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Technical note for the FS-MMSE-IC receiver

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1 Definitions

For any scalars or vectors \mathbf{x} and \mathbf{y} with convenient sizes, we define:

$$\begin{cases} x_n &= \mathbf{e}_n^\dagger \mathbf{x} \\ \mathbf{R}_{\mathbf{x}\mathbf{y}} &= \mathbb{E}_{\mathbf{x},\mathbf{y}}[\mathbf{x}\mathbf{y}^\dagger] \\ \mathbf{R}_{\mathbf{x}} &= \mathbf{R}_{\mathbf{x}\mathbf{x}} \end{cases}$$

where \mathbf{e}_n is the $(n+1)^{\text{th}}$ element of the canonic base. The operator $\mathbb{E}_{\mathbf{x},\mathbf{y}}[\mathbf{x}\mathbf{y}^\dagger]$ represents the statistical average with respect to \mathbf{x} and \mathbf{y} .

For any matrix \mathbf{A} , we define \mathbf{A}^* the conjugate matrix of \mathbf{A} , \mathbf{A}^T its transpose matrix and $\mathbf{A}^\dagger = (\mathbf{A}^T)^*$.

2 Context & Problem formulation

We suppose having the following expressions:

$$\begin{cases} \mathbf{r} &= \mathbf{H}\mathbf{U}\mathbf{s} + \mathbf{w} \\ \mathcal{I}(\mathbf{f}, p) &= \mathbf{f}^\dagger \mathbf{r} + p \\ \mathcal{J}(\mathbf{f}, p) &= \mathbb{E}_{\mathcal{I}(\mathbf{f}, p)} |\mathcal{I}(\mathbf{f}, p) - s_n|^2 \end{cases}$$

where :

- \mathbf{f} : a $(2N \times 1)$ vector,
- p : a scalar,
- \mathbf{H} : a $(2N \times 2N_s)$ deterministic convolution matrix,
- \mathbf{U} : a $(2N_s \times N_s)$ up-sampling matrix,
- $\mathbf{D} = \mathbf{U}^\dagger$: a down-sampling matrix,
- \mathbf{s} : a $(N_s \times 1)$ probabilistic vector such that $\mathbb{E}[\mathbf{s}] = \mathbf{s}^d$ and $\mathbf{R}_{s_n, s_m} = (v_n^d + |s_n^d|^2)\delta(n-m) \implies \mathbb{E}[(\mathbf{s} - \mathbf{s}^d)(\mathbf{s} - \mathbf{s}^d)^\dagger] = \mathbf{V}^d = \text{diag}(\mathbf{v}^d)$,
- \mathbf{w} : a $(2N \times 1)$ probabilistic vector such that $\mathbb{E}[\mathbf{w}] = \mathbf{0}_N$ and $\mathbf{R}_{\mathbf{w}} = \sigma_w^2 \mathbf{I}_N$,

Typically, the vector \mathbf{r} represents a band-limited received signal after the convolution operation of the shaping filter and the channel \mathbf{H} , corrupted by a centered circular white noise \mathbf{w} with average power σ_w^2 . Moreover, this holds if and only if the shaping filter bandwidth is bounded by $\frac{2}{T_s}$. Hence, \mathbf{r} represents a Fractionally-Spaced (FS) system model where $\mathcal{I}(\mathbf{f}, p)$ is the estimator of \mathbf{s} with Mean Square Error (MSE) $\mathcal{J}(\mathbf{f}, p)$. Furthermore, considering the prior information $(\mathbf{s}^d, \mathbf{V}^d)$ on \mathbf{s} leads to an *Interference-Cancellation (IC)* structure.

Problem formulation: We search for $\tilde{s}_n^e = \mathcal{I}(\mathbf{f}_n, p_n)$, $\tilde{v}_n^e = \mathcal{J}(\mathbf{f}_n, p_n)$ such that:

$$(\mathbf{f}_n, p_n) = \underset{(\mathbf{f}, p)}{\operatorname{argmin}} \mathcal{J}(\mathbf{f}, p)$$

Consequently, \tilde{v}_n^e represents the Minimum MSE of the considered system model (FS-MMSE-IC) and with estimator \tilde{s}_n^e of s_n .

