H^∞ interpolation and embedding theorems for rational functions

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Abstract. We consider a Nevanlinna–Pick interpolation problem on finite sequences of the unit disc $\mathbb{D}$ constrained by Hardy and radial-weighted Bergman norms. We find sharp asymptotics on the corresponding interpolation constants. As another application of our techniques we prove embedding theorems for rational functions. We find that the embedding of H^∞ into Hardy or radial-weighted Bergman spaces in $\mathbb{D}$ is invertible on the subset of rational functions of a given degree n whose poles are separated from the unit circle and obtain asymptotically sharp estimates of the corresponding embedding constants.

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1. Introduction

We denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc and by $\mathcal{H}ol(\mathbb{D})$ the space of holomorphic functions in $\mathbb{D}$. We consider the following Banach spaces $X \subset \mathcal{H}ol(\mathbb{D})$:

1. the Hardy spaces $X = H^p = H^p(\mathbb{D})$, $1 \le p \le \infty$; we refer to [7] for the corresponding definition and their general properties;
2. the radial-weighted Bergman spaces $X = A^p\big((1-|z|^2)^\beta \mathrm{d}\mathcal{A}\big) = A^p(\beta)$, $1 \le p < \infty$, $\beta > -1$:

$$X = \Big\{ f \in \mathcal{H}ol(\mathbb{D}) \,:\, \|f\|^p_{A^p(\beta)} = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\beta \mathrm{d}\mathcal{A}(z) < \infty \Big\},$$

where $\mathcal{A}$ is the normalized area measure on $\mathbb{D}$. We refer to [12] for general properties of $A^p(\beta)$. For $\beta = 0$ we shorten the notation to $X = A^p$.

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1.1. Effective H^∞ interpolation

We consider the following problem: given a Banach space $X \subset \mathcal{H}ol\,(\mathbb{D})$ and a finite sequence σ in $\mathbb{D}$, what is the best possible interpolation of the traces $f|_\sigma$, $f \in X$, by functions from the space H^∞? The case $X \subset H^\infty$ is of no interest (such a situation implies the uniform boundedness of the interpolation quantity $c(\sigma,\, X,\, H^\infty)$ below), and so one can suppose that either $H^\infty \subset X$ or X and H^∞ are incomparable. More precisely, our problem is to compute or estimate the following interpolation quantity

$$c(\sigma,\, X,\, H^\infty) = \sup_{f\in X,\, \|f\|_X \le 1} \inf\bigl\{\|g\|_\infty :\, g \in H^\infty,\, g|_\sigma = f|_\sigma\bigr\}.$$

It is discussed in [26] that the classical interpolation problems, those of Nevanlinna–Pick and Carathéodory–Schur (see [16, p. 231]) on one hand and Carleson's free interpolation (see [17, p. 158]) on the other hand, are of this nature. For general Banach spaces X containing H^∞ as a dense subset, $c\,(\sigma,\, X,\, H^\infty)$ is expressed as

$$c(\sigma,\, X,\, H^\infty) = \sup_{f\in X\cap H^\infty,\, \|f\|_X \le 1} \|f\|_{H^\infty/B_\sigma H^\infty},$$

where B_σ is the finite Blaschke product

$$B_\sigma = \prod_{\lambda\in\sigma} b_\lambda, \qquad b_\lambda = \frac{\lambda - z}{1 - \overline{\lambda} z},$$

b_λ being the elementary Blaschke factor associated to a $\lambda \in \mathbb{D}$. We denote by $\sigma_{n,\,\lambda} = (\lambda,\, ...,\, \lambda) \in \mathbb{D}^n$ the one-point sequence of multiplicity n corresponding to a given $\lambda \in \mathbb{D}$.

It is a natural problem (related, e.g., to matrix analysis) to study the asymptotic behaviour of $c\,(\sigma,\, X,\, H^\infty)$ when the set σ approaches the boundary and its cardinality tends to infinity. We put

$$C_{n,\,r}(X,\, H^\infty) = \sup\bigl\{c\,(\sigma,\, X,\, H^\infty) :\, \sigma \in \mathbb{D}^n,\ \max_{\lambda\in\sigma} |\lambda| \le r\bigr\}.$$

Initially motivated by a question posed in an applied context in [4, 5], asymptotically sharp estimates of $C_{n,\,r}(X,\, H^\infty)$ were derived in [26] for the cases $X = H^p$, $p \in 2\mathbb{N}$, and $X = A^2$.

Theorem 1.1. [26] *Given $n \ge 1$, $r \in [0,\, 1)$, $p \in 2\mathbb{N}$ and $\lambda \in \mathbb{D}$ with $|\lambda| \le r$, we have*

$$a_p \left(\frac{n}{1-|\lambda|}\right)^{\frac{1}{p}} \le c(\sigma_{n,\,\lambda},\, H^p, H^\infty) \le C_{n,\,r}(H^p,\, H^\infty) \le b_p \left(\frac{n}{1-r}\right)^{\frac{1}{p}}, \tag{1.1}$$

$$a \cdot \frac{n}{1-|\lambda|} \le c(\sigma_{n,\,\lambda},\, A^2,\, H^\infty) \le C_{n,\,r}(A^2,\, H^\infty) \le b \cdot \frac{n}{1-r}, \tag{1.2}$$

where a_p, b_p are constants depending only on p and a, b are some absolute constants.

Remark 1.2. The right-hand side inequality in (1.1) is established in [26] for any $p \in [1, \infty)$. The proof makes use of a deep interpolation result between Hardy spaces by P. Jones [13], which we avoid in the present paper.

From now on, for two positive functions a and b, we say that a is dominated by b, denoted by $a \lesssim b$, if there is a constant $c > 0$ such that $a \leq cb$; and we say that a and b are comparable, denoted by $a \asymp b$, if both $a \lesssim b$ and $b \lesssim a$.

The following conjecture for general Banach spaces X (of analytic functions of moderate growth in $\mathbb{D}$) was formulated in [26]:

$$C_{n,\,r}\,(X,\,H^\infty) \asymp \varphi_X\left(1 - \frac{1-r}{n}\right), \tag{1.3}$$

where $\varphi_X(t)$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X. One of the main results of [25] verifies the conjecture (1.3) for the case $X = A^2(\beta)$, $\beta \in \mathbb{Z}_+$. More recently an upper bound on $c(\sigma_{n,\,\lambda},\, A^p(\beta),\, H^\infty)$ with $1 \leq p \leq 2$, and $\beta > -1$ was derived in [27].

In this paper we

1. strengthen (1.1) by proving the left-hand side inequality for any $p \in [1, +\infty)$ and by providing a simple and direct proof of the right-hand side one;
2. prove conjecture (1.3) for all radial-weighted Bergman spaces $X = A^p(\beta)$ (see Theorem 2.1 below);
3. apply Theorem 2.1 to spectral estimates on norms of functions of matrices (see Subsection 1.2 for details and Corollary 2.2 for the corresponding statement);
4. show that the embedding of H^∞ into $A^p(\beta)$ is invertible on the subset of rational functions of a given degree n whose poles are separated from the unit circle and obtain an asymptotically sharp estimate for the embedding constant (see Subsection 1.3 for details and Theorem 2.3 for the corresponding statement).

1.2. Motivations from matrix analysis

Let $\mathcal{M}_n$ be the set of complex $n \times n$ matrices and let $\|T\|$ denote the operator norm of $T \in \mathcal{M}_n$ associated with the Hilbert norm on $\mathbb{C}^n$. We denote by $\sigma = \sigma(T)$ the spectrum of T, by m_T its minimal polynomial, and by $|m_T|$ the degree of m_T. In our discussion we will assume that $\|T\| \leq 1$ and call such T a *contraction*. Let $\mathcal{C}_n \subset \mathcal{M}_n$ denote the set of all contractions. For a finite sequence σ in $\mathbb{D}$, we denote by P_σ the monic polynomial with zero set σ (counted with multiplicities). For a finite sequence σ in $\mathbb{D}$ and $f \in \mathcal{H}ol(\mathbb{D})$, V. Pták and N. Young [19] introduced the quantity

$$\mathcal{M}(f,\,\sigma) = \sup\{\|f(T)\| : T \in \mathcal{C}_n,\, m_T = P_\sigma\}.$$

Note that interesting cases occur for f such that:

1. $f|_\sigma = z^k|_\sigma$ (estimates on the norm of the powers of an $n \times n$ matrix, see for example [18]);
2. $f|_\sigma = z^{-1}|_\sigma$ (estimates on condition numbers and the norm of inverses of $n \times n$ matrices, see [15]);
3. $f|_\sigma = (\zeta - z)^{-1}|_\sigma$ (estimates on the norm of the resolvent of an $n \times n$ matrix, see for example [15, 21]).

