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# Obstruction to the bilinear control of the Gross-Pitaevskii equation: an example with an unbounded potential <sup>\*</sup>

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**Abstract:** In 1982, Ball, Marsden, and Slemrod proved an obstruction to the controllability of linear dynamics with a bounded bilinear control term. This note presents an example of nonlinear dynamics with respect to the state for which this obstruction still holds while the control potential is not bounded.

*Keywords:* Nonlinear control system, controllability, bilinear control, Gross-Pitaevskii equation

## 1. INTRODUCTION AND RESULTS

### 1.1 Introduction

On the Euclidean space  $\mathbf{R}^3$  endowed with its natural norm  $|\cdot|$ , we study the control problem:

$$\begin{cases} i\partial_t\psi + H\psi = u(t)K(x)\psi - \sigma|\psi|^2\psi, & (t, x) \in \mathbf{R} \times \mathbf{R}^3, \\ \psi(0, x) = \psi_0(x), \end{cases} \quad (1)$$

where  $H = -\Delta + |x|^2$  is the Hamiltonian of the quantum harmonic oscillator on  $\mathbf{R}^3$ ,  $u : \mathbf{R} \rightarrow \mathbf{R}$  is the control,  $K : \mathbf{R}^3 \rightarrow \mathbf{R}$  is a given potential and  $\sigma \in \{0, 1\}$ .

The Sobolev spaces based on the domain of the harmonic oscillator are instrumental in the study of dynamics (1). They are defined, for  $s \geq 0$  and  $p \geq 1$  by

$$\begin{aligned} \mathcal{W}^{s,p} &= \mathcal{W}^{s,p}(\mathbf{R}^3) = \{f \in L^p(\mathbf{R}^3), H^{s/2}f \in L^p(\mathbf{R}^3)\}, \\ \mathcal{H}^s &= \mathcal{H}^s(\mathbf{R}^3) = \mathcal{W}^{s,2}. \end{aligned}$$

The natural norms are denoted by  $\|f\|_{\mathcal{W}^{s,p}}$  and up to equivalence of norms (see *e.g.* Lemma 2.4 of Yaajima and Zhang (2004)), for  $1 < p < +\infty$ , we have

$$\|f\|_{\mathcal{W}^{s,p}} = \|H^{s/2}f\|_{L^p} \equiv \|(-\Delta)^{s/2}f\|_{L^p} + \|\langle x \rangle^s f\|_{L^p}, \quad (2)$$

with the notation  $\langle x \rangle = (1 + |x|^2)^{1/2}$ .

### 1.2 Ball-Marsden-Slemrod obstructions

The dynamical system (1) is called the bilinear Gross-Pitaevskii equation. It is a nonlinear version of dynamics of the type  $\dot{\psi} = A\psi + u(t)B\psi$  where  $A$  and  $B$  are linear operators in a Banach space  $X$  and  $u : \mathbf{R} \rightarrow \mathbf{R}$  is a real-valued control which involve a control term  $uB\psi$  that is bilinear in  $(u, \psi)$ . Such dynamics play a major role in physics and are the subject of a vast literature Khapalov (2010). In Ball et al. (1982), Ball, Marsden, and Slemrod

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have proven that if  $A$  generates a  $C^0$  semi-group in  $X$  and if  $B$  is bounded on  $X$ , then the attainable set from any source  $\psi_0$  in  $X$  with  $L^r$  controls,  $r > 1$ , is contained in a countable union of compact sets of  $X$ . This represents a deep obstruction to the controllability of bilinear control systems in infinite dimensional Banach spaces, since this result implies that the attainable set is meager in Baire sense and has empty interior.

The original result of Ball et al. (1982) (and its adaptation to the Schrödinger equation in Turinici (2000)) has been extended to the case of  $L^1$  controls in Boussaïd et al. (2017). More recently, the case where  $A$  is nonlinear has been investigated in Chambrion and Thomann (2018b) (for the Klein-Gordon equation) and Chambrion and Thomann (2018a) (for the Gross-Pitaevskii equation (1)).

In (Chambrion and Thomann, 2018a, Theorem 1.6) we showed in particular that if  $K \in \mathcal{W}^{1,\infty}(\mathbf{R}^3)$ , the dynamics (1) is non controllable. Under this assumption, the map

$$\begin{aligned} \mathcal{H}^1(\mathbf{R}^3) &\longrightarrow \mathcal{H}^1(\mathbf{R}^3) \\ \psi &\longmapsto K\psi, \end{aligned}$$

is continuous and this was used in the heart of the proof.

