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VINCENT BANSAYE, BERTRAND CLOEZ, PIERRE GABRIEL, AND ALINE MARGUET

Abstract. We consider non-conservative positive semigroups and obtain necessary and sufficient conditions for uniform exponential contraction in weighted total variation norm. This ensures the existence of Perron eigenelements and provides quantitative estimates of spectral gaps, complementing Krein-Rutman theorems and generalizing recent results relying on probabilistic approaches. The proof is based on a non-homogenous h-transform of the semigroup and the construction of Lyapunov functions for this latter. It exploits then the classical necessary and sufficient conditions of Harris' theorem for conservative semigroups. We apply these results and obtain exponential convergence of birth and death processes conditioned on survival to their quasi-stationary distribution, as well as estimates on exponential relaxation to stationary profiles in growth-fragmentation PDEs.

Contents

1. Introduction

We are interested in the asymptotic behavior of positive semigroups acting on a weighted space of measures. For conservative semigroups, Harris' ergodic theorem [42, 54, 40] provides a

Date: March 10, 2019.
2010 *Mathematics Subject Classification.* Primary 47A35; Secondary 35B40, 47D06, 60J80, 92D25.
Key words and phrases. Positive semigroups; measure solutions; ergodicity; Krein-Rutman theorem; branching processes; growth-fragmentation; quasi-stationary distribution.

necessary and sufficient condition for the existence of a unique invariant probability measure and the exponential convergence of the semigroup to this invariant measure for the weighted total variation distance. It combines a Doeblin condition on small sets that provides a contraction and a Lyapunov function that pushes back the mass to these small sets. For non-conservative semigroups, the Perron-Frobenius [62, 35] and Krein-Rutman [46] theory gives conditions ensuring the existence of a steady distribution which grows or decreases exponentially fast, see [14, 43, 24]. We propose a Harris type theorem in that case. We obtain necessary and sufficient conditions to guarantee the existence of a (unique) Perron solution and the exponential convergence of the non-conservative semigroup to this solution, uniformly with respect to the weighted total variation distance.

We start by stating the framework of our study. Let $\mathcal{X}$ be a measurable space. For any measurable function $\varphi : \mathcal{X} \to (0,\infty)$ we denote by $\mathcal{B}(\varphi)$ the space of measurable functions $f : \mathcal{X} \to \mathbb{R}$ which are dominated by φ, *i.e.* such that the quantity

$$\|f\|_{\mathcal{B}(\varphi)} = \sup_{x\in\mathcal{X}} \frac{|f(x)|}{\varphi(x)}$$

is finite. Endowed with this weighted supremum norm, $\mathcal{B}(\varphi)$ is a Banach space. Let $\mathcal{B}_+(\varphi) \subset \mathcal{B}(\varphi)$ be its positive cone, namely the subset of nonnegative functions.

We denote by $\mathcal{M}_+(\varphi)$ the cone of positive measures on $\mathcal{X}$ which integrate φ, *i.e.* the set of positive measures μ on $\mathcal{X}$ such that the quantity

$$\mu(\varphi) = \int_{\mathcal{X}} \varphi \, d\mu$$

is finite. The space of *weighted signed measures* $\mathcal{M}(\varphi)$ is defined here by

$$\mathcal{M}(\varphi) = \mathcal{M}_+(\varphi) \times \mathcal{M}_+(\varphi)\big/_{\sim}$$

where $(\mu_1,\mu_2) \sim (\tilde\mu_1,\tilde\mu_2)$ if $\mu_1 + \tilde\mu_2 = \mu_2 + \tilde\mu_1$. An element μ of $\mathcal{M}(\varphi)$ acts on $\mathcal{B}(\varphi)$ through

$$\mu(f) = \mu_1(f) - \mu_2(f),$$

where (μ_1,μ_2) is any representative of the equivalence class μ. This motivates the notation $\mu = \mu_1 - \mu_2$ to mean that μ is the equivalence class of (μ_1,μ_2). Clearly $\mathcal{M}(\varphi)$ is canonically isomorphic to the space of finite signed measures on $\mathcal{X}$, but it is not a subspace except if φ is bounded from below by a positive constant. Through this isomorphism the Hahn-Jordan decomposition of signed measures ensures that for any $\mu \in \mathcal{M}(\varphi)$ there exists a unique couple $(\mu_+,\mu_-) \in \mathcal{M}_+(\varphi) \times \mathcal{M}_+(\varphi)$ of mutually singular measures such that $\mu = \mu_+ - \mu_-$. We endow $\mathcal{M}(\varphi)$ with the weighted total variation norm

