



HAL
open science

Convergence analysis of a hp-finite element approximation of the time-harmonic Maxwell equations with impedance boundary conditions in domains with an analytic boundary

Serge Nicaise, Jérôme Tomezyk

► To cite this version:

Serge Nicaise, Jérôme Tomezyk. Convergence analysis of a hp-finite element approximation of the time-harmonic Maxwell equations with impedance boundary conditions in domains with an analytic boundary. 2019. hal-02063271

HAL Id: hal-02063271

<https://hal.science/hal-02063271>

Preprint submitted on 11 Mar 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Convergence analysis of a hp -finite element approximation of the time-harmonic Maxwell equations with impedance boundary conditions in domains with an analytic boundary

Serge Nicaise, Jérôme Tomezyk*

March 1, 2019

Abstract

We consider a non conforming hp -finite element approximation of a variational formulation of the time-harmonic Maxwell equations with impedance boundary conditions proposed in [5, §4.5.d]. The advantages of this formulation is that the variational space is embedded in H^1 as soon as the boundary is smooth enough (in particular it holds for domains with an analytic boundary) and standard shift theorem from [8] can be applied since the associated boundary value problem is elliptic. Finally in order to perform a wavenumber explicit error analysis of our problem, a splitting lemma and an estimation of the adjoint approximation quantity are proved by adapting to our system the results from [16, 17] obtained for the Helmholtz equation. Some numerical tests that illustrate our theoretical results are also presented. Analytic regularity results with bounds explicit in the wavenumber of the solution of a general elliptic system with lower order terms depending on the wavenumber are need and hence proved.

AMS (MOS) subject classification 35J57, 35B65, 65N12, 65N30

Key Words Maxwell equations, absorbing boundary conditions, smooth domains, finite elements

1 Introduction

In this paper we are interested in the time-harmonic Maxwell equations for electromagnetic waves in a bounded, simply connected domain Ω of \mathbb{R}^3 with an analytic boundary filled by an isotropic homogeneous material and an absorbing boundary condition (also called Leontovich condition). All together, the boundary value problem takes the form

$$(1.1) \quad \begin{cases} \operatorname{curl} \mathbf{E} - ik\mathbf{H} = \mathbf{0} & \text{and} & \operatorname{curl} \mathbf{H} + ik\mathbf{E} = \mathbf{J} & \text{in } \Omega, \\ \mathbf{H} \times \mathbf{n} - \lambda_{\text{imp}} \mathbf{E}_t = \mathbf{0} & & & \text{on } \partial\Omega. \end{cases}$$

Here \mathbf{E} is the electric part and \mathbf{H} is the magnetic part of the electromagnetic field, and the real number k corresponds to the wave number and is, for the moment, supposed to be non-negative. The right-hand side \mathbf{J} is the current density which – in the absence of free electric charges – is divergence free, namely

$$\operatorname{div} \mathbf{J} = 0 \quad \text{in } \Omega.$$

*Univ. Polytechnique Hauts-de-France, EA 4015 - LAMAV - FR CNRS 2956, F-59313 Valenciennes, France, Serge.Nicaise,Jerome.Tomezyk@uphf.fr

As usual, \mathbf{n} is the unit vector normal to $\partial\Omega$ pointing outside Ω and $\mathbf{E}_t = \mathbf{E} - (\mathbf{E} \cdot \mathbf{n}) \mathbf{n}$ is the tangential component of \mathbf{E} . The impedance λ_{imp} is a smooth (analytic) function defined on $\partial\Omega$ satisfying

$$(1.2) \quad \lambda_{\text{imp}} : \partial\Omega \rightarrow \mathbb{R}, \quad \text{such that } \forall x \in \partial\Omega, \lambda_{\text{imp}}(x) > 0,$$

see for instance [23, 22]. The case $\lambda_{\text{imp}} \equiv 1$ is also called the Silver-Müller boundary condition [1].

In practice absorbing boundary conditions are used to reduce an unbounded domain of calculations into a bounded one, see [23, 22].

As variational formulation, a first attempt is to eliminate \mathbf{H} by the relation $\mathbf{H} = \frac{1}{ik} \text{curl } \mathbf{E}$, that transforms the impedance condition in the form

$$(\text{curl } \mathbf{E}) \times \mathbf{n} - ik\lambda_{\text{imp}} \mathbf{E}_t = \mathbf{0} \quad \text{on } \partial\Omega.$$

Unfortunately such a boundary condition has no meaning in $\mathbf{H}(\text{curl}, \Omega)$, hence a solution is to introduce the subspace

$$\mathbf{H}_{\text{imp}}(\Omega) = \{\mathbf{u} \in \mathbf{H}(\text{curl}; \Omega) : \gamma_0 \mathbf{u}_t \in \mathbf{L}^2(\partial\Omega)\}.$$

Then eliminating \mathbf{H} in the second identity of (1.1), and multiplying by a test function, we arrive at

$$(1.3) \quad \int_{\Omega} (\text{curl } \mathbf{E} \cdot \text{curl } \bar{\mathbf{E}}' - k^2 \mathbf{E} \cdot \bar{\mathbf{E}}') dx - ik \int_{\partial\Omega} \lambda_{\text{imp}} \mathbf{E}_t \cdot \bar{\mathbf{E}}'_t d\sigma \\ = ik \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{E}}' dx, \quad \forall \mathbf{E}' \in \mathbf{H}_{\text{imp}}(\Omega).$$

Error analyses of (1.3) using Nédélec elements are available in [22, 10], but no explicit dependence with respect to k is proved. Moreover there is no hope to get easily regularity results of the solution by applying the theory of elliptic boundary value problems to the system associated with (1.3) because it is not elliptic (see [5, §4.5.d]). Let us mention recent results from [21], where explicit wavenumber error analyses using hp -Nédélec elements are obtained for the Maxwell system set in a ball with transparent boundary conditions.

A second attempt, proposed in [5, §4.5.d] for smooth boundaries and inspired from [23, §5.4.3], is to keep the full electromagnetic field and use the variational space

$$(1.4) \quad \mathbf{V} = \{(\mathbf{E}, \mathbf{H}) \in (\mathbf{H}(\text{curl}, \Omega) \cap \mathbf{H}(\text{div}, \Omega))^2 : \mathbf{H} \times \mathbf{n} = \lambda_{\text{imp}} \mathbf{E}_t \text{ on } \partial\Omega\},$$

considering the impedance condition in (1.1) as an essential boundary condition. Hence the proposed variational formulation is: Find $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ such that

$$(1.5) \quad \mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}), (\mathbf{E}', \mathbf{H}')) = \int_{\Omega} \left(ik \mathbf{J} \cdot \bar{\mathbf{E}}' + \mathbf{J} \cdot \text{curl } \bar{\mathbf{H}}' \right) dx, \quad \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V},$$

with the choice

$$\mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}), (\mathbf{E}', \mathbf{H}')) = a_{k,s}(\mathbf{E}, \mathbf{E}') + a_{k,s}(\mathbf{H}, \mathbf{H}') - ik \int_{\partial\Omega} (\lambda_{\text{imp}} \mathbf{E}_t \cdot \bar{\mathbf{E}}'_t + \frac{1}{\lambda_{\text{imp}}} \mathbf{H}_t \cdot \bar{\mathbf{H}}'_t) d\sigma,$$

with a positive real parameter s that may depend on k but is assumed to be in the fixed interval $[1, 2]$ (see [24] and section 2 below for some details) and

$$a_{k,s}(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\text{curl } \mathbf{u} \cdot \text{curl } \bar{\mathbf{v}} + s \text{div } \mathbf{u} \text{div } \bar{\mathbf{v}} - k^2 \mathbf{u} \cdot \bar{\mathbf{v}}) dx.$$

The natural norm $\|\cdot\|_k$ of \mathbf{V} associated with problem (1.5) is defined by

$$\begin{aligned} \|(\mathbf{E}, \mathbf{H})\|_k^2 &= \|\operatorname{curl} \mathbf{E}\|_\Omega^2 + \|\operatorname{div} \mathbf{E}\|_\Omega^2 + k^2 \|\mathbf{E}\|_\Omega^2 \\ &\quad + \|\operatorname{curl} \mathbf{H}\|_\Omega^2 + \|\operatorname{div} \mathbf{H}\|_\Omega^2 + k^2 \|\mathbf{H}\|_\Omega^2. \end{aligned}$$

This new formulation (1.5) has the advantage that its associated boundary value problem is an elliptic system (see [5, §4.5.d]), hence standard shift regularity results can be used. Nevertheless, this problem is still difficult to solve numerically as the wave number k is large, because oscillatory solutions exist and because of the so-called *pollution effect* [13, 14]: when the number of oscillations inside the propagation domain is important, the numerical solution is only meaningful under restrictive conditions on the mesh size. This effect is manifested by a gap between the error of the best approximation of the finite element scheme and the error of the numerical solution that is actually produced. This gap becomes more important as the frequency increases, unless additional discretization points per wavelength or higher order elements are employed. This problem, typical for wave type equations, is also related to a lack of stability of the finite element scheme, since the associated sesquilinear forms are not coercive. Consequently the quasi-optimality of the finite element solution in the energy norm is not guaranteed for arbitrary meshes, but is achieved only in an asymptotic range, i.e., for small enough mesh sizes, that depends on the frequency and the discretization order.

The behaviour of the asymptotic range with respect to the frequency, the mesh size, and the discretization order is the key to understand the efficiency of a finite element method. For the Helmholtz equation in domain with analytic boundaries, the asymptotic range for hp -finite element methods has been characterized in a sequence of papers by J.M. Melenk and collaborators [9, 16, 17]. For less regular boundaries, similar asymptotic ranges can be achieved using an expansion of the solution in powers of k [2].

The goal of the present paper is therefore to perform a similar analysis for the second variational problem of the time-harmonic Maxwell equations with impedance boundary conditions set on analytical domains. The advantage of this formulation is that it is well-posed in $H^1(\Omega)^6$, see [1, 5], and that the associated system is elliptic [5], therefore analytic regularity results can be reached. Nevertheless several difficulties appear: the first one is that the impedance boundary condition cannot be easily imposed in the finite element space, hence we here propose a non conforming approximation that consists in adding penalisation terms on the boundary. Secondly, following the approach from [16, 17], we split up the solution of (1.5) into a regular but oscillating part and a rough component that behaves nicely for large frequencies. This decomposition allows then to estimate the adjoint approximation quantity, hence to prove well-posedness of the discrete problem as well as to obtain some error estimates. Note that the estimation of the regular part heavily depends on analytic regularity of the solution of an elliptic system with lower order terms depending on the wavenumber k with bounds that explicitly depends on k . These bounds are obtained by combining analytic estimates of the same problem corresponding to $k = 0$ with bootstrapping and induction arguments.

Our paper is organized as follows: The variational formulation of the original problem and some useful properties are recalled in section 2. In section 3 a (non conforming) hp -finite element approximation is proposed and the Schatz argument is adapted to bound the error by the best approximation error. The next section 4 is devoted to the proof of the wavenumber explicit error analyses, the basis blocks being a splitting lemma and an estimation of the adjoint approximation quantity. Some numerical tests that confirm our theoretical analysis are presented in section 5. Analytic regularity results with bounds explicit in the wavenumber are postponed to Appendix A since we prove such results for general elliptic systems.

Let us finish this section with some notations used in the remainder of the paper. For a bounded domain D , the usual norm and semi-norm of $H^t(D)$ ($t \geq 0$) are denoted by $\|\cdot\|_{t,D}$ and $|\cdot|_{t,D}$, respectively. For $t = 0$, we will drop the index t . For shortness, we further write $\mathbf{H}^t(D) = H^t(D)^3$. Here and below γ_0 is a generic notation for the trace operator from $H^t(D)$ to $H^{t-\frac{1}{2}}(\partial D)$, for all $t > \frac{1}{2}$. The notation (\cdot, \cdot) corresponds to the $L^2(\Omega)^m$ -inner product, with $m = 1, 3$ or 6 .

Furthermore, the notation $A \lesssim B$ (resp. $A \gtrsim B$) means the existence of a positive constant¹ C_1 (resp. C_2), which is independent of A, B , the wave number k , the parameter s , any mesh size h , and any polynomial degree p such that $A \leq C_1 B$ (resp. $A \geq C_2 B$). The notation $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$ hold simultaneously.

2 The continuous problem

In this section, we briefly recall some useful results concerning the problem (1.5). First we recall that \mathbf{V} is continuously embedded into $(\mathbf{H}^1(\Omega))^2$, see for instance [1] or Lemma 4.5.5 of [5]. Consequently, if the impedance function λ_{imp} satisfies (1.2) and $-k^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in Ω , then for any $\mathbf{F} \in \mathbf{V}'$, the problem

$$(2.1) \quad \mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}); (\mathbf{E}', \mathbf{H}')) = \langle \mathbf{F}; (\mathbf{E}', \mathbf{H}') \rangle, \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V},$$

has a unique solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$. Furthermore for the particular choice

$$\langle \mathbf{F}; (\mathbf{E}', \mathbf{H}') \rangle = \int_{\Omega} \left(i\omega \mathbf{J} \cdot \bar{\mathbf{E}}' + \mathbf{J} \cdot \text{curl} \bar{\mathbf{H}}' \right) dx,$$

with $\mathbf{J} \in \mathbf{L}^2(\Omega)$, problem (2.1) reduces to (1.5). Hence under the previous assumptions and if $\mathbf{J} \in \mathbf{H}(\text{div}; \Omega)$, this last problem has a unique solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$, that owing to Lemma 4.5.9 of [5] is moreover solution of the original problem (1.1).

Now given two functions $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, we denote by $(\mathbf{E}, \mathbf{H}) = \mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$, the unique solution of

$$(2.2) \quad \mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}); (\mathbf{E}', \mathbf{H}')) = \int_{\Omega} \left(\mathbf{f}_1 \cdot \bar{\mathbf{E}}' + \mathbf{f}_2 \cdot \bar{\mathbf{H}}' \right) dx, \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V},$$

which corresponds to (2.1) with \mathbf{F} given by

$$\langle \mathbf{F}; (\mathbf{E}', \mathbf{H}') \rangle = ((\mathbf{f}_1, \mathbf{f}_2), (\mathbf{E}', \mathbf{H}')) = \int_{\Omega} \left(\mathbf{f}_1 \cdot \bar{\mathbf{E}}' + \mathbf{f}_2 \cdot \bar{\mathbf{H}}' \right) dx, \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V}.$$

Note also that the general considerations from [5, §4.5.d] implies that (\mathbf{E}, \mathbf{H}) is actually the solution of the elliptic boundary value system

$$(2.3) \quad \left\{ \begin{array}{l} L_{k,s}(\mathbf{E}) = \mathbf{f}_1 \\ L_{k,s}(\mathbf{H}) = \mathbf{f}_2 \\ \text{div } \mathbf{E} = 0 \\ \text{div } \mathbf{H} = 0 \\ T(\mathbf{E}, \mathbf{H}) = 0 \\ B_k(\mathbf{E}, \mathbf{H}) = 0 \end{array} \right\} \begin{array}{l} \text{in } \Omega, \\ \\ \\ \text{on } \partial\Omega, \end{array}$$

¹by a constant we mean a real number

where

$$\begin{aligned} L_{k,s}(\mathbf{u}) &= \operatorname{curl} \operatorname{curl} \mathbf{u} - s \nabla \operatorname{div} \mathbf{u} - k^2 \mathbf{u}, \\ T(\mathbf{E}, \mathbf{H}) &= \mathbf{H} \times \mathbf{n} - \lambda_{\text{imp}} \mathbf{E}_t, \\ B_k(\mathbf{E}, \mathbf{H}) &= (\operatorname{curl} \mathbf{H}) \times \mathbf{n} + \frac{1}{\lambda_{\text{imp}}} (\operatorname{curl} \mathbf{E})_t - \frac{ik}{\lambda_{\text{imp}}} \mathbf{H}_t + ik \mathbf{E} \times \mathbf{n}. \end{aligned}$$

The basic block for a wavenumber explicit error analysis of problem (2.2) is a so-called stability estimate at the energy level; for the Helmholtz equation, see [6, 9, 11], while for problem (1.3), see [12, 24]. Hence we make the following definition.

Definition 2.1 *We will say that system (2.3) satisfies the k -stability property with exponent $\alpha \geq 1$ (independent of k and s) if there exists $k_0 > 0$ such that for all $k \geq k_0$ and all $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, the solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ of (2.2) satisfies*

$$(2.4) \quad \|(\mathbf{E}, \mathbf{H})\|_k \lesssim k^\alpha (\|\mathbf{f}_1\|_{0,\Omega} + \|\mathbf{f}_2\|_{0,\Omega}).$$

Note that we have shown in [24, Lemma 5.6] that (2.4) always holds with $\alpha = 2$ for an appropriate choice of $s \in [1, 2]$. Note further that if Ω is star-shaped with respect to a point and $\operatorname{div} \mathbf{f}_1 = \operatorname{div} \mathbf{f}_2 = 0$, then (2.4) holds with $\alpha = 1$, see [12, Theorem 3.3] or [24, Theorem 5.3].

3 The discrete problem

3.1 The hp -nonconforming finite element method

To approximate problem (2.2) we will use a nonconforming finite element methods, because we can not impose the impedance boundary condition (the essential boundary condition) in the finite element space. Futhermore we cannot build an interpolation operator which preserves the essential condition. So, we have decided to penalize this condition.

Let \mathcal{T}_h be a partition of Ω into "simplicial" elements which are the image of the reference tetrahedron, denoted by \hat{K} , via an element map $F_K : \hat{K} \rightarrow K$ that satisfies (see Assumption 5.1 in [18]) the next assumption:

Hypothesis 3.1 (*Quasi-uniform regular triangulation*) *For each $K \in \mathcal{T}_h$, there exist mappings R_K and A_K which verify $F_K = R_K \circ A_K$, $\hat{K} = A_K(K)$ with (recalling that J_f is the Jacobian of f)*

- A_K is an affine transformation and R_K is a C^∞ transformation,
- $\|J_{A_K}\|_{\infty, \hat{K}} \leq C_{\text{affine}} h$, $\|(J_{A_K})^{-1}\|_{\infty, \hat{K}} \leq C_{\text{affine}} h^{-1}$,
- $\|(J_{R_K})^{-1}\|_{\infty, \hat{K}} \leq C_{\text{metric}}$, $\|\nabla^n R_K\|_{\infty, \hat{K}} \leq C_{\text{metric}} \beta^n n!$, $\forall n \in \mathbb{N}$,

with C_{affine} , C_{metric} , $\beta > 0$ independent of the maximal meshsize $h = \max_{K \in \mathcal{T}_h} h_K$, where h_K is the diamenter of the element K .

