Technical note for the FS-MMSE-IC receiver

Titouan Petitpied, R Tajan, G. Ferré, Pascal Chevalier, S Traverso

Technical note for the FS-MMSE-IC receiver

T. Petitpied[T †], R. Tajan[T], G. Ferré[T], P. Chevalier[†‡], S. Traverso[†]

—

[T] IMS, Univ. Bordeaux, Bordeaux INP, CNRS (UMR 5218), F-33405 Talence (FRANCE)

[†] Thales, 4 avenue des Louvresses, 92230 Gennevilliers (FRANCE)

[‡] CEDRIC laboratory – Conservatoire National des Arts et Métiers
292 rue Saint-Martin, 75141 Paris Cedex 3 (FRANCE)

—

Email: forename.name@{[T]ims-bordeaux.fr, [†]thalesgroup.com, [‡]cnam.fr}

February 2019

1 Definitions

For any scalars or vectors $\mathbf{x}$ and $\mathbf{y}$ with convenient sizes, we define:

$$\begin{cases} x_n &= \mathbf{e}_n^\dagger \mathbf{x} \\ \mathbf{R}_{\mathbf{xy}} &= \mathbb{E}_{\mathbf{x},\mathbf{y}}[\mathbf{x}\mathbf{y}^\dagger] \\ \mathbf{R}_{\mathbf{x}} &= \mathbf{R}_{\mathbf{xx}} \end{cases}$$

where $\mathbf{e}_n$ is the $(n+1)^{\text{th}}$ element of the canonic base. The operator $\mathbb{E}_{\mathbf{x},\mathbf{y}}[\mathbf{x}\mathbf{y}^\dagger]$ represents the statistical average with respect to $\mathbf{x}$ and $\mathbf{y}$.

For any matrix $\mathbf{A}$, we define $\mathbf{A}^*$ the conjugate matrix of $\mathbf{A}$, $\mathbf{A}^T$ its transpose matrix and $\mathbf{A}^\dagger = \left(\mathbf{A}^T\right)^*$.

2 Context & Problem formulation

We suppose having the following expressions:

$$\begin{cases} \mathbf{r} &= \mathbf{HUs} + \mathbf{w} \\ \mathcal{I}(\mathbf{f}, p) &= \mathbf{f}^\dagger \mathbf{r} + p \\ \mathcal{J}(\mathbf{f}, p) &= \mathbb{E}_{\mathcal{I}(\mathbf{f},p)} \left|\mathcal{I}(\mathbf{f}, p) - s_n\right|^2 \end{cases}$$

where :

- $\mathbf{f}$: a $(2N \times 1)$ vector,
- p : a scalar,
- $\mathbf{H}$: a $(2N \times 2N_s)$ deterministic convolution matrix,
- $\mathbf{U}$: a $(2N_s \times N_s)$ up-sampling matrix,
- $\mathbf{D} = \mathbf{U}^\dagger$: a down-sampling matrix,
- $\mathbf{s}$: a $(N_s \times 1)$ probabilistic vector such that $\mathbb{E}[\mathbf{s}] = \mathbf{s}^d$ and $\mathbf{R}_{s_n,s_m} = (v_n^d + |s_n^d|^2)\delta(n-m) \Longrightarrow \mathbb{E}\left[\left(\mathbf{s}-\mathbf{s}^d\right)\left(\mathbf{s}-\mathbf{s}^d\right)^\dagger\right] = \mathbf{V}^d = \text{diag}(\mathbf{v}^d)$,
- $\mathbf{w}$: a $(2N \times 1)$ probabilistic vector such that $\mathbb{E}[\mathbf{w}] = \mathbf{0}_N$ and $\mathbf{R}_{\mathbf{w}} = \sigma_w^2 \mathbf{I}_N$,