Given a Blaschke sequence σ in $\mathbb{D}$ and $f \in H^\infty$ it is possible to evaluate $\mathcal{M}(f, \sigma)$ as follows:

$$\mathcal{M}(f, \sigma) = \|f\|_{H^\infty/B_\sigma H^\infty} = \|f(M_{B_\sigma})\|, \tag{1.4}$$

where M_{B_σ} is the compression of the multiplication operation by z to the model space K_{B_σ}, see Subsection 3.1 for the definitions. This formula is due to N. K. Nikolski [15, Theorem 3.4] while the last equality is a well-known corollary of Commutant Lifting Theorem of B. Sz.-Nagy and C. Foiaş [14, 11, 20].

Let $X \subset \mathcal{H}ol(\mathbb{D})$ be a Banach space containing H^∞. The above equality on $\mathcal{M}(f, \sigma)$ naturally extends to any $f \in X$ as follows. There exists an analytic polynomial p interpolating f on the finite set σ. Therefore for any $T \in \mathcal{C}_n$ with $m_T = P_\sigma$ and $\sigma \subset \mathbb{D}$, we have $f(T) = p(T)$ (since $f = p + m_T h$ for some $h \in \mathcal{H}ol(\mathbb{D})$). Hence,

$$\begin{aligned} \mathcal{M}(f, \sigma) &= \mathcal{M}(p, \sigma) = \|p\|_{H^\infty/B_\sigma H^\infty} \\ &= \|p(M_{B_\sigma})\| = \|f(M_{B_\sigma})\|. \end{aligned}$$

Here we used (1.4) applied to p. Moreover

$$\begin{aligned} \|p\|_{H^\infty/B_\sigma H^\infty} &= \inf\{\|p + B_\sigma h\|_\infty : \ h \in H^\infty\} \\ &= \inf\{\|g\|_\infty : \ g|_\sigma = p|_\sigma, \ g \in H^\infty\} \\ &= \inf\{\|g\|_\infty : \ g|_\sigma = f|_\sigma, \ g \in H^\infty\}. \end{aligned}$$

We conclude that

$$\mathcal{M}(f, \sigma) = \inf\left\{\|g\|_\infty : \ g \in H^\infty, \ g|_\sigma = f|_\sigma\right\}.$$

Therefore, given a division-closed Banach space $X \subset \mathcal{H}ol\,(\mathbb{D})$ containing H^∞ and a finite sequence σ in $\mathbb{D}$, it turns out that

$$\sup_{\|f\|_X \le 1} \mathcal{M}(f, \sigma) = c\,(\sigma, X, H^\infty)$$

and so, for all $n \ge 1$, $r \in (0, 1)$,

$$\sup_{\|f\|_X \le 1} \left\{\mathcal{M}(f, \sigma) : \ \sigma \in \mathbb{D}^n, \ \max_{\lambda \in \sigma} |\lambda| \le r\right\} = \mathcal{C}_{n,\,r}(X, H^\infty).$$

1.3. Embedding theorems for rational functions

In [9, 10] the following phenomenon was discovered: sharp embedding theorems are invertible on the set of rational functions of a given degree. Let $n \geq 1$, let $\mathcal{P}_n$ be the space of complex analytic polynomials of degree less or equal than n and let

$$\mathcal{R}_n = \{P/Q : \ P, Q \in \mathcal{P}_n, \ Q(\zeta) \neq 0 \ \text{ for } |\zeta| \leq 1\}$$

be the set of rational functions of degree at most n with poles outside of the closed unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} : \ |z| \leq 1\}$. Recall that the Hardy–Littlewood embedding theorem [7, Theorem 1.1] says that $H^q \subset A^p(\beta)$ for any $p > 1$, $\beta > -1$ and $q = \frac{p}{2+\beta}$. Given two Banach spaces of analytic functions in the disc which contain $\mathcal{R}_n$, denote by $\mathcal{E}_n(X, Y)$ the best possible constant such that

$$\|f\|_X \leq \mathcal{E}_n(X, Y)\|f\|_Y, \qquad f \in \mathcal{R}_n. \tag{1.5}$$

Dyn'kin [10, Theorem 4.1] proved that the Hardy–Littlewood embedding theorem is invertible on $\mathcal{R}_n$; namely,

$$\mathcal{E}_n(H^q, A^p(\beta)) \asymp n^{\frac{1+\beta}{p}},$$

when $q = \frac{p}{2+\beta}$.

Note that for many choices of X and Y we have $\mathcal{E}_n(X, Y) = +\infty$ for every $n \in \mathbb{N}$, since the poles of the rational functions are allowed to be arbitrarily close to the unit circle $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$. This is for example the case when $X = H^\infty$ and $Y = H^p$ or $Y = A^p$, $1 \leq p < +\infty$ (to see this one can consider the function $f(z) = (1 - rz)^{-1}$ as $r \to 1^-$). This observation suggests to consider a more general problem when one replaces the class $\mathcal{R}_n$ in (1.5) by $\mathcal{R}_{n,\,r}$ (for any fixed $r \in [0,\, 1)$) defined by

$$\mathcal{R}_{n,\,r} = \Big\{P/Q : \ P, Q \in \mathcal{P}_n, \ Q(\zeta) \neq 0 \ \text{ for } \ |\zeta| < \frac{1}{r}\Big\},$$

i.e., by the set of all rational functions of degree at most $n \geq 1$ without poles in $\frac{1}{r}\mathbb{D}$. The quantity $\mathcal{E}_n(X, Y)$ (when it is infinite) is replaced by the best possible constant $\mathcal{E}_{n,\,r}(X, Y)$ such that

$$\|f\|_X \leq \mathcal{E}_{n,\,r}(X, Y)\|f\|_Y, \qquad f \in \mathcal{R}_{n,\,r},$$

and we can study the asymptotic dependence on the parameters r and n as $r \to 1^-$ and $n \to \infty$. Recently, the authors [3, Theorem 2.4] proved S. M. Nikolskii-type inequalities for rational functions (whose poles do not belong to $\mathbb{T}$) which can be formulated here as

$$\mathcal{E}_{n,\,r}(H^q, H^p) \asymp \left(\frac{n}{1-r}\right)^{\frac{1}{p}-\frac{1}{q}}, \qquad 0 \leq p < q \leq \infty. \tag{1.6}$$

1.4. Outline of the paper

The paper is organized as follows. Section 2 states our main results. Section 3 is devoted to the main ingredients and tools employed in the proofs. In particular, we recall the so-called *theory of model spaces* which plays a central role here (see Subsection 3.1) and discuss the strategy of the proofs of main results (Subsection 3.2). Section 4 contains sharp asymptotic estimates for the norms of derivatives of reproducing kernels in various function spaces. In Sections 5 and 6 we prove, respectively, the upper and the lower bounds in Theorems 2.1 and 2.3.

2. Main results

Theorem 2.1. *Let $n \geq 1$, $r \in [0,\, 1)$, $p \in [1,\, +\infty)$ and $\beta > -1$. Then we have*

$$C_{n,\,r}\,(H^p,\, H^\infty) \asymp \left(\frac{n}{1-r}\right)^{\frac{1}{p}} \tag{2.1}$$

with constants depending only on p, and

$$C_{n,\,r}\,(A^p(\beta),\, H^\infty) \asymp \left(\frac{n}{1-r}\right)^{\frac{2+\beta}{p}} \tag{2.2}$$

with constants depending only on p and β.