Let $\mathbf{S}_{h,p}$ be the hp -FEM space (without constraint on the boundary)

$$(3.1) \quad \mathbf{S}_{h,p} = \mathcal{S}_{h,p}(\Omega)^6,$$

with

$$(3.2) \quad \mathcal{S}_{h,p}(\Omega) = \{v \in H^1(\Omega) \mid v|_K \circ F_K \in \mathbb{P}^p, \forall K \in \mathcal{T}_h\}.$$

As we cannot add the essential boundary condition to our finite element space, we will use a discrete sesquilinear form, where we penalise this boundary condition. Therefore we define the discrete sesquilinear form $\mathbf{a}_{k,s,h,p}(\cdot, \cdot) : H^1(\Omega)^6 \times H^1(\Omega)^6 \rightarrow \mathbb{C}$ as follow

$$\begin{aligned} \mathbf{a}_{k,s,h,p}(\mathbf{u}, \mathbf{v}) &= \mathbf{a}_{k,s}(\mathbf{u}, \mathbf{v}) - \int_{\partial\Omega} (\operatorname{curl} \mathbf{E} \times \mathbf{n} + ik\mathbf{E}_t) \overline{\left(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H}' \times \mathbf{n}\right)} d\sigma \\ &\quad - \int_{\partial\Omega} \left(\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H} \times \mathbf{n}\right) \overline{(\operatorname{curl} \mathbf{E}' \times \mathbf{n} + ik\mathbf{E}'_t)} d\sigma \\ &\quad + \frac{p^2}{h} \sum_{f \in \mathcal{E}^B} \alpha_f \int_f \left(\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H} \times \mathbf{n}\right) \overline{\left(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H}' \times \mathbf{n}\right)} d\sigma, \end{aligned}$$

with $\mathbf{u} = (\mathbf{E}, \mathbf{H})$ and $\mathbf{v} = (\mathbf{E}', \mathbf{H}')$, and where \mathcal{E}^B is the set of faces of the triangulation included into $\partial\Omega$. Note that the last term of this right-hand side is a penalization term, while the two other added ones are introduced to guaranbee the consistency of the approximation scheme. The parameters α_f are positive constants that will be fixed large enough to ensure the coercivity of the form $\mathbf{a}_{k,s,h,p}$ (cf. (3.5) below).

Let us first check the consistency of the formulation, that is

Lemma 3.2 *Let $\mathbf{f} \in L^2(\Omega)^6$ and $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$ (i.e., solution of (2.2)), then*

$$\mathbf{a}_{k,s,h,p}(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega)^2.$$

Proof. Indeed, as $\mathbf{u} = (\mathbf{E}, \mathbf{H})$ satisfies $\mathbf{H} \times \mathbf{n} - \lambda_{\text{imp}} \mathbf{E}_t = 0$ on $\partial\Omega$, one has

$$\mathbf{a}_{k,s,h,p}(\mathbf{u}, \mathbf{v}) = \mathbf{a}_{k,s}(\mathbf{u}, \mathbf{v}) - \int_{\partial\Omega} (\operatorname{curl} \mathbf{E} \times \mathbf{n} + ik\mathbf{E}_t) \cdot \overline{\left(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H}' \times \mathbf{n}\right)} d\sigma.$$

As $\mathbf{f} \in L^2(\Omega)^6$ then $(\mathbf{E}, \mathbf{H}) \in \mathbf{H}^2(\Omega)^2$ (cf. [5]) and by Green's formula,

$$\begin{aligned} \int_{\Omega} (\operatorname{curl} \mathbf{E} \cdot \operatorname{curl} \overline{\mathbf{E}'} + s \operatorname{div} \mathbf{E} \operatorname{div} \overline{\mathbf{E}'} - k^2 \mathbf{E} \cdot \overline{\mathbf{E}'}) dx &= \int_{\Omega} L_{k,s} \mathbf{E} \cdot \overline{\mathbf{E}'} dx \\ &\quad + \int_{\partial\Omega} \left(\operatorname{curl} \mathbf{E} \times \mathbf{n} \cdot \overline{\mathbf{E}'_t} + s \operatorname{div} \mathbf{E} \overline{\mathbf{E}' \cdot \mathbf{n}} \right) d\sigma. \end{aligned}$$

Applying the previous identity to \mathbf{E} and \mathbf{H} , noticing that $\operatorname{div} \mathbf{E} = \operatorname{div} \mathbf{H} = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned} \mathbf{a}_{k,s,h,p}(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} (L_{k,s}(\mathbf{E}) \cdot \overline{\mathbf{E}'} + L_{k,s}(\mathbf{H}) \cdot \overline{\mathbf{H}'}) dx - \int_{\partial\Omega} ((\operatorname{curl} \mathbf{E} \times \mathbf{n}) + ik\mathbf{E}_t) \cdot \overline{\left(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H}' \times \mathbf{n}\right)} d\sigma \\ &\quad + \int_{\partial\Omega} \left((\operatorname{curl} \mathbf{E} \times \mathbf{n}) \cdot \overline{\mathbf{E}'_t} + s \operatorname{div} \mathbf{E} \overline{\mathbf{E}' \cdot \mathbf{n}} \right) d\sigma + \int_{\partial\Omega} \left((\operatorname{curl} \mathbf{H} \times \mathbf{n}) \cdot \overline{\mathbf{H}'_t} + s \operatorname{div} \mathbf{H} \overline{\mathbf{H}' \cdot \mathbf{n}} \right) d\sigma \\ &\quad - ik \int_{\partial\Omega} \left(\lambda_{\text{imp}} \mathbf{E}_t \cdot \overline{\mathbf{E}'_t} + \frac{1}{\lambda_{\text{imp}}} \mathbf{H}_t \cdot \overline{\mathbf{H}'_t} \right) d\sigma \\ &= \int_{\Omega} L_{k,s}(\mathbf{E}) \cdot \overline{\mathbf{E}'} dx + \int_{\Omega} L_{k,s}(\mathbf{H}) \cdot \overline{\mathbf{H}'} dx + \int_{\partial\Omega} B_k(\mathbf{E}, \mathbf{H}) \cdot \overline{\mathbf{H}'_t} d\sigma. \end{aligned}$$

As $B_k(\mathbf{E}, \mathbf{H}) = 0$, we conclude the consistency of the problem. ■

The discrete norm (related to the space $\mathbf{S}_{h,p}$) associated with the discrete sesquilinear form $\mathbf{a}_{k,s,h,p}$ is

$$\|\mathbf{u}\|_{k,h,p}^2 = \|\mathbf{u}\|_k^2 + \frac{p^2}{h} \sum_{f \in \mathcal{E}^B} \alpha_f \left\| \mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H} \times \mathbf{n} \right\|_f^2.$$

Remark 3.3 We can remark that for all $\mathbf{v} \in \mathbf{V}$, $\|\mathbf{v}\|_{k,h,p} = \|\mathbf{v}\|_k$.

In order to compensate the negative term in $\mathbf{a}_{k,s,h,p}(\cdot, \cdot)$, we introduce the sesquilinear form $\mathbf{b}_{k,s,h,p}(\cdot, \cdot) = \mathbf{a}_{k,s,h,p}(\cdot, \cdot) + 2k^2(\cdot, \cdot)$, which turns to be continuous and coercive. Before proving these properties, we introduce a useful technical lemma.

Lemma 3.4 Let $\mathbf{E}, \mathbf{E}', \mathbf{H}' \in \mathcal{S}_{h,p}(\Omega)^3$, then

$$(3.3) \quad \left| \int_{\partial\Omega} (-\text{curl } \mathbf{E} \times \mathbf{n} + ik\mathbf{E}_t) \cdot \overline{\left(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H}' \times \mathbf{n} \right)} d\sigma \right| \lesssim \frac{p}{\sqrt{h}} \left(\int_{\Omega} (|\text{curl } \mathbf{E}|^2 + k^2|\mathbf{E}|^2) dx \right)^{\frac{1}{2}} \times \left(\sum_{f \in \mathcal{E}^B} \left\| \mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H}' \times \mathbf{n} \right\|_f^2 d\sigma \right)^{\frac{1}{2}}.$$

Proof. First, by Cauchy-Schwarz inequality, we have

$$\left| \int_{\partial\Omega} (-\text{curl } \mathbf{E} \times \mathbf{n} + ik\mathbf{E}_t) \cdot \overline{\left(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H}' \times \mathbf{n} \right)} d\sigma \right| \lesssim \sum_{f \in \mathcal{E}^B} \left[\left(\int_f (|\text{curl } \mathbf{E} \times \mathbf{n}|^2 + k^2|\mathbf{E}_t|^2) d\sigma \right)^{\frac{1}{2}} \times \left(\left\| \mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H}' \times \mathbf{n} \right\|_f^2 \right)^{\frac{1}{2}} \right].$$

By using a covariant transformation, which preserves the curl, namely

$$(3.4) \quad \text{curl } \mathbf{E}(x) = \frac{DF_K(\hat{x})}{J_{F_K}(\hat{x})} \hat{\text{curl}} \hat{\mathbf{E}}(\hat{x}), \text{ for } x = F_K(\hat{x}),$$

with an inverse trace inequality (cf. Lemma 4.3 of [19]), we have

$$\int_f (|\text{curl } \mathbf{E} \times \mathbf{n}|^2 + k^2|\mathbf{E}_t|^2) d\sigma \lesssim \frac{p^2}{h} \int_{K_f} (|\text{curl } \mathbf{E}|^2 + k^2|\mathbf{E}|^2) dx,$$

where $K_f \in \mathcal{T}_h$ is the unique tetrahedron such that $f \subset \partial K_f$. The conclusion follows from the two above inequalities. ■

Now, we can show the coercivity of $\mathbf{b}_{k,s,h,p}$. Let $\mathbf{u} = (\mathbf{E}, \mathbf{H}) \in \mathbf{S}_{h,p}$ be fixed. Then

$$\begin{aligned} \Re(\mathbf{b}_{k,s,h,p}(\mathbf{u}, \mathbf{u})) &= \|\mathbf{u}\|_k^2 - 2\Re \left(\int_{\partial\Omega} (\text{curl } \mathbf{E} \times \mathbf{n} - ik\mathbf{E}_t) \cdot \overline{\left(\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H} \times \mathbf{n} \right)} d\sigma \right) \\ &\quad + \frac{p^2}{h} \Re \left(\sum_{f \in \mathcal{E}^B} \alpha_f \int_f \left| \mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H} \times \mathbf{n} \right|^2 d\sigma \right). \end{aligned}$$

We then need to estimate $A = \Re \left(\int_{\partial\Omega} (-\operatorname{curl} \mathbf{E} \times \mathbf{n} + ik\mathbf{E}_t) \cdot \overline{\left(\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H} \times \mathbf{n} \right)} d\sigma \right)$. But Lemma 3.4 and Young's inequality yield

$$\begin{aligned} A &\lesssim \frac{p}{\sqrt{h}} \left(\int_{\Omega} (|\operatorname{curl} \mathbf{E}|^2 + k^2 |\mathbf{E}|^2) dx \right)^{\frac{1}{2}} \left(\sum_{f \in \mathcal{E}^B} \left\| \mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H} \times \mathbf{n} \right\|_f^2 \right)^{\frac{1}{2}} \\ &\lesssim \frac{\epsilon}{2} \int_{\Omega} (|\operatorname{curl} \mathbf{E}|^2 + k^2 |\mathbf{E}|^2) dx + \frac{1}{2\epsilon} \frac{p^2}{h} \sum_{f \in \mathcal{E}^B} \left\| \mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H} \times \mathbf{n} \right\|_f^2, \end{aligned}$$

for all $\epsilon > 0$. Hence there exists a positive constant C such that

$$\Re(\mathbf{b}_{k,s,h,p}(\mathbf{u}, \mathbf{u})) \geq \|\mathbf{u}\|_k^2 - C\epsilon(\|\operatorname{curl} \mathbf{E}\|_{\Omega}^2 + k^2 \|\mathbf{E}\|_{\Omega}^2) + \frac{p^2}{h} \sum_{f \in \mathcal{E}^B} \left(\alpha_f - \frac{C}{\epsilon} \right) \left\| \mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}} \mathbf{H} \times \mathbf{n} \right\|_f^2,$$

for all $\epsilon > 0$. We then fix $\epsilon = \frac{1}{2C}$ and therefore by choosing $\alpha_f > 0$ large enough such that $\alpha_f \geq \frac{2C}{\epsilon} = 4C^2$, we deduce that

$$(3.5) \quad \Re(\mathbf{b}_{k,s,h,p}(\mathbf{u}, \mathbf{u})) \gtrsim \|\mathbf{u}\|_{k,h,p}^2.$$

The continuity of $\mathbf{b}_{k,s,h,p}$, namely

$$(3.6) \quad |\mathbf{b}_{k,s,h,p}(\mathbf{u}, \mathbf{v})| \lesssim \|\mathbf{u}\|_{k,h,p} \|\mathbf{v}\|_{k,h,p}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{S}_{h,p},$$

directly follows from the continuity of $\mathbf{a}_{k,s}$ and Lemma 3.4. Note that this argument also allows to show the continuity of $\mathbf{a}_{k,s,h,p}$.

Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega)^6$, we define the following approximated problem: Find $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$ such that

$$(3.7) \quad \mathbf{a}_{k,s,h,p}(\mathbf{u}_{h,p}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{S}_{h,p}.$$

Such $\mathbf{u}_{h,p}$, if it exists, is called a Galerkin solution.

We will now show that under an appropriate condition, (3.7) has a unique solution $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$ and give some error estimates.

Lemma 3.5 *Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega)^6$, $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$ and if $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$ is a solution of (3.7), then we have*

$$(3.8) \quad \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \lesssim \inf_{\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}} \|\mathbf{u} - \mathbf{v}_{h,p}\|_{k,h,p} + k \sup_{\mathbf{w}_{h,p} \in \mathbf{S}_{h,p}} \frac{|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{w}_{h,p})|}{\|\mathbf{w}_{h,p}\|_{\Omega}}.$$

Proof. Let $\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}$ be arbitrary, then by the triangle inequality, we have

$$\|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \leq \|\mathbf{u} - \mathbf{v}_{h,p}\|_{k,h,p} + \|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|_{k,h,p}.$$

Moreover

$$\begin{aligned} \|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|_{k,h,p}^2 &\lesssim \mathcal{R}(\mathbf{b}_{k,s,h,p}(\mathbf{v}_{h,p} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})) \\ &\lesssim |\mathbf{b}_{k,s,h,p}(\mathbf{v}_{h,p} - \mathbf{u}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})| + |\mathbf{b}_{k,s,h,p}(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})|. \end{aligned}$$

By the fact that $\mathbf{b}_{k,s,h,p} = \mathbf{a}_{k,s,h,p} + 2k^2(\cdot, \cdot)$ and the Galerkin orthogonality, we have

$$\begin{aligned} \|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|_{k,h,p}^2 &\lesssim |\mathbf{b}_{k,s,h,p}(\mathbf{v}_{h,p} - \mathbf{u}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})| + 2k^2 |(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})| \\ &\lesssim \|\mathbf{v}_{h,p} - \mathbf{u}\|_{k,h,p} \|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|_{k,h,p} + k^2 |(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})|. \end{aligned}$$

We then have

$$\|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \lesssim \|\mathbf{u} - \mathbf{v}_{h,p}\|_{k,h,p} + k \frac{|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})|}{\|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|_{k,h,p}}.$$

We conclude by the bound

$$\frac{|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})|}{\|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|_{k,h,p}} \leq \sup_{\mathbf{w}_{h,p} \in \mathbf{S}_{h,p}} \frac{|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{w}_{h,p})|}{\|\mathbf{w}_{h,p}\|_{k,h,p}},$$

and then by taking the infimum on $\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}$. ■

In order to control the second term of the right-hand side of (3.8), we introduce the quantity $\eta(\mathbf{S}_{h,p})$, called adjoint approximation quantity (cf. [16, 20, 2]):

$$(3.9) \quad \eta(\mathbf{S}_{h,p}) = \sup_{\mathbf{f} \in L^2(\Omega)^6} \inf_{\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}} \frac{\|\mathbb{S}_{k,s}^*(\mathbf{f}) - \mathbf{v}_{h,p}\|_{k,h,p}}{\|\mathbf{f}\|_{\Omega}},$$

where $\mathbb{S}_{k,s}^*(\mathbf{f}) = \overline{\mathbb{S}_{k,s}(\bar{\mathbf{f}})}$ is the adjoint operator of $\mathbb{S}_{k,s}(\mathbf{f})$.

Now we will use the Schatz argument (Aubin-Nitsche trick for the Helmholtz equation) [25] in order to bring out $\eta(\mathbf{S}_{h,p})$ and $\|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p}$ in (3.8) and obtain the following theorem.

Theorem 3.6 *There exists a positive constant C such that if $\eta(\mathbf{S}_{h,p}) < \frac{1}{kC}$, then for any $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega)^6$, if $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$ and if $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$ is a solution of (3.7), then*

$$(3.10) \quad \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \lesssim \inf_{\mathbf{v} \in \mathbf{S}_{h,p}} \|\mathbf{u} - \mathbf{v}\|_{k,h,p},$$

$$(3.11) \quad \|\mathbf{u} - \mathbf{u}_{h,p}\|_{\Omega} \lesssim \eta(\mathbf{S}_{h,p}) \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p}.$$

Proof. Let $\phi = \mathbb{S}_{k,s}^*(\mathbf{w}_{h,p})$, with $\mathbf{w}_{h,p} \in \mathbf{S}_{h,p}$, then for any $\phi_{h,p} \in \mathbf{S}_{h,p}$ owing to the continuity of $\mathbf{a}_{k,s,h,p}$ and the Galerkin orthogonality, one has

$$\begin{aligned} |(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{w}_{h,p})| &= |\mathbf{a}_{k,s,h,p}(\mathbf{u} - \mathbf{u}_{h,p}, \phi)| \\ &= |\mathbf{a}_{k,s,h,p}(\mathbf{u} - \mathbf{u}_{h,p}, \phi - \phi_{h,p})| \\ &\lesssim \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \|\phi - \phi_{h,p}\|_{k,h,p}. \end{aligned}$$

By the definition of $\eta(\mathbf{S}_{h,p})$ we can conclude that

$$(3.12) \quad k \frac{|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{w}_{h,p})|}{\|\mathbf{w}_{h,p}\|_{\Omega}} \lesssim k\eta(\mathbf{S}_{h,p}) \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p}.$$

We obtain by Lemma 3.5 and (3.12) the existence of a constant $C > 0$ such that

$$(1 - Ck\eta(\mathbf{S}_{h,p})) \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \lesssim \inf_{\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}} \|\mathbf{u} - \mathbf{v}_{h,p}\|_{k,h,p}.$$

This means that (3.10) holds as soon as $1 - Ck\eta(\mathbf{S}_{h,p})$ is positive.