3 Problem resolution

3.1 Expression of the MSE minimizers

$$\begin{aligned} \mathcal{J}(\mathbf{f}, p) &= \mathbb{E}_{\mathcal{I}(\mathbf{f}, p)} |\mathcal{I}(\mathbf{f}, p) - s_n|^2 \\ &= \mathbf{f}^\dagger \mathbf{R}_r \mathbf{f} + \mathbf{f}^\dagger \mathbb{E}[\mathbf{r}] p^* - \mathbf{f}^\dagger \mathbf{R}_{rs_n} + p \mathbb{E}[\mathbf{r}^\dagger] \mathbf{f} + |p|^2 - p \mathbb{E}[s_n^*] - \mathbf{R}_{s_n r} \mathbf{f} - \mathbb{E}[s_n] p^* + \mathbf{R}_{s_n} \end{aligned}$$

Minimizing $\mathcal{J}(\mathbf{f}, p)$ leads us to nullify its the first partial derivatives.

$$\begin{aligned} \frac{\partial \mathcal{J}(\mathbf{f}, p)}{\partial \mathbf{f}} = 0 &\iff 2\mathbf{R}_r \mathbf{f}_n + 2\mathbb{E}[\mathbf{r}] p_n^* - 2\mathbf{R}_{rs_n} = 0 \\ &\iff \mathbf{f}_n = \mathbf{R}_r^{-1} (\mathbf{R}_{rs_n} - \mathbb{E}[\mathbf{r}] p_n^*) \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathcal{J}(\mathbf{f}, p)}{\partial p^*} = 0 &\iff 2\mathbb{E}[\mathbf{r}^\dagger] \mathbf{f}_n + 2p_n^* - 2\mathbb{E}[s_n^*] = 0 \\ &\iff p_n = \mathbb{E}[s_n] - \mathbf{f}_n^\dagger \mathbb{E}[\mathbf{r}] \end{aligned}$$

We check that such an extremum corresponds to a minimum :

$$\begin{cases} \frac{\partial^2 \mathcal{J}(\mathbf{f}, p)}{\partial \mathbf{f}^2} = 2\mathbf{R}_r \geq 0 \\ \frac{\partial^2 \mathcal{J}(\mathbf{f}, p)}{\partial p^{*2}} = 2 \geq 0 \end{cases}$$

3.2 Computation of the MSE minimizers

Let us first compute :

$$\begin{aligned} \mathbb{E}[s_n] &= s_n^d \\ \mathbb{E}[\mathbf{r}] &= \mathbf{H} \mathbf{U} s_n^d \\ \mathbf{R}_r &= \mathbf{H} \mathbf{U} \mathbf{R}_s \mathbf{D} \mathbf{H}^\dagger + \sigma_w^2 \mathbf{I}_N \\ \mathbf{R}_{rs_n} &= \mathbf{H} \mathbf{U} \mathbf{R}_{s s_n} \end{aligned}$$

Then, the MSE minimizers can be computed:

$$\begin{aligned}
\mathbf{f}_n^\dagger &= (\mathbf{R}_{s_n \mathbf{r}} - p_n \mathbb{E}[\mathbf{r}^\dagger]) \mathbf{R}_{\mathbf{r}}^{-1\dagger} \\
&= \left(\mathbf{R}_{s_n \mathbf{r}} - \mathbb{E}[s_n] \mathbb{E}[\mathbf{r}^\dagger] + \mathbf{f}_n^\dagger |\mathbb{E}[\mathbf{r}]|^2 \right) \mathbf{R}_{\mathbf{r}}^{-1\dagger} \\
&= (\mathbf{R}_{s_n \mathbf{r}} - \mathbb{E}[s_n] \mathbb{E}[\mathbf{r}^\dagger]) \mathbf{R}_{\mathbf{r}}^{-1\dagger} (\mathbf{I}_N - |\mathbb{E}[\mathbf{r}]|^2 \mathbf{R}_{\mathbf{r}}^{-1\dagger})^{-1} \\
&= (\mathbf{R}_{s_n \mathbf{r}} - \mathbb{E}[s_n] \mathbb{E}[\mathbf{r}^\dagger]) (\mathbf{R}_{\mathbf{r}}^{-1\dagger} - |\mathbb{E}[\mathbf{r}]|^2)^{-1} \\
&= (\mathbf{R}_{s_n \mathbf{s}} - \mathbb{E}[s_n] \mathbb{E}[\mathbf{s}^\dagger]) \mathbf{D} \mathbf{H}^\dagger \left(\mathbf{H} \mathbf{U} (\mathbf{R}_{\mathbf{s}} - \mathbb{E}[\mathbf{s} \mathbf{s}^\dagger]) \mathbf{D} \mathbf{H}^\dagger + \sigma_w^2 \mathbf{I}_{2N} \right)^{-1} \\
&= v_n^d \mathbf{e}_n^\dagger \mathbf{D} \mathbf{H}^\dagger (\mathbf{H} \mathbf{U} \mathbf{V}^d \mathbf{D} \mathbf{H}^\dagger + \sigma_w^2 \mathbf{I}_{2N})^{-1}
\end{aligned}$$