$$\|\mu\|_{\mathcal{M}(\varphi)} = |\mu|(\varphi) = \mu_+(\varphi) + \mu_-(\varphi) = \sup_{\|f\|_{\mathcal{B}(\varphi)}\le 1} |\mu(f)|$$

which makes it a Banach space, the canonical isomorphism with the space of signed measures being actually an isometry if the latter is endowed with the standard total variation norm.

We consider two measurable functions $V : \mathcal{X} \to (0,\infty)$ and $\psi : \mathcal{X} \to (0,\infty)$ and a positive semigroup $(M_t)_{t\ge 0}$ acting both on $\mathcal{B}(V)$ (on the right $f \mapsto M_t f$) and $\mathcal{M}(V)$ (on the left $\mu \mapsto \mu M_t$). We also assume the standard duality relation $(\mu M_t)f = \mu(M_t f)$ and we denote this common value by $\mu M_t f$. For two functions $f, g : \Omega \to \mathbb{R}$, we use the notation $f \lesssim g$ when there exists a constant $C > 0$ such that $f \le Cg$ on Ω. We now state our assumptions on $M = (M_t)_{t\ge 0}$.

Assumption A. *There exist $\tau, T > 0$, $\beta > \alpha > 0$, $\theta \ge 0$, $(c,d) \in (0,1]^2$, $K \subset \mathcal{X}$ and ν a probability measure on $\mathcal{X}$ supported by K such that*

(A0) $\psi \le V$ *on* $\mathcal{X}$ *and* $V \lesssim \psi$ *on* K; $MV \lesssim V$ *and* $M\psi \gtrsim \psi$ *on* $[0,T]\times\mathcal{X}$,

(A1) $M_\tau V \le \alpha V + \theta \mathbf{1}_K \psi$,

(A2) $M_\tau \psi \ge \beta \psi$,

(A3) For all $x \in K$ *and* $f \in \mathcal{B}_+(V/\psi)$,

$$M_\tau(f\psi)(x) \ge c\,\nu(f)\,M_\tau\psi(x),$$

(A4) For any integer n,

$$d \sup_{x\in K} \frac{M_{n\tau}\psi(x)}{\psi(x)} \le \nu\left(\frac{M_{n\tau}\psi}{\psi}\right).$$

Assumption **A** is an extension of the criterion for exponential convergence of conservative semigroups for the weighted total variation distance [40], see forthcoming Theorem 2.1. They are linked to classical assumptions for spectral gaps of non-conservative semigroups, see in particular [58, 7, 21]. They relax these assumptions and provide necessary conditions for exponential convergence in total variation distance.
More precisely, one can compare **A** to the assumptions of [58, Theorem 5.3], where (A2) corresponds to (2), while (1) and (4) seem to be relaxed and simplified here. In particular Doeblin's condition (A3) is less restrictive than assuming strong positivity or irreducibility.
Assumption **A** provides an extension of the conditions of [7] in the homogeneous case. In that latter, Doeblin's condition (A3) was required on the whole space $\mathcal{X}$, which imposes the "coming back from infinity" property. It does not hold for instance in the two applications we consider in the present paper.
Our assumptions are very similar to [21]. Our approach is different, as explained below. In particular, we relax the boundedness of ψ required in [21] and we obtain necessary conditions for weighted exponential convergence. As a by product, we can capture new regimes, see the two applications and comments in Section 4.1 for convergence to quasi-stationary distribution.

The main result of the paper can be stated as followed. Its proof is postponed to Section 3.5.