It remains to estimate the L^2 norm. First by the definition of $\mathbb{S}_{k,s}^*$ and the Galerkin orthogonality, one has

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_{h,p}\|_{\Omega}^2 &= \mathbf{a}_{k,s,h,p}(\mathbf{u} - \mathbf{u}_{h,p}, \mathbb{S}_{k,s}^*(\mathbf{u} - \mathbf{u}_{h,p})) \\ &= \mathbf{a}_{k,s,h,p}(\mathbf{u} - \mathbf{u}_{h,p}, \mathbb{S}_{k,s}^*(\mathbf{u} - \mathbf{u}_{h,p}) - \mathbf{v}_{h,p}), \end{aligned}$$

for all $\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}$. By the continuity of $\mathbf{a}_{k,s,h,p}$ and the definition of $\eta(\mathbf{S}_{h,p})$, we conclude that

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_{h,p}\|_{\Omega}^2 &\leq C_c \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \|\mathbb{S}_{k,s}^*(\mathbf{u} - \mathbf{u}_{h,p}) - \mathbf{v}_{h,p}\|_{k,h,p} \\ &\leq C_c \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \eta(\mathbf{S}_{h,p}) \|\mathbf{u} - \mathbf{u}_{h,p}\|_{\Omega}. \end{aligned}$$

which proves (3.11). ■

Corollary 3.7 *Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega)^6$ and $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$. If $\eta(\mathbf{S}_{h,p}) < \frac{1}{kC}$, then problem (3.7) has a unique solution $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$.*

Proof. As $\mathbf{S}_{h,p}$ is finite-dimensional, problem (3.7) is a linear system. So, we just need to prove uniqueness to have existence. Let $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$ be such that $\mathbf{a}_{k,s,h,p}(\mathbf{u}_{h,p}, \mathbf{v}) = 0, \forall \mathbf{v} \in \mathbf{S}_{h,p}$. By Theorem 3.6 and if $\eta(\mathbf{S}_{h,p}) < \frac{1}{kC}$, we have (since $\mathbf{0}$ is the unique solution of (2.2) with $\mathbf{f} = \mathbf{0}$)

$$\|\mathbf{u}_{h,p}\|_{k,h,p} \lesssim \inf_{\mathbf{v} \in \mathbf{S}_{h,p}} \|\mathbf{v}\|_{k,h,p} = 0,$$

which shows the uniqueness. ■

We have shown that under the condition $\eta(\mathbf{S}_{h,p}) < \frac{1}{kC}$, there exists a unique (discrete) solution $\mathbf{u}_{h,p}$ to (3.7), this solution may then be called $\mathbb{S}_{k,s,h,p}(\mathbf{f})$. In the next sections, we will give reasonable conditions between k , h and p such that this condition holds. But before, we recall some interpolation error estimates.

3.2 Some interpolation error estimates

We will use the same interpolation operators as in the papers [20] and [16]. These operators are built from the following definition:

Definition 3.8 *(element-by-element construction, from [20])*

Let \hat{K} be the reference simplex of \mathbb{R}^3 . A polynomial Π is said to permit an element-by-element construction of polynomial of degree p for $u \in H^s(\hat{K})$, $s > \frac{3}{2}$, if

- (i) $\Pi(V) = u(V)$ for each vertices of \hat{K} ,
- (ii) for each edge e of \hat{K} , $\Pi|_e \in \mathcal{P}_p$ is the unique minimizer of

$$\Pi \rightarrow p^{\frac{1}{2}} \|\mathbf{u} - \Pi\|_e + \|\mathbf{u} - \Pi\|_{H_{00}^{\frac{1}{2}}(e)},$$

$$\text{where } \Pi \text{ verifies (i) and } \|v\|_{H_{00}^{\frac{1}{2}}(e)}^2 = \|v\|_{\frac{1}{2},e}^2 + \left\| \frac{v}{\sqrt{\text{dist}(\cdot, \partial e)}} \right\|_e^2,$$

(iii) for each face f of \hat{K} , $\Pi|_f \in \mathcal{P}_p$ is the unique minimizer of

$$\Pi \rightarrow p \|u - \Pi\|_f + \|u - \Pi\|_{1,f},$$

where Π verifies (i) and (ii).

J. M. Melenk and S. Sauter propose in [20] (see [16] for more details) two interpolants satisfying the conditions (i) to (iii) from Definition 3.8, the first one for general $H^s(\Omega)$ functions ($s > \frac{3}{2}$) and the second one more specific for analytic functions.

Lemma 3.9 *Let $v \in H^m(\Omega)$ with $m > 0$, and h_K the diameter of an element K , then we have*

$$\begin{aligned} |v|_{m,K} &\lesssim h_K^{\frac{d}{2}(1-m)} |\hat{v}|_{m,\hat{K}}, \\ |\hat{v}|_{m,\hat{K}} &\lesssim h_K^{\frac{d}{2}(m-1)} |v|_{m,K}, \end{aligned}$$

and, for $\hat{\mathbf{u}} \in \mathbf{H}^t(\hat{K})^2$, with $p+1 \geq t > \frac{3}{2}$, there exists $\hat{\Pi}_p \hat{\mathbf{u}} \in \mathbf{S}_{h,p}$ (satisfying the conditions (i) to (iii) from Definition 3.8), such that

$$\begin{aligned} \left\| \hat{\mathbf{u}} - \hat{\Pi}_p \hat{\mathbf{u}} \right\|_{t',\hat{K}} &\lesssim p^{-(t-t')} |\hat{\mathbf{u}}|_{t,\hat{K}}, \quad \forall t' \in [0, t], \\ \left\| \hat{\mathbf{u}} - \hat{\Pi}_p \hat{\mathbf{u}} \right\|_{t',\hat{f}} &\lesssim p^{-(t-1/2-t')} |\hat{\mathbf{u}}|_{t,\hat{K}}, \quad \forall t' \in [0, t-1/2]. \end{aligned}$$

Combining the two above results, for all $\mathbf{u} \in \mathbf{H}^t(\Omega)^2$, we obtain

$$\begin{aligned} \|\mathbf{u} - \Pi_p \mathbf{u}\|_{t',K} &\lesssim \left(\frac{h}{p}\right)^{t-t'} |\mathbf{u}|_{t,K}, \quad \forall t' \in [0, t], \\ \|\mathbf{u} - \Pi_p \mathbf{u}\|_{t',f} &\lesssim \left(\frac{h}{p}\right)^{t-t'-1/2} |\mathbf{u}|_{t,K}, \quad \forall t' \in [0, t-1/2], \end{aligned}$$

as well as

$$\|\mathbf{u} - \Pi_p \mathbf{u}\|_{t',\Omega} \lesssim \left(\frac{h}{p}\right)^{t-t'} |\mathbf{u}|_{t,\Omega}, \quad \forall t' \in [0, t].$$

Proof. The proof of this lemma can be found in [16, Theorem B.4] (applied to each component of the vector fields). ■

Lemma 3.10 *For $\beta > 0$, there exists $\sigma > 0$ such that for all analytic function \mathbf{u}_A satisfying*

$$|\mathbf{u}_A|_{n,K} \leq (2\beta \max(n, k))^n C_K, \quad \forall n \in \mathbb{N} : n \geq 2,$$

for all $K \in \mathcal{T}_h$ and some $C_K > 0$ (independent of n and k), there exists $\Pi_p \mathbf{u}_A \in \mathbf{S}_{h,p}$ (which respect to Definition 3.8) such that for $q \in \{0, 1, 2\}$,

$$\|\mathbf{u}_A - \Pi_p \mathbf{u}_A\|_{q,K} \lesssim h^{-q} C_K \left(\left(\frac{h}{h+\sigma}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1} \right).$$

Proof. With a scaling argument, we can apply Lemma C.3 of [16] to each component. ■

4 Wavenumber explicit error analyses

Here, following the approach from [16, 17], we will split up the solution of the adjoint problem (appearing in the definition of $\eta(\mathbf{S}_{h,p})$) in a H^2 -part and an analytical part. This decomposition allows to give an estimate of $k\eta(\mathbf{S}_{h,p})$, which depends on k , h and p and obtain some error estimates.

4.1 A splitting lemma

The aim of this part is to split the solution $\mathbf{u} = (\mathbf{E}, \mathbf{H})$ of problem (2.3) in two parts: an analytical part but strongly oscillating and a part only in $\mathbf{H}^2(\Omega)^2$ but weakly oscillating.

We start by introducing some technical tools:

- First, a frequency splitting, based on Fourier transform, which will be applied to the right-hand side \mathbf{f}_i ($i \in \{1, 2\}$). More precisely, we will split up \mathbf{f}_i in two parts, one part just in L^2 and the other one being analytic.
- Second, we will introduce two auxiliary problems and give a stability result for these problems.

4.1.1 Frequency splitting

The frequency splitting is done with the help of the Fourier transform and an extension operator. We recall that for a compactly supported function $u \in L^2(\mathbb{R}^3)$, its Fourier transform is

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} u(x) dx,$$

and this mapping can be extended into an isometry from $L^2(\mathbb{R}^3)$ into itself. Hence we denote by \mathcal{F}^{-1} its inverse transformation.

Let $\eta > 0$, we denote by $\chi_{\eta k}$ the indicator function of the ball $B_{\eta k}(0)$. Then, we define the low-pass frequency projection

$$(4.1) \quad L_{\mathbb{R}^d}(f) = \mathcal{F}^{-1}(\chi_{\eta k} \mathcal{F}(f)),$$

and the high-pass frequency projection

$$(4.2) \quad H_{\mathbb{R}^d}(f) = \mathcal{F}^{-1}((1 - \chi_{\eta k}) \mathcal{F}(f)), \forall f \in L^2(\mathbb{R}^d).$$

For $f \in L^2(\Omega)$, we set

$$E_{\Omega}(f) = \begin{cases} f & \text{in } \Omega, \\ 0 & \text{outside } \Omega. \end{cases}$$

as well as

$$\begin{aligned} L_{\Omega}(f) &= L_{\mathbb{R}^d}(E_{\Omega}(f))|_{\Omega}, \\ H_{\Omega}(f) &= H_{\mathbb{R}^d}(E_{\Omega}(f))|_{\Omega}. \end{aligned}$$

Theorem 4.1 *Let $\eta > 0$ be the parameter which is in the definition of $H_{\mathbb{R}^d}$ and $L_{\mathbb{R}^d}$, then for all $0 \leq t' \leq t$, $p \in \mathbb{N}^*$, and for each $f \in H^t(\mathbb{R}^3)$, we have*

$$\begin{aligned} \|H_{\mathbb{R}^3}(f)\|_{t', \mathbb{R}^3} &\lesssim (\eta k)^{t'-t} \|f\|_{t, \mathbb{R}^3}, \\ |L_{\mathbb{R}^3}(f)|_{p, \mathbb{R}^3} &\leq (\eta k)^p \|f\|_{\mathbb{R}^3}, \end{aligned}$$

while for all $f \in L^2(\Omega)$, we have

$$\begin{aligned}\|H_\Omega(f)\|_\Omega &\lesssim \|f\|_\Omega, \\ |L_\Omega(f)|_{p,\Omega} &\lesssim (\eta k)^p \|f\|_\Omega,\end{aligned}$$

with a constant independent of p .

Proof. Cf. Lemmas 4.2 and 4.3 of [20]. ■

4.1.2 Auxiliary problems

We will introduce two well-known problems which are useful for our splitting of \mathbf{u} .

The first problem is to consider $\mathbf{E} = N_k(\mathbf{f})$ solution of

$$(4.3) \quad \operatorname{curl} \operatorname{curl} \mathbf{E} - s \nabla \operatorname{div} \mathbf{E} - k^2 \mathbf{E} = \mathbf{f} \text{ in } \mathbb{R}^3.$$

As usual, $N_k(\mathbf{f})$ is obtained with the help of the Green function G (here, it is a matrix) of this problem, namely the distribution that satisfies

$$\operatorname{curl} \operatorname{curl} G(x) - s \nabla \operatorname{div} G(x) - k^2 G(x) = \delta_x \operatorname{Id}_3.$$

Applying the Fourier transform to this identity, direct calculations show that \hat{G} satisfies

$$M(\xi) \hat{G}(\xi) = \operatorname{Id}_3,$$

with

$$M(\xi) = \begin{pmatrix} |\xi|^2 - k^2 - (1-s)\xi_1^2 & -(1-s)\xi_1\xi_2 & -(1-s)\xi_1\xi_3 \\ -(1-s)\xi_1\xi_2 & |\xi|^2 - k^2 - (1-s)\xi_2^2 & -(1-s)\xi_2\xi_3 \\ -(1-s)\xi_1\xi_3 & -(1-s)\xi_2\xi_3 & |\xi|^2 - k^2 - (1-s)\xi_3^2 \end{pmatrix}.$$

Therefore

$$\hat{G}(\xi) = M(\xi)^{-1} \operatorname{Id}_3.$$

By direct calculations, we check that the eigenvalues of $M(\xi)^{-1}$ are $\frac{1}{s|\xi|^2 - k^2}$ and $\frac{1}{|\xi|^2 - k^2}$. Recalling that $s \in [1, 2]$, we get

$$(4.4) \quad \|M(\xi)^{-1}\| = \frac{1}{|\xi|^2 - k^2} \text{ if } |\xi| > k.$$

For $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}^3)$, we define $N_k(\mathbf{f})$ as the convolution product of G with \mathbf{f} , namely

$$N_k(\mathbf{f})(x) = (G * \mathbf{f})(x) = \int_{\mathbb{R}^3} G(x-y) \mathbf{f}(y) dy,$$

which verifies (4.3).

Now we want to estimate the norm of $N_k(H_\Omega \mathbf{f})$.

Lemma 4.2 *Let $\mathbf{f} \in L^2(\Omega)^3$, if $\mathbf{E} = N_k(H_\Omega \mathbf{f})$ then for all $q \in (0, 1)$, there exists $\eta > 0$ (appearing in the definition of L_Ω) such that*

$$(4.5) \quad \|\mathbf{E}\|_k \leq qk^{-1} \|\mathbf{f}\|_\Omega, \quad \|\mathbf{E}\|_{2,\Omega} \lesssim \|\mathbf{f}\|_\Omega.$$

Proof. We recall that $\mathbf{E} = G * (H_\Omega \mathbf{f})$ and fix $\eta > 1$. We start by estimating the L^2 norm of \mathbf{E} :

$$\begin{aligned} \|\mathbf{E}\|_{\mathbb{R}^3} &= \|\mathcal{F}(G * H_\Omega f)\|_{\mathbb{R}^3} \\ &= \left\| \hat{G}(1 - \chi_{\eta k}) \hat{\mathbf{f}} \right\|_{\mathbb{R}^3} \\ &= \left(\int_{\mathbb{R}^3} \left| M(\xi)^{-1} (1 - \chi_{\eta k}(\xi)) \hat{\mathbf{f}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{R}^3 \setminus B(\eta k)} \left| \frac{1}{|\xi|^2 - k^2} \right|^2 \left| \hat{\mathbf{f}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}}, \end{aligned}$$

this last estimate following from (4.4). As $|\xi| \geq \eta k$ on $\mathbb{R}^3 \setminus B(\eta k)$, we deduce that

$$\|\mathbf{E}\|_{\mathbb{R}^3} \leq \frac{1}{\eta^2 - 1} k^{-2} \|\mathbf{f}\|_{\Omega}.$$

Now, we estimate the H^1 norm of \mathbf{E} :

$$\begin{aligned} |\mathbf{E}|_{1, \mathbb{R}^3} &= \left(\sum_{i=1}^3 \int_{\mathbb{R}^3} \left| \xi_i M(\xi)^{-1} (1 - \chi_{\eta k}(\xi)) \hat{\mathbf{f}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^3 \int_{\mathbb{R}^3 \setminus B(\eta k)} \left| \frac{\xi_i}{|\xi|^2 - k^2} \right|^2 \left| \hat{\mathbf{f}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}}. \end{aligned}$$

As before we deduce that

$$|\mathbf{E}|_{1, \mathbb{R}^3} \leq \frac{1}{\eta - \frac{1}{\eta}} k^{-1} \|\mathbf{f}\|_{\Omega}.$$

We end up with the H^2 norm of \mathbf{E} :

$$\begin{aligned} |\mathbf{E}|_{2, \mathbb{R}^3} &= \left(\sum_{i,j=1}^3 \int_{\mathbb{R}^3} \left| \xi_i \xi_j M(\xi)^{-1} (1 - \chi_{\eta k}(\xi)) \hat{\mathbf{f}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i,j=1}^3 \int_{\mathbb{R}^3 \setminus B(\eta k)} \left| \frac{\xi_i \xi_j}{|\xi|^2 - k^2} \right|^2 \left| \hat{\mathbf{f}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}}. \end{aligned}$$

And again we obtain

$$|\mathbf{E}|_{2, \mathbb{R}^3} \leq \frac{1}{1 - \frac{1}{\eta^2}} \|\mathbf{f}\|_{\Omega}.$$

Hence, we have proved (4.5), for η large enough. \blacksquare

Now, we will study the second problem, namely: For $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega)^6$, we consider

$(\mathbf{V}_1, \mathbf{V}_2) = \mathbb{S}_{k,s}^+(\mathbf{f})$ solution of

$$(4.6) \quad \left\{ \begin{array}{l} L_{k,s}^+(\mathbf{V}_1) = L_{k,s}^+(N_k(H_\Omega \mathbf{f}_1)) \\ L_{k,s}^+(\mathbf{V}_2) = L_{k,s}^+(N_k(H_\Omega \mathbf{f}_2)) \\ \operatorname{div} \mathbf{V}_1 = 0 \\ \operatorname{div} \mathbf{V}_2 = 0 \\ T(\mathbf{V}_1, \mathbf{V}_2) = 0 \\ B_k(\mathbf{V}_1, \mathbf{V}_2) = 0 \end{array} \right\} \begin{array}{l} \text{in } \Omega, \\ \\ \\ \text{on } \partial\Omega. \end{array}$$

where $L_{k,s}^+(\mathbf{E}) = -\Delta \mathbf{E} + (1-s)\nabla \operatorname{div} \mathbf{E} + k^2 \mathbf{E}$. The existence of this solution as well as norm estimates are the goal of the next lemma.