In view of the discussion in Subsection 1.2, the following corollary is immediate:

Corollary 2.2. *Let $n \geq 1$, $r \in [0,\, 1)$, $p \in [1,\, +\infty)$ and $\beta > -1$. Then we have*

$$\sup_{\|f\|_{H^p}\leq 1} \left\{\mathcal{M}(f,\, \sigma) :\, \sigma \in \mathbb{D}^n,\, \max_{\lambda\in\sigma} |\lambda| \leq r\right\} \asymp \left(\frac{n}{1-r}\right)^{\frac{1}{p}}$$

with constants depending only on p, and

$$\sup_{\|f\|_{A^p(\beta)}\leq 1} \left\{\mathcal{M}(f,\, \sigma) :\, \sigma \in \mathbb{D}^n,\, \max_{\lambda\in\sigma} |\lambda| \leq r\right\} \asymp \left(\frac{n}{1-r}\right)^{\frac{2+\beta}{p}}$$

with constants depending only on p and β.

The techniques employed to prove Theorem 2.1 make use of the theory of model spaces and their reproducing kernels. They naturally lead to asymptotically sharp estimates of the embedding constants $\mathcal{E}_{n,\,r}(H^\infty,\, A^p(\beta))$, $1 \leq p < \infty$, $\beta > -1$.

Theorem 2.3. *Let $n \geq 1$, $r \in [0,\, 1)$, $p \in [1,\, +\infty)$ and $\beta > -1$. Then we have*

$$\mathcal{E}_{n,\,r}(H^\infty,\, A^p(\beta)) \asymp \left(\frac{n}{1-r}\right)^{\frac{2+\beta}{p}} \tag{2.3}$$

with constants depending only on p and β.

3. Main ingredients

In this section we give the main ingredients and tools we use in the proofs of Theorem 2.1 and Theorem 2.3. We begin with the definition of model spaces.

3.1. Model spaces

Let Θ be an *inner function*, i.e., $\Theta \in H^\infty$ and $|\Theta(\xi)| = 1$ for a.e. $\xi \in \mathbb{T}$. We define the model subspace K_Θ of the Hardy space H^2 by

$$K_\Theta = H^2 \cap \left(\Theta H^2\right)^\perp = H^2 \ominus \Theta H^2.$$

By the famous theorem of Beurling, these and only these subspaces of H^2 are invariant with respect to the backward shift operator S^* defined by

$$S^* f = \frac{f - f(0)}{z}.$$

We refer to [16] for the general theory of the spaces K_Θ and their numerous applications. Given $\sigma \in \mathbb{D}^n$, put $B = B_\sigma$ and consider the model subspace K_B. Let us first establish the relation between $\mathcal{R}_n$, $\mathcal{R}_{n,\,r}$, and model spaces K_B. It is well known that if

$$\sigma = (\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_t, ..., \lambda_t) \in \mathbb{D}^n,$$

where every λ_s is repeated according to its multiplicity n_s, $\sum_{s=1}^t n_s = n$, then

$$K_B = \overline{\operatorname{span}}\{k_{\lambda_s,\,j} : 1 \le s \le t,\, 0 \le j \le n_s - 1\},$$

where for $\lambda \neq 0$, $k_{\lambda,\,j} = \left(\frac{d}{d\overline{\lambda}}\right)^j k_\lambda$ and $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$ is the standard Cauchy kernel at the point λ, whereas $k_{0,\,j} = z^j$. Thus the subspace K_B consists of rational functions of the form P/Q, where $P \in \mathcal{P}_{n-1}$ and $Q \in \mathcal{P}_n$, with the poles $1/\overline{\lambda}_1, \ldots, 1/\overline{\lambda}_n$ of corresponding multiplicities (including possible poles at ∞). Hence, if $f \in \mathcal{R}_n$ and $1/\overline{\lambda}_1, \ldots, 1/\overline{\lambda}_n$ are the poles of f, then $f \in K_{zB}$ with $\sigma = (\lambda_1, \ldots, \lambda_n)$.

For any inner function Θ the reproducing kernel of the model space K_Θ corresponding to a point $\zeta \in \mathbb{D}$ is of the form

$$k_\zeta^\Theta(z) = \frac{1 - \overline{\Theta(\zeta)}\Theta(z)}{1 - \overline{\zeta}z} = (1 - \overline{\Theta(\zeta)}\Theta(z))k_\zeta(z).$$

We recall the definition of the Malmquist–Walsh family $(e_j)_{j=1}^n$ for a sequence $\sigma = (\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$ (see [17, p. 117]):

$$e_1 = \left(1 - |\lambda_1|^2\right) k_{\lambda_1}, \qquad e_j = \left(1 - |\lambda_j|^2\right)^{\frac{1}{2}} \left(\prod_{i=1}^{j-1} b_{\lambda_i}\right) k_{\lambda_j}, \quad j = 2 \ldots n.$$

Note that $(e_j)_{j=1}^n$ is an orthonormal basis of K_B for $B = B_\sigma$. The *model operator* M_B evoked above in Section 1.2 is the compression of the shift operator $S : f \mapsto zf$ on K_B, i.e., $M_B f = P_B S f$, $f \in K_B$, where P_B is the orthogonal projection on K_B.

3.2. Upper bounds in Theorems 2.1 **and** 2.3**.**
In this subsection we outline the strategy of the proof of Theorems 2.1 and 2.3.

From now on we denote by $\langle \cdot, \cdot \rangle$ the Cauchy sesquilinear form:

$$\langle h, g\rangle = \sum_{k\geq 0} \hat{h}(k)\overline{\hat{g}(k)},$$

which makes sense for any $h = \sum_{k\geq 0} \hat{h}(k)z^k \in \mathcal{H}ol(\mathbb{D})$ and $g = \sum_{k\geq 0} \hat{g}(k)z^k$ analytic in the disc $(1+\delta)\mathbb{D}$ for some $\delta > 0$. If $h, g \in H^2$, then $\langle \cdot, \cdot \rangle$ coincides with the usual scalar product in $L^2(\mathbb{T})$,

$$\langle h, g\rangle = \int_{\mathbb{T}} h(u)\overline{g(u)}\mathrm{d}m(u),$$

where m is the normalized Lebesgue measure on $\mathbb{T}$. Also, denote by (h, g) the scalar product on A^2 defined by

$$(h, g) = \int_{\mathbb{D}} h(u)\overline{g(u)}\mathrm{d}\mathcal{A}(u), \qquad h, g \in A^2.$$

As in [27, 26], we will use the following interpolation operator:

$$f \mapsto P_B f = \sum_{k=1}^{n} \langle f, e_k\rangle e_k, \tag{3.1}$$

where $(e_k)_{k=1}^n$ is the Malmquist–Walsh basis of K_B. If $f \in H^2$, then this is the usual orthogonal projection of f onto K_B. However the formula $P_B f = \sum_{k=1}^n \langle f, e_k\rangle e_k$ correctly defines this operator for any $f \in \mathcal{H}ol(\mathbb{D})$.

3.2.1. The upper bounds in Theorem 2.1**.** For $X = H^p$, $1 \leq p < +\infty$, the proof is simple. We have

$$|P_B f(\zeta)| = |\left\langle f, k_\zeta^B\right\rangle| \leq \|f\|_{H^p}\|k_\zeta^B\|_{H^q},$$

where q is the conjugate of p. It remains to use the estimate for $\|k_\zeta^B\|_{H^q}$, $\zeta \in \mathbb{T}$, given in Proposition 4.1.

To relate $P_B f(\zeta) = \left\langle f, k_\zeta^B\right\rangle$ to the norm of f in a Bergman space $A^p(\beta)$, we use the simplest form of the Green formula,

$$\langle \varphi, \psi\rangle = (\varphi', S^*\psi) + \varphi(0)\overline{\psi(0)}, \tag{3.2}$$

which is true, in particular, when φ is analytic in some disc $(1+\delta)\mathbb{D}$, $\delta > 0$, and $\psi \in H^\infty$. We then apply it to $\varphi = k_\zeta^B$ and $\psi = f \in X \cap H^\infty$.

If $-1 < \beta \leq 0$, then we apply the Hölder inequality and it remains to estimate the norm of the derivative $(k_\zeta^B)'$ in the dual Bergman space. This norm is estimated in Proposition 4.1.