Theorem 1.1. *(i) Let* (V,ψ) *be a couple of measurable functions from* $\mathcal{X}$ *to* $(0,\infty)$ *which satisfies Assumption* **A**. *Then, there exists a unique triplet* $(\gamma,h,\lambda) \in \mathcal{M}_+(V)\times\mathcal{B}_+(V)\times\mathbb{R}$ *of eigenelements of* M *with* $\gamma(h) = \|h\|_{\mathcal{B}(V)} = 1$, *i.e. satisfying for all* $t \ge 0$

$$\gamma M_t = \mathrm{e}^{\lambda t}\gamma \qquad \text{and} \qquad M_t h = \mathrm{e}^{\lambda t} h. \tag{1.1}$$

Moreover, there exists $C,\omega > 0$ *such that for all* $t \ge 0$ *and* $\mu \in \mathcal{M}(V)$,

$$\left\|\mathrm{e}^{-\lambda t}\mu M_t - \mu(h)\gamma\right\|_{\mathcal{M}(V)} \le C\,\|\mu\|_{\mathcal{M}(V)}\,\mathrm{e}^{-\omega t}. \tag{1.2}$$

(ii) Assume that there exist a positive measurable function V, *a triplet* $(\gamma,h,\lambda) \in \mathcal{M}_+(V)\times\mathcal{B}_+(V)\times\mathbb{R}$, *and constants* $C,\omega > 0$ *such that* (1.1) *and* (1.2) *hold. Then, the couple* $(V, h/\|h\|_{\mathcal{B}(V)})$ *satisfies Assumption* **A**.

In case (i) we have additionally the estimates $\log(\beta)/\tau \le \lambda \le \log(\alpha+\theta)/\tau$ and $(\psi/V)^q\psi \lesssim h \lesssim V$ for some $q > 0$. Besides, as explained in the next section and in the forthcoming proofs, one can also provide quantitative bounds for all the constants above, expressed using the constants of Assumption **A**.
Assumption **A** happens to be necessary for exponential convergence. We also provide in this

paper more convenient sufficient conditions. In the conservative case, Foster-Lyapunov inequality (A1) is usually checked through a so-called drift condition on the generator [55]. In Section 2.4, we give the counterpart in the non-conservative case. Loosely speaking, writing $\mathcal{L}$ the generator of $(M_t)_{t\geq 0}$, we prove that

$$\mathcal{L}V \leq aV + \zeta\psi, \qquad b\psi \leq \mathcal{L}\psi \leq \xi\psi,$$

entail that (A0), (A1) and (A2) hold. Furthermore, (A3) and (A4) can be easily checked under irreducibility conditions when $\mathcal{X}$ is discrete, while in the continuous setting coupling methods can be invoked.

Spectral results for semigroups have been obtained recently using stability theorems for the associated Markov processes. Let us mention in particular works associated with growth-fragmentation process [13] and Feynman-Kac semigroups [34]. The results here are related but the proofs are different. We use a non-homogeneous Markov process rather than a penalization of an homogeneous Markov process arising through Feynman-Kac formula. In particular, we stress that our results provide the existence of eigenelements without needing the application of Krein-Rutman theorem. Furthermore, our contraction method is extendable to non-homogeneous setting and applications and in this vein we refer to [7] for the case of Doeblin conditions. Let us finally mention [70, 71] for related works on exponential convergence for subconservative semigroups. It provides in particular relevant sufficient conditions for (A3) and (A4) when $K = \mathcal{X}$.

The paper is structured as follows. In Section 2, we give the main statements and ingredients of the paper, which both lead to the proof of Theorem 1.1 and provide additional quantitative estimates. We first recall the classical conservative Harris theorem. Then, we introduce the non-homogeneous conservative auxiliary semigroup which allows us to extend this result to non-conservative cases by proving the existence of eigenelements and quantifying the spectral gap. We end this section by stating some handy sufficient conditions. Section 3 contains the proofs of these statements. In Section 4 we derive new results for two well studied examples, namely the random walk on integers absorbed in 0 and the growth-fragmentation equation.

2. Main statements and outline of the proof

In this section, we explain the main steps of the proof of Theorem 1.1 and give more quantitative estimates, especially about the spectral gap. We also introduce the key objects and ideas of the paper. First, we recall the well-known Harris ergodic theorem, for conservative semigroups, that we slightly adapt to our purposes, see forthcoming Theorem 2.1. Then, we introduce the non-homogeneous conservative semigroup $P^{(t)}$ via a h-transform. We provide for this semigroup both a Lyapunov function (Lemma 2.2) and a Doeblin condition on small sets (Lemma 2.3) using Assumption **A**. Theorem 2.1 can then be applied to this conservative semigroup and forthcoming Proposition 2.4 yields a contraction principle for $P^{(t)}$. We can then prove the existence of eigenelements (γ, h, λ) and control the growth of the mass $M_t\psi$ by $\mathrm{e}^{\lambda t}$ and conclude. Finally, we give in Section 2.4 sufficient conditions for verifying Assumption **A**, which will be useful for forthcoming applications. Let us stress that in the lemmas of Sections 2 and 3, Assumption **A** is implicitly supposed to be verified.