The main result of this note, Theorem 1 below, provides an example of potential  $K \notin L^\infty(\mathbf{R}^3)$  where this condition is violated, but where the obstruction to controllability result still holds true.

### 1.3 Main result

Our main result reads as follows

*Theorem 1.* Let  $K(x) = \log(|x|)\mathbf{1}_{\{|x| \leq 1\}}$  and  $\psi_0 \in \mathcal{H}^1(\mathbf{R}^3)$ . Assume that  $u \in \bigcup_{r>1} L^r_{loc}(\mathbf{R})$ , then the equation (1) admits a global flow  $\psi(t) = \Phi^u(t)(\psi_0)$ .

Moreover, for every  $\psi_0 \in \mathcal{H}^1(\mathbf{R}^3)$ , the attainable set

$$\bigcup_{t \in \mathbf{R}} \bigcup_{\substack{u \in L_{loc}^r(\mathbf{R}), \\ r > 1}} \{\Phi^u(t)(\psi_0)\} \quad (3)$$

is a countable union of compact subsets of  $\mathcal{H}^1(\mathbf{R}^3)$ .

In this paper, the solutions to (1) are understood in the mild sense

$$\begin{aligned} \psi(t) &= e^{itH} \psi_0 - i \int_0^t u(\tau) e^{i(t-\tau)H} (K\psi(\tau)) d\tau \\ &\quad + i\sigma \int_0^t e^{i(t-\tau)H} (|\psi|^2 \psi) d\tau. \end{aligned}$$

#### 1.4 Content of the paper

The rest of this note provides a proof of Theorem 1. The proof crucially relies on classical Strichartz estimates, which we recall in Section 2. The proof itself is split in two parts. The global well-posedness of the problem (1) is established in Section 3.1, using among other some energy estimates. The proof of the obstruction result follows the strategy used in the paper Ball et al. (1982) and is given in Section 3.2.

## 2. STRICHARTZ ESTIMATES

As in Chambrion and Thomann (2018a), the Strichartz estimates play a major role in the argument, let us recall them in the three-dimensional case. A couple  $(q, r) \in [2, +\infty]^2$  is called admissible if

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2},$$

and if one defines

$$X_T^1 := \bigcap_{(q,r) \text{ admissible}} L^q([-T, T]; \mathcal{W}^{1,r}(\mathbf{R}^3)),$$

then for all  $T > 0$  there exists  $C_T > 0$  so that for all  $\psi_0 \in \mathcal{H}^1(\mathbf{R}^3)$  we have

$$\|e^{itH} \psi_0\|_{X_T^1} \leq C_T \|\psi_0\|_{\mathcal{H}^1(\mathbf{R}^3)}. \quad (4)$$

Using interpolation theory one can prove that

$$X_T^1 = L^\infty([-T, T]; \mathcal{H}^1(\mathbf{R}^3)) \cap L^2([-T, T]; \mathcal{W}^{1,6}(\mathbf{R}^3)),$$

so that one can define

$$\|\psi\|_{X_T^1} = \|\psi\|_{L^\infty([-T, T]; \mathcal{H}^1(\mathbf{R}^3))} + \|\psi\|_{L^2([-T, T]; \mathcal{W}^{1,6}(\mathbf{R}^3))}.$$

We will also need the inhomogeneous version of Strichartz: for all  $T > 0$ , there exists  $C_T > 0$  so that for all admissible couple  $(q, r)$  and function  $F \in L^{q'}([T, T]; \mathcal{W}^{1,r'}(\mathbf{R}^3))$ ,

$$\left\| \int_0^t e^{i(t-\tau)H} F(\tau) d\tau \right\|_{X_T^1} \leq C_T \|F\|_{L^{q'}([-T, T], \mathcal{W}^{1,r'}(\mathbf{R}^3))}, \quad (5)$$

where  $q'$  and  $r'$  are the Hölder conjugate of  $q$  and  $r$ . We refer to Poiret (2012) for a proof. Let us point out that (5) implies that

$$\begin{aligned} &\left\| \int_0^t e^{i(t-\tau)H} F(\tau) d\tau \right\|_{X_T^1} \leq \\ &C_T \left( \|F_1\|_{L^1([-T, T], \mathcal{H}^1(\mathbf{R}^3))} + \|F_2\|_{L^2([-T, T], \mathcal{W}^{1, \frac{6}{5}}(\mathbf{R}^3))} \right) \end{aligned} \quad (6)$$

for any  $F_1, F_2$  such that  $F_1 + F_2 = F$ , which will prove useful.