To treat the case $\beta > 0$, we need a modified Green formula. Recall that the fractional differentiation operator D_α, $-1 < \alpha < \infty$, is defined by $D_\alpha(z^m) = \frac{\Gamma(m+2+\alpha)}{(m+1)!\Gamma(2+\alpha)}z^m$, $m = 0, 1, 2, \ldots$, and extended linearly to the

whole space $\mathcal{H}ol(\mathbb{D})$ (see [12, Lemma 1.17]). Then, for a function f analytic in a neighborhood of $\overline{\mathbb{D}}$ and $-1 < \alpha < \infty$, we have

$$\int_{\mathbb{D}} f(u)\overline{g(u)}\mathrm{d}\mathcal{A}(u) = (\alpha+1)\int_{\mathbb{D}} D_\alpha f(u)\overline{g(u)}\left(1-|u|^2\right)^\alpha \mathrm{d}\mathcal{A}(u) \tag{3.3}$$

for any $g \in H^\infty$ (see [12, Lemma 1.20]). Note that even for $l \in \mathbb{N}$, $D_l f$ differs from the usual derivative $f^{(l)}$. However,

$$\|D_l f\|_{A^p(\beta)} \asymp \|f^{(l)}\|_{A^p(\beta)} + \sum_{j=0}^{l-1} |f^{(j)}(0)| \tag{3.4}$$

for any Bergman space $A^p(\beta)$.

Formula (3.3) reduces the problem to estimates of the Bergman norms of $(k_\zeta^B)^{(l)}$, $l \in \mathbb{N}$, which are again given in Proposition 4.1.

3.2.2. The upper bound in Theorem 2.3. To prove the upper bound in Theorem 2.3 it is sufficient to note that given $f \in \mathcal{R}_{n,\,r}$ with poles $1/\overline{\lambda}_1, \ldots, 1/\overline{\lambda}_n$ (repeated according to multiplicities and satisfying $|\lambda_i| \le r$ for all $i = 1 \ldots n$), we have $f \in K_{zB}$ where $B = B_\sigma$, $\sigma = (\lambda_1, \ldots, \lambda_n)$. Therefore

$$f(\zeta) = \left\langle f,\, k_\zeta^{zB} \right\rangle .$$

This means that f pointwise coincides with the interpolation operator (3.1) and we can apply the same reasoning as above with zB instead of B.

3.3. Lower bounds

The lower bound problem in Theorem 2.1 is treated by using the "worst" interpolation n-tuple $\sigma = \sigma_{\lambda,\,n} = (\lambda, \ldots, \lambda) \in \mathbb{D}^n$, a one-point set of multiplicity n (a Carathéodory–Schur type interpolation problem). The "worst" interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} z^k$ transplanted from the origin to λ. The lower bound in (2.3) is achieved by rational functions of the same kind (i.e., whose poles are concentrated at the same point $1/\bar{\lambda}$).

4. Estimates of norms of reproducing kernels

We will use the following simple Bernstein-type inequality for rational functions (see, e.g., [3, Theorem 2.3]). Let $\sigma = (\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$, let $B = B_\sigma$ be the corresponding finite Blaschke product and $r = \max_{\lambda\in\sigma} |\lambda|$. Given $l \in \mathbb{N}$ we have

$$\|f^{(l)}\|_\infty \lesssim \left(\frac{n}{1-r}\right)^l \|f\|_\infty \tag{4.1}$$

for any $f \in K_B$.

We need to introduce an additional scale of Banach spaces of holomorphic functions in $\mathbb{D}$. The weighted Bloch space $\mathcal{B}_\alpha$, $0 \le \alpha \le 1$, consists of functions $f \in \mathcal{H}ol(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|\,(1-|z|)^\alpha < \infty$$

(which is in fact a seminorm).

Proposition 4.1. *Let $\sigma = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$ and $B = B_\sigma$ be the corresponding finite Blaschke product, $r = \max_{\lambda\in\sigma} |\lambda|$ and $z \in \overline{\mathbb{D}}$. The following inequalities hold:*

1. *Given $q \in (1, +\infty]$ we have*
$$\|k_z^B\|_{H^q} \lesssim \left(\frac{n}{1-r}\right)^{1-\frac{1}{q}}. \tag{4.2}$$
2. *Given $l \in \mathbb{N}$ and $0 \le \alpha \le 1$ we have*
$$\|(k_z^B)^{(l)}\|_{\mathcal{B}_\alpha} \lesssim \left(\frac{n}{1-r}\right)^{l+2-\alpha}. \tag{4.3}$$
3. *Given $q \in (1, +\infty)$ and $\gamma \in (-1, q-1]$ we have*
$$\|(k_z^B)'\|_{A^q(\gamma)} \lesssim \left(\frac{n}{1-r}\right)^{2-\frac{\gamma+2}{q}}. \tag{4.4}$$
4. *Given $q \in (1, +\infty)$ and $\gamma \in (-1, q]$ we have*
$$\|(k_z^B)''\|_{A^q(\gamma)} \lesssim \left(\frac{n}{1-r}\right)^{3-\frac{\gamma+2}{q}}. \tag{4.5}$$

All involved constants may depend on q, α and γ but do not depend on n, r and z.

Proof. Note that all above norms (in appropriate powers) are subharmonic functions in z. Thus, it suffices to prove the inequalities only in the case $z = \zeta \in \mathbb{T}$. So in what follows we assume that $\zeta \in \mathbb{T}$.

Proof of (4.2)**.** For $q = 2$,

$$\begin{aligned}\|k_\zeta^B\|_{H^2}^2 &= \frac{1-|B(\zeta)|^2}{1-|\zeta|^2} = \sum_{j=1}^n \frac{1-|\lambda_j|^2}{|1-\overline{\lambda_j}\zeta|^2}|B_{j-1}(\zeta)|^2 \\ &\le \sum_{j=1}^n \frac{1+|\lambda_j|}{1-|\lambda_j|} \le \frac{1+r}{1-r}n.\end{aligned}$$

Here $B_j = \prod_{i=1}^j b_{\lambda_i}$, $B_0 \equiv 1$. The estimate for $q = \infty$ follows from

$$|k_\zeta^B(u)| \le \|k_\zeta^B\|_{H^2}\|k_u^B\|_{H^2} \lesssim \frac{n}{1-r}$$

for any u, $\zeta \in \overline{\mathbb{D}}$. If $q \in [2, \infty)$, then we write

$$\|k_\zeta^B\|_{H^q}^q \le \|k_\zeta^B\|_{H^2}^2\|k_\zeta^B\|_\infty^{q-2} \lesssim \left(\frac{n}{1-r}\right)^{q-1}.$$

Finally for $q \in (1, 2)$ we apply the following result by W. Cohn [8, Lemma 4.2]. Denoting by p the conjugate exponent of q (i.e., $\frac{1}{q} + \frac{1}{p} = 1$)

$$\begin{aligned}\|k_\zeta^B\|_{H^q} &= \sup_{g\in H^p,\, \|g\|_{H^p}\le 1} \left|\int_{\mathbb{T}} g(z)\overline{k_\zeta^B(z)}\mathrm{d}m(z)\right| \\ &= \sup_{g\in H^p,\, \|g\|_{H^p}\le 1} \left|\int_{\mathbb{T}} P_B g(z)\overline{k_\zeta^B(z)}\mathrm{d}m(z)\right| \\ &\lesssim \sup_{h\in K_B,\, \|h\|_{H^p}\le C_p} |h(\zeta)|,\end{aligned}$$

where the last inequality is due to the fact that $h = P_B g \in K_B$ and there exists $C_p > 0$ such that $\|P_B g\|_{H^p} \le C_p \|g\|_{H^p}$, $p \in (1, \infty)$.

Applying the inequality $\|h\|_\infty \le \left(\frac{n}{1-r}\right)^{1/p} \|h\|_{H^p}$, $h \in K_B$, which is a special case of (1.6), we obtain (4.2).

