1(M_{R_1,R_2})$ and $\lambda_1(B_{R_1}) = \lambda_6(B_{R_2})$ is an eigenvalue of the Dirichlet Laplacian in $\mathfrak{D}(N,\epsilon)$ for any $\epsilon \in [0, \frac{\pi}{N}]$. Its labelling as eigenvalue is 2 for $0 < \epsilon \le \epsilon_0$ and becomes 6 for ϵ sufficienly close $\frac{\pi}{N}$. In addition, there is a unique ϵ_0^* such that the labelling is 2 for $0 < \epsilon \le \epsilon_0^*$ and becomes > 2 for $\epsilon > \epsilon_0^*$. This follows from the piecewise analyticity of the eigenvalues (Kato's theory) and a more detailed analysis as $\epsilon \to 0$ or $\epsilon \to \frac{\pi}{N}$ (see [5] for the technical details, [4, 15] for related questions and our previous section). The two next subsections recall what will be used for a theoretical verification of our numerical approach in the limits $\epsilon \to 0$ and $\epsilon \to +\frac{\pi}{2}$. We have indeed in these cases enough theoretical information for controlling the method.

8.2 Reminder: the case of the disk (Dirichlet)

Let $j_{\ell,k}$ be the k-th zero of the Bessel function J_ℓ corresponding to the integer $\ell \in \mathbb{N}$. Here is a list of approximate values after the celebrated handbook of [2], p. 409, we keep only the first values including at least all the eigenvalues which are less than approximately 13.

	$\ell =$ 0	1	2	3	4	5	6	7	8
$k = 1$	2.404	3.831	5.135	6.380	7.588	8.771	9.936	11.086	12.225
2	5.520	7.015	8.417	9.761	11.064	12.338	13.589	14.821	16.037
3	8,653	10.173	11.619	13.015	14.372	15.700	17.003	18.287	19.554
4	11.791	13.323	14.796	16.223	17.616	18.980	20.320	21.641	22.942

(8.1)

This leads to the following ordering of the zeros :

$$\begin{aligned} &j_{0,1} < j_{1,1} < j_{2,1} < j_{0,2} < j_{3,1} < j_{1,2} < j_{4,1} < j_{2,2} < j_{0,3} < \dots \\ &\quad\cdots < j_{5,1} < j_{3,2} < j_{6,1} < j_{1,3} < j_{7,1} < j_{2,3} < j_{0,4} < j_{8,1}\,. \end{aligned} \tag{8.2}$$

The corresponding eigenvalues for the disk B_1 of radius 1, with Dirichlet condition are given by $j_{\ell,k}^2$. The multiplicity is 1 if $\ell = 0$ (radial case) and 2 if $\ell \neq 0$. Hence we get for the six first eigenvalues (ordered in increasing order):

$$\begin{array}{l} \lambda_1^D(B_1) \sim 5.78 \\ \lambda_2^D(B_1) = \lambda_3^D(B_1) \sim 14.68 \\ \lambda_4^D(B_1) = \lambda_5^D(B_1) \sim 26.37 \\ \lambda_6^D(B_1) \sim 30.47\,. \end{array} \tag{8.3}$$

With our choice of the pair (R_1, R_2) in the previous subsection, we have

$$R_1 = j_{0,1}/j_{0,2} \sim 0.4356\,. \tag{8.4}$$

8.3 The case of the annulus with $R_1 = 0.4356$

We only keep the eigenvalues which are smaller than 150. Note that the multiplicity is 2 as soon as $\ell > 0$. The precision is relatively good but of no use due to the fact that we also use some approximation of the right R_1 defined in (8.4).

	$\ell =$	0	1	2	3	4	5	6	7	8	
$k = 1$		30.46	32.53	38.68	48.78	62.61	79.91	100.39	123.79	149.90	...
2		123.38	125.60	132.29	143.45	...					
3		...									