In the sequel  $c, C > 0$  denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded. For  $x \in \mathbf{R}^3$ , we write  $\langle x \rangle = (1 + |x|^2)^{1/2}$ . We will sometimes use the notations  $L_T^p = L^p([0, T])$  and  $L_T^p X = L^p([0, T]; X)$  for  $T > 0$ .

## 3. PROOF OF THEOREM 1

### 3.1 Global existence theory for dynamics (1)

Using the reversibility of the equation (1), it is enough to consider non-negative times in the proofs.

The following result will be useful to control the bilinear term in (1).

*Lemma 2.* Let  $K(x) = \log(|x|) \mathbf{1}_{\{|x| \leq 1\}}$  and  $T > 0$ . Then for all  $2 \leq q < \infty$ , there exists  $C_T > 0$  such that for all  $\psi \in X_T^1$

$$\|K\psi\|_{L^q([-T, T]; \mathcal{H}^1(\mathbf{R}^3))} \leq C_T \|\psi\|_{X_T^1}.$$

**Proof.** Firstly by (2) we have

$$\|K\psi\|_{\mathcal{H}^1} \leq c \|\nabla K\psi\|_{L^2} + c \|K\nabla\psi\|_{L^2} + c \|K\langle x \rangle \psi\|_{L^2}. \quad (7)$$

• Let us study the first term in (7). Since  $|\nabla K| \leq C|x|^{-1}$ , we can use the Hardy inequality

$$\||x|^{-1}\psi\|_{L^2(\mathbf{R}^3)} \leq C \|\psi\|_{H^1(\mathbf{R}^3)} \quad (8)$$

(we refer to (Tao, 2006, Lemma A.2) for the general statement and proof of this inequality), and therefore the contribution of the first term reads  $\|\nabla K\psi\|_{L_T^q L^2} \leq$

$$CT^{1/q} \|\psi\|_{L_T^\infty \mathcal{H}^1} \leq C_T \|\psi\|_{X_T^1}.$$

• To bound the contribution of the two last terms in (7), we will use that  $K \in L^p(\mathbf{R}^3)$  for any  $1 \leq p < \infty$ . Given  $2 \leq q < \infty$ , we choose  $r > 2$  such that the couple  $(q, r)$  is (Strichartz) admissible and write, using Hölder

$$\|K\nabla\psi\|_{L^2} \leq \|K\|_{L^p} \|\nabla\psi\|_{L^r} \leq c \|K\|_{L^p} \|\psi\|_{\mathcal{W}^{1,r}},$$

with  $1/p + 1/r = 1/2$ . Thus

$$\|K\nabla\psi\|_{L_T^q L^2} \leq c \|K\|_{L^p} \|\psi\|_{L_T^q \mathcal{W}^{1,r}} \leq c \|K\|_{L^p} \|\psi\|_{X_T^1}.$$

Similarly,

$$\|K\langle x \rangle \psi\|_{L_T^q L^2} \leq c \|K\|_{L^p} \|\psi\|_{L_T^q \mathcal{W}^{1,r}} \leq c \|K\|_{L^p} \|\psi\|_{X_T^1}.$$

We now state a global existence result for (1) adapted to our control problem.

*Proposition 3.* Let  $u \in L_{loc}^1(\mathbf{R})$  and set  $K : x \mapsto \log(|x|) \mathbf{1}_{\{|x| \leq 1\}}$ . Let  $\psi_0 \in \mathcal{H}^1(\mathbf{R}^3)$ , then the equation (1) admits a unique global solution  $\psi \in \mathcal{C}(\mathbf{R}; \mathcal{H}^1(\mathbf{R}^3)) \cap L_{loc}^2(\mathbf{R}; \mathcal{W}^{1,6})$  which moreover satisfies the bounds

$$\begin{aligned} &\|\psi\|_{L^\infty([-T, T]; \mathcal{H}^1(\mathbf{R}^3))} \\ &\leq C (\|\psi_0\|_{\mathcal{H}^1(\mathbf{R}^3)}) \exp \left( c \int_{-T}^T |u(\tau)| d\tau \right), \end{aligned} \quad (9)$$

and

$$\|\psi\|_{L^2([-T, T]; \mathcal{W}^{1,6}(\mathbf{R}^3))} \leq C(T, \|\psi_0\|_{\mathcal{H}^1(\mathbf{R}^3)}) \int_{-T}^T |u(\tau)| d\tau. \quad (10)$$

**Proof.** The proof is in the spirit of the proof of (Chambrion and Thomann, 2018a, Proposition 1.5), but here

we use moreover the Hardy inequality (8) to control the bilinear term in (1).