Proof of (4.3)**.** Clearly, for $\alpha = 1$,

$$\sup_{u\in\mathbb{D}}(1-|u|)\big|(k_\zeta^B)^{(l+1)}(u)\big| = \|(k_\zeta^B)^{(l)}\|_{\mathcal{B}_1} \le \|(k_\zeta^B)^{(l)}\|_\infty \lesssim \left(\frac{n}{1-r}\right)^{l+1}$$

by (4.1). Therefore, for any $0 \le \alpha \le 1$,

$$\|(k_\zeta^B)^{(l)}\|_{\mathcal{B}_\alpha} \le \|(k_\zeta^B)^{(l)}\|_{\mathcal{B}_1}^{\alpha} \|(k_\zeta^B)^{(l+1)}\|_\infty^{1-\alpha} \lesssim \left(\frac{n}{1-r}\right)^{l+2-\alpha},$$

which completes the proof.

Proof of (4.4)**.** For the derivative of k_ζ^B we have

$$(k_\zeta^B)'(z) = \zeta\overline{B(\zeta)}\frac{B(\zeta) - B(z) - (\zeta - z)B'(z)}{(\zeta - z)^2}.$$

Then we can write $\|(k_\zeta^B)'\|_{A^q(\gamma)}^q = I_1 + I_2$, where

$$I_1 = \int_{|z-\zeta|\le\frac{1-r}{n}} \left|\frac{B(\zeta) - B(z) - (\zeta - z)B'(z)}{(\zeta - z)^2}\right|^q (1-|z|)^\gamma \mathrm{d}\mathcal{A}(z)$$

and

$$I_2 = \int_{|z-\zeta|>\frac{1-r}{n}} \left|\frac{B(\zeta) - B(z) - (\zeta - z)B'(z)}{(\zeta - z)^2}\right|^q (1-|z|)^\gamma \mathrm{d}\mathcal{A}(z).$$

Since $|B''(u)| \lesssim \left(\frac{n}{1-r}\right)^2$ for any $u \in \overline{\mathbb{D}}$, it follows that

$$\begin{aligned} I_1 &\lesssim \max_{u\in\overline{\mathbb{D}}} |B''(u)|^q \int_{|z-\zeta|\le\frac{1-r}{n}} (1-|z|)^\gamma \mathrm{d}\mathcal{A}(z) \\ &\lesssim \left(\frac{n}{1-r}\right)^{2q} \left(\frac{1-r}{n}\right)^{\gamma+2} \lesssim \left(\frac{n}{1-r}\right)^{2q-2-\gamma}. \end{aligned}$$

Now we estimate I_2. To this aim we first observe that if we put $w = \big(1-\frac{1-r}{2n}\big)\zeta$, then $|z-\zeta|/2 \le |1-\bar{w}z| \le 3|z-\zeta|/2$ when $|z-\zeta| \ge (1-r)/n$. Hence,

$$\int_{|z-\zeta|>\frac{1-r}{n}} \left|\frac{B(\zeta)-B(z)}{(z-\zeta)^2}\right|^q (1-|z|)^\gamma \mathrm{d}\mathcal{A}(z) \lesssim \int_{\mathbb{D}} \frac{(1-|z|)^\gamma}{|1-\bar{w}z|^{2q}}\mathrm{d}\mathcal{A}(z)$$
$$\asymp \frac{1}{(1-|w|)^{2(q-1)-\gamma}} \asymp \left(\frac{n}{1-r}\right)^{2q-\gamma-2}.$$

Here we use the standard fact (see, e.g., [12, Theorem 1.7]) that for $\alpha > -1$ and $\beta > \alpha + 2$ one has

$$\int_{\mathbb{D}} \frac{(1-|z|)^\alpha}{|1-\bar{w}z|^\beta}\mathrm{d}\mathcal{A}(z) \asymp \frac{1}{(1-|w|)^{\beta-\alpha-2}} \tag{4.6}$$

with constants depending on α and β, but not on $w \in \mathbb{D}$. Note that $\gamma \in (-1, q-1]$ and so in our case the assumptions on exponents are satisfied.

It remains to estimate

$$\int_{|z-\zeta|>\frac{1-r}{n}} \left|\frac{B'(z)}{\zeta-z}\right|^q (1-|z|)^\gamma \mathrm{d}\mathcal{A}(z) \asymp \int_{\mathbb{D}} \frac{|B'(z)|^q}{|1-\bar{w}z|^q}(1-|z|)^\gamma \mathrm{d}\mathcal{A}(z) =: J.$$

Take $\varepsilon > 0$ sufficiently small so that $\varepsilon < \min(q-1, \gamma+1)$, and put $s = \gamma+1-\varepsilon$. Then $s \in (0, q)$. Writing

$$|B'(z)|^q(1-|z|)^\gamma = |B'(z)|^{q-s}(1-|z|)^{\gamma-s}\left(|B'(z)|(1-|z|)\right)^s$$

and observing that $\sup_{u\in\mathbb{D}} |B'(u)|(1-|u|) \le 1$, we get

$$J \le \int_{\mathbb{D}} \frac{|B'(z)|^{q-s}(1-|z|)^{\gamma-s}}{|1-\bar{w}z|^q}\mathrm{d}\mathcal{A}(z) \lesssim \left(\frac{n}{1-r}\right)^{q-s} \int_{\mathbb{D}} \frac{(1-|z|)^{\gamma-s}}{|1-\bar{w}z|^q}\mathrm{d}\mathcal{A}(z)$$

(here we use the inequality $|B'(u)| \lesssim \frac{n}{1-r}$, $u \in \overline{\mathbb{D}}$). Since $\gamma - s = -1 + \varepsilon$ and $q - \gamma + s - 2 = q - 1 - \varepsilon > 0$, we get by (4.6)

$$\int_{\mathbb{D}} \frac{(1-|z|)^{\gamma-s}}{|1-\bar{w}z|^q}\mathrm{d}\mathcal{A}(z) \asymp \frac{1}{(1-|w|)^{q-\gamma+s-2}} \asymp \left(\frac{n}{1-r}\right)^{q-\gamma+s-2},$$

which completes the proof of (4.4).

Proof of (4.5)**.** Note that

$$|(k_\zeta^B)''(z)| = 2\left|\frac{B(\zeta)-B(z)-(\zeta-z)B'(z)-\frac{(\zeta-z)^2}{2}B''(z)}{(\zeta-z)^3}\right|,$$

whence $|(k_\zeta^B)''(z)| \le \sup_{u\in\mathbb{D}} |B'''(u)|/3$, $z \in \mathbb{D}$. Since $|B'''(u)| \lesssim \big(\frac{n}{1-r}\big)^3$ for any $u \in \overline{\mathbb{D}}$, it follows that

$$\big|(k_\zeta^B)''(u)\big| \lesssim \sup_{u\in\mathbb{D}} |B'''(u)| \lesssim \left(\frac{n}{1-r}\right)^3.$$

Therefore

$$\int_{|z-\zeta|\le\frac{1-r}{n}} \left|(k_\zeta^B)''(z)\right|^q (1-|z|)^\gamma \mathrm{d}\mathcal{A}(z) \lesssim \left(\frac{n}{1-r}\right)^{3q} \int_{|z-\zeta|\le\frac{1-r}{n}} (1-|z|)^\gamma \mathrm{d}\mathcal{A}(z)$$
$$\lesssim \left(\frac{n}{1-r}\right)^{3q} \left(\frac{1-r}{n}\right)^{\gamma+2}.$$

It remains to estimate

$$J_1 := \int_{|z-\zeta|>\frac{1-r}{n}} \frac{(1-|z|)^\gamma}{|\zeta-z|^{3q}} \mathrm{d}\mathcal{A}(z), \qquad J_2 := \int_{|z-\zeta|>\frac{1-r}{n}} \frac{|B'(z)|^q}{|\zeta-z|^{2q}} (1-|z|)^\gamma \mathrm{d}\mathcal{A}(z)$$

and

$$J_3 := \int_{|z-\zeta|>\frac{1-r}{n}} \frac{|B''(z)|^q}{|\zeta-z|^q} (1-|z|)^\gamma \mathrm{d}\mathcal{A}(z).$$

The estimate $J_1 \asymp \left(\frac{n}{1-r}\right)^{3q-2-\gamma}$ follows immediately from (4.6). Now let $s = \frac{\gamma+1}{2}$. Then $s \in [0, q]$ and we have

$$J_2 \lesssim \sup_{u\in\mathbb{D}} |B'(u)|^{q-s} \cdot \sup_{u\in\mathbb{D}} \left((1-|u|)^s |B'(u)|^s\right) \cdot \int_{|z-\zeta|>\frac{1-r}{n}} \frac{(1-|z|)^{\gamma-s}}{|\zeta-z|^{2q}} \mathrm{d}\mathcal{A}(z).$$

As in the proof of (4.4), let $w = \zeta\left(1-\frac{1-r}{2n}\right)$. Then, by (4.6),

$$J_2 \lesssim \left(\frac{n}{1-r}\right)^{q-s} \int_{\mathbb{D}} \frac{(1-|z|)^{\gamma-s}}{|1-\bar{w}z|^{2q}} \mathrm{d}\mathcal{A}(z) \asymp \left(\frac{n}{1-r}\right)^{q-s} \frac{1}{(1-|w|)^{2q-\gamma+s-2}}.$$

We used the fact that $2q-\gamma+s-2 = 2q-\frac{3}{2}-\frac{\gamma}{2} > 0$ since $q \ge \gamma$ and $q > 1$.