2.1. Contraction in total variation for conservative operators.

The following result is a direct generalization of [40, Theorem 1.3]. The adaptation of the proof is briefly given in Section 3.1 for completeness. We consider a positive operator P acting both on bounded measurable

functions $f : \mathcal{X} \to \mathbb{R}$ on the right and on bounded measures μ of finite mass on the left, and such that $(\mu P)f = \mu(Pf)$. Note that the right action of P extends trivially to any measurable function $f : \mathcal{X} \to [0, +\infty]$. We assume that P is conservative in the sense that

$$P\mathbf{1} = \mathbf{1},$$

or equivalently, if μ is a probability measure then so does μP.

Theorem 2.1. *Assume that there exist two measurable functions $V_0, V_1 : \mathcal{X} \to [0, \infty)$ and $(\mathfrak{a}, \mathfrak{b}) \in (0,1)^2$, $\mathfrak{c} > 0$, $\mathfrak{R} > 2\mathfrak{c}/(1-\mathfrak{a})$ and a probability measure ν on $\mathcal{X}$ such that:*

- *for all $x \in \mathcal{X}$,*
$$PV_0(x) \le \mathfrak{a}V_1(x) + \mathfrak{c}, \tag{2.1}$$
- *for all $x \in \{V_1 \le \mathfrak{R}\}$,*
$$\delta_x P \ge \mathfrak{b}\nu. \tag{2.2}$$

Then, there exist $\mathfrak{y} \in (0,1)$ and $\kappa > 0$ such that for all probability measures μ_1 and μ_2,

$$\|\mu_1 P - \mu_2 P\|_{\mathcal{M}(1+\kappa V_0)} \le \mathfrak{y} \, \|\mu_1 - \mu_2\|_{\mathcal{M}(1+\kappa V_1)} . \tag{2.3}$$

In particular, for any $\mathfrak{b}' \in (0, \mathfrak{b})$ and $\mathfrak{a}' \in (\mathfrak{a} + 2\mathfrak{c}/\mathfrak{R}, 1)$, one can choose

$$\kappa = \frac{\mathfrak{b}'}{\mathfrak{c}}, \qquad \mathfrak{y} = \max\left\{1 - (\mathfrak{b} - \mathfrak{b}'), \frac{2 + \kappa\mathfrak{R}\mathfrak{a}'}{2 + \kappa\mathfrak{R}}\right\}.$$

Usually, for conservative semigroups and Markov chains [54, 40], Theorem 2.1 is stated and used with one single function $V_0 = V_1$. The contraction in total variation then gives the exponential convergence of the sequence $(\mu P^n)_{n \ge 0}$ to the unique invariant measure. Hereafter, a time-inhomogeneous semigroup is involved and a suitable family of functions V_k is considered.

2.2. **Auxiliary conservative semigroup and Lyapunov functions.** To exploit the previous estimates, we need to consider a relevant conservative semigroup associated to M. Let us fix a positive function $g \in \mathcal{B}(V)$ and a time $t > 0$. For any $0 \le s \le u \le t$, we define the operator $P_{s,u}^{(t,g)}$ acting on bounded measurable functions f through

$$P_{s,u}^{(t,g)} f = \frac{M_{u-s}(f M_{t-u} g)}{M_{t-s} g}. \tag{2.4}$$

We observe that the family $P^{(t,g)} = (P_{s,u}^{(t,g)})_{0 \le s \le u \le t}$ is a non-homogeneous conservative semigroup (or semiflow, or propagator), meaning that for any $0 \le s \le u \le v \le t$,

$$P_{s,u}^{(t,g)} P_{u,v}^{(t,g)} = P_{s,v}^{(t,g)}.$$

It has a probabilistic interpretation in terms of particles systems, see *e.g.* [50] and references below. Roughly speaking, it provides the position of the backward lineage of a particle at time t sampled with a bias g.