Lemma 4.3 *Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega)^6$, then problem (4.6) has a unique solution and for all $q \in]0, 1[$, there exists $\eta > 0$ such that*

$$(4.7) \quad \left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_k \lesssim q^{\frac{1}{2}} k^{-1} \|\mathbf{f}\|_\Omega,$$

$$(4.8) \quad \left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_{2,\Omega} \lesssim \|\mathbf{f}\|_\Omega.$$

Proof. We first notice that the variational formulation of problem (4.6) is

$$\mathbf{b}_{k,s}(\mathbb{S}_{k,s}^+(\mathbf{f}), \mathbf{v}') = ((L_k^+(N_k(H_\Omega \mathbf{f}_1)), L_k^+(N_k(H_\Omega \mathbf{f}_2))), \mathbf{v}'), \forall \mathbf{v}' = (\mathbf{E}', \mathbf{H}') \in \mathbf{V},$$

with

$$\begin{aligned} \mathbf{b}_{k,s}(\mathbf{v}, \mathbf{v}') &= \int_\Omega (\operatorname{curl} \mathbf{V}_1 \cdot \overline{\operatorname{curl} \mathbf{E}'} + s \operatorname{div} \mathbf{V}_1 \overline{\operatorname{div} \mathbf{E}'} + k^2 \mathbf{V}_1 \cdot \overline{\mathbf{E}'}) \, dx \\ &\quad + \int_\Omega (\operatorname{curl} \mathbf{V}_2 \cdot \overline{\operatorname{curl} \mathbf{H}'} + s \operatorname{div} \mathbf{V}_2 \overline{\operatorname{div}(\mathbf{H}')} + k^2 \mathbf{V}_2 \cdot \overline{\mathbf{H}'}) \, dx \\ &\quad + ik \int_{\partial\Omega} \left(\lambda_{\operatorname{imp}}(\mathbf{V}_1)_t \cdot \overline{\mathbf{E}'_t} + \frac{1}{\lambda_{\operatorname{imp}}} (\mathbf{V}_2)_t \cdot \overline{\mathbf{H}'_t} \right) \, d\sigma, \end{aligned}$$

for all $\mathbf{v} = (\mathbf{V}_1, \mathbf{V}_2), \mathbf{v}' = (\mathbf{E}', \mathbf{H}') \in \mathbf{V}$.

The existence and uniqueness of a solution follows from Lax-Milgram lemma since the sesquilinear form $\mathbf{b}_{k,s}$ is coercive and continuous on \mathbf{V} .

Now by taking $\mathbf{v}' = \mathbb{S}_{k,s}^+(\mathbf{f}) = (\mathbf{V}_1, \mathbf{V}_2)$ and the real part, we have:

$$\begin{aligned} \left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_k^2 &= \Re(\mathbf{b}_{k,s}(\mathbb{S}_{k,s}^+(\mathbf{f}), \mathbb{S}_{k,s}^+(\mathbf{f}))) \\ &= \Re \left(\int_\Omega \left(L_{k,s}^+(N_k(H_\Omega \mathbf{f}_1)) \cdot \overline{\mathbf{V}_1} + L_{k,s}^+(N_k(H_\Omega \mathbf{f}_2)) \cdot \overline{\mathbf{V}_2} \right) \, dx \right). \end{aligned}$$

But by Green's formula, for $i = 1$ or 2 , we notice that

$$\begin{aligned}
\left| \int_{\Omega} L_{k,s}^+(N_k(H_{\Omega}\mathbf{f}_i)) \cdot \overline{\mathbf{V}}_i \, dx \right| &= \left| \int_{\Omega} (\operatorname{curl} N_k(H_{\Omega}\mathbf{f}_i) \cdot \operatorname{curl} \overline{\mathbf{V}}_i + s \operatorname{div} N_k(H_{\Omega}\mathbf{f}_i) \operatorname{div} \overline{\mathbf{V}}_i) \, dx \right. \\
&\quad + \int_{\Omega} k^2 N_k(H_{\Omega}\mathbf{f}_i) \cdot \overline{\mathbf{V}}_i \, dx + \int_{\partial\Omega} \operatorname{curl}(N_k(H_{\Omega}\mathbf{f}_i)) \times \mathbf{n} \cdot \overline{\mathbf{V}}_i \, d\sigma \\
&\quad \left. + \int_{\partial\Omega} \operatorname{div} N_k(H_{\Omega}\mathbf{f}_i) \overline{\mathbf{V}}_i \cdot \mathbf{n} \, d\sigma \right| \\
&\lesssim \|N_k(H_{\Omega}\mathbf{f}_i)\|_k \left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_k \\
&\quad + \left| \int_{\partial\Omega} \operatorname{div} N_k(H_{\Omega}\mathbf{f}_i) \overline{\mathbf{V}}_i \cdot \mathbf{n} \, d\sigma \right| + \left| \int_{\partial\Omega} \operatorname{curl} N_k(H_{\Omega}\mathbf{f}_i) \times \mathbf{n} \cdot \overline{\mathbf{V}}_i \, d\sigma \right|.
\end{aligned}$$

Now, we must estimate the boundary term. First Cauchy-Schwarz's inequality yields

$$\begin{aligned}
\left| \int_{\partial\Omega} \operatorname{div} N_k(H_{\Omega}\mathbf{f}_i) \overline{\mathbf{V}}_i \cdot \mathbf{n} \, d\sigma \right| &\lesssim \|\operatorname{div} N_k(H_{\Omega}\mathbf{f}_i)\|_{\partial\Omega} \|\mathbf{V}_i\|_{\partial\Omega}, \\
\left| \int_{\partial\Omega} \operatorname{curl} N_k(H_{\Omega}\mathbf{f}_i) \times \mathbf{n} \cdot \overline{\mathbf{V}}_i \, d\sigma \right| &\lesssim \|\operatorname{curl} N_k(H_{\Omega}\mathbf{f}_i)\|_{\partial\Omega} \|\mathbf{V}_i\|_{\partial\Omega}.
\end{aligned}$$

Second by a trace estimate and Young's inequality, we have

$$\begin{aligned}
\|\mathbf{V}_i\|_{\partial\Omega} &\lesssim \|\mathbf{V}_i\|_{\Omega}^{\frac{1}{2}} \|\mathbf{V}_i\|_{1,\Omega}^{\frac{1}{2}} \\
&\lesssim k^{-\frac{1}{2}} \left(k \|\mathbf{V}_i\|_{\Omega} + \|\mathbf{V}_i\|_{1,\Omega} \right) \\
&\lesssim k^{-\frac{1}{2}} \left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_k.
\end{aligned}$$

Thirdly by Lemma 4.2, we also get

$$\begin{aligned}
\|\operatorname{curl} N_k(H_{\Omega}\mathbf{f}_i)\|_{\partial\Omega} &\lesssim \|\operatorname{curl} N_k(H_{\Omega}\mathbf{f}_i)\|_{\Omega}^{\frac{1}{2}} \|\operatorname{curl} N_k(H_{\Omega}\mathbf{f}_i)\|_{1,\Omega}^{\frac{1}{2}} \\
&\lesssim \|N_k(H_{\Omega}\mathbf{f}_i)\|_k^{\frac{1}{2}} \|N_k(H_{\Omega}\mathbf{f}_i)\|_{2,\Omega}^{\frac{1}{2}} \\
&\lesssim q^{\frac{1}{2}} k^{-\frac{1}{2}} \|\mathbf{f}_i\|_{\Omega}.
\end{aligned}$$

In the same way, we obtain

$$\|\operatorname{div} N_k(H_{\Omega}\mathbf{f}_i)\|_{\partial\Omega} \lesssim q^{\frac{1}{2}} k^{-\frac{1}{2}} \|\mathbf{f}_i\|_{\Omega}.$$

These estimates lead to

$$\begin{aligned}
\left| \int_{\partial\Omega} \operatorname{div} N_k(H_{\Omega}\mathbf{f}_i) \overline{\mathbf{V}}_i \cdot \mathbf{n} \, d\sigma \right| &\lesssim q^{\frac{1}{2}} k^{-1} \|\mathbf{f}_i\|_{\Omega} \left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_k, \\
\left| \int_{\partial\Omega} \operatorname{curl} N_k(H_{\Omega}\mathbf{f}_i) \times \mathbf{n} \cdot \overline{\mathbf{V}}_i \, d\sigma \right| &\lesssim q^{\frac{1}{2}} k^{-1} \|\mathbf{f}_i\|_{\Omega} \left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_k.
\end{aligned}$$

Hence, by the previous estimates and Lemma 4.2, we have

$$\left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_k^2 \lesssim q^{\frac{1}{2}} k^{-1} \|\mathbf{f}\|_{\Omega} \left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_k,$$

which proves (4.7).

To estimate the H^2 norm of $\mathbb{S}_{k,s}^+(\mathbf{f})$, we apply Theorem 2.D of [5] (the constant being independent of s since the ellipticity of $L_{k,s}^+$ is continuous in $s \in [1, 2]$) to get

$$\left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_{2,\Omega} \lesssim k^2 \left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_{\Omega} \lesssim k \left\| \mathbb{S}_{k,s}^+(\mathbf{f}) \right\|_k,$$

which proves (4.8) owing to (4.7). ■

4.1.3 The splitting result

Now, we can state the main result of this part, namely the following decomposition theorem:

Theorem 4.4 *Assume that the k -stability property (2.4) holds with $\alpha \geq 1$. Let $\mathbf{u} = (\mathbf{E}, \mathbf{H}) = \mathbb{S}_{k,s}(\mathbf{f})$, where $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega)^6$, then there exist $\mathbf{u}_{\mathcal{A}}$ an analytical function and \mathbf{u}_{H^2} a H^2 function such that:*

$$\mathbf{u} = \mathbf{u}_{\mathcal{A}} + \mathbf{u}_{H^2},$$

with

$$(4.9) \quad \|\mathbf{u}_{\mathcal{A}}\|_k \lesssim k^\alpha \|\mathbf{f}\|_{\Omega},$$

$$(4.10) \quad |\mathbf{u}_{\mathcal{A}}|_{p,\Omega} \lesssim K^p \max(p, k)^p k^{\alpha-1} \|\mathbf{f}\|_{\Omega}, \quad \forall p \in \mathbb{N}, p \geq 2,$$

$$(4.11) \quad \|\mathbf{u}_{H^2}\|_k \lesssim k^{-1} \|\mathbf{f}\|_{\Omega},$$

$$(4.12) \quad \|\mathbf{u}_{H^2}\|_{2,\Omega} \lesssim \|\mathbf{f}\|_{\Omega},$$

for some constant $K \geq 1$.

To prove this theorem, we will need the following lemma:

Lemma 4.5 *Under the assumption of Theorem 4.4, let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega)^6$. Then $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$ admits the splitting*

$$\mathbf{u} = \mathbf{u}_{H^2} + \mathbf{u}_{\mathcal{A}} + \tilde{\mathbf{u}},$$

where $\tilde{\mathbf{u}} = \mathbb{S}_{k,s}(\tilde{\mathbf{f}})$ for some $\tilde{\mathbf{f}} \in L^2(\Omega)^6$ with

$$\left\| \tilde{\mathbf{f}} \right\|_{\Omega} \leq q' \|\mathbf{f}\|_{\Omega},$$

for some $q' \in (0, 1)$ and the following estimates hold

$$\|\mathbf{u}_{\mathcal{A}}\|_k \lesssim k^\alpha \|\mathbf{f}\|_{\Omega},$$

$$|\mathbf{u}_{\mathcal{A}}|_{p,\Omega} \lesssim K^p \max(p, k)^p k^{\alpha-1} \|\mathbf{f}\|_{\Omega}, \quad \forall p \in \mathbb{N} : p \geq 2,$$

$$\|\mathbf{u}_{H^2}\|_k \lesssim k^{-1} \|\mathbf{f}\|_{\Omega},$$

$$\|\mathbf{u}_{H^2}\|_{2,\Omega} \lesssim \|\mathbf{f}\|_{\Omega}.$$

Proof. We set

$$\mathbf{u}_{\mathcal{A}} = \mathbb{S}_{k,s}(L_{\Omega}(\mathbf{f})) \text{ and } \mathbf{u}_{H^2} = \mathbb{S}_{k,s}^+(\mathbf{f}).$$

Then, we see that

$$\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_{\mathcal{A}} - \mathbf{u}_{H^2}$$

verifies

$$(4.13) \quad \left\{ \begin{array}{l} L_{k,s}(\tilde{\mathbf{E}}) = \tilde{\mathbf{f}}_1, \\ L_{k,s}(\tilde{\mathbf{H}}) = \tilde{\mathbf{f}}_2, \\ \operatorname{div} \tilde{\mathbf{E}} = 0 \\ \operatorname{div} \tilde{\mathbf{H}} = 0 \\ T(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) = 0 \\ B(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) = 0 \end{array} \right\} \begin{array}{l} \text{in } \Omega, \\ \\ \\ \text{on } \partial\Omega. \end{array}$$

where $\tilde{\mathbf{f}} = 2k^2(\mathbb{S}_{k,s}^+(\mathbf{f}) - N_k(H_\Omega(\mathbf{f})))$.

Now, we will estimate the different norms. First the estimate on the norms of \mathbf{u}_{H^2} directly follows from Lemma 4.3. Secondly by Lemmas 4.3 and 4.2, we have

$$\|\tilde{\mathbf{f}}\|_\Omega = 2k^2 \left(\|\mathbb{S}_{k,s}^+(\mathbf{f})\|_\Omega + \|N_k(H_\Omega(\mathbf{f}))\|_\Omega \right) \leq Cq^{\frac{1}{2}} \|\mathbf{f}\|_\Omega \leq q' \|\mathbf{f}\|_\Omega,$$

where $q' = Cq^{\frac{1}{2}}$ that belongs to $]0, 1[$ for q small enough.

To estimate $\|\mathbf{u}_{\mathcal{A}}\|_k$, we simply use the k -stability property (2.4) to get

$$(4.14) \quad \|\mathbf{u}_{\mathcal{A}}\|_k \lesssim k^\alpha \|L_\Omega(\mathbf{f})\|_\Omega \lesssim k^\alpha \|\mathbf{f}\|_\Omega.$$

To estimate $|\mathbf{u}_{\mathcal{A}}|_{p,\Omega}$ with $p \geq 2$, we apply Theorem A.1 below and (4.14) to get

$$\begin{aligned} |\mathbf{u}_{\mathcal{A}}|_{p,\Omega} &\lesssim K^p \max(p, k)^p (\|\mathbf{f}\|_\Omega + k^{-1} \|\mathbf{u}_{\mathcal{A}}\|_k) \\ &\lesssim K^p \max(p, k)^p k^{\alpha-1} \|\mathbf{f}\|_\Omega. \end{aligned}$$

Lemma 4.3 directly furnishes the estimate of the norms of \mathbf{u}_{H^2} . ■

Now, we can prove Theorem 4.4.

Proof of Theorem 4.4. Let $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$, we apply the previous lemma, and obtain that there exists $q' \in (0, 1)$ such that

$$\mathbf{u} = \mathbf{u}_{\mathcal{A}}^1 + \mathbf{u}_{H^2}^1 + \tilde{\mathbf{u}}^1,$$

where $\tilde{\mathbf{u}}^1 = \mathbb{S}_{k,s}(\tilde{\mathbf{f}}^1)$ with $\|\tilde{\mathbf{f}}^1\|_\Omega \leq q' \|\mathbf{f}\|_\Omega$.

We iterate this procedure to get

$$\begin{aligned} \mathbf{u} &= \sum_{i=1}^{\infty} \mathbf{u}_{\mathcal{A}}^i + \sum_{i=1}^{\infty} \mathbf{u}_{H^2}^i \\ &= \mathbf{u}_{\mathcal{A}} + \mathbf{u}_{H^2}. \end{aligned}$$

We then have the right estimates by the previous lemma and the fact that $q' < 1$ (so that the associated geometric series converge). ■

4.2 Estimation of $k\eta(\mathbf{S}_{h,p})$

The approximation quantity $\eta(\mathbf{S}_{h,p})$ will be estimated by using the decomposition theorem applied to the adjoint problem.

Theorem 4.6 *Assume that the k -stability property (2.4) holds with $\alpha \geq 1$ and that $\frac{kh}{p} \lesssim 1$. Let $\mathbf{S}_{h,p}$ be previously defined, then we have*

$$(4.15) \quad k\eta(\mathbf{S}_{h,p}) \lesssim \left(\frac{kh}{\sqrt{p}} + k^\alpha \left(p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p \right) \right).$$

Proof. For any $\mathbf{f} \in L^2(\Omega)^6$, we apply the decomposition theorem 4.4 to $\mathbf{u} = \mathbb{S}_{k,s}^*(\mathbf{f})$ and obtain

$$\mathbf{u} = \mathbf{u}_{H^2} + \mathbf{u}_{\mathcal{A}}.$$

The analytical part highly dependent on k , while the H^2 part is less dependent on k , so we will estimate separately the two parts.