To estimate J_3 note that $|B''(u)| \le 2(1-|u|)^{-2}$, $u \in \mathbb{D}$. Let $\varepsilon \in (0, \min(q-1, \gamma+1))$ and put $s = \frac{\gamma+1-\varepsilon}{2}$. Then $s \in (0, q)$, $\gamma - 2s = -1+\varepsilon$ and $q-\gamma+2s-2 = q-1-\varepsilon > 0$. Now

$$J_3 \le \sup_{u\in\mathbb{D}} |B''(u)|^{q-s} \cdot \sup_{u\in\mathbb{D}} \left((1-|u|)^{2s} |B''(u)|^s\right) \cdot \int_{|z-\zeta|>\frac{1-r}{n}} \frac{(1-|z|)^{\gamma-2s}}{|1-\bar{w}z|^q} \mathrm{d}\mathcal{A}(z)$$
$$\asymp \left(\frac{n}{1-r}\right)^{2q-2s} \left(\frac{1}{1-|w|}\right)^{q-\gamma+2s-2} \asymp \left(\frac{n}{1-r}\right)^{3q-\gamma-2}.$$

This completes the proof of (4.5). □

Corollary 4.2. *Let $l \in \mathbb{N}$, $l \ge 2$, $q \in (1, \infty)$ and $\gamma \in (-1, q]$. We have*

$$\|(k_\zeta^B)^{(l)}\|_{A^q(\gamma)} \lesssim \left(\frac{n}{1-r}\right)^{l+1-\frac{\gamma+2}{q}}.$$

Proof. An application of [2, Theorem 1.3] yields

$$\|(k_\zeta^B)^{(l)}\|_{A^q(\gamma)} \lesssim \left(\frac{n}{1-r}\right)^{l-2} \|(k_\zeta^B)''\|_{A^q(\gamma)}.$$

Now the result follows from inequality (4.5). □

5. Proofs of the upper bounds in Theorems 2.1 and 2.3

5.1. The upper bounds in (2.1): a direct proof

We start by giving an easier proof than the one in [26, Theorem 2.3] of the upper bound in (2.1) for the case $1 \leq p \leq +\infty$. The main drawback of the proof in [26] is that it makes use of a strong interpolation result between Hardy spaces by P. Jones [13]. The proof below is a two-line corollary of H^p-norms estimates of reproducing kernel of model spaces.

Proof. For any $f \in H^p$,

$$|P_B f(\zeta)| = |\left\langle f,\, k_\zeta^B \right\rangle| \leq \|f\|_{H^p} \|k_\zeta^B\|_{H^q}, \qquad \zeta \in \overline{\mathbb{D}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Taking the supremum over all $\zeta \in \overline{\mathbb{D}}$, we obtain from (4.2) that

$$c\left(\sigma,\, H^p,\, H^\infty\right) \lesssim \left(\frac{n}{1-r}\right)^{\frac{1}{p}},$$

for any $\sigma = (\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$. □

5.2. The upper bound in (2.2)

In the following proof, given $\sigma \in \mathbb{D}^n$ we will assume first that $f \in H^\infty$ and bound $\|f\|_{H^\infty / B_\sigma H^\infty}$ in terms of $\|f\|_{A^p(\beta)}$. The corresponding upper bound for $c(\sigma,\, A^p\left(\beta\right),\, H^\infty)$ will follow by density.

Proof. **Case 1: $\beta \leq 0$.** First we prove the upper bound for $\beta \in (-1, 0]$. Let $f \in A^p\left(\beta\right) \cap H^\infty$ be such that $\|f\|_{A^p(\beta)} \leq 1$. Let

$$g(\zeta) = (P_B f)(\zeta) = \langle f,\, k_\zeta^B \rangle.$$

Applying the Green formula (3.2) to $\varphi = k_\zeta^B$ and $\psi = f$ we obtain

$$\overline{g(\zeta)} - \overline{f(0)} k_\zeta^B(0) = ((k_\zeta^B)',\, S^* f) = \int_{\mathbb{D}} (k_\zeta^B)'(u) \overline{S^* f(u)} \mathrm{d}\mathcal{A}(u). \tag{5.1}$$

We first assume that $p > 1$ so that its conjugate exponent q (i.e., $\frac{1}{p} + \frac{1}{q} = 1$) is finite. Applying the Hölder inequality to $(1 - |u|)^{\beta/p} |S^* f(u)|$ and $(1 - |u|)^{-\beta/p} \left|(k_\zeta^B)'(u)\right|$ with exponents p and q, we obtain

$$\left|((k_\zeta^B)',\, S^* f)\right| \leq \|S^* f\|_{A^p(\beta)} \|(k_\zeta^B)'\|_{A^q(-(q-1)\beta)},$$

and the estimate for $\|g\|_\infty$ follows by a direct application of (4.4) with $\gamma = -(q-1)\beta \in [0, q-1)$ (note that S^* is bounded from $A^p(\beta)$ onto itself and $|k_\zeta^B(0)| \leq 2$).

If $p = 1$ then

$$|((k_\zeta^B)',\, S^* f)| \lesssim \|f\|_{A^1(\beta)} \|k_\zeta^B\|_{\mathcal{B}_{-\beta}} \lesssim \|f\|_{A^1(\beta)} \left(\frac{n}{1-r}\right)^{\beta+2}$$

by (4.3) with $l = 0$ and $\alpha = -\beta$.

Case 2: $\beta > 0$**.** Now we prove the upper bound for $\beta > 0$. Applying (5.1) and (3.3) with $l = \left[\frac{\beta}{p}\right] + 1$ we get

$$\left((k_\zeta^B)',\, S^*f\right) = \int_{\mathbb{D}} D_l\left((k_\zeta^B)'\right)(u)\overline{S^*f(u)}(1-|u|^2)^l \mathrm{d}\mathcal{A}(u). \tag{5.2}$$

Again we first assume that $p > 1$ so that its conjugate exponent q is finite. Writing $(1-|u|^2)^l = (1-|u|^2)^{\frac{\beta}{p}+l-\frac{\beta}{p}}$ and applying the Hölder inequality to $(1-|u|^2)^{\frac{\beta}{p}}\overline{S^*f(u)}$ and $(1-|u|^2)^{l-\frac{\beta}{p}} D_l\left(\left(k_\zeta^B\right)'\right)(u)$ we get

$$|((k_\zeta^B)',\, S^*f)| \le \|S^*f\|_{A^p(\beta)} \left(\int_{\mathbb{D}} (1-|u|)^{q\alpha} |D_l\left((k_\zeta^B)'\right)(u)|^q \mathrm{d}\mathcal{A}(u)\right)^{\frac{1}{q}},$$

where $\alpha = l - \frac{\beta}{p}$. It remains to apply Corollary 4.2 with $\gamma = q\alpha \in [0, q]$: the result follows since

$$\|D_l\left((k_\zeta^B)'\right)\|_{A^q(q\alpha)} \asymp \|(k_\zeta^B)^{(l+1)}\|_{A^q(q\alpha)}.$$