(8.5)

8.4 The case of the quarter of a disk (NND) with D-cracks

When $\epsilon = 0$, the two first eigenvalues of this problem with Neumann on the two radii ($\theta = 0, \frac{\pi}{2}$) are equal (due to our choice of R_1) and correspond to $\lambda_6^D(B_1) \sim 30.47$.

The third eigenvalue is either the fourth Dirichlet eigenvalue in B_{R_1} or the second NND eigenvalue in the annulus. By dilation, this would be in the first case

$$(j_{0,2}/j_{0,1})^2 \lambda_2^{NND}(B_1) = (j_{0,2}/j_{0,1})^2 \lambda_4^D(B_1) \sim 139.05\,.$$

We are actually in the second case, the next eigenvalues for the (NND) problem corresponding for $\epsilon = 0$ to the pairs (k, ℓ) for the annulus: $(1, 2)$, $(1, 4)$, $(1, 6)$ and $(2, 0)$.

When $\epsilon = \frac{\pi}{2}$, we should recover the eigenvalues of the Dirichlet problem in B_1 which have the right symmetry. We get:

$$\begin{array}{l} \lambda_1^{NND}(\widehat{\mathfrak{D}}(2, \frac{\pi}{2})) = \lambda_1^D(\mathfrak{D}(2, \frac{\pi}{2})) \sim 5.76 \\ \lambda_2^{NND}(\widehat{\mathfrak{D}}(2, \frac{\pi}{2})) = \lambda_4^D(\mathfrak{D}(2, \frac{\pi}{2})) \sim 26.42 \\ \lambda_3^{NND}(\widehat{\mathfrak{D}}(2, \frac{\pi}{2})) = \lambda_6^D(\mathfrak{D}(2, \frac{\pi}{2})) \sim 30.47\,. \end{array} \tag{8.6}$$

We recover the result predicted by our two terms asymptotics in (7.2).

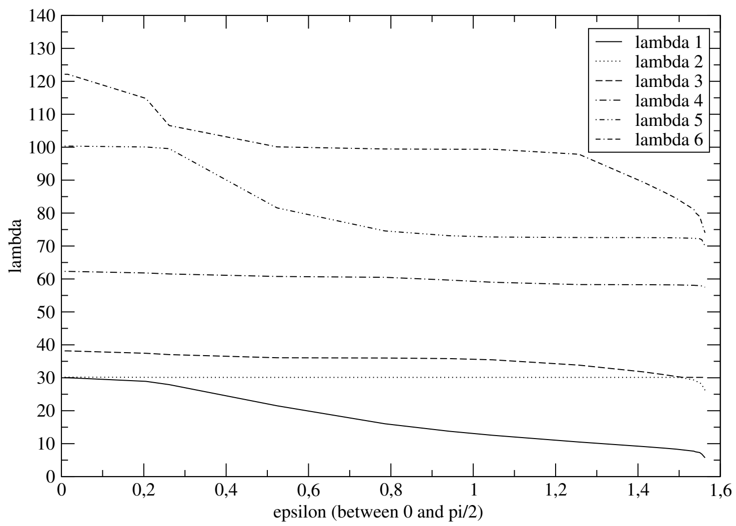

Figure 4: Case Neumann-Neumann: six first eigenvalues ($R_1 = 0.4356$)