For \mathbf{u}_{H^2} , we use the same construction as in Theorem B.4 of [16] (Lemma 3.9), hence there exists $\mathbf{w}_{H^2} (= \Pi_p \mathbf{u}_{H^2}) \in \mathbf{S}_{h,p}$ such that

$$\|\mathbf{u}_{H^2} - \mathbf{w}_{H^2}\|_{t,\Omega} \lesssim \left(\frac{h}{p} \right)^{2-t} \|\mathbf{u}_{H^2}\|_{2,\Omega},$$

for all $0 \leq t < 2$. Hence

$$k \|\mathbf{u}_{H^2} - \mathbf{w}_{H^2}\|_k \lesssim \left(\frac{hk}{p} + \left(\frac{hk}{p} \right)^2 \right) \|\mathbf{f}\|_\Omega.$$

We now have to estimate the boundary term in $\|\mathbf{u}_{H^2} - \mathbf{w}_{H^2}\|_{k,h,p}$. This essentially follows from Lemma 3.9 and the estimate (4.12).

$$(4.16) \quad \begin{aligned} \frac{p^2}{h} \sum_{f \in \mathcal{E}^B} \alpha_f \left\| \left(\mathbf{E} - \mathbf{w}_{H^2}^1 \right)_t - \frac{1}{\lambda_{\text{imp}}} \left(\mathbf{H} - \mathbf{w}_{H^2}^2 \right) \times \mathbf{n} \right\|_f^2 &\lesssim \frac{p^2}{h} \sum_{f \in \mathcal{E}^B} \alpha_f \|\mathbf{u}_{H^2} - \mathbf{w}_{H^2}\|_f^2 \\ &\lesssim \frac{p^2}{h} \left(\frac{h}{p} \right)^3 \sum_{f \in \mathcal{E}^B} \alpha_f |\mathbf{u}_{H^2}|_{K_f}^2 \\ &\lesssim \left(\frac{h^2}{p} \right) \sum_{f \in \mathcal{E}^B} \alpha_f |\mathbf{u}_{H^2}|_{2,K_f}^2 \\ &\lesssim \left(\left(\frac{h}{\sqrt{p}} \right) \|\mathbf{f}\|_\Omega \right)^2. \end{aligned}$$

We hence obtain

$$(4.17) \quad k \|\mathbf{u}_{H^2} - \mathbf{w}_{H^2}\|_{k,h,p} \lesssim \left(\frac{kh}{\sqrt{p}} \right) \|\mathbf{f}\|_\Omega.$$

We now estimate the analytical part. The estimate (4.10) gives us

$$|\mathbf{u}_{\mathcal{A}}|_{n,\Omega} \leq C(\gamma \max(n, k))^n k^{\alpha-1} \|\mathbf{f}\|_\Omega, \quad \forall n \in \mathbb{N}, n \geq 2.$$

We then define C_K by

$$C_K^2 = \sum_{n \in \mathbb{N}: n \geq 2} \frac{\|\nabla^n \mathbf{u}_{\mathcal{A}}\|_K^2}{(2\gamma \max\{n, k\})^{2n}},$$

to have

$$|\mathbf{u}_{\mathcal{A}}|_{n,K} \leq (2\gamma \max\{n, k\})^n C_K, \forall n \in \mathbb{N} : n \geq 2,$$

but also

$$(4.18) \quad \sum_{K \in \mathcal{T}} C_K^2 \leq C k^{2(\alpha-1)} \|\mathbf{f}\|_{\Omega}^2.$$

We use Lemma 3.10 (cf. Lemma C.3 of [16]), to get, for $\sigma > 0$, the following estimate, for $q = 0, 1, 2$, with $\mathbf{w}_{\mathcal{A}} = \Pi_{\mathcal{A}} \mathbf{u}_{\mathcal{A}}$:

$$(4.19) \quad \|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_{q,K} \leq C h^{-q} C_K \left(\left(\frac{h}{h+\sigma} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right).$$

This estimate for $q = 0$ and 1 leads to

$$\begin{aligned} k^2 \|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_k^2 &= k^2 \sum_{K \in \mathcal{T}} (|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}|_{1,K}^2 + k^2 \|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_K^2) \\ &\lesssim k^2 (h^{-1} + k)^2 \left(\left(\frac{h}{h+\sigma} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right)^2 \left(\sum_{K \in \mathcal{T}} C_K^2 \right). \end{aligned}$$

Simple calculations yield

$$\begin{aligned} (h^{-1} + k) \left(\left(\frac{h}{h+\sigma} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right) &\lesssim (1 + kh) \left(\frac{h}{h+\sigma} \right)^p + \left(\frac{k}{p} + \frac{k^2 h}{p} \right) \left(\frac{kh}{\sigma p} \right)^p \\ &\lesssim \left(\frac{1}{p} + \frac{kh}{p} \right) \left(p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p \right) \\ &\lesssim p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p, \end{aligned}$$

recalling that $\frac{kh}{p} \lesssim 1$. These two estimates and (4.18) give

$$(4.20) \quad k \|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_k \lesssim \left(p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p \right) k^{\alpha} \|\mathbf{f}\|_{\Omega}.$$

As before we need to estimate the boundary term in $\|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_{k,h,p}$:

$$B = \frac{p^2}{h} \sum_{f \in \mathcal{E}^B} \alpha_f \left\| (\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}})_t - \frac{1}{\lambda_{\text{imp}}} (\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}) \times \mathbf{n} \right\|_f^2.$$

By using the trace estimate

$$\|v\|_{\partial K}^2 \leq C \left(\|v\|_K \|v\|_{1,K} + h^{-1} \|v\|_K^2 \right),$$

we get

$$\begin{aligned} B &\lesssim \frac{p^2}{h} \sum_{f \in \mathcal{E}^B} \alpha_f \|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_{\partial K_f}^2 \\ &\lesssim \frac{p^2}{h} \sum_{f \in \mathcal{E}^B} \alpha_f \left(\|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_{K_f} \|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_{1,K_f} + h^{-1} \|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_{K_f}^2 \right). \end{aligned}$$

By (4.19) with $q = 0$ or 1 , we obtain

$$B \lesssim \frac{p^2}{h^2} \left(\left(\frac{h}{h+\sigma} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right)^2 \left(\sum_{f \in \mathcal{E}^B} C_{K_f}^2 \right).$$

Again simple calculations yield

$$\frac{p}{h} \left(\left(\frac{h}{h+\sigma} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right) \lesssim p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p.$$

These two estimates and (4.18) give

$$B \lesssim \left(p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p \right)^2 k^{2(\alpha-1)} \|\mathbf{f}\|_{\Omega}^2.$$

Combining this estimate with (4.20), we get

$$(4.21) \quad k \|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_{k,h,p} \lesssim \left(p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p \right) k^{\alpha} \|\mathbf{f}\|_{\Omega}.$$

We can now estimate $k\eta(\mathbf{S}_{h,p})$, indeed the triangle inequality yields

$$k \|\mathbf{u} - \mathbf{w}_{H^2} - \mathbf{w}_{\mathcal{A}}\|_{k,h,p} \leq k \|\mathbf{u}_{H^2} - \mathbf{w}_{H^2}\|_{k,h,p} + k \|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\|_{k,h,p}.$$

By (4.17) and (4.21), we deduce that

$$k \|\mathbf{u} - \mathbf{w}_{H^2} - \mathbf{w}_{\mathcal{A}}\|_{k,h,p} \lesssim \left(\frac{kh}{\sqrt{p}} + k^{\alpha} \left(p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p \right) \right) \|\mathbf{f}\|_{\Omega},$$

which proves (4.15) because $\mathbf{w}_{H^2} + \mathbf{w}_{\mathcal{A}}$ belongs to $\mathbf{S}_{h,p}$. ■

Remark 4.7 In the previous proof, we can see that the term $\frac{kh}{\sqrt{p}}$ in the right-hand side of (4.15) appears because of the penalisation term (see (4.16)). Since this term is, up to the factor $\frac{h}{\sqrt{p}}$, bounded by the H^2 -norm of \mathbf{u}_{H^2} in a neighborhood of the boundary, we believe that this penalisation term is neglectable and that the term $\frac{kh}{\sqrt{p}}$ can be replaced by $\frac{kh}{p}$. This fact is confirmed by our numerical experiments.

In the same manner, we obtain the following convergence result:

Theorem 4.8 *Assume that $k\eta(\mathbf{S}_{h,p}) \leq \frac{1}{C}$. Let \mathbf{u} be the solution of (2.3) and $\mathbf{u}_{h,p}$ the solution of (3.7). Then, we have*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} &\lesssim \frac{h}{\sqrt{p}} + k^{\alpha-1} p \left(\frac{h}{h+\sigma} \right)^p + k^{\alpha} \left(\frac{kh}{\sigma p} \right)^p, \\ \|\mathbf{u} - \mathbf{u}_{h,p}\|_{0,\Omega} &\lesssim \left(\frac{h}{\sqrt{p}} + k^{\alpha-1} p \left(\frac{h}{h+\sigma} \right)^p + k^{\alpha} \left(\frac{kh}{\sigma p} \right)^p \right)^2. \end{aligned}$$

Proof. We use Theorem 3.6 and the same decomposition technique as for the estimate of $\eta(\mathbf{S}_{h,p})$. \blacksquare

For practical purposes, we formulate explicit conditions that guarantee $k\eta(\mathbf{S}_{h,p}) \leq \frac{1}{C}$ (compare with [16, Corollary 5.6]).

Theorem 4.9 *There exist three positive constants C_1 , C_2 and k_0 , such that if $k > k_0$ and*

$$(4.22) \quad \frac{kh}{\sqrt{p}} \leq C_1 \text{ and } \ln k \leq C_2 p,$$

then $k\eta(\mathbf{S}_{h,p}) \leq \frac{1}{C}$.

Proof. We just need to find some positive constants C_1 and C_2 , such that

$$(4.23) \quad \begin{cases} \frac{kh}{\sqrt{p}} \leq C_1 \\ \ln k \leq C_2 p \end{cases} \Rightarrow k\eta(\mathbf{S}_{h,p}) \leq \frac{1}{C}.$$

By (4.15), it is sufficient to control its right-hand side, namely to show that

$$(4.24) \quad \frac{kh}{\sqrt{p}} + k^\alpha \left(p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p \right) \leq \frac{1}{C} = C'.$$

We will first show that for all $C > 0$, $\gamma \in (0, 1)$ and $\delta \geq 0$, there exist $\beta > 0$ and $k_0 > 0$ such that if

$$(4.25) \quad \beta \ln k \leq \frac{p}{2} \text{ and } k > k_0,$$

then we have

$$(4.26) \quad k^\alpha p^\delta \gamma^p \leq C.$$

First we want to find $\beta > 0$ such that

$$(4.27) \quad \frac{\alpha}{|\ln \gamma|} \ln k - \frac{\ln C}{|\ln \gamma|} \leq \beta \ln k,$$

or equivalently

$$-\frac{\ln C}{|\ln \gamma|} \leq \left(\beta - \frac{\alpha}{|\ln \gamma|} \right) \ln k.$$

Consequently for $\beta > \frac{\alpha}{|\ln \gamma|} + 1$ and $k_0 \geq e^{-\frac{\ln C}{|\ln \gamma|}}$, (4.27) is valid.

Second, there exists $p_0 \geq 0$ such that for $p \geq p_0$ we have

$$(4.28) \quad \frac{p}{2} \leq p - \frac{\delta \ln p}{|\ln \gamma|}.$$

By (4.25), (4.27) and (4.28), we obtain

$$\frac{\alpha}{|\ln \gamma|} \ln k - \frac{\ln C}{|\ln \gamma|} \leq p - \frac{\delta \ln p}{|\ln \gamma|}.$$

And, since $\ln \gamma < 0$,

$$\alpha \ln k + (\ln \gamma)p + \delta \ln p \leq \ln C.$$

By taking the exponential, we get (4.26) with $\beta > \max\left(\frac{\alpha}{|\ln \gamma|} + 1, \frac{p_0}{2 \ln k_0}\right)$.

Now, we can control each term of the left hand-side of (4.24):

1. $\frac{kh}{\sqrt{p}} \leq \frac{C'}{3}$.
2. there exist $C_3 > 0$ and $k_{0,1} \geq e^{-\frac{\ln \frac{C'}{3}}{|\ln \gamma|}}$ with $\gamma = \frac{\text{diam}(\Omega)}{\text{diam}(\Omega) + \sigma}$ and $\delta = 1$ such that if $C_3 \ln k \leq p$ and $k > k_{0,1}$, then we have

$$k^\alpha p \left(\frac{h}{h + \sigma} \right)^p \leq k^\alpha p \left(\frac{\text{diam}(\Omega)}{\text{diam}(\Omega) + \sigma} \right)^p \leq \frac{C'}{3}.$$

3. there exist $C_4 > 0$ and $k_{0,2} \geq e^{-\frac{\ln \frac{C'}{3}}{|\ln \gamma|}}$ with $\gamma = \frac{1}{2}$ and $\delta = 0$, such that if $C_4 \ln k \leq p$ and $k > k_{0,2}$, then we have

$$k^{\alpha+1} \left(\frac{1}{2} \right)^p \leq \frac{C'}{3}.$$

Hence if $\frac{kh}{\sigma p} \leq \frac{kh}{\sigma \sqrt{p}} \leq \frac{1}{2}$, then

$$k^{\alpha+1} \left(\frac{kh}{\sigma p} \right)^p \leq \frac{C'}{3}.$$

Hence, (4.23) holds with $C_1 = \min\left(\frac{C'}{3}, \frac{\sigma}{2}\right)$, $\frac{1}{C_2} = \max(C_3, C_4)$ and $k_0 \geq \max(k_{0,1}, k_{0,2})$. ■

Remark 4.10 From the above proof, we see that C_1 and k_0 depend on $\frac{1}{C}$ (in such a way that if C is large, then C_1 is small and k_0 is large), while C_2 depends only on α , $\text{diam}(\Omega)$ and σ .

5 Some numerical tests

For the sake of simplicity, we restrict ourselves to the TE/TH polarization of the problem (2.3). In other words, we take

$$\Omega = D \times \mathbb{R},$$

where D is a two-dimensional disk and assume that the solution of our problem is independent of the third variable. In such a case, the original problem splits up into a TE polarization problem in (E_1, E_2, H_3) in D , and a TH polarization one in (H_1, H_2, E_3) in D , whose variational formulations are fully similar to (2.1). Namely, the TE problem reads (with $\lambda_{\text{imp}} = 1$): Find $(\mathbf{E}, q) \in \mathbf{V}$ such that

$$(5.1) \quad a_{k,s}((\mathbf{E}, q), (\mathbf{U}, v)) = (\mathbf{f}, (U, v)), \forall (\mathbf{U}, v) \in \mathbf{V},$$

with

$$\begin{aligned} a_{k,s}((\mathbf{E}, q), (\mathbf{U}, v)) &= \int_D (\text{curl } \mathbf{E} \cdot \text{curl } \bar{\mathbf{U}} + s \text{div } \mathbf{E} \text{div } \bar{\mathbf{U}} - k^2 \mathbf{E} \cdot \bar{\mathbf{U}}) dx \\ &\quad + \int_D (\nabla q \cdot \nabla \bar{v} - k^2 q \bar{v}) dx + ik \int_{\partial D} (\mathbf{E}_t \bar{\mathbf{U}}_t + q \bar{v}) d\sigma, \end{aligned}$$

and

$$\mathbf{V} = \{(\mathbf{U}, v) \mid \mathbf{U} \in H(\text{curl}, D) \cap H(\text{div}, D), v \in H^1(D) \text{ and } \mathbf{U}_t + v = 0 \text{ on } \partial D\}.$$

In our tests (with the discrete space $\mathcal{S}_{h,p}(D)^3$ and the sesquilinear form $a_{k,s,h,p}$ corresponding to the TE-formulation), we take $D = B(0,1)$ and use meshes built with the help of quadrangles of order 2. To illustrate our results, we consider two exact solutions, the first one is given by

$$\mathbf{E}_{ex}(x, y) = \begin{pmatrix} y \\ -x \end{pmatrix} q_{ex}(x, y) \text{ and } q_{ex}(x, y) = e^{ik(x^2+y^2)^{\frac{1-p}{2}}},$$

that belongs to $H^2(D)$ but is not in $H^3(D)$, while as second example we consider

$$\mathbf{E}_{ex}(x, y) = \begin{pmatrix} y \\ -x \end{pmatrix} q_{ex}(x, y) \text{ and } q_{ex}(x, y) = e^{ikx},$$

that, in this case, is analytical. In both cases, we compute the right-hand side of (2.2) accordingly. In our numerical experiments, we have chosen $s = 14.3$ and $\alpha_f = 10$, because they yield satisfactory numerical results. Figure 1 corresponds to the tests for the first solution, while Figure 2 corresponds to the tests for the one.

First to validate our method, we have computed the error in the norm $\|\cdot\|_{k,h,p}$ and compare it with the projection error $\|u - \mathbb{P}_{h,p}u\|_{k,h,p}$, where $\mathbb{P}_{h,p}$ is the orthogonal projection on $\mathbf{S}_{h,p}$ for the inner product associated with the norm $\|\cdot\|_{k,h,p}$, namely for $(\mathbf{U}, v) \in \mathbf{V}$, $\mathbb{P}_{h,p}(\mathbf{U}, v) \in \mathbf{S}_{h,p}$ is the unique solution of

$$(\mathbb{P}_h(\mathbf{U}, v), (\mathbf{U}'_h, v'_h))_{k,h,p} = ((\mathbf{U}, v), (\mathbf{U}'_h, v'_h))_{k,h,p}, \quad \forall (\mathbf{U}'_h, v'_h) \in \mathbf{S}_{h,p},$$

where

$$\begin{aligned} ((\mathbf{U}, v), (\mathbf{U}', v'))_{k,h,p} &= \int_D (\text{curl } \mathbf{U} \cdot \text{curl } \bar{\mathbf{U}}' + s \text{div } \mathbf{U} \text{div } \bar{\mathbf{U}}' + k^2 \mathbf{U} \cdot \bar{\mathbf{U}}') dx \\ &+ \int_D (\nabla v \cdot \nabla \bar{v}' + k^2 v \cdot \bar{v}') dx \\ &+ \frac{10p^2}{h} \int_{\partial D} (\mathbf{U}_t + q)(\bar{\mathbf{U}}'_t + \bar{v}) d\sigma. \end{aligned}$$

In Figures 1(a) and 1(b), we have depicted the two errors for the non smooth solution with $k = 30$ or 60 , $h = \frac{\pi}{10}$ or $h = \frac{\pi}{20}$ and various values of p , there we see that for p large enough we enter in the asymptotic regime (since both errors are almost equal) and the convergence rate is around 1.1 as theoretically expected. Similarly for the analytical solution, we can see in Figures 2(a) and 2(b), the convergence rate seems to be exponential. Let us notice that in the asymptotic regime, the error seems to reach a lower bound for the largest degrees of freedom. This can be explained by the fact that the error due to the variational crime (caused by the nonconformity of our meshes) becomes predominant with respect to the approximation error.

The second main result from section 4 states that if (4.22) holds, then

$$(5.2) \quad \|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{S}_{k,s,h,p}(\mathbf{f}_1, \mathbf{f}_2)\|_{k,h,p} \lesssim \|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{P}_{h,p}\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_{k,h,p}.$$

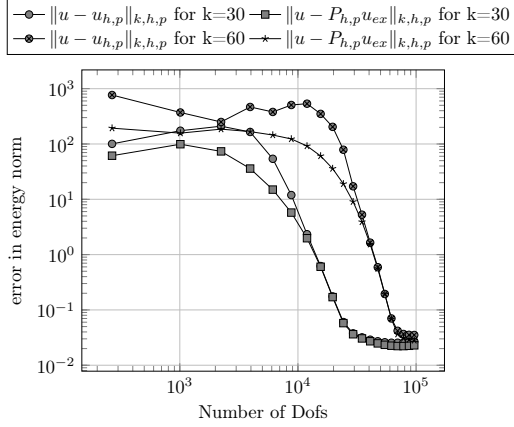
In order to see if this bound is sharp or not, we compute $\mathbb{S}_{k,s,h,p}(\mathbf{f}_1, \mathbf{f}_2)$ and $\mathbb{P}_{h,p}\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ for different values of h, p , and k . For different values of k, h , and p , we denote by p^* the smallest value p_0 such that

$$(5.3) \quad \|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{S}_{k,s,p,h}(\mathbf{f}_1, \mathbf{f}_2)\|_{k,h,p} \leq 2\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{P}_{h,p}\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_{k,h,p}, \quad \forall p \leq p_0.$$

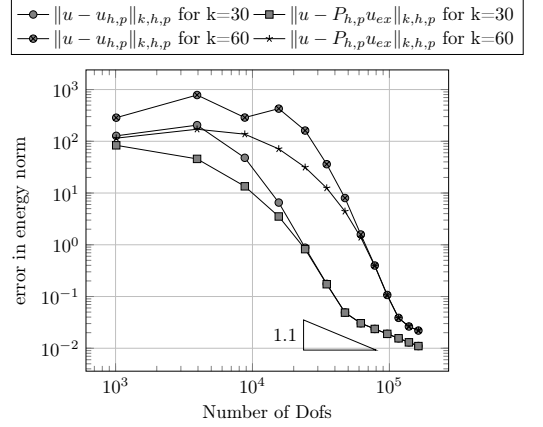
The value of p^* for a given pair (k, h) is obtained by inspecting the ratio

$$\frac{\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{S}_{k,s,p,h}(\mathbf{f}_1, \mathbf{f}_2)\|_{k,h,p}}{\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{P}_{h,p}\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_{k,h,p}}.$$

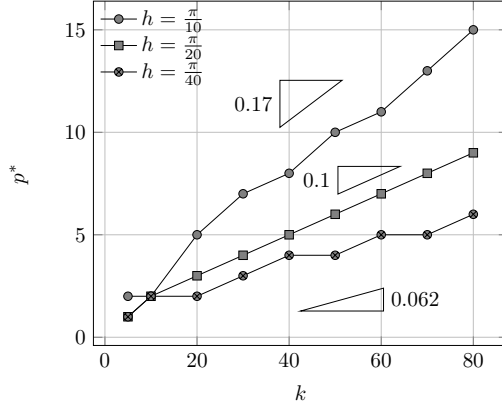
Condition (5.3) states that the finite element solution must be quasi optimal in the $\|\cdot\|_{k,h,p}$ norm, uniformly in k (with the arbitrary constant 2).