If $p = 1$ then, by (4.3),

$$|((k_\zeta^B)',\, S^*f)| \lesssim \|f\|_{A^1(\beta)} \|(k_\zeta^B)^{(l)}\|_{\mathcal{B}_\alpha} \lesssim \|f\|_{A^1(\beta)} \left(\frac{n}{1-r}\right)^{\beta+2}.$$

□

5.3. The upper bound in (2.3)

Proof. The upper bound in Theorem 2.3 follows directly from the following observation: let $f \in \mathcal{R}_n$ and $1/\overline{\lambda}_1, \ldots, 1/\overline{\lambda}_n$ are the poles of f (repeated according to multiplicities), then $f \in K_{zB}$ with $\sigma = (\lambda_1, \ldots, \lambda_n)$. In particular we have $f = P_{\tilde{B}} f = \left\langle f,\, k_\zeta^{\tilde{B}} \right\rangle$, where $\tilde{B}(z) = zB(z)$. Now we can repeat the above proof for $\tilde{B}$ instead of B. □

6. Proof of the lower bounds

In this section we estimate from below the interpolation constant $c(\sigma,\ X,\ H^\infty)$ for the one-point interpolation sequence $\sigma_{\lambda,\,n} = (\lambda, \lambda, ..., \lambda) \in \mathbb{D}^n$:

$$c(\sigma_{n,\,\lambda},\ X,\ H^\infty) = \sup\{\|f\|_{H^\infty/b_\lambda^n H^\infty} :\ f \in X \cap H^\infty,\ \|f\|_X \le 1\},$$

where $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf\{\|f + b_\lambda^n g\|_\infty :\ g \in X \cap H^\infty\}$ (recall that $b_\lambda(z) = \frac{\lambda - z}{1-\overline{\lambda}z}$). Since the spaces $X = H^p$, $A^p(\beta)$ and H^∞ are rotation invariant we have $c\,(\sigma_{n,\,\lambda}, X,\, H^\infty) = c\,(\sigma_{n,\,\mu}, X,\, H^\infty)$ for every λ, μ with $|\lambda| = |\mu| = r$. Without loss of generality we can thus suppose that $\lambda = -r$.

6.1. The lower bounds in Theorem 2.1

Recall that we need to prove the following estimates:

$$c\,(\sigma_{n,\,-r},\,H^p,H^\infty)\gtrsim\left(\frac{n}{1-r}\right)^{\frac{1}{p}} \tag{6.1}$$

and

$$c\,(\sigma_{n,\,-r},\,A^p(\beta),\,H^\infty)\gtrsim\left(\frac{n}{1-r}\right)^{\frac{2+\beta}{p}} \tag{6.2}$$

for any $n\geq 1$, $r\in[0,\,1)$, $p\in[1,\,+\infty)$ and $\beta>-1$.

Proof. For $N\in\mathbb{N}$, we consider the test function

$$\varphi_n:=Q_n^N, \tag{6.3}$$

where

$$Q_n=\frac{1-r^2}{(1+rz)^2}\left(\sum_{k=0}^{n-1}b_{-r}^k(z)\right)^2=\frac{1-r^2}{(1+rz)^2}D_n^2(b_{-r}(z))$$

and $D_n(z)=\sum_{j=0}^{n-1}z^j$ is the (analytic part of) the n^{th} Dirichlet kernel. We have

$$c\,(\sigma_{n,\,-r},\,X,H^\infty)\geq\frac{\|\varphi_n\|_{H^\infty/b_{-r}^nH^\infty}}{\|\varphi_n\|_X}.$$

Thus we need to obtain an upper estimate for $\|\varphi_n\|_X$ and a lower one for $\|\varphi_n\|_{H^\infty/b_{-r}^nH^\infty}$.

Step 1. Upper estimate for $\|\varphi_n\|_{H^p}$**,** $N=1$**.** Note that $Q_n=(\sum_{k=0}^{n-1}e_k)^2$, where e_k are the elements of the Malmquist–Walsh basis. Hence, $\|Q_n\|_{H^1}=n$. Now we compute $\|Q_n\|_\infty$. Note that $Q_n\circ b_{-r}$ is a polynomial of degree $2n-2$ with positive coefficients: indeed,

$$Q_n\circ b_{-r}=\left(\sum_{k=0}^{n-1}z^k\frac{(1-r^2)^{1/2}}{1+rb_{-r}(z)}\right)^2=\left(1-r^2\right)^{-1}\left(1+(1+r)\sum_{k=1}^{n-1}z^k+rz^n\right)^2.$$

In particular,

$$\|Q_n\|_\infty=\|Q_n\circ b_{-r}\|_\infty=Q_n\circ b_{-r}(1)=n^2\frac{1+r}{1-r}, \tag{6.4}$$

and for any $p\geq 1$, $\|Q_n\|_{H^p}^p\leq\|Q_n\|_{H^1}\|Q_n\|_\infty^{p-1}$. Thus

$$\|Q_n\|_{H^p}\leq n^{2-\frac{1}{p}}\left(\frac{1+r}{1-r}\right)^{1-\frac{1}{p}}. \tag{6.5}$$

Step 2. Upper estimate for $\|\varphi_n\|_{A^p(\beta)}$. Assume that $\beta\in(l-1,\,l]$ where $l\geq 0$ is an integer and put $N=l+2$. We will prove that

$$\|\varphi_n\|_{A^p(\beta)}\lesssim\frac{n^{2N-\frac{\beta+2}{p}}}{(1-r)^{N-\frac{\beta+2}{p}}}. \tag{6.6}$$

The change of variable $w = b_{-r}(z)$ (equivalently, $z = b_{-r}(w)$) gives

$$\int_{\mathbb{D}} f(b_{-r}(z))|b'_{-r}(z)|^2 \mathrm{d}\mathcal{A}(z) = \int_{\mathbb{D}} f(w)\mathrm{d}\mathcal{A}(w)$$

for any function f summable with respect to $\mathcal{A}$. Then we have

$$\begin{aligned}\|Q_n^N\|^p_{A^p(\beta)} &= \frac{1}{(1-r^2)^{pN-2-\beta}} \int_{\mathbb{D}} |1+rw|^{2pN-4-2\beta}|D_n(w)|^{2Np}(1-|w|^2)^\beta \mathrm{d}\mathcal{A}(w) \\ &\lesssim \frac{1}{(1-r)^{pN-2-\beta}} \int_{\mathbb{D}} |D_n(w)|^{2Np}(1-|w|^2)^\beta \mathrm{d}\mathcal{A}(w)\end{aligned}$$

since $pN \geq N \geq \beta + 2$ and so $2pN - 4 - 2\beta \geq 0$. It remains to see that

$$\int_{\mathbb{D}} |D_n(w)|^{2Np}(1-|w|^2)^\beta \mathrm{d}\mathcal{A}(w) \lesssim n^{2pN-2-\beta}.$$

Indeed, for $p = 1$ we have by a very rough estimate

$$\|D_n^N\|^2_{A^2(\beta)} = \sum_{k=0}^{(n-1)N} \frac{|\widehat{D_n^N}(k)|^2}{k^{1+\beta}} \lesssim \sum_{k=1}^{(n-1)N} \frac{k^{2N-2}}{k^{1+\beta}} \lesssim n^{2N-2\beta-2},$$

while for $p \in [1, \infty)$,

$$\begin{aligned}\int_{\mathbb{D}} |D_n(w)|^{2Np}(1-|w|^2)^\beta \mathrm{d}\mathcal{A}(w) &\leq \|D_n^{2N}\|_\infty^{p-1}\|D_n^N\|^2_{A^2(\beta)} \\ &\lesssim n^{2N(p-1)} n^{2N-2\beta-2} = n^{2pN-2-\beta}.\end{aligned}$$

This completes the proof of (6.6).