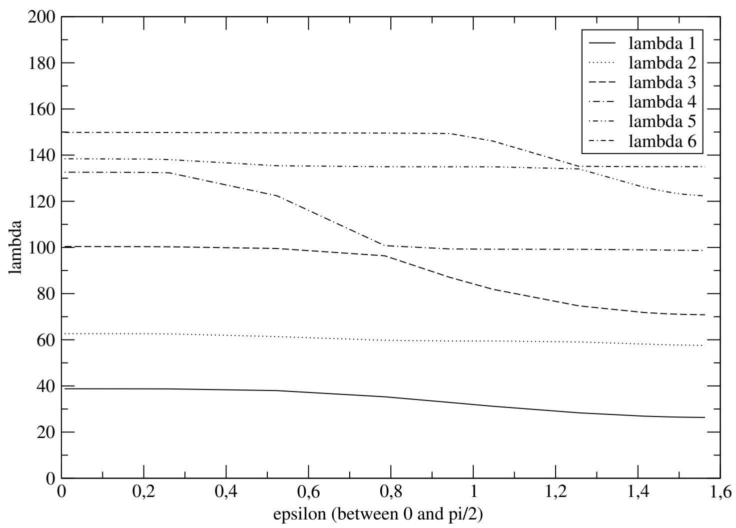

Figure 5: Case Dirichlet-Dirichlet: six first eigenvalues ($R_1 = 0.4356$)

8.5 The case of the quarter of a disk (DDD) with D-cracks

For $\epsilon = \frac{\pi}{2}$, the first three eigenvalues correspond to $(k, \ell) = (1, 2)$, $(k, \ell) = (1, 4)$, $(k, \ell) = (2, 2)$ in the tabular of Subsection 8.2 and correspond to the approximate values: 26.41 ; 57.61 ; 68.89.
For $\epsilon = 0$, we should recover the eigenvalue corresponding to the fourth Dirichlet eigenvalue in B_{R_1} i.e. 139.05 (which is also an eigenvalue of the NND problem) with a labelling 5. This suggests that the four first eigenvalues correspond to eigenvalues of the annulus with the right symmetry as confirmed by our computations in Subsection 8.4. They correspond to the pairs for the annulus $(1, 2)$, $(1, 4)$, $(1, 6)$ and $(2, 2)$. The sixth one corresponding to the pair $(1, 8)$.

8.6 The case of the quarter of a disk (DND) with D-cracks

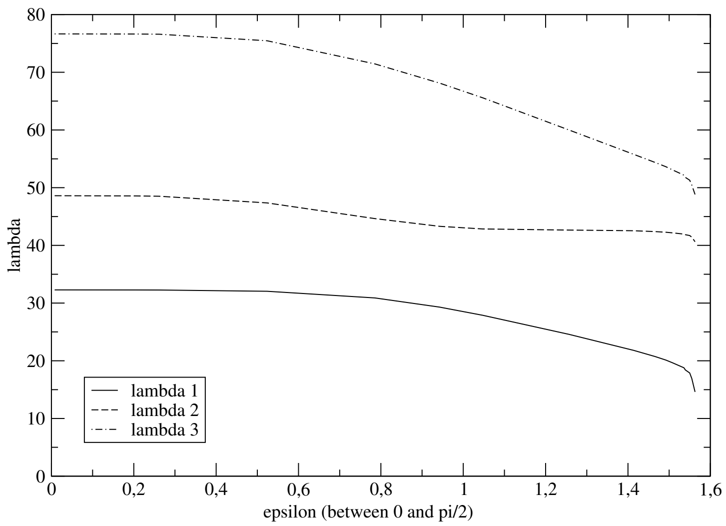

Figure 6: Case Dirichlet-Neumann: three first eigenvalues ($R_1 = 0.4356$)

Here we keep the same R_1, R_2 and assume Dirichlet for $\theta = 0$ and Neumann for $\theta = \frac{\pi}{2}$ and we are mainly interested in the ground state energy. For $\epsilon = 0$, the first eigenvalue is the second eigenvalue ($\ell = 1$) either in B_{R_1} or in the annulus $B_1 \setminus \overline{B_{R_1}}$. In the first case, this would be

$$(j_{0,2}/j_{0,1})^2 \lambda_2^{DND}(\widehat{\mathfrak{D}}(2,0)) = (j_{0,2}/j_{0,1})^2 \lambda_2^D(B_1) \sim 77.21,$$

which appears with labelling 3.

Hence, we have to look at the first DND-eigenvalue of the annulus corresponding to $(k, \ell) = (1, 1)$, which is approximately 32.53. Note that the second eigenvalue is obtained for $(k, \ell) = (1, 3)$, and is approximately 48.78.