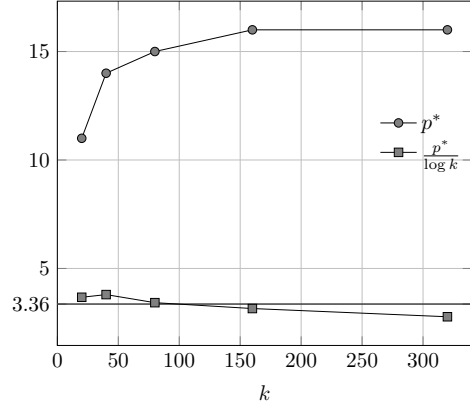
(a) Convergence curves with $h = \frac{\pi}{10}$ and $k = 30$ and 60



(b) Convergence curves with $h = \frac{\pi}{20}$ and $k = 30$ and 60



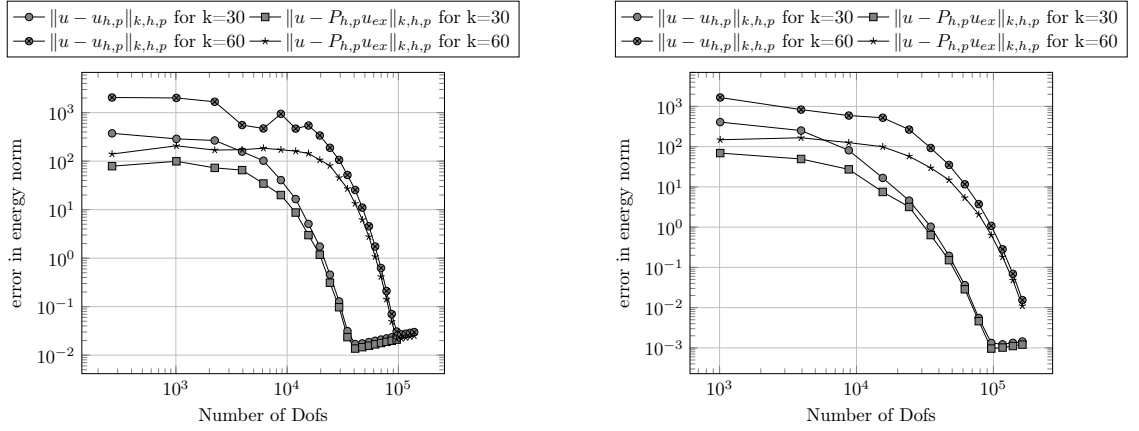
(c) p^* for different values of k and h .



(d) p^* for different values of k , with $kh = 5\pi$

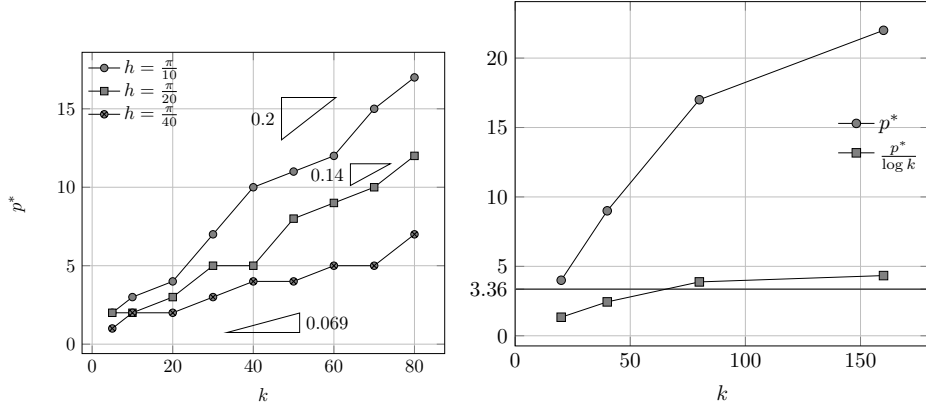
Figure 1: First experiment with $\mathbf{E}_{ex}(x, y) = (y, -x)q_{ex}(x, y)$ and $q_{ex}(x, y) = e^{ikr^{1.1}}$

We have compute $p^*(k)$ in two different ways: First, we have chosen the mesh size h independent of k . So, three values of the meshsize $h = \frac{\pi}{10}$, $\frac{\pi}{20}$ and $\frac{\pi}{40}$ have been fixed and we have computed the value of p^* for k varying from 5 to 80. The graph of $p^*(k)$ is represented in Figures 1(c) and 2(c). There we observe that in both cases $p^*(k) \sim k$, which is better than conditions (4.22) since for h bounded from below, these conditions are equivalent to $p \geq Ck^2$ for C large enough (but in accordance with our conjecture from Remark 4.7). Moreover, the slope seems to depend linearly on h , in other words, the condition on $p^*(k)$ seems to be $p^*(k) = Chk$. Secondly, we fix the product



(a) Convergence curves with $h = \frac{\pi}{10}$ and $k = 30$ and 60

(b) Convergence curves with $h = \frac{\pi}{20}$ and $k = 30$ and 60



(c) p^* for different values of k and h .

(d) p^* for different values of k , with $kh = 5\pi$

Figure 2: Second experiment with $\mathbf{E}_{ex} = (y, -x)q_{ex}(x, y)$ and $q_{ex}(x, y) = e^{ikx}$

kh to be constant (equal to 5π) with k varying from 20 to 320 for the non smooth solution (and 160 for the smooth solution) and again compute p^* as before. In that case, the conditions (4.22) are satisfied if $p \geq C \ln k$ for C large enough. This is confirmed experimentally since Figures 1(d) and 2(d) show a behavior of p^* of the order of $\ln k$.

Note finally that the numerical tests presented in this section are performed with the help of XLife++, a FEM library developed in C++ by P.O.E.M.S. (Ensta) and I.R.M.A.R. (Rennes) laboratories.

A Analytic regularity with bounds explicit in the wavenumber

In this section, we will prove the analytical regularity for the solution of the problem (2.3) with estimates explicit in the wavenumber k . For that purpose, the right-hand side \mathbf{f} is supposed to be an analytic function such that

$$(A.1) \quad |\mathbf{f}|_{p,\Omega} \leq C_{\mathbf{f}} \lambda_{\mathbf{f}}^p \max(p, k)^p, \quad \forall p \in \mathbb{N}.$$

Theorem A.1 *Let $\Omega \subset \mathbb{R}^n$, $n \leq 3$, be a bounded domain with an analytical boundary, and (L, D, B) an elliptic system in the sense of Definition 2.2.27 of [5] with L (resp. D and B) a $N \times N$ (resp. $N_0 \times N$ and $N_0 \times N$ with $N_0, N_1 \in \mathbb{N}^*$ such that $N_0 + N_1 = N$) system of differential operators of order 2 (resp. 0 and 1) with $N \in \mathbb{N}^*$ and $k > 1$. Let \mathbf{f} be an analytical function which verifies hypothesis (A.1) and \mathbf{G} a matrix with analytical coefficients. If \mathbf{u} is a solution of*

$$(A.2) \quad \begin{cases} L(\mathbf{u}) &= \mathbf{f} + k^2 \mathbf{u} & \text{in } \Omega, \\ D(\mathbf{u}) &= 0 & \text{on } \partial\Omega, \\ B(\mathbf{u}) &= k \mathbf{G} \mathbf{u} & \text{on } \partial\Omega, \end{cases}$$

then we have

$$|\mathbf{u}|_{p,\Omega} \leq C_{\mathbf{u}} K^p \max(p, k)^p, \quad \forall p \in \mathbb{N}, p \geq 2,$$

with $C_{\mathbf{u}} = C(C_{\mathbf{f}} + \|\mathbf{u}\|_{\Omega} + k^{-1} \|\mathbf{u}\|_{1,\Omega})$.

Corollary A.2 *When $\Omega \subset \mathbb{R}^3$ is a bounded domain with an analytical boundary and if we take $L = (L_{0,s}, L_{0,s})$, $D = T$, $B = (\text{div}, \text{div}, B_0)$ and $\mathbf{G} \mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ -i \mathbf{E} \times \mathbf{n} - \frac{i}{\lambda_{\text{imp}}} \mathbf{H}_t \end{pmatrix}$, then if \mathbf{u} is a solution of (2.3) with \mathbf{f} verifying the same hypothesis as in Theorem A.1. Then we have*

$$|\mathbf{u}|_{p,\Omega} \leq C_{\mathbf{u}} K^p \max(p, k)^p, \quad \forall p \in \mathbb{N}, p \geq 2,$$

with $C_{\mathbf{u}} = C(C_{\mathbf{f}} + \|\mathbf{u}\|_{\Omega} + k^{-1} \|\mathbf{u}\|_{1,\Omega})$.

Proof. The proof is the same as the previous Theorem, but in this case, L depends on s , which in practice depends on k . As s is supposed to be in the compact set $[1, 2]$, the ellipticity constant can be bounded independently from s . Hence, the estimates (A.14) and (A.21) below (standard elliptic regularity results in balls or half balls) remain valid with some constants independent of k . ■

Remark A.3 Theorem A.1 is applicable for the Helmholtz equation with the standard absorbing boundary conditions (of Robin type), see [17, p. 1225]. But it is also applicable for the time-harmonic elastodynamic system in an isotropic medium with the so-called Lysmer-Kuhlemeyer absorbing boundary conditions [15, 4, 7].

In order to prove this theorem, we will first introduce two auxiliary lemmas which give us regularity results in half balls with a boundary condition on the flat part (Lemma A.8) and balls without boundary condition (Lemma A.10).

By a covering of Ω by some well chosen balls, we can apply these two auxiliary lemmas to obtain Theorem A.1.

A.1 Analytic regularity near the boundary

Let $B_R^+ = B(0, R) \cap \{x | x_n > 0\}$ and $\Gamma_R = \{x \in \overline{B_R^+} | x_n = 0\}$, with $R \in (0, 1]$.

Let \mathbf{f} be an analytical function and \mathbf{G} a matrix with analytical coefficients defined in $\overline{B_R^+}$ such that

$$(A.3) \quad \|\partial^\alpha \mathbf{f}\|_{\overline{B_R^+}} \leq C_{\mathbf{f}} \lambda_{\mathbf{f}}^{|\alpha|} \max(|\alpha|, k)^{|\alpha|}, \forall \alpha \in \mathbb{N}^n,$$

$$(A.4) \quad \|\partial^\alpha \mathbf{G}\|_{\infty, \overline{B_R^+}} \leq C_{\mathbf{G}} \lambda_{\mathbf{G}}^{|\alpha|} |\alpha|!, \forall \alpha \in \mathbb{N}^n,$$

for some $k \geq 1$ for some positive constants $C_{\mathbf{f}}$, $\lambda_{\mathbf{f}}$, $C_{\mathbf{G}}$ and $\lambda_{\mathbf{G}}$ independent of k .

Let $\mathbf{u} \in \mathbf{H}^2(B_R^+)$ be a solution of

$$(A.5) \quad \begin{cases} L(\mathbf{u}) &= \mathbf{f} + k^2 \mathbf{u} & \text{in } B_R^+, \\ D(\mathbf{u}) &= 0 & \text{on } \Gamma_R, \\ B(\mathbf{u}) &= k \mathbf{G} \mathbf{u} & \text{on } \Gamma_R, \end{cases}$$

where (L, D, B) is an elliptic system with analytical coefficients (in the above sense), with T (resp. B) an operator of order 0 (resp. 1).

For further purposes, we define a few norms

$$\begin{aligned} |\mathbf{u}|_{p,q,B_R^+} &:= \max_{\substack{|\alpha|=p \\ \alpha_n \leq q}} \|\partial^\alpha \mathbf{u}\|_{B_R^+}, \\ \llbracket \mathbf{u} \rrbracket_{p,q,B_R^+} &:= \max_{0 \leq \rho \leq \frac{R}{2p}} \rho^p |\mathbf{u}|_{p,q,B_{R-\rho}^+}, \text{ for all } p > 0, \\ \llbracket \mathbf{u} \rrbracket_{0,0,B_R^+} &:= \|\mathbf{u}\|_{B_R^+}, \\ \rho_*^2 \llbracket \mathbf{u} \rrbracket_{p,q,B_R^+} &:= \max_{0 \leq \rho \leq \frac{R}{2(p+1)}} \rho^{p+2} |\mathbf{u}|_{p,q,B_{R-(p+1)\rho}^+}, \\ |\mathbf{u}|_{p,\frac{1}{2},\Gamma_R} &:= \max_{\alpha' \in \mathbb{N}^{n-1}; |\alpha'|=p} \|\partial^{\alpha'} \mathbf{u}\|_{\frac{1}{2},\Gamma_R}, \\ \rho_*^{\frac{3}{2}} \llbracket \mathbf{u} \rrbracket_{p,\frac{1}{2},\Gamma_R} &:= \max_{0 \leq \rho \leq \frac{R}{2(p+1)}} \rho^{p+\frac{3}{2}} |\mathbf{u}|_{p,\frac{1}{2},\Gamma_{R-(p+1)\rho}}, \end{aligned}$$

for all $p, q \in \mathbb{N}$, $q \leq p$.

We will first estimate the norm of the tangential derivatives (and the normal derivative up to 2) by using standard analytic regularity of elliptic systems. Then, we will be able to estimate the complete norm $\llbracket \mathbf{u} \rrbracket_{p,q,B_R^+}$. So we start with an estimation of the norm of tangential derivatives $\llbracket \mathbf{u} \rrbracket_{p,2,B_R^+}$. Before let us prove the next technical results that allow to pass from a sum on the multi-indices into a sum on their lengths.

Lemma A.4 *Let h be a mapping from \mathbb{N} into $[0, \infty)$ and a multi-index $\alpha' \in \mathbb{N}^{n-1}$, for $n = 2$ or 3 . Then we have*

$$(A.6) \quad \sum_{\beta' \in \mathbb{N}^{n-1}; \beta' \leq \alpha'} h(|\beta'|) \leq 2 \sum_{p=0}^{|\alpha'|} h(p) e^{|\alpha'|-p}.$$

Proof. The estimate (A.6) being trivial for $n = 2$, we only need to consider the case $n = 3$. In this case, without loss of generality, we can assume that $\alpha' = (\alpha_1, \alpha_2)$ is such that $\alpha_2 \leq \alpha_1$. Now

since in the left-hand side of (A.6) the summand depends only on the length of β' , we may write

$$(A.7) \quad \sum_{\beta' \in \mathbb{N}^2; \beta' \leq \alpha'} h(|\beta'|) = \sum_{p=0}^{|\alpha'|} h(p) N_p,$$

where N_p is the number of pairs $\beta' = (\beta_1, \beta_2) \leq \alpha'$ of length p that can be explicitly computed:

$$N_p = \begin{cases} p+1 & \text{if } 0 \leq p \leq \alpha_2, \\ \alpha_2 + 1 & \text{if } \alpha_2 \leq p \leq \alpha_1, \\ |\alpha'| - p + 1 & \text{if } \alpha_1 \leq p \leq |\alpha'|. \end{cases}$$

Since

$$x \leq e^x, \quad \forall x \in [0, \infty),$$

we easily see that

$$N_p \leq 2e^{|\alpha'| - p}, \quad \forall p \in \{0, \dots, |\alpha'|\}.$$

This estimate and (A.7) yield (A.6). \blacksquare

Corollary A.5 *Let h be a mapping from \mathbb{N} into $[0, \infty)$ and a multi-index $\alpha \in \mathbb{N}^n$ with $\alpha_n \leq 1$. Then we have*

$$(A.8) \quad \sum_{\beta \in \mathbb{N}^n; \beta \leq \alpha} h(|\beta|) \leq 2\left(1 + \frac{1}{e}\right) \sum_{p=0}^{|\alpha|} h(p) e^{|\alpha| - p}.$$

Proof. If $n = 1$, (A.8) is direct, so we assume that $n = 2$ or 3 . If $\alpha_n = 0$, the assertion is a direct consequence of (A.6), while if $\alpha_n = 1$, we write

$$\sum_{\beta \in \mathbb{N}^n; \beta \leq \alpha} h(|\beta|) = \sum_{\beta = (\beta', 0) \in \mathbb{N}^n; \beta' \leq \alpha'} h(|\beta'|) + \sum_{\beta = (\beta', 1) \in \mathbb{N}^n; \beta' \leq \alpha'} h(|\beta'| + 1).$$

Then we apply the estimate (A.6) to each term of this right-hand side to get

$$\sum_{\beta \in \mathbb{N}^n; \beta \leq \alpha} h(|\beta|) = 2 \sum_{p=0}^{|\alpha'|} h(p) e^{|\alpha'| - p} + 2 \sum_{p=0}^{|\alpha'|} h(p+1) e^{|\alpha'| - p}.$$

We conclude by performing a simple change of unknowns in the second sum of this right-hand side and adding some non negative terms. \blacksquare