Using twice the Woodbury identity on any $(K_1 \times K_2)$ matrix A and $(K_2 \times K_1)$ matrix B :

$$\begin{aligned}
\mathbf{A}(\mathbf{B} \mathbf{A} + \mathbf{I}_{K_2})^{-1} &= \mathbf{A}(\mathbf{B} \mathbf{I}_{K_1} \mathbf{A} + \mathbf{I}_{K_2})^{-1} \\
&= \mathbf{A}(\mathbf{I}_{K_2} - \mathbf{B}(\mathbf{I}_{K_1} + \mathbf{A} \mathbf{B})^{-1} \mathbf{A}) \\
&= (\mathbf{I}_{K_1} - \mathbf{A} \mathbf{B}(\mathbf{I}_{K_1} + \mathbf{A} \mathbf{B})^{-1}) \mathbf{A} \\
&= (\mathbf{A} \mathbf{B} \mathbf{I}_{K_1} + \mathbf{I}_{K_1})^{-1} \mathbf{A} \\
&= (\mathbf{A} \mathbf{B} + \mathbf{I}_{K_1})^{-1} \mathbf{A}
\end{aligned}$$

We identify $\begin{cases} \mathbf{A} = \sigma_w^{-2} \mathbf{D} \mathbf{H}^\dagger \\ \mathbf{B} = \mathbf{H} \mathbf{U} \mathbf{V}^d \end{cases}$ leading to:

$$\begin{aligned}
\mathbf{f}_n^\dagger &= v_n^d \mathbf{e}_n^\dagger \mathbf{D} \mathbf{H}^\dagger (\mathbf{H} \mathbf{U} \mathbf{V}^d \mathbf{D} \mathbf{H}^\dagger + \sigma_w^2 \mathbf{I}_{2N})^{-1} \\
&= v_n^d \mathbf{e}_n^\dagger (\mathbf{D} \mathbf{H}^\dagger \mathbf{H} \mathbf{U} \mathbf{V}^d + \sigma_w^2 \mathbf{I}_N)^{-1} \mathbf{D} \mathbf{H}^\dagger \\
&= v_n^d \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{D} \mathbf{H}^\dagger
\end{aligned}$$

where $\boldsymbol{\Sigma} = \mathbf{G} \mathbf{V}^d + \sigma_w^2 \mathbf{I}_N$ is the MMSE equalization matrix and $\mathbf{G} = \mathbf{D} \mathbf{H}^\dagger \mathbf{H} \mathbf{U}$ is also called *global filter* and gathers the up-sampling, channel and matched filtering and down-sampling operations.

$$\begin{aligned}
p_n &= \mathbb{E}[s_n] - \mathbf{f}_n^\dagger \mathbb{E}[\mathbf{r}] \\
&= s_n^d - v_n^d \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{G} \mathbf{s}^d
\end{aligned}$$

3.3 Computation of the FS-MMSE-IC and its estimator

The FS-MMSE-IC estimator is given by:

$$\begin{aligned}
\tilde{s}_n^e &= \mathbf{f}_n^\dagger \mathbf{r} + p_n \\
&= s_n^d + v_n^d \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} (\mathbf{D} \mathbf{H}^\dagger \mathbf{r} - \mathbf{G} \mathbf{s}^d) \\
&= s_n^d + v_n^d \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \mathbf{G} \mathbf{s}^d)
\end{aligned}$$

where $\mathbf{y} = \mathbf{D} \mathbf{H}^\dagger \mathbf{r}$ represents the received signal after matched filtering and down-sampling.