Typically, the vector $\mathbf{r}$ represents a band-limited received signal after the convolution operation of the shaping filter and the channel $\mathbf{H}$, corrupted by a centered circular white noise $\mathbf{w}$ with average power σ_w^2. Moreover, this holds if and only if the shaping filter bandwidth is bounded by $\frac{2}{T_s}$. Hence, $\mathbf{r}$ represents a Fractionanally-Spaced (FS) system model where $\mathcal{I}(\mathbf{f}, p)$ is the estimator of $\mathbf{s}$ with Mean Square Error (MSE) $\mathcal{J}(\mathbf{f}, p)$. Furthermore, considering the prior information $(\mathbf{s}^d, \mathbf{V}^d)$ on $\mathbf{s}$ leads to an *Interference-Cancellation (IC)* structure.

Problem formulation: We search for $\tilde{s}_n^e = \mathcal{I}(\mathbf{f}_n, p_n)$, $\tilde{v}_n^e = \mathcal{J}(\mathbf{f}_n, p_n)$ such that:

$$(\mathbf{f}_n, p_n) = \underset{(\mathbf{f},p)}{\operatorname{argmin}}\, \mathcal{J}(\mathbf{f}, p)$$

Consequently, $\tilde{v}_n^e$ represents the Minimum MSE of the considered system model (FS-MMSE-IC) and with estimator $\tilde{s}_n^e$ of s_n.

3 Problem resolution

3.1 Expression of the MSE minimizers

$$\begin{aligned}\mathcal{J}(\mathbf{f}, p) &= \mathbb{E}_{\mathcal{I}(\mathbf{f},p)} \left|\mathcal{I}(\mathbf{f}, p) - s_n\right|^2 \\ &= \mathbf{f}^\dagger \mathbf{R_r} \mathbf{f} + \mathbf{f}^\dagger \mathbb{E}[\mathbf{r}] p^* - \mathbf{f}^\dagger \mathbf{R}_{\mathbf{r}s_n} + p\mathbb{E}[\mathbf{r}^\dagger]\mathbf{f} + |p|^2 - p\mathbb{E}[s_n^*] - \mathbf{R}_{s_n\mathbf{r}}\mathbf{f} - \mathbb{E}[s_n]p^* + \mathbf{R}_{s_n}\end{aligned}$$

Minimizing $\mathcal{J}(\mathbf{f}, p)$ leads us to nullify its the first partial derivatives.

$$\begin{aligned}\frac{\partial \mathcal{J}(\mathbf{f}, p)}{\partial \mathbf{f}} = 0 &\Longleftrightarrow 2\mathbf{R_r}\mathbf{f}_n + 2\mathbb{E}[\mathbf{r}]p_n^* - 2\mathbf{R}_{\mathbf{r}s_n} = 0 \\ &\Longleftrightarrow \mathbf{f}_n = \mathbf{R_r}^{-1}\left(\mathbf{R}_{\mathbf{r}s_n} - \mathbb{E}[\mathbf{r}]p_n^*\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{J}(\mathbf{f}, p)}{\partial p^*} = 0 &\Longleftrightarrow 2\mathbb{E}[\mathbf{r}^\dagger]\mathbf{f}_n + 2p_n^* - 2\mathbb{E}[s_n^*] = 0 \\ &\Longleftrightarrow p_n = \mathbb{E}[s_n] - \mathbf{f}_n^\dagger \mathbb{E}[\mathbf{r}]\end{aligned}$$

We check that such an extremum corresponds to a minimum :

$$\begin{cases}\frac{\partial^2 \mathcal{J}(\mathbf{f},p)}{\partial \mathbf{f}^2} = 2\mathbf{R_r} \geq 0 \\ \frac{\partial^2 \mathcal{J}(\mathbf{f},p)}{\partial p^{*2}} = 2 \geq 0\end{cases}$$