*Energy bound:* Assume for a moment that the solution exists on a time interval  $[0, T]$ . For  $0 \leq t \leq T$ , we define

$$\begin{aligned} E(t) &= \int_{\mathbf{R}^3} (\bar{\psi}H\psi + |\psi|^2 + \frac{\sigma}{2}|\psi|^4) dx \\ &= \int_{\mathbf{R}^3} (|\nabla\psi|^2 + |x|^2|\psi|^2 + |\psi|^2 + \frac{\sigma}{2}|\psi|^4) dx. \end{aligned}$$

Then, using that  $\partial_t \bar{\psi} = -i(H\bar{\psi} + \sigma|\psi|^2\bar{\psi}) + iu(t)K(x)\bar{\psi}$ , we get

$$\begin{aligned} E'(t) &= 2\Re \int_{\mathbf{R}^3} \partial_t \bar{\psi} (\psi + H\psi + \sigma|\psi|^2\psi) dx \\ &= -2u(t)\Im \int_{\mathbf{R}^3} K\bar{\psi}H\psi dx \\ &= 2u(t)\Im \int_{\mathbf{R}^3} \bar{\psi}\nabla K \cdot \nabla\psi dx. \end{aligned}$$

Observing that  $|\nabla K| \leq C|x|^{-1}$ , by the Hardy inequality (8) we get

$$\begin{aligned} E'(t) &\leq 2|u(t)| \|\psi\nabla K\|_{L^2} \|\nabla\psi\|_{L^2} \\ &\leq C|u(t)| \||x|^{-1}\psi\|_{L^2} \|\nabla\psi\|_{L^2} \\ &\leq C|u(t)| \|\psi\|_{\mathcal{H}^1}^2 \\ &\leq C|u(t)|E(t). \end{aligned}$$

Thus, using that  $\sigma \geq 0$ , we deduce the bound (9).

*Local existence and global existence:* We consider the map

$$\begin{aligned} \Phi(\psi)(t) &= e^{itH}\psi_0 + i\sigma \int_0^t e^{i(t-\tau)H} (|\psi|^2\psi) d\tau \\ &\quad - i \int_0^t u(\tau) e^{i(t-\tau)H} (K\psi) d\tau, \end{aligned}$$

and we will show that it is a contraction in the space

$$B_{T,R} := \{\|\psi\|_{X_T^1} \leq R\},$$

with  $R > 0$  and  $T > 0$  to be fixed.

By the Strichartz inequalities (4) and (6)

$$\begin{aligned} \|\Phi(\psi)\|_{X_T^1} &\leq c\|\psi_0\|_{\mathcal{H}^1} + c\||\psi|^2\psi\|_{L_T^1\mathcal{H}^1} + c\|uK\psi\|_{L_T^2\mathcal{W}^{1,6/5}(\mathbf{R}^3)}. \end{aligned}$$

Then by (Chambrion and Thomann, 2018a, Lemma A.1)

$$\begin{aligned} \|K\psi\|_{\mathcal{W}^{1,6/5}(\mathbf{R}^3)} &\leq C\|K\|_{L^3(\mathbf{R}^3)}\|\psi\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ &\quad + C\|K\|_{\mathcal{W}^{1,3/2}(\mathbf{R}^3)}\|\psi\|_{L^6(\mathbf{R}^3)}. \end{aligned}$$

Next, by the Sobolev embedding  $\mathcal{H}^1(\mathbf{R}^3) \subset L^6(\mathbf{R}^3)$ , from the previous line we get

$$\|K\psi\|_{\mathcal{W}^{1,6/5}(\mathbf{R}^3)} \leq C\|\psi\|_{\mathcal{H}^1(\mathbf{R}^3)}, \quad (11)$$

which in turn implies

$$\begin{aligned} \|\Phi(\psi)\|_{X_T^1} &\leq c\|\psi_0\|_{\mathcal{H}^1} + c\|\psi\|_{L_T^\infty\mathcal{H}^1}\|\psi\|_{L_T^2L^\infty}^2 + c\|u\|_{L_T^1}\|\psi\|_{L_T^\infty\mathcal{H}^1}. \end{aligned}$$