Step 3. Lower estimate for $\|\varphi_n\|_{H^\infty/b^n_{-r}H^\infty}$. Put $\Psi_n := \varphi_n \circ b_{-r}$. Clearly,

$$\|\varphi_n\|_{H^\infty/b^n_{-r}H^\infty} = \|\Psi_n\|_{H^\infty/z^n H^\infty}.$$

We will show that

$$\|\Psi_n\|_{H^\infty/z^n H^\infty} \gtrsim \frac{n^{2N}}{(1-r)^N}. \tag{6.7}$$

Denote by F_n the n-th Fejer kernel, $F_n(z) = \frac{1}{2\pi}\sum_{|j|\leq n}\left(1 - \frac{|j|}{n}\right)z^j$, and denote by $*$ the usual convolution operation in $L^1(\mathbb{T})$. Then, for any $g \in L^\infty(\mathbb{T})$, we have $\|g * F_n\|_\infty \leq \|g\|_\infty \|F_n\|_{H^1} = \|g\|_\infty$. On the other hand, since $\widehat{g * h}(j) = \hat{g}(j)\hat{h}(j)$ and $\widehat{F}_n(j) = 0$ for every $j \geq n$, we have

$$g * F_n = \Psi_n * F_n$$

for any $g \in H^\infty$ such that $\hat{g}(k) = \widehat{\Psi}_n(k)$, $k = 0, 1, , \ldots, n-1$. Hence, for any such g, $\|g\|_\infty \geq \|\Psi_n * F_n\|_\infty$ and so

$$\begin{aligned}\|\Psi_n\|_{H^\infty/z^n H^\infty} = \inf\left\{\|g\|_\infty :\ g \in H^\infty,\ \hat{g}(k) = \hat{\Psi}_n(k),\ 0 \leq k \leq n-1\right\} \\ \geq \|\Psi_n * F_n\|_\infty \geq (\Psi_n * F_n)(1).\end{aligned}$$

Note that the convolution with F_n gives us the Cesàro mean of the partial sums of the Fourier series. Denote by S_j the j-th partial sum for Ψ_n at 1. Recall that

$$\Psi_n(z) = \frac{1}{(1-r^2)^N}\left(1 + (1+r)\sum_{k=1}^{n-1} z^k + rz^n\right)^{2N}.$$

Since all Taylor coefficients for Ψ_n are positive, we have

$$\begin{aligned} S_j(1) &\geq \frac{1}{(1-r^2)^N}\left(1 + (1+r)\sum_{k=1}^{[(2N)^{-1}j]} z^k\right)^{2N}\Bigg|_{z=1} \\ &\geq \frac{(1+(1+r)[(2N)^{-1}j])^{2N}}{(1-r^2)^N} \gtrsim \frac{j^{2N}}{(1-r)^N} \end{aligned}$$

with the constants depending on N only. Hence,

$$(\Psi_n * F_n)(1) = \frac{1}{n}\sum_{j=0}^{n-1} S_j(1) \gtrsim \frac{n^{2N}}{(1-r)^N},$$

which proves (6.7).

Step 4. Completion of the proof. The estimate (6.1) follows from (6.5) and (6.7) (with $N = 1$ and $\varphi_n = Q_n$). Combining (6.6) and (6.7) we arrive at the estimate (6.2). □

6.2. The lower bounds in Theorem 2.3

Proof. We prove the lower bound for $\mathcal{E}_{n,\,r}(H^\infty,\, A^p(\beta))$ in (2.3). We put $N = l + 2$, where $l \geq 0$ is the integer such that $\beta \in (l-1,\, l]$, and consider the test function φ_m defined in (6.3) with $m = \left[\frac{n}{2N}\right]$ (assuming that $n > 2N$). Therefore

$$\varphi_m = \left(\sum_{k=0}^{m-1}(1-|r|^2)^{1/2} b_{-r}^k\,(1+rz)^{-1}\right)^{2N} \in \mathcal{R}_{n,\,r}.$$

We know from (6.4) that

$$\|Q_m^N\|_\infty = m^{2N}\left(\frac{1+r}{1-r}\right)^N,$$

and it follows from (6.6) that

$$\|Q_m^N\|_{A^p(\beta)} \lesssim \frac{m^{2N-\frac{\beta+2}{p}}}{(1-r)^{N-\frac{\beta+2}{p}}} \lesssim \frac{m^{-\frac{\beta+2}{p}}}{(1-r)^{-\frac{\beta+2}{p}}}\|Q_m^N\|_\infty$$

which completes the proof. □

References

[1] J. Agler, J. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, AMS: Graduate studies in mathematics 44, 2002.

[2] A. Baranov, R. Zarouf, *A Bernstein-type inequality for rational functions in weighted Bergman spaces*, Bull. Sci. Math. **137** (2013), 541–556.

[3] A. Baranov, R. Zarouf, *A model space approach to some classical inequalities for rational functions*, J. Math. Anal. Appl. **418** (2014), 1, 121–141.

[4] L. Baratchart, *Rational and meromorphic approximation in L_p of the circle: system-theoretic motivations, critical points and error rates.* In N. Papamichael, S. Ruscheweyh, and E. Saff, Eds, Computational Methods and Function Theory, 45–78, World Scientific Publish. Co, 1999.

[5] L. Baratchart, F. Wielonsky, *Rational approximation problem in the real Hardy space H_2 and Stieltjes integrals: a uniqueness theorem*, Constr. Approx. **9** (1993), 1–21.

[6] J. Bergh, J. Löfström, *Interpolation Spaces*, Springer-Verlag, Berlin, 1976.

[7] P. L. Duren, *Theory of H^p Spaces,* Academic Press, New York, 1970.

[8] W. S. Cohn, *Radial limits and star invariant subspaces of bounded mean oscillation*, Amer. J. Math. **108** (1986), 3, 719–749.

[9] E. M. Dyn'kin, *Rational functions in Bergman spaces*, Complex analysis, operators, and related topics, Oper. Theory Adv. Appl., Vol. 113, Birkhäuzer, Basel, 77–94, 2000.

[10] E. M. Dyn'kin, *Inequalities for rational functions*, J. Approx. Theory **91** (1997), 349–367.

[11] F. Foiaş, A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Birkhuser, Basel, 1990.

[12] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman Spaces*, Graduate Texts in Mathematics, 199, Springer-Verlag, New York, 2000.

[13] P. W. Jones, *L^∞ estimates for the $\bar{\partial}$ problem in the half plane*, Acta Math. **150** (1983), 137–152.

[14] S. B. Nagy, C. Foiaş, *Commutants de certains opérateurs*, Act. Sci. Math. **29** (1968), 1–17.

[15] N. Nikolski, *Condition Numbers of Large Matrices and Analytic Capacities*, St. Petersburg Math. J. **17** (2006), 641–682.

[16] N. K. Nikolski, *Operators, Functions, and Systems: an Easy Reading*, Vol.1, Amer. Math. Soc. Monographs and Surveys, 2002.

[17] N. K. Nikolski, *Treatise on the shift operator*, Springer-Verlag, Berlin etc., 1986.

[18] V. Pták, *A maximum problem for matrices*, Lin. Alg. Appl. **28** (1979), 193–204.

[19] V. Pták, N. J. Young, *Functions of operators and the spectral radius*, Linear Algebra Appl. **29** (1980), 357–392.

[20] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 289–299.

[21] O. Szehr, R. Zarouf, *Maximum of the resolvent over matrices with given spectrum*, J. Funct. Anal. **272** (2017), 2, 819–847.

[22] H. Triebel, *Spaces of Besov–Hardy–Sobolev Type*, Teubner Verlag, Leipzig, 1978.

[23] N. J. Young, *Norms of powers of matrices with constrained spectra*, Linear Algebra and Appl. **23** (1979), 227–244.

[24] R. Zarouf, *Interpolation avec contraintes sur des ensembles finis du disque*, C. R. Acad. Sci. Paris, Ser. I **347** (2009), 785–790.

[25] R. Zarouf, *Effective H^∞ interpolation constrained by Hardy and Bergman norms,* Ann. Funct. Anal. **2** (2011), 2, 59–74.

[26] R. Zarouf, *Effective H^∞ interpolation*, Houston J. Math. **39** (2013), 2, 487–514.

[27] R. Zarouf, *Carathéodory–Schur H^∞ interpolation constrained by weighted Bergman norms*, Harmonic Analysis, Function Theory Operator Theory and Applications, Theta Foundation, 2018, 261–267.

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