3.2 Computation of the MSE minimizers

Let us first compute :

$$\begin{aligned}\mathbb{E}[s_n] &= s_n^d \\ \mathbb{E}[\mathbf{r}] &= \mathbf{HUs}^d \\ \mathbf{R_r} &= \mathbf{HUR_sDH}^\dagger + \sigma_w^2 \mathbf{I}_N \\ \mathbf{R}_{\mathbf{r}s_n} &= \mathbf{HUR}_{\mathbf{s}s_n}\end{aligned}$$

Then, the MSE minimizers can be computed:

$$
\begin{aligned}
\mathbf{f}_n^\dagger &= \left(\mathbf{R}_{s_n\mathbf{r}} - p_n\mathbb{E}[\mathbf{r}^\dagger]\right)\mathbf{R}_{\mathbf{r}}^{-1\dagger} \\
&= \left(\mathbf{R}_{s_n\mathbf{r}} - \mathbb{E}[s_n]\mathbb{E}[\mathbf{r}^\dagger] + \mathbf{f}_n^\dagger \left|\mathbb{E}[\mathbf{r}]\right|^2\right)\mathbf{R}_{\mathbf{r}}^{-1\dagger} \\
&= \left(\mathbf{R}_{s_n\mathbf{r}} - \mathbb{E}[s_n]\mathbb{E}[\mathbf{r}^\dagger]\right)\mathbf{R}_{\mathbf{r}}^{-1\dagger}(\mathbf{I}_N - \left|\mathbb{E}[\mathbf{r}]\right|^2\mathbf{R}_{\mathbf{r}}^{-1\dagger})^{-1} \\
&= \left(\mathbf{R}_{s_n\mathbf{r}} - \mathbb{E}[s_n]\mathbb{E}[\mathbf{r}^\dagger]\right)(\mathbf{R}_{\mathbf{r}}^{-1\dagger} - \left|\mathbb{E}[\mathbf{r}]\right|^2)^{-1} \\
&= \left(\mathbf{R}_{s_n\mathbf{s}} - \mathbb{E}[s_n]\mathbb{E}[\mathbf{s}^\dagger]\right)\mathbf{D}\mathbf{H}^\dagger\left(\mathbf{H}\mathbf{U}(\mathbf{R}_{\mathbf{s}} - \mathbb{E}\left|\mathbf{s}\right|^2)\mathbf{D}\mathbf{H}^\dagger + \sigma_w^2\mathbf{I}_{2N}\right)^{-1} \\
&= v_n^d\mathbf{e}_n^\dagger\mathbf{D}\mathbf{H}^\dagger\left(\mathbf{H}\mathbf{U}\mathbf{V}^d\mathbf{D}\mathbf{H}^\dagger + \sigma_w^2\mathbf{I}_{2N}\right)^{-1}
\end{aligned}
$$

Using twice the Woodbury identity on any $(K_1 \times K_2)$ matrix A and $(K_2 \times K_1)$ matrix B:

$$
\begin{aligned}
\mathbf{A}(\mathbf{B}\mathbf{A} + \mathbf{I}_{K_2})^{-1} &= \mathbf{A}(\mathbf{B}\mathbf{I}_{K_1}\mathbf{A} + \mathbf{I}_{K_2})^{-1} \\
&= \mathbf{A}(\mathbf{I}_{K_2} - \mathbf{B}(\mathbf{I}_{K_1} + \mathbf{A}\mathbf{B})^{-1}\mathbf{A}) \\
&= (\mathbf{I}_{K_1} - \mathbf{A}\mathbf{B}(\mathbf{I}_{K_1} + \mathbf{A}\mathbf{B})^{-1})\mathbf{A} \\
&= (\mathbf{A}\mathbf{B}\mathbf{I}_{K_1} + \mathbf{I}_{K_1})^{-1}\mathbf{A} \\
&= (\mathbf{A}\mathbf{B} + \mathbf{I}_{K_1})^{-1}\mathbf{A}
\end{aligned}
$$