By the Gagliardo-Nirenberg and Sobolev inequalities on  $\mathbf{R}^3$ ,

$$\|\psi\|_{L^\infty} \leq C\|\psi\|_{L^6}^{1/2}\|\psi\|_{\mathcal{W}^{1,6}}^{1/2} \leq C\|\psi\|_{\mathcal{H}^1}^{1/2}\|\psi\|_{\mathcal{W}^{1,6}}^{1/2},$$

thus  $\|\psi\|_{L_T^2L^\infty} \leq cT^{1/4}\|\psi\|_{L_T^\infty\mathcal{H}^1}^{1/2}\|\psi\|_{L_T^2\mathcal{W}^{1,6}}^{1/2}$ , and for  $\psi \in B_{T,R}$  we get

$$\|\Phi(\psi)\|_{X_T^1} \leq c\|\psi_0\|_{\mathcal{H}^1} + cT^{1/2}R^3 + cR\|u\|_{L_T^1}.$$

We now choose  $R = 2c\|\psi_0\|_{\mathcal{H}^1}$ . Then for  $T > 0$  small enough,  $\Phi$  maps  $B_{T,R}$  into itself. With similar estimates we can show that  $\Phi$  is a contraction in  $B_{T,R}$ , namely

$$\|\Phi(\psi_1) - \Phi(\psi_2)\|_{X_T^1} \leq [cT^{1/2}R^2 + c\|u\|_{L_T^1}]\|\psi_1 - \psi_2\|_{X_T^1}.$$

As a conclusion there exists a unique fixed point to  $\Phi$ , which is a local solution to (1).

The local time of existence only depends on  $u$  and on the  $\mathcal{H}^1$ -norm. Therefore one can use the energy bound to show the global existence.

*Proof of the bound (10):* The proof follows the main lines of (Chambrion and Thomann, 2018a, Bound (1.18)), hence we do not detail it here.

### 3.2 Meagerness of the attainable set

Let  $\epsilon > 0$  and let  $u, u_n \in L^{1+\epsilon}([0, T]; \mathbf{R})$  such that  $u_n \rightharpoonup u$  weakly in  $L^{1+\epsilon}([0, T]; \mathbf{R})$ . This implies a bound  $\|u_n\|_{L_T^{1+\epsilon}} \leq C(T)$  for some  $C(T) > 0$ , uniformly in  $n \geq 1$ . We have

$$\begin{aligned} \psi(t) &= e^{itH}\psi_0 - i \int_0^t u(\tau) e^{i(t-\tau)H} (K\psi(\tau)) d\tau \\ &\quad + i\sigma \int_0^t e^{i(t-\tau)H} (|\psi|^2\psi) d\tau, \end{aligned}$$

and

$$\begin{aligned} \psi_n(t) &= e^{itH}\psi_0 - i \int_0^t u_n(\tau) e^{i(t-\tau)H} (K\psi_n(\tau)) d\tau \\ &\quad + i\sigma \int_0^t e^{i(t-\tau)H} (|\psi_n|^2\psi_n) d\tau. \end{aligned}$$

We set  $z_n = \psi - \psi_n$ , then  $z_n$  satisfies

$$z_n = \mathcal{L}(\psi, \psi_n) + \mathcal{N}(\psi, \psi_n), \quad (12)$$

with

$$\begin{aligned} \mathcal{L}(\psi, \psi_n) &= -i \int_0^t (u(\tau) - u_n(\tau)) e^{i(t-\tau)H} (K\psi) d\tau \\ &\quad - i \int_0^t u_n(\tau) e^{i(t-\tau)H} (K(\psi - \psi_n)) d\tau \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}(\psi, \psi_n) &= i\sigma \int_0^t e^{i(t-\tau)H} ((\psi - \psi_n)(\psi + \psi_n)\bar{\psi}) d\tau \\ &\quad + i\sigma \int_0^t e^{i(t-\tau)H} ((\bar{\psi} - \bar{\psi}_n)\psi_n^2) d\tau. \end{aligned}$$

Let us prove that  $z_n \rightarrow 0$  in  $L^\infty([0, T]; \mathcal{H}^1(\mathbf{R}^3))$ .

*Lemma 4.* Denote by

$$\epsilon_n := \left\| \int_0^t (u_n(\tau) - u(\tau)) e^{i(t-\tau)H} (K\psi(\tau)) d\tau \right\|_{L_T^\infty \mathcal{H}^1(\mathbf{R}^3)}.$$

Then  $\epsilon_n \rightarrow 0$ , when  $n \rightarrow +\infty$ .