The particular case $g = \mathbf{1}$:

$$P_{s,u}^{(t,\mathbf{1})} f = \frac{M_{u-s}(f M_{t-u}\mathbf{1})}{M_{t-s}\mathbf{1}},$$

corresponds to uniform sampling and has been successfully used in the study of semigroups, see [26, 27, 5, 20, 50, 7]. In particular [20, 7] obtain quantitative uniform exponential estimates of M in the particular case $K = \mathcal{X}$. More precisely, when $K = \mathcal{X}$, [20] guarantees the equivalence between assumptions (A3)-(A4) for M and the Doeblin assumption (2.2) for $P^{(t,\mathbf{1})}$. However,

for several examples these conditions do not hold on the whole space $\mathcal{X}$, see the applications of Section 4. Such conditions are valid only on some (compact) subset $K \subset \mathcal{X}$. A counterpart of Harris ergodic theorem for non-conservative semigroup is then expected and obtained below.

Whenever possible, the right positive eigenfunction of the semigroup provides another relevant choice for g. More precisely, if there exist a positive function h and a real number λ such that $M_t h = \mathrm{e}^{\lambda t} h$ for any $t \geq 0$, then (2.4) simplifies for $g = h$. Indeed,

$$P_s f = P^{(t,h)}_{u,u+s} f = \frac{M_s(hf)}{\mathrm{e}^{\lambda s} h}$$

is a conservative homogeneous semigroup P. This transformation is usually called (Doob) h-transform and we refer to [29, 60], [30, chapter VIII] or [65, Section 39 p.83]. It allows to derive a Markov process from a non-conservative semigroup. This transformation provides a powerful tool for the analysis of branching processes and absorbed Markov process, see *e.g.* respectively [47, 32, 22] and [29, 23]. Let us explain here how to apply Theorem 2.1 with $(P_s)_{s\geq 0}$ and get the asymptotic behavior of M. Inequality (2.1) with $V_0 = V_1$ and $P = P_\tau$ reads $P_\tau V_0 \leq \mathfrak{a} V_0 + \mathfrak{c}$ and that latter writes $M_\tau(V_0 h) \leq \mathfrak{a} V_0 h + \mathfrak{c} h$ in terms of the original semigroup. This inequality is guaranteed by (A1) by setting $V = V_0 h$ and $\psi = h$. Additionally, Equation (2.2) corresponds exactly to (A3). Thus, in cases we know a positive eigenelement h, checking (A1) and (A3) with (V, h) allows to derive ergodic estimates for M using P and $V_0 = V/h$.

In this paper, we deal with the general case and consider a positive function ψ satisfying (A2). We introduce the following non-homogeneous conservative semigroup

$$P^{(t)}_{s,u} f = P^{(t,\psi)}_{s,u} f = \frac{M_{u-s}(f M_{t-u}\psi)}{M_{t-s}\psi}. \tag{2.5}$$

In general and at least for our applications, ψ will be not an eigenelement or $\mathbf{1}$. Nevertheless, the analogy with the h-transform above suggests to look for Lyapunov functions of the form $V_0 = V/\psi$. The family of functions $V/M_{k\tau}\psi$ actually readily satisfies (2.1). But their level sets may degenerate as $k \to \infty$, which raises a problem to check (2.2). We compensate the magnitude of $M_{k\tau}\psi$ and introduce the functions

$$V_k = \nu\left(\frac{M_{k\tau}\psi}{\psi}\right)\frac{V}{M_{k\tau}\psi}. \tag{2.6}$$

The two following lemmas, which are proved in Section 3.3, ensure that $(V_k)_{k\geq 0}$ indeed provides relevant Lyapunov functions with Doeblin's condition satisfied on the sublevel sets for $P^{(t)}$.