For $\epsilon = \frac{\pi}{2}$, we get as ground state

$$\begin{array}{l}\lambda_1^{DND}(\widehat{\mathfrak{D}}(2, \frac{\pi}{2})) = \lambda_2^D(B_1) \sim 14.67 \\ \lambda_2^{DND}(\widehat{\mathfrak{D}}(2, \frac{\pi}{2})) = \lambda_7^D(B_1) \sim 40.70 \\ \lambda_3^{DND}(\widehat{\mathfrak{D}}(2, \frac{\pi}{2})) = \lambda_9^D(B_1) \sim 49\,.\end{array}$$

We also recover the behavior announced in (7.1). For $\epsilon = 0$, we recover the pairs $(1, 1)$, $(1, 3)$ and $(1, 5)$ of the annulus.

8.7 The case of the quarter of a disk (NDD) with D-cracks

Here we keep the same pair R_1, R_2 and assume Neumann for $\theta = 0$ and Dirichlet for $\theta = \frac{\pi}{2}$ and we are mainly interested in the ground state energy.

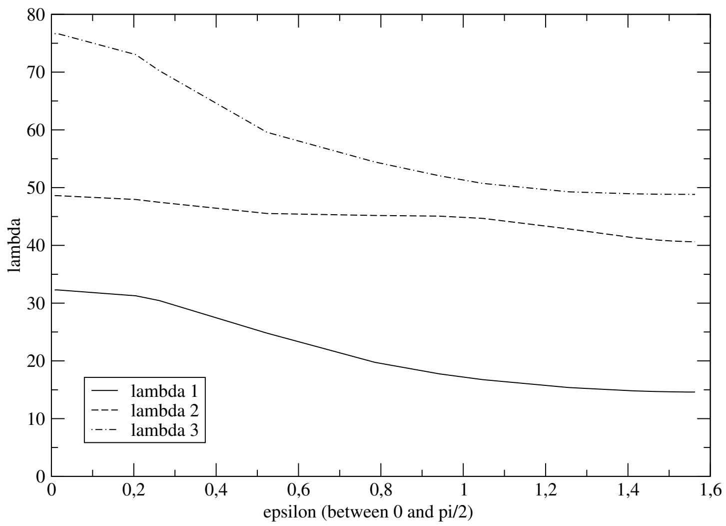

Figure 7: Case Neumann-Dirichlet: three first eigenvalues ($R_1 = 0.4356$)

For $\epsilon = 0$, we recover as for (DND) the pairs $(1, 1)$, $(1, 3)$ and $(1, 5)$ of the annulus.

8.8 Comparison between (NDD) and (DND) with D-cracks

For $\epsilon = 0$ and $\frac{\pi}{2}$ the theory says that the two spectra coincide. We recall from Section 6, that the union of these two spectra corresponds to the odd eigenfunctions on $\mathfrak{D}(2, \epsilon)$ which are antisymmetric by inversion.
For the ground state energies, the two curves do not cross and have different curvature properties. This strongly suggests that they are only equal for $\epsilon = 0$ and $\frac{\pi}{2}$. Some crossing (two points) is observed for the curves corresponding to the second eigenvalues. No crossing is observed for the curves corresponding to the third eigenvalues.