The FS-MMSE-IC \tilde{v}_n^e of the estimator \tilde{s}_n^e can be computed by first deriving:

$$\begin{aligned}
\mathbf{R}_{\tilde{s}_n^e} &= |s_n^d|^2 + 2(s_n^d)^* v_n^d \mathbf{e}_n^\dagger \operatorname{Re}(\boldsymbol{\Sigma}^{-1} \mathbb{E}_{\mathbf{y}}[(\mathbf{y} - \mathbf{G}\mathbf{s}^d)]) + (v_n^d)^2 \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbb{E}_{\mathbf{y}}[(\mathbf{y} - \mathbf{G}\mathbf{s}^d)(\mathbf{y} - \mathbf{G}\mathbf{s}^d)^\dagger] \boldsymbol{\Sigma}^{-1} \mathbf{e}_n \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbb{E}_{\mathbf{s}, \mathbf{w}} |\mathbf{G}(\mathbf{s} - \mathbf{s}^d) + \mathbf{D}\mathbf{H}^\dagger \mathbf{w}|^2 \boldsymbol{\Sigma}^{-1} \mathbf{e}_n \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{D}\mathbf{H}^\dagger \mathbb{E}_{\mathbf{s}, \mathbf{w}} |\mathbf{H}\mathbf{U}(\mathbf{s} - \mathbf{s}^d) + \mathbf{w}|^2 \boldsymbol{\Sigma}^{-1} \mathbf{e}_n \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{D}\mathbf{H}^\dagger (\mathbf{H}\mathbf{U}\mathbf{V}^d \mathbf{D}\mathbf{H}^\dagger + \sigma_w^2 \mathbf{I}_{2N}) \mathbf{H}\mathbf{U} \boldsymbol{\Sigma}^{-1} \mathbf{e}_n \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} (\mathbf{D}\mathbf{H}^\dagger \mathbf{H}\mathbf{U}\mathbf{V}^d + \sigma_w^2 \mathbf{I}_N) \mathbf{D}\mathbf{H}^\dagger \mathbf{H}\mathbf{U} \boldsymbol{\Sigma}^{-1} \mathbf{e}_n \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{G} \mathbf{e}_n
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{\tilde{s}_n^e, s_n} &= \mathbb{E}_{\mathbf{y}}[(s_n^d + v_n^d \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \mathbf{G}\mathbf{s}^d)) s_n^*] \\
&= |s_n^d|^2 + v_n^d \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{G} \mathbb{E}_{\mathbf{s}}[(\mathbf{s} - \mathbf{s}^d) s_n^*] \\
&= |s_n^d|^2 + (v_n^d)^2 \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{G} \mathbf{e}_n \\
&= \mathbf{R}_{\tilde{s}_n^e}
\end{aligned}$$

$$\mathbf{R}_{s_n} = |s_n^d|^2 + v_n^d$$

This leads to the FS-MMSE-IC \tilde{v}_n^e expression:

$$\begin{aligned}
\tilde{v}_n^e &= \mathbb{E}_{\tilde{s}_n^e} |\tilde{s}_n^e - s_n|^2 \\
&= \mathbf{R}_{\tilde{s}_n^e} - \mathbf{R}_{\tilde{s}_n^e, s_n} - \mathbf{R}_{s_n, \tilde{s}_n^e} + \mathbf{R}_{s_n} \\
&= \mathbf{R}_{s_n} - \mathbf{R}_{s_n, \tilde{s}_n^e} \\
&= |s_n^d|^2 + v_n^d - |s_n^d|^2 - (v_n^d)^2 \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{G} \mathbf{e}_n \\
&= v_n^d (1 - v_n^d c_n)
\end{aligned}$$

where $c_n = \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{G} \mathbf{e}_n$.