**Proof.** We proceed by contradiction. Assume that there exists  $\epsilon > 0$ , a subsequence of  $u_n$  (still denoted by  $u_n$ ) and a sequence  $t_n \rightarrow t \in [0, T]$  such that

$$\left\| \int_0^{t_n} (u_n(\tau) - u(\tau)) e^{i(t_n-\tau)H} (K\psi(\tau)) d\tau \right\|_{\mathcal{H}^1(\mathbf{R}^3)} \geq \epsilon. \quad (13)$$

Up to a subsequence, we can assume that for all  $n \geq 1$ ,  $t_n \leq t$  or  $t_n \geq t$ . We only consider the first case, since the second is similar. By the Minkowski inequality and the unitarity of  $e^{i\tau H}$

$$\begin{aligned} & \left\| \int_0^{t_n} (u_n(\tau) - u(\tau)) (e^{i(t_n-\tau)H} - e^{i(t-\tau)H}) (K\psi(\tau)) d\tau \right\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ & \leq \int_0^{t_n} |u_n(\tau) - u(\tau)| \\ & \quad \times \left\| (e^{i(t_n-\tau)H} - e^{i(t-\tau)H}) (K\psi(\tau)) \right\|_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \\ & = \int_0^{t_n} |u_n(\tau) - u(\tau)| \left\| (e^{it_n H} - e^{itH}) (K\psi(\tau)) \right\|_{\mathcal{H}^1(\mathbf{R}^3)} d\tau. \end{aligned}$$

Then by Hölder

$$\begin{aligned} & \left\| \int_0^{t_n} (u_n(\tau) - u(\tau)) (e^{i(t_n-\tau)H} - e^{i(t-\tau)H}) (K\psi(\tau)) d\tau \right\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ & \leq \|u_n - u\|_{L_T^{1+\epsilon}} \left\| (e^{it_n H} - e^{itH}) (K\psi(\tau)) \right\|_{L_{\tau \in [0, T]}^{q_\epsilon} \mathcal{H}^1(\mathbf{R}^3)}, \end{aligned}$$

where  $1 < q_\epsilon < \infty$  is such that  $1/(1+\epsilon) + 1/q_\epsilon = 1$ . Now, by Lemma 2, we have

$$\|K\psi\|_{L_T^{q_\epsilon} \mathcal{H}^1(\mathbf{R}^3)} \leq C_{T, \epsilon} \|\psi\|_{X_T^1} < \infty. \quad (14)$$

Now we apply (Chambrion and Thomann, 2018a, Lemma 3.2) (with  $d = 3$  and  $s = 1$ ) together with the previous lines, and we get that

$$\left\| \int_0^{t_n} (u_n(\tau) - u(\tau)) (e^{i(t_n-\tau)H} - e^{i(t-\tau)H}) (K\psi(\tau)) d\tau \right\|_{\mathcal{H}^1(\mathbf{R}^3)} \quad (15)$$

tends to 0 as  $n \rightarrow +\infty$ .

By the Minkowski inequality, the unitarity of  $e^{i\tau H}$  and the Hölder inequality

$$\begin{aligned} & \left\| \int_{t_n}^t (u_n(\tau) - u(\tau)) e^{i(t-\tau)H} (K\psi(\tau)) d\tau \right\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ & \leq \int_{t_n}^t |u_n(\tau) - u(\tau)| \|K\psi(\tau)\|_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \\ & \leq \|u_n - u\|_{L_T^{1+\epsilon}} \|K\psi\|_{L_{\tau \in [t_n, t]}^{q_\epsilon} \mathcal{H}^1(\mathbf{R}^3)} \\ & \leq |t - t_n|^{1/q_\epsilon} \|u_n - u\|_{L_T^{1+\epsilon}} \|K\psi\|_{L_T^{2q_\epsilon} \mathcal{H}^1(\mathbf{R}^3)}. \end{aligned} \quad (16)$$

Using Lemma 2 and the fact that  $\|u_n - u\|_{L_T^{1+\epsilon}} \leq C$ , we deduce that the term (16) tends to 0. We combine this with (15) to deduce

$$\begin{aligned} & \left\| \int_0^{t_n} (u_n(\tau) - u(\tau)) e^{i(t_n-\tau)H} (K\psi(\tau)) d\tau \right. \\ & \quad \left. - \int_0^t (u_n(\tau) - u(\tau)) e^{i(t-\tau)H} (K\psi(\tau)) d\tau \right\|_{\mathcal{H}^1(\mathbf{R}^3)} \rightarrow 0. \end{aligned} \quad (17)$$