Lemma A.6 *There exist a positive constant C (depending on n), a positive constant $C_{tr,R}$ (depending only on $R \leq 1$), and a positive constant $\lambda'_G \geq \lambda_G$ such that for all $l \in \mathbb{N}$ and any $\mathbf{u} \in \mathbf{H}^{l+1}(B_R^+)$, we have*

$$(A.9) \quad \rho_{*}^{\frac{3}{2}} \|\mathbf{G}\mathbf{u}\|_{l, \frac{1}{2}, \Gamma_R} \leq CC_{tr,R} C_G \sum_{p=0}^{l+1} (\lambda'_G)^{l+1-p} \max(l+1, k)^{l+1-p} \|\mathbf{u}\|_{p, 2, B_R^+}.$$

Proof. \mathbf{G} being a matrix with analytical coefficients defined in B_R^+ , by a standard trace theorem, there exists a positive constant $C_{tr,R}$ depending only on $R \leq 1$ such that

$$(A.10) \quad \rho_*^{\frac{3}{2}} \llbracket \mathbf{G}\mathbf{u} \rrbracket_{l, \frac{1}{2}, \Gamma_R} \leq C_{tr,R} \left(\llbracket \mathbf{G}\mathbf{u} \rrbracket_{l,0,B_R^+} + \llbracket \mathbf{G}\mathbf{u} \rrbracket_{l+1,1,B_R^+} \right).$$

We now estimate each term of this right-hand side. First for any $|\alpha| \leq l+1$, Leibniz's rule and the assumption (A.4) yields

$$\begin{aligned} \|\partial^\alpha \mathbf{G}\mathbf{u}\|_{B_R^+} &\leq n \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta} \mathbf{G}\|_{\infty B_R^+} \|\partial^\beta \mathbf{u}\|_{B_R^+} \\ &\leq n C_{\mathbf{G}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \lambda_{\mathbf{G}}^{|\alpha|-|\beta|} (|\alpha| - |\beta|)! \|\partial^\beta \mathbf{u}\|_{B_R^+}. \end{aligned}$$

As one easily checks that

$$(A.11) \quad \frac{p!}{q!} \leq p^{p-q}, \quad \forall p, q \in \mathbb{N} : q \leq p,$$

together with the combinatorial inequality (that can be shown using the combinatorial interpretation of binomial coefficients, see [3, p. 328] or [5, p. 48])

$$\frac{\beta!}{\gamma!(\beta-\gamma)!} \leq \frac{|\beta|!}{|\gamma|!(|\beta|-|\gamma|)!},$$

we deduce that

$$(A.12) \quad \|\partial^\alpha \mathbf{G}\mathbf{u}\|_{B_R^+} \leq n C_{\mathbf{G}} \sum_{\beta \leq \alpha} \lambda_{\mathbf{G}}^{|\alpha|-|\beta|} \max(|\alpha|, k)^{|\alpha|-|\beta|} \|\partial^\beta \mathbf{u}\|_{B_R^+}.$$

Therefore, we may write

$$\begin{aligned} \llbracket \mathbf{G}\mathbf{u} \rrbracket_{l,0,B_R^+} &= \max_{0 \leq \rho \leq \frac{R}{2l}} \rho^l \max_{\substack{\alpha_n=0 \\ |\alpha|=l}} \|\partial^\alpha \mathbf{G}\mathbf{u}\|_{B_{R-l\rho}^+} \\ &\leq n C_{\mathbf{G}} \max_{0 \leq \rho \leq \frac{R}{2l}} \rho^l \max_{\substack{\alpha_n=0 \\ |\alpha|=l}} \sum_{\beta \leq \alpha} \lambda_{\mathbf{G}}^{l-|\beta|} \max(l, k)^{l-|\beta|} \|\partial^\beta \mathbf{u}\|_{B_{R-l\rho}^+}. \end{aligned}$$

As $R \leq 1$ and as $|\beta| \leq l$, we have $\rho^l \leq \rho^{|\beta|}$, and then

$$\llbracket \mathbf{G}\mathbf{u} \rrbracket_{l,0,B_R^+} \leq n C_{\mathbf{G}} \max_{\substack{\alpha_n=0 \\ |\alpha|=l}} \sum_{\beta \leq \alpha} \lambda_{\mathbf{G}}^{l-|\beta|} \max(l, k)^{l-|\beta|} \max_{0 \leq \rho \leq \frac{R}{2|\beta|}} \rho^{|\beta|} \|\partial^\beta \mathbf{u}\|_{B_{R-|\beta|\rho}^+}.$$

In the above estimate as $\beta \leq \alpha$ and $\alpha_n = 0$, β_n is trivially equal to zero, and we deduce that

$$\llbracket \mathbf{G}\mathbf{u} \rrbracket_{l,0,B_R^+} \leq n C_{\mathbf{G}} \max_{\substack{\alpha_n=0 \\ |\alpha|=l}} \sum_{\beta \leq \alpha} \lambda_{\mathbf{G}}^{l-|\beta|} \max(l, k)^{l-|\beta|} \llbracket \mathbf{u} \rrbracket_{|\beta|,0,B_R^+}.$$

Applying Lemma A.4 to the sum on β (recalling that $\alpha_n = 0$), we deduce that

$$(A.13) \quad \llbracket \mathbf{G}\mathbf{u} \rrbracket_{l,0,B_R^+} \leq 2n C_{\mathbf{G}} \sum_{p=0}^l (e\lambda_{\mathbf{G}})^{l-p} \max(l, k)^{l-p} \llbracket \mathbf{u} \rrbracket_{p,0,B_R^+}.$$

Similarly, using (A.12) we have

$$\begin{aligned} \llbracket \mathbf{G}\mathbf{u} \rrbracket_{l+1,1,B_R^+} &\leq \max_{0 \leq \rho \leq \frac{R}{2(l+1)}} \rho^{l+1} \max_{\substack{\alpha_n \leq 1 \\ |\alpha| = l+1}} \|\partial^\alpha \mathbf{G}\mathbf{u}\|_{B_{R-(l+1)\rho}^+} \\ &\leq nC_{\mathbf{G}} \max_{0 \leq \rho \leq \frac{R}{2(l+1)}} \rho^{l+1} \max_{\substack{\alpha_n \leq 1 \\ |\alpha| = l+1}} \sum_{\beta \leq \alpha} \lambda_{\mathbf{G}}^{|\alpha|-|\beta|} K^{|\alpha|-|\beta|} \max(|\alpha|, k)^{|\alpha|-|\beta|} \|\partial^\beta \mathbf{u}\|_{B_{R-(l+1)\rho}^+}. \end{aligned}$$

Since $|\alpha| = l + 1$, we get as before

$$\begin{aligned} \llbracket \mathbf{G}\mathbf{u} \rrbracket_{l+1,1,B_R^+} &\leq nC_{\mathbf{G}} \max_{0 \leq \rho \leq \frac{R}{2(l+1)}} \rho^{l+1} \max_{\substack{\alpha_n \leq 1 \\ |\alpha| = l+1}} \sum_{\beta \leq \alpha} \lambda_{\mathbf{G}}^{l+1-|\beta|} \max(l+1, k)^{l+1-|\beta|} \|\partial^\beta \mathbf{u}\|_{B_{R-(l+1)\rho}^+} \\ &\leq nC_{\mathbf{G}} \max_{\substack{\alpha_n \leq 1 \\ |\alpha| = l+1}} \sum_{\beta \leq \alpha} \lambda_{\mathbf{G}}^{l+1-|\beta|} \max(l+1, k)^{l+1-|\beta|} \llbracket \mathbf{u} \rrbracket_{|\beta|,1,B_R^+}. \end{aligned}$$

Applying Corollary A.5 to the summation on β , we conclude that

$$\llbracket \mathbf{G}\mathbf{u} \rrbracket_{l+1,1,B_R^+} \leq 2\left(1 + \frac{1}{e}\right) nC_{\mathbf{G}} \sum_{p=0}^{l+1} (e\lambda_{\mathbf{G}})^{l+1-p} \max(l+1, k)^{l+1-p} \llbracket \mathbf{u} \rrbracket_{p,1,B_R^+}.$$

This estimate and (A.13) in the estimate (A.10) lead to (A.9) with $\lambda'_{\mathbf{G}} = e\lambda_{\mathbf{G}}$ (as $\llbracket \mathbf{u} \rrbracket_{|\beta|,j,B_R^+} \leq \llbracket \mathbf{u} \rrbracket_{|\beta|,2,B_R^+}$, for $j = 0$ or 1). ■

Now we can estimate the different derivatives.

Lemma A.7 *Let $\mathbf{u} \in \mathbf{H}^2(B_R^+)$ be a solution of (A.5) with \mathbf{f} and \mathbf{G} analytic and satisfying (A.3)-(A.4). Then there exist $K > 1$ and $C_R > 1$ such that for all $p \geq 2$,*

$$\llbracket \mathbf{u} \rrbracket_{p,2,B_R^+} \leq C_{\mathbf{u}}(B_R^+) K^p \max(p, k)^p,$$

with $C_{\mathbf{u}}(B_R^+) = C_R(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_R^+} + k^{-1} \|\mathbf{u}\|_{1,B_R^+})$.

Proof. We will prove this result by induction, by applying a standard analytic regularity result (i.e. Proposition 2.6.6 of [5]), which gives us a real number $A \geq 1$ such that for all $p \geq 2$

$$(A.14) \quad \llbracket \mathbf{u} \rrbracket_{p,2,B_R^+} \leq \sum_{l=0}^{p-2} A^{p-1-l} \left(\rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,0,B_R^+} + \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R} \right) + A^{p-1} \sum_{l=0}^1 \llbracket \mathbf{u} \rrbracket_{l,l,B_R^+}.$$

Initialization: For $p = 2$, by (A.14), we have

$$\begin{aligned} \llbracket \mathbf{u} \rrbracket_{2,2,B_R^+} &\leq A \left(\rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{0,0,B_R^+} + \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{0,\frac{1}{2},\Gamma_R} \right) + A \sum_{l=0}^1 \llbracket \mathbf{u} \rrbracket_{l,l,B_R^+} \\ &\leq A \left(\rho_*^2 \llbracket \mathbf{f} + k^2 \mathbf{u} \rrbracket_{0,0,B_R^+} + \rho_*^{\frac{3}{2}} \llbracket k\mathbf{G}\mathbf{u} \rrbracket_{0,\frac{1}{2},\Gamma_R} \right) + A \sum_{l=0}^1 \llbracket \mathbf{u} \rrbracket_{l,l,B_R^+} \\ &\leq A \left(\|\mathbf{f}\|_{B_R^+} + k^2 \|\mathbf{u}\|_{B_R^+} + k \|\mathbf{G}\mathbf{u}\|_{\frac{1}{2},\Gamma_R} \right) + A \sum_{l=0}^1 \|\mathbf{u}\|_{l,B_R^+} \\ &\leq A \left(\|\mathbf{f}\|_{B_R^+} + (k^2 + 1) \|\mathbf{u}\|_{B_R^+} + kC_{tr,R} \|\mathbf{G}\mathbf{u}\|_{1,B_R^+} + \|\mathbf{u}\|_{1,B_R^+} \right), \end{aligned}$$

with the positive constant $C_{tr,R}$ introduced before. By noticing that

$$kC_{tr,R} \|\mathbf{G}\mathbf{u}\|_{1,B_R^+} \leq CkC_{tr,R}kC_{\mathbf{G}}(\|\mathbf{u}\|_{1,B_R^+} + \lambda_{\mathbf{G}} \|\mathbf{u}\|_{B_R^+}),$$

we then have

$$\begin{aligned} \llbracket \mathbf{u} \rrbracket_{2,2,B_R^+} &\leq A \left(\|\mathbf{f}\|_{B_R^+} + (k^2 + 1 + CC_{tr,R}C_{\mathbf{G}}\lambda_{\mathbf{G}}k) \|\mathbf{u}\|_{B_R^+} + (CC_{tr,R}C_{\mathbf{G}}k + 1) \|\mathbf{u}\|_{1,B_R^+} \right) \\ &\leq Ak^2 \left(C_{\mathbf{f}} + (2 + CC_{tr,R}C_{\mathbf{G}}\lambda_{\mathbf{G}}) \|\mathbf{u}\|_{B_R^+} + (CC_{tr,R}C_{\mathbf{G}} + 1)k^{-1} \|\mathbf{u}\|_{1,B_R^+} \right) \\ &\leq Ak^2 \max(2 + CC_{tr,R}C_{\mathbf{G}}\lambda_{\mathbf{G}}, CC_{tr,R}C_{\mathbf{G}} + 1)(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_R^+} + k^{-1} \|\mathbf{u}\|_{1,B_R^+}) \\ &\leq C_{\mathbf{u}}(B_R^+) \max(2, k)^2 \leq C_{\mathbf{u}}(B_R^+)K^2 \max(2, k)^2, \end{aligned}$$

with $C_R \geq A \max(2 + CC_{tr,R}C_{\mathbf{G}}\lambda_{\mathbf{G}}, CC_{tr,R}C_{\mathbf{G}} + 1) \geq 1$ and since $K \geq 1$.

Induction hypothesis: For all $2 \leq p' \leq p$, we have

$$(A.15) \quad \llbracket \mathbf{u} \rrbracket_{p',2,B_R^+} \leq C_{\mathbf{u}}(B_R^+)K^{p'} \max(p', k)^{p'}.$$

We will show this estimate for $p + 1$: Using (A.14), we may write

$$(A.16) \quad \llbracket \mathbf{u} \rrbracket_{p+1,2,B_R^+} \leq \sum_{l=0}^{p-1} A^{p-l} \left(\rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,0,B_R^+} + \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R} \right) + A^p \sum_{l=0}^1 \llbracket \mathbf{u} \rrbracket_{l,l,B_R^+}.$$

Now we need to estimate each term of this right-hand side. We start by estimating $\rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,0,B_R^+}$ for $l \leq p - 1$: First we notice that

$$\rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,0,B_R^+} \leq \llbracket \mathbf{f} + k^2 \mathbf{u} \rrbracket_{l,0,B_R^+} \leq \llbracket \mathbf{f} \rrbracket_{l,0,B_R^+} + k^2 \llbracket \mathbf{u} \rrbracket_{l,2,B_R^+}.$$

By the induction hypothesis (A.15), we then have

$$\begin{aligned} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,0,B_R^+} &\leq C_{\mathbf{f}} \lambda_{\mathbf{f}}^l \max(l, k)^l + k^2 C_{\mathbf{u}}(B_R^+) K^l \max(l, k)^l \\ &\leq k^2 \max(l, k)^l C_{\mathbf{u}}(B_R^+) K^l \left(\frac{\lambda_{\mathbf{f}}^l}{k^2 K^l} + 1 \right). \end{aligned}$$

As $l + 2 \leq p + 1$, this estimate directly implies that

$$\begin{aligned} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,0,B_R^+} &\leq \max(p + 1, k)^{p+1} C_{\mathbf{u}}(B_R^+) K^l \left(\left(\frac{\lambda_{\mathbf{f}}}{K} \right)^l + 1 \right) \\ &\leq 2 \max(p + 1, k)^{p+1} C_{\mathbf{u}}(B_R^+) K^l, \end{aligned}$$

for $K > \lambda_{\mathbf{f}}$. Multiplying this estimate by A^{p-l} and summing on l , one gets

$$\begin{aligned} \sum_{l=0}^{p-1} A^{p-l} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,0,B_R^+} &\leq C_{\mathbf{u}}(B_R^+) K^{p+1} \max(p + 1, k)^{p+1} \frac{2}{K} \sum_{l=0}^{p-1} A^{p-l} K^{l-p} \\ &\leq C_{\mathbf{u}}(B_R^+) K^{p+1} \max(p + 1, k)^{p+1} \frac{2}{K} \sum_{l=0}^{p-1} \left(\frac{A}{K} \right)^{p-l}. \end{aligned}$$

If $K \geq 2A$, then $\sum_{l=0}^{p-1} \left(\frac{A}{K}\right)^{p-l} \leq \sum_{l=1}^{\infty} \left(\frac{A}{K}\right)^l \leq 1$, which yields

$$(A.17) \quad \sum_{l=0}^{p-1} A^{p-l} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,0,B_R^+} \leq C_{\mathbf{u}}(B_R^+) K^{p+1} \max(p+1, k)^{p+1} \frac{2}{K}.$$

Estimation of $\rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l, \frac{1}{2}, \Gamma_R}$: By the boundary condition on \mathbf{u} , we have

$$\rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l, \frac{1}{2}, \Gamma_R} = k \rho_*^{\frac{3}{2}} \llbracket \mathbf{G}\mathbf{u} \rrbracket_{l, \frac{1}{2}, \Gamma_R},$$

and by the estimate (A.9), we get

$$(A.18) \quad \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l, \frac{1}{2}, \Gamma_R} \leq k C C_{tr,R} C_{\mathbf{G}} \sum_{p'=0}^{l+1} (\lambda'_{\mathbf{G}})^{l+1-p'} \max(l+1, k)^{l+1-p'} \llbracket \mathbf{u} \rrbracket_{p', 2, B_R^+}.$$

The induction hypothesis (A.15) then leads to

$$\begin{aligned} \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l, \frac{1}{2}, \Gamma_R} &\leq k C C_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) \sum_{p'=0}^{l+1} (\lambda'_{\mathbf{G}})^{l+1-p'} K^{p'} \max(l+1, k)^{l+1-p'} \max(p', k)^{p'} \\ &\leq k C C_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) \max(l+1, k)^{l+1} K^{l+1} \sum_{p'=0}^{l+1} \left(\frac{\lambda'_{\mathbf{G}}}{K}\right)^{l+1-p'}. \end{aligned}$$

Hence for $K \geq 2\lambda'_{\mathbf{G}}$ (recalling that $l+2 \leq p+1$ and that $\sum_{p'=0}^{l+1} \left(\frac{\lambda'_{\mathbf{G}}}{K}\right)^{l+1-p'} \leq 2$), we deduce

$$\rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l, \frac{1}{2}, \Gamma_R} \leq 2 C C_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) K^{l+1} \max(p+1, k)^{p+1}.$$

Multiplying this estimate by A^{p-l} and summing on l , we get

$$\begin{aligned} \sum_{l=0}^{p-1} A^{p-l} \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l, \frac{1}{2}, \Gamma_R} &\leq C_{\mathbf{u}}(B_R^+) K^{p+1} \max(p+1, k)^{p+1} 2 C C_{tr,R} C_{\mathbf{G}} \sum_{l=0}^{p-1} A^{p-l} K^{l-p} \\ &\leq C_{\mathbf{u}}(B_R^+) K^{p+1} \max(p+1, k)^{p+1} \frac{2 C C_{tr,R} C_{\mathbf{G}} A}{K} \sum_{l=0}^{p-1} \left(\frac{A}{K}\right)^l. \end{aligned}$$

Again, for $K \geq 2A$, we arrive at

$$(A.19) \quad \sum_{l=0}^{p-1} A^{p-l} \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l, \frac{1}{2}, \Gamma_R} \leq C_{\mathbf{u}}(B_R^+) K^{p+1} \max(p+1, k)^{p+1} \frac{4 C C_{tr,R} C_{\mathbf{G}} A}{K}.$$

Finally using the definition of $C_{\mathbf{u}}(B_R^+)$, we directly check that

$$(A.20) \quad \sum_{l=0}^1 \llbracket \mathbf{u} \rrbracket_{l, l, B_R^+} \leq \frac{k}{C_R} C_{\mathbf{u}}(B_R^+),$$

and therefore (since we assume that $K \geq 2A$)

$$A^p \sum_{l=0}^1 \llbracket \mathbf{u} \rrbracket_{l,l,B_R^+} \leq \frac{1}{C_R} C_{\mathbf{u}}(B_R^+) K^p \max(p+1, k)^{p+1}.$$

In summary, using this estimate, (A.17), and (A.19) in (A.16), we have obtained that

$$\llbracket \mathbf{u} \rrbracket_{p+1,2,B_R^+} \leq C_{\mathbf{u}}(B_R^+) K^{p+1} \max(p+1, k)^{p+1} \frac{\left(2 + 4CC_{tr,R}C_{\mathbf{G}}A + \frac{1}{C_R}\right)}{K}.$$

This yields (A.15) for $p+1$ if

$$K \geq \max\left(\lambda_{\mathbf{f}}, 2A, 2\lambda'_{\mathbf{G}}, 2 + 4CC_{tr,R}C_{\mathbf{G}}A + \frac{1}{C_R}\right).$$

■

Now, we will show an equivalent lemma but which also estimates the norm of the normal derivatives of higher order.