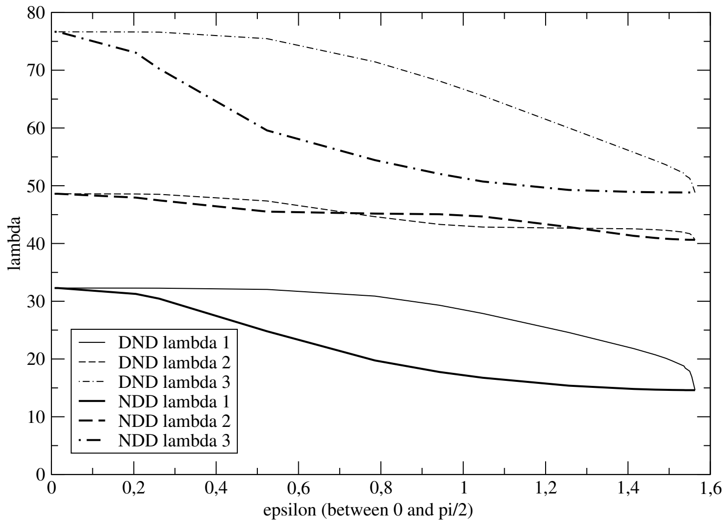

Figure 8: Case Neumann-Dirichlet and case Dirichlet-Neumann: three first eigenvalues ($R_1 = 0.4356$)

8.9 The complete spectrum in $\mathfrak{D}(\epsilon)$.

Putting the whole spectrum together, we see clearly in Figure 8, as mentioned in the introduction of Section 6, why there was no hope to get the multiplicity 3 in the case $N = 2$ by the successful approach presented for $N > 2$. We have indeed a first crossing but it only leads to an eigenvalue of multiplicity 2. Look at Figure 2 corrresponding to $N = 3$ for an interesting comparison.

8.10 On the numerical approach

Here we detail the numerical method used to obtain the different figures.

We look for the numerical computation of the eigenvalue problem :

$$-\Delta u(x, y) = \lambda u(x, y) \tag{8.7}$$

in the case of a domain $\Omega = \{(x, y) \in \mathbb{R}^2, x = r\cos(\theta), y = r\sin(\theta), r \in]0, 1[$ and $\theta \in]0, \pi/2[\}$ (a quarter of a disk).

In polar coordinates, Problem (8.7) becomes :

$$-\Delta u(r, \theta) = \lambda u(r, \theta) \tag{8.8}$$

where $\Delta u(r, \theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ (singularity for $r = 0$). For the boundary conditions we impose Dirichlet boundary conditions on $r = 1$. Moreover, we impose Dirichlet boundary conditions on a line (D-crack) corresponding to

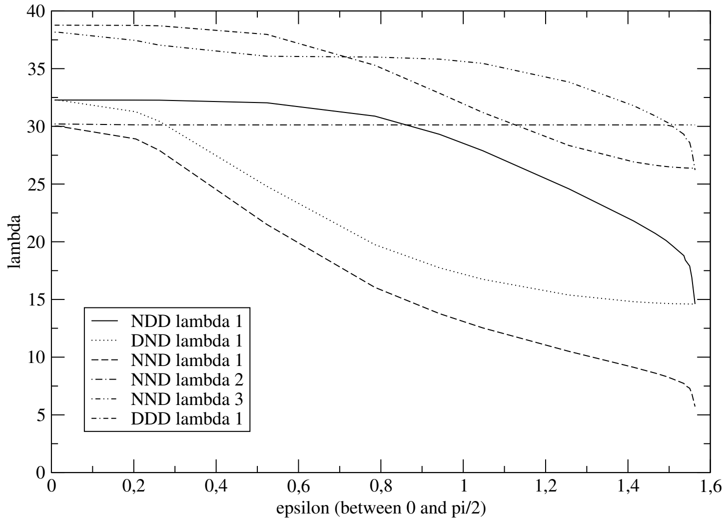

Figure 9: Six first eigenvalues ($R_1 = 0.4356$)

$r = R_1$, with $0 < R_1 < 1$, and $\theta \in [\epsilon, \pi/2]$, where $0 < \epsilon < \pi/2$.