Let us now prove that  $\int_0^t (u_n(\tau) - u(\tau)) e^{i(t-\tau)H} (K\psi(\tau)) d\tau$  tends to 0 in  $\mathcal{H}^1(\mathbf{R}^3)$ , to reach a contradiction with (13). We set  $v(\tau) = e^{i(t-\tau)H} (K\psi(\tau))$ . Then by the unitarity of  $H$ , we have  $\|v(\tau)\|_{\mathcal{H}^1} = \|K\psi(\tau)\|_{\mathcal{H}^1}$ , thus by (14),  $v \in L^{q_\epsilon}([0, T]; \mathcal{H}^1(\mathbf{R}^3))$ . We expand  $v$  on the Hermite functions  $(h_k)_{k \geq 0}$  (which are the eigenfunctions of  $H$ ) which form a Hilbertian basis of  $L^2(\mathbf{R}^3)$

$$v(\tau, x) = \sum_{k=0}^{+\infty} \alpha_k(\tau) h_k(x),$$

so that we have  $\|v(\tau, \cdot)\|_{\mathcal{H}^1}^2 = \sum_{k=0}^{+\infty} (2k+1) |\alpha_k(\tau)|^2$  and

$$\|v\|_{L_T^{q_\epsilon} \mathcal{H}^1} = \left[ \int_0^T \left( \sum_{k=0}^{+\infty} (2k+1) |\alpha_k(\tau)|^2 \right)^{q_\epsilon/2} d\tau \right]^{1/q_\epsilon}.$$

This implies in particular that

$$\left( \sum_{k=0}^{+\infty} (2k+1) |\alpha_k(\tau)|^2 \right)^{q_\epsilon/2} \in L_T^{q_\epsilon}. \quad (18)$$

Denote by  $\rho = \sup_{n \geq 0} \|u_n - u\|_{L_T^{1+\epsilon}}$ . We claim that there exists  $M > 0$  large enough such that the function  $g(\tau, x) = \sum_{k=0}^M \alpha_k(\tau) h_k(x)$  satisfies  $\|v - g\|_{L^{q_\epsilon}([0, T]; \mathcal{H}^1(\mathbf{R}^3))} \leq \epsilon/(4\rho)$ . Actually,

$$\|v - g\|_{L_T^{q_\epsilon} \mathcal{H}^1} = \left[ \int_0^T \left( \sum_{k=M+1}^{+\infty} (2k+1) |\alpha_k(\tau)|^2 \right)^{q_\epsilon/2} d\tau \right]^{1/q_\epsilon}$$

tends to zero when  $M$  tends to 0, by the Lebesgue theorem and (18), hence the claim.

We have

$$\int_0^t (u_n(\tau) - u(\tau)) g(\tau) d\tau = \sum_{k=0}^M h_k \int_0^t (u_n(\tau) - u(\tau)) \alpha_k(\tau) d\tau.$$

Then, by (18), for all  $k \geq 0$ ,  $\alpha_k \in L_T^{q_\epsilon}$ , which implies

$$\begin{aligned} & \left\| \int_0^t (u_n(\tau) - u(\tau)) g(\tau) d\tau \right\|_{\mathcal{H}^1(\mathbf{R}^3)}^2 \\ & = \sum_{k=0}^M (2k+1)^s \left| \int_0^t (u_n(\tau) - u(\tau)) \alpha_k(\tau) d\tau \right|^2 \rightarrow 0, \end{aligned}$$

by the weak convergence of  $(u_n)$ . Finally, for  $n$  large enough,

$$\begin{aligned} & \left\| \int_0^t (u_n(\tau) - u(\tau)) v(\tau) d\tau \right\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ & \leq \frac{\epsilon}{4\rho} \|u_n - u\|_{L_T^{1+\epsilon}} + \left\| \int_0^t (u_n(\tau) - u(\tau)) g(\tau) d\tau \right\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ & \leq \frac{\epsilon}{2} \end{aligned}$$

which together with (13) and (17) gives the contradiction.