We identify $\begin{cases}\mathbf{A} = \sigma_w^{-2}\mathbf{D}\mathbf{H}^\dagger \\ \mathbf{B} = \mathbf{H}\mathbf{U}\mathbf{V}^d\end{cases}$ leading to:

$$
\begin{aligned}
\mathbf{f}_n^\dagger &= v_n^d\mathbf{e}_n^\dagger\mathbf{D}\mathbf{H}^\dagger\left(\mathbf{H}\mathbf{U}\mathbf{V}^d\mathbf{D}\mathbf{H}^\dagger + \sigma_w^2\mathbf{I}_{2N}\right)^{-1} \\
&= v_n^d\mathbf{e}_n^\dagger\left(\mathbf{D}\mathbf{H}^\dagger\mathbf{H}\mathbf{U}\mathbf{V}^d + \sigma_w^2\mathbf{I}_N\right)^{-1}\mathbf{D}\mathbf{H}^\dagger \\
&= v_n^d\mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}\mathbf{D}\mathbf{H}^\dagger
\end{aligned}
$$

where $\mathbf{\Sigma} = \mathbf{G}\mathbf{V}^d + \sigma_w^2\mathbf{I}_N$ is the MMSE equalization matrix and $\mathbf{G} = \mathbf{D}\mathbf{H}^\dagger\mathbf{H}\mathbf{U}$ is also called *global filter* and gathers the up-sampling, channel and matched filtering and down-sampling operations.

$$
\begin{aligned}
p_n &= \mathbb{E}[s_n] - \mathbf{f}_n^\dagger\mathbb{E}[\mathbf{r}] \\
&= s_n^d - v_n^d\mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}\mathbf{G}\mathbf{s}^d
\end{aligned}
$$

3.3 Computation of the FS-MMSE-IC and its estimator

The FS-MMSE-IC estimator is given by:

$$
\begin{aligned}
\tilde{s}_n^e &= \mathbf{f}_n^\dagger\mathbf{r} + p_n \\
&= s_n^d + v_n^d\mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}\left(\mathbf{D}\mathbf{H}^\dagger\mathbf{r} - \mathbf{G}\mathbf{s}^d\right) \\
&= s_n^d + v_n^d\mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}\left(\mathbf{y} - \mathbf{G}\mathbf{s}^d\right)
\end{aligned}
$$

where $\mathbf{y} = \mathbf{D}\mathbf{H}^\dagger\mathbf{r}$ represents the received signal after matched filtering and down-sampling.

The FS-MMSE-IC $\tilde{v}_n^e$ of the estimator $\tilde{s}_n^e$ can be computed by first deriving:

$$\begin{aligned}
\mathbf{R}_{\tilde{s}_n^e} &= |s_n^d|^2 + 2(s_n^d)^* v_n^d \mathbf{e}_n^\dagger \operatorname{Re}\left(\mathbf{\Sigma}^{-1}\mathbb{E}_{\mathbf{y}}[\left(\mathbf{y}-\mathbf{G}\mathbf{s}^d\right)]\right) + (v_n^d)^2 \mathbf{e}_n^\dagger \mathbf{\Sigma}^{-1}\mathbb{E}_{\mathbf{y}}[\left(\mathbf{y}-\mathbf{G}\mathbf{s}^d\right)\left(\mathbf{y}-\mathbf{G}\mathbf{s}^d\right)^\dagger]\mathbf{\Sigma}^{-1\dagger}\mathbf{e}_n \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \mathbf{\Sigma}^{-1}\mathbb{E}_{\mathbf{s},\mathbf{w}}\left|\mathbf{G}(\mathbf{s}-\mathbf{s}^d) + \mathbf{D}\mathbf{H}^\dagger\mathbf{w}\right|^2 \mathbf{\Sigma}^{-1\dagger}\mathbf{e}_n \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \mathbf{\Sigma}^{-1}\mathbf{D}\mathbf{H}^\dagger\mathbb{E}_{\mathbf{s},\mathbf{w}}\left|\mathbf{H}\mathbf{U}(\mathbf{s}-\mathbf{s}^d) + \mathbf{w}\right|^2 \mathbf{\Sigma}^{-1\dagger}\mathbf{e}_n \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \mathbf{\Sigma}^{-1}\mathbf{D}\mathbf{H}^\dagger\left(\mathbf{H}\mathbf{U}\mathbf{V}^d\mathbf{D}\mathbf{H}^\dagger + \sigma_w^2\mathbf{I}_{2N}\right)\mathbf{H}\mathbf{U}\mathbf{\Sigma}^{-1\dagger}\mathbf{e}_n \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \mathbf{\Sigma}^{-1}\left(\mathbf{D}\mathbf{H}^\dagger\mathbf{H}\mathbf{U}\mathbf{V}^d + \sigma_w^2\mathbf{I}_N\right)\mathbf{D}\mathbf{H}^\dagger\mathbf{H}\mathbf{U}\mathbf{\Sigma}^{-1\dagger}\mathbf{e}_n \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \mathbf{\Sigma}^{-1}\mathbf{G}\mathbf{e}_n
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{\tilde{s}_n^e, s_n} &= \mathbb{E}_{\mathbf{y}}[\left(s_n^d + v_n^d\mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}(\mathbf{y}-\mathbf{G}\mathbf{s}^d)\right)s_n^*] \\
&= |s_n^d|^2 + v_n^d\mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}\mathbf{G}\mathbb{E}_{\mathbf{s}}[\left(\mathbf{s}-\mathbf{s}^d\right)s_n^*] \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \mathbf{\Sigma}^{-1}\mathbf{G}\mathbf{e}_n \\
&= \mathbf{R}_{\tilde{s}_n^e}
\end{aligned}$$

$$\mathbf{R}_{s_n} = |s_n^d|^2 + v_n^d$$

This leads to the FS-MMSE-IC $\tilde{v}_n^e$ expression:

$$\begin{aligned}
\tilde{v}_n^e &= \mathbb{E}_{\tilde{s}_n^e}\left|\tilde{s}_n^e - s_n\right|^2 \\
&= \mathbf{R}_{\tilde{s}_n^e} - \mathbf{R}_{\tilde{s}_n^e, s_n} - \mathbf{R}_{s_n, \tilde{s}_n^e} + \mathbf{R}_{s_n} \\
&= \mathbf{R}_{s_n} - \mathbf{R}_{s_n, \tilde{s}_n^e} \\
&= |s_n^d|^2 + v_n^d - |s_n^d|^2 - (v_n^d)^2\mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}\mathbf{G}\mathbf{e}_n \\
&= v_n^d\left(1 - v_n^d c_n\right)
\end{aligned}$$

where $c_n = \mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}\mathbf{G}\mathbf{e}_n$.

3.4 Unbiased FS-MMSE-IC and its estimator

We can note that the estimator $\tilde{s}_n^e$ of s_n has a residual bias :

$$\begin{aligned}
\mathbb{E}\left[s_n^e|\mathbb{E}[s_n]=\mu\right] &= \mathbb{E}_{\mathbf{s}_{\sim\{n\}},\mathbf{w}}[s_n^d + v_n^d\mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}\left(\mathbf{y}-\mathbf{G}\mathbf{s}^d\right)|\mathbb{E}[s_n]=\mu] \\
&= s_n^d + v_n^d\mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}\mathbf{G}\mathbb{E}_{\mathbf{s}_{\sim\{n\}}}\left[\mathbf{s}-\mathbf{s}^d|\mathbb{E}[s_n]=\mu\right] \\
&= s_n^d + (\mu - s_n^d)v_n^d c_n \\
&= v_n^d c_n\mu + s_n^d(1 - v_n^d c_n)
\end{aligned}$$

where the vector $\mathbf{s}_{\sim\{n\}}$ represents all the components of $\mathbf{s}$ but s_n.