For the numerical discretization of the Laplacian in polar coordinates we use a second order centered finite difference scheme :

$$\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\delta r^2}\right) + \frac{1}{r_i}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\delta r}\right) + \frac{1}{r_i^2}\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\delta \theta^2}\right) \tag{8.9}$$

where $u_{i,j}$ is a numerical approximation of $u(r_i, \theta_j)$ on the grid $r_i = i\,\delta r$ and $\theta_j = j\,\delta\theta$ for $i, j = 1, \ldots, M-1$, where δr and $\delta\theta$ are the steps in each direction $\delta r = \frac{R_2}{M}$ and $\delta\theta = \frac{\pi}{2M}$ ($i = 0, M$ and $j = 0, M$ correspond to the boundary conditions). After discretization we obtain a non symmetric tridiagonal $M \times M$ matrix A. For the simulations we have retained $M = 180$.

To compute the eigenvalues of the previous matrix A we use the function DGEEV of the Lapack library. To validate the code we have considered the case of the unit disk with Dirichlet conditions, computing the six first eigenvalues to compare with (8.3). This allows us in particular the treatment of the singularity of the coefficients of the operator appearing at $r = 0$. We can also control some limits as $\epsilon \to 0$ and $\epsilon \to \frac{\pi}{2}$ where again we have theoretical values or numerical values obtained by different methods.

For the numerical tests we have considered $R_1 = 0.4356$, corresponding to an approximation of the radius of the nodal line of the second radial eigenfunction in B_1.

A Asymptotic additivity of the capacity

We recall the definition of the condenser capacity of a compact set $K \subset \Omega$, relative to Ω:

$$\mathrm{Cap}_\Omega(K) := \inf\left\{\int_\Omega |\nabla v|^2 \,;\, v \in \Gamma_K\right\}. \tag{A.1}$$

Here Γ_K is the closed convex subset of $H_0^1(\Omega)$ consisting of the functions v satisfying $v \geq 1$ in the following sense: there exists a sequence (v_n) of functions in $H_0^1(\Omega)$ such that $v_n \geq 1$ almost everywhere in an open neighborhood of K and $v_n \to v$ in $H_0^1(\Omega)$ (see for instance Definition 3.3.19 in [10]). By the Projection Theorem in the Hilbert space $H_0^1(\Omega)$, there exists a unique $V_K \in \Gamma_K$ realizing the infimum, called the capacitary potential. From the minimization property, it follows immediately that V_K is harmonic in $\Omega \setminus K$. Furthermore, V_K is non-negative in Ω (see for instance Item 3 of Theorem 3.3.21 in [10]).

Let us now fix an integer $N \geq 1$, N distinct points $x_1, \dots, x_N$ in Ω, and N families of compact subsets of Ω, $(K_i^\varepsilon)_{\varepsilon>0}$ for $i \in \{1, \dots, N\}$. We assume that, for all $i \in \{1, \dots, N\}$, $(K_i^\varepsilon)_{\varepsilon>0}$ concentrates to x_i as $\varepsilon \to 0$, that is to say, for any open neighborhood U of x_i, there exists $\varepsilon_U > 0$ such that $K_i^\varepsilon \subset U$ for $\varepsilon \leq \varepsilon_U$. We use the notation $K^\varepsilon := \cup_{i=1}^N K_i^\varepsilon$. We remark that for all $i \in \{1, \dots, N\}$, $K_i^\varepsilon \subset \overline{B}(x_i, r_i^\varepsilon)$, with $\lim_{\varepsilon \to 0} r_i^\varepsilon = 0$. By monotonicity of the capacity,

$$\mathrm{Cap}_\Omega(K_i^\varepsilon) \leq \mathrm{Cap}_\Omega(\overline{B}(x_i, r_i^\varepsilon)) \to 0 \text{ as } \varepsilon \to 0\,.$$

Furthermore, by subadditivity of the capacity,

$$\mathrm{Cap}_\Omega(K^\varepsilon) \leq \sum_{i=1}^N \mathrm{Cap}_\Omega(K_i^\varepsilon) \to 0 \text{ as } \varepsilon \to 0\,. \tag{A.2}$$

Let us now show that in this situation, the capacity is asymptotically additive.