Thanks to (Chambrion and Thomann, 2018a, Lemma A.3) we get

$$\begin{aligned} & \|\mathcal{N}(\psi, \psi_n)(t)\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ & \leq \int_0^t \|(\psi - \psi_n)(\psi + \psi_n)\bar{\psi}\|_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \\ & \quad + \int_0^t \|(\bar{\psi} - \bar{\psi}_n)\psi_n^2\|_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \\ & \leq \int_0^t \|z_n\|_{\mathcal{H}^1(\mathbf{R}^3)} (\|\psi\|_{\mathcal{W}^{1,6}}^2 + \|\psi_n\|_{\mathcal{W}^{1,6}}^2) d\tau. \end{aligned} \quad (19)$$

By (12) we have

$$\begin{aligned} H^{1/2}z_n(t) &= -iH^{1/2} \int_0^t (u(\tau) - u_n(\tau))e^{i(t-\tau)H}(K\psi)d\tau \\ & - i \int_0^t u_n(\tau)e^{i(t-\tau)H}H^{1/2}(Kz_n)d\tau + H^{1/2}\mathcal{N}(\psi, \psi_n)(t). \end{aligned}$$

Thus from the Strichartz inequality (6) we deduce

$$\begin{aligned} \|z_n(t)\|_{\mathcal{H}^1(\mathbf{R}^3)} & \leq \epsilon_n + \|u_n H^{1/2}(Kz_n)\|_{L_t^2 L^{6/5}(\mathbf{R}^3)} \\ & \quad + \|\mathcal{N}(\psi, \psi_n)(t)\|_{\mathcal{H}^1(\mathbf{R}^3)}. \end{aligned} \quad (20)$$

By (11)

$$\|H^{1/2}(Kz_n)\|_{L^{6/5}(\mathbf{R}^3)} = \|Kz_n\|_{\mathcal{W}^{1,6/5}(\mathbf{R}^3)} \leq C\|z_n\|_{\mathcal{H}^1(\mathbf{R}^3)},$$

which in turn implies

$$\begin{aligned} & \|u_n H^{1/2}(Kz_n)\|_{L_t^2 L^{6/5}(\mathbf{R}^3)} \\ & \leq C \left( \int_0^t |u_n(\tau)| \|z_n(\tau)\|_{\mathcal{H}^1(\mathbf{R}^3)}^2 d\tau \right)^{1/2} \end{aligned} \quad (21)$$

As a conclusion, from (19), (20) and (21) we infer

$$\begin{aligned} \|z_n(t)\|_{\mathcal{H}^1(\mathbf{R}^3)} & \leq \epsilon_n + C \left( \int_0^t |u_n(\tau)| \|z_n(\tau)\|_{\mathcal{H}^1(\mathbf{R}^3)}^2 d\tau \right)^{1/2} \\ & \quad + \int_0^t \|z_n(\tau)\|_{\mathcal{H}^1(\mathbf{R}^3)} (\|\psi(\tau)\|_{\mathcal{W}^{1,6}}^2 + \|\psi_n(\tau)\|_{\mathcal{W}^{1,6}}^2) d\tau. \end{aligned}$$

Then by the Grönwall inequality, for all  $0 \leq t \leq T$  and (10)

$$\begin{aligned} & \|z_n(t)\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ & \leq \left( \epsilon_n + C \left( \int_0^t |u_n(\tau)| \|z_n(\tau)\|_{\mathcal{H}^1(\mathbf{R}^3)}^2 d\tau \right)^{1/2} \right) \\ & \quad \times e^{C \int_0^t (\|\psi(\tau)\|_{\mathcal{W}^{1,6}}^2 + \|\psi_n(\tau)\|_{\mathcal{W}^{1,6}}^2) d\tau} \\ & \leq C_1(T) \left( \epsilon_n + \left( \int_0^t |u_n(\tau)| \|z_n(\tau)\|_{\mathcal{H}^1(\mathbf{R}^3)}^2 d\tau \right)^{1/2} \right). \end{aligned}$$

We square the previous inequality and get

$$\|z_n(t)\|_{\mathcal{H}^1(\mathbf{R}^3)}^2 \leq 2C_1^2(T) \left( \epsilon_n^2 + \int_0^t |u_n(\tau)| \|z_n(\tau)\|_{\mathcal{H}^1(\mathbf{R}^3)}^2 d\tau \right).$$

By the Grönwall inequality again we deduce

$$\begin{aligned} & \|z_n\|_{L_T^\infty \mathcal{H}^1(\mathbf{R}^3)} \\ & \leq 2C_1(T)\epsilon_n \exp \left( C_1^2(T) \int_0^T |u_n(\tau)| d\tau \right) \leq C_2(T)\epsilon_n, \end{aligned}$$

and this latter quantity tends to 0 when  $n \rightarrow +\infty$ .

## 4. CONCLUSION

This note provides an example of a Ball-Marsden-Slemrod like obstruction to the controllability of a nonlinear partial differential equation with a bilinear control term. The novelty of the result lies in the unboundedness of the bilinear control term.

The possible relations of this obstruction result and the concepts introduced in Boussaïd et al. (2013) will be the subject of further investigations in future works.

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