Removing the additive term $s_n^d(1-v_n^d c_n)$ and dividing the multiplicative term $v_n^d c_n$ leads to:

$$\begin{aligned}
s_n^e &= (v_n^d c_n)^{-1}\left(\tilde{s}_n^e - s_n^d(1-v_n^d c_n)\right) \\
&= s_n^d + c_n^{-1}\mathbf{e}_n^\dagger\mathbf{\Sigma}^{-1}\left(\mathbf{y}-\mathbf{G}\mathbf{s}^d\right)
\end{aligned}$$

We compute the new statistical matrices:

$$\begin{aligned}
\mathbf{R}_{s_n^e} &= |s_n^d|^2 + c_n^{-2}\mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1}\mathbb{E}_{\mathbf{y}}[\left(\mathbf{y}-\mathbf{G}\mathbf{s}^d\right)\left(\mathbf{y}-\mathbf{G}\mathbf{s}^d\right)^\dagger]\boldsymbol{\Sigma}^{-1\dagger}\mathbf{e}_n \\
&= |s_n^d|^2 + c_n^{-1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{s_n^e,s_n} &= \mathbb{E}_{\mathbf{y}}[\left(s_n^d + c_n^{-1}\mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\mathbf{G}\mathbf{s}^d)\right) s_n^*] \\
&= |s_n^d|^2 + c_n^{-1}\mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1}\mathbf{G}\mathbb{E}_{\mathbf{s}}[\left(\mathbf{s}-\mathbf{s}^d\right) s_n^*] \\
&= |s_n^d|^2 + v_n^d
\end{aligned}$$

$$\mathbf{R}_{s_n} = |s_n^d|^2 + v_n^d$$

And finally we find we unbiased FS-MMSE-IC:

$$\begin{aligned}
v_n^e &= \mathbb{E}_{s_n^e}\left|s_n^e - s_n\right|^2 \\
&= \mathbf{R}_{s_n^e} - \mathbf{R}_{s_n^e,s_n} - \mathbf{R}_{s_n,s_n^e} + \mathbf{R}_{s_n} \\
&= |s_n^d|^2 + c_n^{-1} - |s_n^d|^2 - v_n^d - |s_n^d|^2 - v_n^d + |s_n^d|^2 + v_n^d \\
&= c_n^{-1} - v_n^d
\end{aligned}$$

4 Conclusion

The biased FS-MMSE-IC receiver is given by:

$$\begin{cases}
\tilde{s}_n^e &= s_n^d + v_n^d\mathbf{e}_n^\dagger\boldsymbol{\Sigma}^{-1}\left(\mathbf{y}-\mathbf{G}\mathbf{s}^d\right) \\
\tilde{v}_n^e &= v_n^d\left(1 - v_n^d c_n\right)
\end{cases}$$

where $c_n = \mathbf{e}_n^\dagger\boldsymbol{\Sigma}^{-1}\mathbf{G}\mathbf{e}_n$ is a scalar, $\mathbf{y} = \mathbf{D}\mathbf{H}^\dagger\mathbf{r}$ is the receive signal after matched filtering and down-sampling, $\mathbf{G} = \mathbf{D}\mathbf{H}^\dagger\mathbf{H}\mathbf{U}$ is the global effect including up-sampling, channel and matched filtering and down-sampling, $\boldsymbol{\Sigma} = \mathbf{G}\mathbf{V}^d + \sigma_w^2\mathbf{I}_N$ is the MMSE equalization matrix.

Moreover, the unbiased FS-MMSE-IC receiver is given by:

$$\begin{cases}
s_n^e &= s_n^d + c_n^{-1}\mathbf{e}_n^\dagger\boldsymbol{\Sigma}^{-1}\left(\mathbf{y}-\mathbf{G}\mathbf{s}^d\right) \\
v_n^e &= c_n^{-1} - v_n^d
\end{cases}$$

Consequently, the FS-MMSE-IC receiver leads to a FS matched filter followed by a symbol time equalizer.