Lemma A.8 *Let $\mathbf{u} \in \mathbf{H}^2(B_R^+)$ be a solution of (A.5) with \mathbf{f} and \mathbf{G} analytic and satisfying (A.3)-(A.4). Then there exist $K_1, K_2 \geq 1$ such that for all $p, q \geq 2$ with $q \leq p$, we have*

$$\llbracket \mathbf{u} \rrbracket_{p,q,B_R^+} \leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^q \max(p, k)^p,$$

with $C_{\mathbf{u}}(B_R^+) = C_R(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_R^+} + k^{-1} \|\mathbf{u}\|_{1,B_R^+})$.

Proof. Again, we will show this lemma by induction and by using a standard analytical regularity result for elliptic problem (i.e. proposition 2.6.7 of [5]), which gives us

$$(A.21) \quad \llbracket \mathbf{u} \rrbracket_{p,q,B_R^+} \leq \sum_{l=0}^{p-2} A^{p-1-l} \left\{ \sum_{\nu=0}^{\min(l,q-2)} B^{q-1-\nu} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+} + B^{q-1} \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R} \right\} \\ + A^{p-1} B^{q-1} \sum_{l=0}^1 \llbracket \mathbf{u} \rrbracket_{l,l,B_R^+},$$

for some positive constants A and $B \geq 1$. The induction is done on q , the initialization step $q = 2$ is obtained from Lemma A.7, by taking $K_1 \geq K$ and $K_2 \geq 1$.

The induction hypothesis is: For all $p \geq 3$, $2 \leq q' \leq q \leq p-1$, it holds

$$(A.22) \quad \llbracket \mathbf{u} \rrbracket_{p,q',B_R^+} \leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q'} \max(p, k)^p.$$

We use the estimate (A.21) to get

$$(A.23) \quad \llbracket \mathbf{u} \rrbracket_{p,q+1,B_R^+} \leq \sum_{l=0}^{p-2} A^{p-1-l} \left\{ \sum_{\nu=0}^{\min(l,q-1)} B^{q-\nu} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+} + B^q \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R} \right\} \\ + A^{p-1} B^q \sum_{l=0}^1 \llbracket \mathbf{u} \rrbracket_{l,l,B_R^+}.$$

We start with the estimate of $\rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+}$. By the induction hypothesis (A.22), we may write

$$\begin{aligned} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+} &\leq \llbracket \mathbf{f} \rrbracket_{l,\nu,B_R^+} + k^2 \llbracket \mathbf{u} \rrbracket_{l,\nu,B_R^+} \\ &\leq C_{\mathbf{f}} \lambda_{\mathbf{f}}^l \max(l, k)^l + k^2 C_{\mathbf{u}}(B_R^+) K_1^l K_2^\nu \max(l, k)^l \\ &\leq C_{\mathbf{u}}(B_R^+) K_1^l K_2^\nu k^2 \max(l, k)^l \left(\left(\frac{\lambda_{\mathbf{f}}}{K_1} \right)^l \frac{1}{k^2 K_2^\nu} + 1 \right) \\ &\leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p, k)^p \frac{2}{K_1 K_2} K_1^{l-p+1} K_2^{\nu-q}, \end{aligned}$$

if $K_1 \geq \lambda_{\mathbf{f}}$. Multiplying this estimate by $A^{p-1-l} B^{q-\nu}$ and summing on ν and l , one gets

$$\begin{aligned} &\sum_{l=0}^{p-2} A^{p-1-l} \sum_{\nu=0}^{\min(l,q-1)} B^{q-\nu} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+} \\ &\leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p, k)^p \frac{2}{K_1 K_2} \sum_{l=0}^{p-2} A^{p-1-l} \sum_{\nu=0}^{\min(l,q-1)} B^{q-\nu} K_1^{l-p+1} K_2^{\nu-q} \\ &\leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p, k)^p \frac{2}{K_1 K_2} \sum_{l=0}^{p-2} \left(\frac{A}{K_1} \right)^{p-1-l} \sum_{\nu=0}^{\min(l,q-1)} \left(\frac{B}{K_2} \right)^{q-\nu}. \end{aligned}$$

Choosing $K_1 \geq 2A$ and $K_2 \geq 2B$, we conclude that

$$(A.24) \quad \sum_{l=0}^{p-2} A^{p-1-l} \sum_{\nu=0}^{\min(l,q-1)} B^{q-\nu} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+} \leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p, k)^p \frac{8}{K_1 K_2}.$$

Estimation of $\rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R}$ for $l \leq p-2$: We use the estimate (A.9) and the induction hypothesis (A.22) to get

$$\rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R} \leq k C C_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) K_2^2 \sum_{p'=0}^{l+1} (\lambda'_{\mathbf{G}})^{l+1-p'} K_1^{p'} \max(l+1, k)^{l+1-p'} \max(p', k)^{p'}.$$

In the above right-hand side as $l+2 \leq p$ and $p' \leq p-1$, we obtain

$$\rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R} \leq C C_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) K_2^2 \max(p, k)^p K_1^{l+1} \sum_{p'=0}^{l+1} \left(\frac{\lambda'_{\mathbf{G}}}{K_1} \right)^{l+1-p'}.$$

For $K_1 \geq 2\lambda'_{\mathbf{G}}$, we deduce that

$$(A.25) \quad \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R} \leq 2 C C_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) K_2^2 \max(p, k)^p K_1^{l+1}.$$

Multiplying this estimate by $A^{p-1-l} B^q$ and summing on l , as before one gets (since $K_1 \geq 2A$ and $K_2 \geq 2B$)

$$\sum_{l=0}^{p-2} A^{p-1-l} B^q \rho_*^{\frac{3}{2}} * \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R} \leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p, k)^p \left(\frac{4 C C_{tr,R} C_{\mathbf{G}} A K_2}{K_1} \right),$$

Finally using (A.20), one has

$$A^{p-1}B^q \sum_{l=0}^1 \llbracket \mathbf{u} \rrbracket_{l,l,B_R^+} \leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p, k)^p \frac{1}{C_R K_1 K_2}.$$

Using this estimate and the estimates (A.24)-(A.25) into (A.16), we can conclude that

$$\begin{aligned} \llbracket \mathbf{u} \rrbracket_{p,q+1,B_R^+} &\leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \left(\frac{8}{K_1 K_2} + \frac{4CC_{tr,R}C_{\mathbf{G}}K_2}{K_1} + \frac{1}{C_R K_1 K_2} \right) \\ &\leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p, k)^p, \end{aligned}$$

for K_1 and K_2 large enough. ■

Remark A.9 In Lemma A.8, if we take $p = q$, we obtain

$$\llbracket \mathbf{u} \rrbracket_{p,p,B_R^+} \leq C_{\mathbf{u}}(B_R^+) K^p \max(p, k)^p,$$

with $K = K_1 K_2$.

A.2 Interior analytic regularity

Let $B_R = B(0, R)$, L an elliptic system of order 2 defined in B_R , and $k > 1$. Here we consider a solution \mathbf{u} of

$$(A.26) \quad L(\mathbf{u}) = \mathbf{f} + k^2 \mathbf{u} \text{ in } B_R.$$

We now define the following semi-norms

$$\begin{aligned} \llbracket \mathbf{u} \rrbracket_{p,B_R} &:= \max_{0 < \rho < \frac{R}{2p}} \max_{|\alpha|=p} \rho^p \|\partial^\alpha \mathbf{u}\|_{B_{R-p\rho}}, \\ \rho_*^2 \llbracket \mathbf{u} \rrbracket_{p,B_R} &:= \max_{0 < \rho < \frac{R}{2p}} \max_{|\alpha|=p} \rho^{p+2} \|\partial^\alpha \mathbf{u}\|_{B_{R-p\rho}}. \end{aligned}$$

We suppose that \mathbf{f} is analytic with

$$(A.27) \quad \|\partial^\alpha \mathbf{f}\|_{B_R} \leq C_{\mathbf{f}} \lambda_{\mathbf{f}}^p \max(|\alpha|, k)^{|\alpha|}, \quad \forall \alpha \in \mathbb{N}^n,$$

for some positive constants $C_{\mathbf{f}}$ and $\lambda_{\mathbf{f}}$ independent of k .

Lemma A.10 Let $\mathbf{u} \in \mathbf{H}^2(B_R)$ be a solution of (A.26) with f satisfying (A.27). Then there exists $K \geq 1$ such that

$$\llbracket \mathbf{u} \rrbracket_{p,B_R} \leq C_{\mathbf{u}}(B_R) K^p \max(p, k)^p,$$

with $C_{\mathbf{u}}(B_R) = C_R(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_R} + k^{-1} \|\mathbf{u}\|_{1,B_R})$, for $C_R \geq 1$.

Proof. The proof is exactly the same as the one of Lemma A.7 when we use Proposition 1.6.3 of [5] (a standard interior regularity result) instead of Proposition 2.6.6 of [5]. ■

A.3 Proof of Theorem A.1

The first step of the proof is to consider a covering of Ω by some balls, which verifies

$$\Omega \subset \cup_{j=1}^N \hat{B}_j \subset \cup_{j=1}^N B_j,$$

where $B_j = B(x_j, \xi_j)$ and $\hat{B}_j = B(x_j, \frac{\xi_j}{2})$, with $\xi_j > 0$ small enough such that $\overline{B}(x_j, \xi_j) \subset \Omega$ if $x_j \in \Omega$. This yields

$$\begin{aligned} |\mathbf{u}|_{p,\Omega} &\lesssim \sum_{i=1}^N |\mathbf{u}|_{p,\hat{B}_i \cap \Omega} \\ &\lesssim \sum_{1 \leq i \leq N: x_i \in \Omega} |\mathbf{u}|_{p,\hat{B}_i} + \sum_{1 \leq i \leq N: x_i \in \partial\Omega} |\mathbf{u}|_{p,\hat{B}_i \cap \Omega}. \end{aligned}$$

In the case of an interior ball, namely for i such that $x_i \in \Omega$, we simply perform a translation to apply Lemma A.10. Hence, the operator L does not change and we directly have

$$|\mathbf{u}|_{p,\hat{B}_i} \lesssim p^n \llbracket \mathbf{u} \rrbracket_{p,B_i} \lesssim p^n C_{\mathbf{u}}(B_i) K^p \max(p, k)^p.$$

By the definition of $C_{\mathbf{u}}(B_i)$, we then arrive at

$$(A.28) \quad |\mathbf{u}|_{p,\hat{B}_i} \lesssim \left(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_i} + k^{-1} \|\mathbf{u}\|_{1,B_i} \right) (eK)^p \max(p, k)^p.$$

In the case when a ball intersects the boundary of Ω , namely for each i such that $x_i \in \partial\Omega$, we apply a change of variables which allow to pass from $B_i \cap \Omega$ to $B_{\xi_i}^+$. First thanks to a Faà-di-Bruno formula, we obtain (see [5, (1.b)])

$$|\mathbf{u}|_{p,\hat{B}_i \cap \Omega} \lesssim c_i^{p+1} \sum_{l=0}^p \frac{k!}{l!} |\hat{\mathbf{u}}|_{l,B_{\xi_i}^+},$$

with a positive constant c_i which depends only on the transformation that allows to pass from $B_i \cap \Omega$ to $B_{\xi_i}^+$. Then we can apply Lemma A.8 (see Remark A.9) and get

$$|\mathbf{u}|_{p,\hat{B}_i \cap \Omega} \lesssim e^p c_i^{p+1} C_{\hat{\mathbf{u}}}(B_{\xi_i}^+) \sum_{l=0}^p \frac{k!}{l!} K^l \max(l, k)^l.$$

Using (A.11), and a change of variables (in $C_{\hat{\mathbf{u}}}(B_{\xi_i}^+)$ and again Faà-di-Bruno formula) we obtain

$$|\mathbf{u}|_{p,\hat{B}_i \cap \Omega} \lesssim e^p c_i^{p+1} \left(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_i \cap \Omega} + k^{-1} \|\mathbf{u}\|_{1,B_i \cap \Omega} \right) \max(p, k)^p \sum_{l=0}^p K^l.$$

This yields

$$|\mathbf{u}|_{p,\hat{B}_i \cap \Omega} \lesssim \frac{c_i K}{K-1} \left(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_i \cap \Omega} + k^{-1} \|\mathbf{u}\|_{1,B_i \cap \Omega} \right) (c_i eK)^p \max(p, k)^p.$$

The combination of this estimate with (A.28) yield the result.

References

- [1] H. Barucq and B. Hanouzet. Etude asymptotique du système de Maxwell avec la condition aux limites absorbante de Silver-Müller II. *C. R. Acad. Sci. Paris Sér. I*, 316:1019–1024, 1993.
- [2] T. Chaumont-Frelet and S. Nicaise. Wavenumber explicit convergence analysis for finite element discretizations of general wave propagation problems. Technical report, Lamav, Univ. Valenciennes, 2017.
- [3] J. Chazarain and A. Piriou. *Introduction to the theory of linear partial differential equations*, volume 14 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam-New York, 1982. Translated from the French.
- [4] R. Clayton and B. Engquist. Absorbing boundary conditions for acoustic and elastic wave equations. *Bulletin of the Seismological Society of America*, 67(6):1529–1540, 1977.
- [5] M. Costabel, M. Dauge, and S. Nicaise. *Corner Singularities and Analytic Regularity for Linear Elliptic Systems. Part I: Smooth domains*. <http://hal.archives-ouvertes.fr/hal-00453934/en/> Online version of Chapters 1 to 5. (2010).
- [6] P. Cummings and X. Feng. Sharp regularity coefficient estimates for complex-valued acoustic and elastic Helmholtz equations. *Math. Models Methods Appl. Sci.*, 16(1):139–160, 2006.
- [7] M. Darbas, C. Geuzaine, and V. Mattesi. High-order absorbing boundary conditions for 2d elastic scattering problems. *Comput. Math. Appl.*, 2019. to appear.
- [8] M. Dauge. *Elliptic boundary value problems on corner domains – smoothness and asymptotics of solutions*, volume 1341 of *Lecture Notes in Mathematics*. Springer, Berlin, 1988.
- [9] S. Esterhazy and J. Melenk. On stability of discretizations of the Helmholtz equation. In *Lecture Notes in Computational Science and Engineering*, pages 285–324. Springer-Verlag Berlin Heidelberg, 2012.
- [10] G. N. Gatica and S. Meddahi. Finite element analysis of a time harmonic Maxwell problem with an impedance boundary condition. *IMA J. Numer. Anal.*, 32(2):534–552, 2012.
- [11] U. Hetmaniuk. Stability estimates for a class of Helmholtz problems. *Commun. Math. Sci.*, 5(3):665–678, 2007.
- [12] R. Hiptmair, A. Moiola, and I. Perugia. Stability results for the time-harmonic Maxwell equations with impedance boundary conditions. *Math. Models Methods Appl. Sci.*, 21(11):2263–2287, 2011.
- [13] F. Ihlenburg and I. Babuška. Finite element solution of the Helmholtz equation with high wave number part i: the h-version of the fem. *Computers Math. Applic.*, 30(9):9–37, 1995.
- [14] F. Ihlenburg and I. Babuška. Finite element solution of the Helmholtz equation with high wave number part ii: the h-p version of the fem. *SIAM J. Numer. Anal.*, 34(1):315–358, 1997.
- [15] J. Lysmer and R. Kuhlemeyer. Finite dynamic model for infinite media. *J. Eng. Mech. Div., ASCE*, 95:859–877, 1969.

- [16] J. Melenk and S. Sauter. Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions. *Mathematics of Computation*, 79(272):1871–1914, 2010.
- [17] J. Melenk and S. Sauter. Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation. *SIAM J. Numer. Anal.*, 49(3):1210–1243, 2011.
- [18] J. M. Melenk. Mapping properties of combined field Helmholtz boundary integral operators. *SIAM Journal on Mathematical Analysis*, 44(4):2599–2636, 2012.
- [19] J. M. Melenk, A. Parsania, and S. Sauter. General dg-methods for highly indefinite helmholtz problems. *Journal of Scientific Computing*, 57(3):536–581, 2013.
- [20] J. M. Melenk and S. Sauter. Wave-number explicit convergence analysis for Galerkin discretizations of the Helmholtz equations (extended version). 2011. <https://doi.org/10.5167/uzh-49739>.
- [21] J. M. Melenk and S. Sauter. Wavenumber-explicit hp-FEM analysis for Maxwell’s equations with transparent boundary conditions. *arXiv preprint arXiv:1803.01619v1*, 2018.
- [22] P. Monk. *Finite element methods for Maxwell’s equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, 2003.
- [23] J.-C. Nédélec. *Acoustic and electromagnetic equations*, volume 144 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001.
- [24] S. Nicaise and J. Tomezyk. The time-harmonic Maxwell equations with impedance boundary conditions in polyhedral domains. In U. Langer, D. Pauly, and S. Repin, editors, *Maxwell’s equations: analysis and numerics*. De Gruyter, 2019.
- [25] A. Schatz. An observation concerning Ritz-Galerkin methods with indefinite bilinear forms. *Mathematics of Computation*, 28(128):959–962, 1974.