3.4 Unbiased FS-MMSE-IC and its estimator

We can note that the estimator \tilde{s}_n^e of s_n has a residual bias :

$$\begin{aligned}
\mathbb{E}[s_n^e | \mathbb{E}[s_n] = \mu] &= \mathbb{E}_{\mathbf{s}_{\sim\{n\}}, \mathbf{w}} [s_n^d + v_n^d \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \mathbf{G}\mathbf{s}^d) | \mathbb{E}[s_n] = \mu] \\
&= s_n^d + v_n^d \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{G} \mathbb{E}_{\mathbf{s}_{\sim\{n\}}} [\mathbf{s} - \mathbf{s}^d | \mathbb{E}[s_n] = \mu] \\
&= s_n^d + (\mu - s_n^d) v_n^d c_n \\
&= v_n^d c_n \mu + s_n^d (1 - v_n^d c_n)
\end{aligned}$$

where the vector $\mathbf{s}_{\sim\{n\}}$ represents all the components of \mathbf{s} but s_n .

Removing the additive term $s_n^d (1 - v_n^d c_n)$ and dividing the multiplicative term $v_n^d c_n$ leads to:

$$\begin{aligned}
s_n^e &= (v_n^d c_n)^{-1} (\tilde{s}_n^e - s_n^d (1 - v_n^d c_n)) \\
&= s_n^d + c_n^{-1} \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \mathbf{G}\mathbf{s}^d)
\end{aligned}$$

We compute the new statistical matrices:

$$\begin{aligned}\mathbf{R}_{s_n^e} &= |s_n^d|^2 + c_n^{-2} \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbb{E}_{\mathbf{y}}[(\mathbf{y} - \mathbf{G}\mathbf{s}^d)(\mathbf{y} - \mathbf{G}\mathbf{s}^d)^\dagger] \boldsymbol{\Sigma}^{-1} \mathbf{e}_n \\ &= |s_n^d|^2 + c_n^{-1}\end{aligned}$$

$$\begin{aligned}\mathbf{R}_{s_n^e, s_n} &= \mathbb{E}_{\mathbf{y}}[(s_n^d + c_n^{-1} \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \mathbf{G}\mathbf{s}^d)) s_n^*] \\ &= |s_n^d|^2 + c_n^{-1} \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{G} \mathbb{E}_{\mathbf{s}}[(\mathbf{s} - \mathbf{s}^d) s_n^*] \\ &= |s_n^d|^2 + v_n^d\end{aligned}$$

$$\mathbf{R}_{s_n} = |s_n^d|^2 + v_n^d$$

And finally we find we unbiased FS-MMSE-IC:

$$\begin{aligned}v_n^e &= \mathbb{E}_{s_n^e} |s_n^e - s_n|^2 \\ &= \mathbf{R}_{s_n^e} - \mathbf{R}_{s_n^e, s_n} - \mathbf{R}_{s_n, s_n^e} + \mathbf{R}_{s_n} \\ &= |s_n^d|^2 + c_n^{-1} - |s_n^d|^2 - v_n^d - |s_n^d|^2 - v_n^d + |s_n^d|^2 + v_n^d \\ &= c_n^{-1} - v_n^d\end{aligned}$$

4 Conclusion

The biased FS-MMSE-IC receiver is given by:

$$\begin{cases} \tilde{s}_n^e &= s_n^d + v_n^d \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \mathbf{G}\mathbf{s}^d) \\ \tilde{v}_n^e &= v_n^d (1 - v_n^d c_n) \end{cases}$$

where $c_n = \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{G} \mathbf{e}_n$ is a scalar, $\mathbf{y} = \mathbf{D}\mathbf{H}^\dagger \mathbf{r}$ is the receive signal after matched filtering and down-sampling, $\mathbf{G} = \mathbf{D}\mathbf{H}^\dagger \mathbf{H}\mathbf{U}$ is the global effect including up-sampling, channel and matched filtering and down-sampling, $\boldsymbol{\Sigma} = \mathbf{G}\mathbf{V}^d + \sigma_w^2 \mathbf{I}_N$ is the MMSE equalization matrix.

Moreover, the unbiased FS-MMSE-IC receiver is given by:

$$\begin{cases} s_n^e &= s_n^d + c_n^{-1} \mathbf{e}_n^\dagger \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \mathbf{G}\mathbf{s}^d) \\ v_n^e &= c_n^{-1} - v_n^d \end{cases}$$

Consequently, the FS-MMSE-IC receiver leads to a FS matched filter followed by a symbol time equalizer.