Proposition A.1. *If* $\mathrm{Cap}_\Omega(K^\varepsilon) > 0$ *for all* $\varepsilon > 0$*, we have, as* $\varepsilon \to 0$*,*

$$\mathrm{Cap}_\Omega(K^\varepsilon) \sim \sum_{i=1}^N \mathrm{Cap}_\Omega(K_i^\varepsilon). \tag{A.3}$$

Proof. Taking into account Inequality (A.2), we only have to prove

$$\limsup_{\varepsilon \to 0} \frac{\sum_{i=1}^N \mathrm{Cap}_\Omega(K_i^\varepsilon)}{\mathrm{Cap}_\Omega(K^\varepsilon)} \leq 1.$$

Let us set $V_\varepsilon := V_{K^\varepsilon}$. We claim that for any (fixed) K compact subset of $\Omega \setminus \{x_1, \dots, x_N\}$, V_ε converges to 0 as ε tends to 0, uniformly in K. Indeed, for ε small enough, $K \subset \Omega \setminus K^\varepsilon$, so that V_ε is harmonic in an open neighborhood U of K. Let us fix $r > 0$ such that $\overline{B}(x, r) \subset U$ for all $x \in K$. From the Mean Value Formula, for all $x \in K$,

$$0 \leq V_\varepsilon(x) = \frac{1}{\omega_d r^d} \int_{B(x,r)} V_\varepsilon \leq \frac{1}{(\omega_d r^d)^{\frac{1}{2}}} \|V_\varepsilon\|_{L^2(\Omega)}.$$

The claim follows from the fact that V_ε tends to 0 in $L^2(\Omega)$.

Let us now fix $R > 0$ such that the closed balls $\overline{B}_i := \overline{B}(x_i, R)$ are contained in Ω and mutually disjoint. By the above claim, $\delta_\varepsilon \to 0$ when $\varepsilon \to 0$, where

$$\delta_\varepsilon := \max_{i\in\{1,\dots,N\}} \max_{\partial \overline{B}_i} V_\varepsilon .$$

For $i \in \{1, \dots, N\}$, we define

$$v_i^\varepsilon := \frac{1}{1-\delta_\varepsilon} \left(V_\varepsilon - \delta_\varepsilon\right)_+ \mathbf{1}_{\overline{B}_i}.$$

We have $v_i^\varepsilon \in H_0^1(\Omega)$, and furthermore $v_i^\varepsilon \in \Gamma_{K_i^\varepsilon}$. Indeed, let us pick a sequence (φ_n) converging in $H_0^1(\Omega)$ to V_ε and such that, for all n, $\varphi_n \geq 1$ almost everywhere in a neighborhood of K_i^ε. Setting

$$\psi_n := \frac{1}{1-\delta_\varepsilon} \left(\varphi_n - \delta_\varepsilon\right)_+ \mathbf{1}_{\overline{B}_i},$$

we get $\psi_n \in H_0^1(\Omega)$ for all n and (ψ_n) converges to v_i^ε in $H_0^1(\Omega)$. Furthermore, for all n, $\varphi_n(x) \geq 1$ implies $\psi_n(x) \geq 1$, so that $\psi_n \geq 1$ almost everywhere in a neighborhood of K_i^ε. It follows that

$$\mathrm{Cap}_\Omega(K_i^\varepsilon) \leq \int_\Omega \left|\nabla v_i^\varepsilon\right|^2 .$$

Summing for i ranging from 1 to N, we find

$$\begin{aligned} \textstyle\sum_{i=1}^N \mathrm{Cap}_\Omega(K_i^\varepsilon) &\leq \textstyle\sum_{i=1}^N \int_\Omega \left|\nabla v_i^\varepsilon\right|^2 \\ &\leq \tfrac{1}{(1-\delta_\varepsilon)^2} \textstyle\int_\Omega \left|\nabla V_\varepsilon\right|^2 \\ &= \tfrac{1}{(1-\delta_\varepsilon)^2} \mathrm{Cap}_\Omega(K^\varepsilon). \quad \square \end{aligned}$$

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