Lemma 2.2. *For all $k \geq 0$ and $n \geq m \geq k+1$, we have*

$$P^{(n\tau)}_{k\tau,m\tau} V_{n-m} \leq \mathfrak{a} V_{n-k} + \mathfrak{c},$$

where

$$\mathfrak{a} = \frac{\alpha}{\beta} \in (0,1), \qquad \mathfrak{c} = \frac{\theta}{c(\beta-\alpha)} \geq 0.$$

Lemma 2.3. *Let $\mathfrak{R} > 0$ and set*

$$\mathfrak{p} = \left\lfloor \frac{\log\left(\frac{2\mathfrak{R}(\alpha+\theta)}{(\beta-\alpha)d}\right)}{\log(\beta/\alpha)} \right\rfloor + 1. \tag{2.7}$$

Then, there exists a family of probability measures $\{\nu_{k,n},\ k \leq n\}$ such that for all $0 \leq k \leq n - \mathfrak{p}$ and $x \in \{V_{n-k} \leq \mathfrak{R}\}$,

$$\delta_x P^{(n\tau)}_{k\tau,(k+\mathfrak{p})\tau} \geq \mathfrak{b}\, \nu_{k,n},$$

where $\mathfrak{b} \in (0,1]$ *is defined by*

$$\mathfrak{b} = \frac{(cd(\beta-\alpha))^2\beta}{2\theta(\alpha+\theta)(\alpha R+\theta)} \frac{1}{\sum_{j=1}^{\ell}(\mathfrak{a}/cr)^j}$$

with $R = \sup_K V/\psi$ *and* $r = (\beta/(\alpha(R+\theta/(\beta-\alpha))+\theta))^2$.

We can now state the key contraction result.

Proposition 2.4. *Let* (V,ψ) *be a couple of measurable functions from* $\mathcal{X}$ *to* $(0,\infty)$ *which satisfies Assumption* **A**. *Let* $\mathfrak{R} > 2\mathfrak{c}/(1-\mathfrak{a})$, $\mathfrak{b}' \in (0,\mathfrak{b})$, $\mathfrak{a}' \in (\mathfrak{a}+2\mathfrak{c}/\mathfrak{R},1)$ *and set*

$$\kappa = \frac{\mathfrak{b}'}{\mathfrak{c}}, \qquad \mathfrak{y} = \max\left\{1-(\mathfrak{b}-\mathfrak{b}'), \frac{2+\kappa\mathfrak{R}\mathfrak{a}'}{2+\kappa\mathfrak{R}}\right\}.$$

Then, for any $\mu_1,\mu_2 \in \mathcal{M}_+(V/\psi)$ *and any integers* k *and* n *such that* $0 \le k \le n-\mathfrak{p}$,

$$\left\|\mu_1 P^{(n\tau)}_{k\tau,(k+\mathfrak{p})\tau} - \mu_2 P^{(n\tau)}_{k\tau,(k+\mathfrak{p})\tau}\right\|_{\mathcal{M}(1+\kappa V_{n-k-\mathfrak{p}})} \le \mathfrak{y}\left\|\mu_1-\mu_2\right\|_{\mathcal{M}(1+\kappa V_{n-k})}.$$

2.3. **Quantitative estimates for non-conservative semigroups.** We now derive from $P^{(t)}$ the expected estimates for the original semigroup M, under the conditions of Proposition 2.4. Using notation introduced in the previous section, we set

$$\rho = \max\left\{\mathfrak{y}, \mathfrak{a}^{\mathfrak{p}}\right\} = \max\left\{1-(\mathfrak{b}-\mathfrak{b}'), \frac{2+\kappa\mathfrak{R}\mathfrak{a}'}{2+\kappa\mathfrak{R}}, \mathfrak{a}^{\mathfrak{p}}\right\} \in (0,1), \tag{2.8}$$

with $\mathfrak{b}' \in (0,\mathfrak{b})$ and $\mathfrak{a}' \in (\mathfrak{a}+2\mathfrak{c}/\mathfrak{R},1)$. Let us first consider the existence of eigenelements.

Lemma 2.5. *There exists* $h \in \mathcal{B}_+(V)$ *and* $\lambda \in \mathbb{R}$ *such that for all* $t \ge 0$

$$M_t h = \mathrm{e}^{\lambda t} h.$$

Moreover,

$$\left(\frac{\psi}{V}\right)^q \psi \lesssim h \lesssim V,$$

with $q = \log(cr)/\log(\mathfrak{a}) > 0$ *and there exists* $C > 0$ *such that for all* $\mu \in \mathcal{M}_+(V)$ *and* $t \ge 0$,

$$\left|\mu(h) - \frac{\mu M_t\psi}{\nu(M_t\psi/\psi)}\right| \le C\frac{\mu(V)^2}{\mu(\psi)}\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}. \tag{2.9}$$

Lemma 2.6. *There exists* $\gamma \in \mathcal{M}_+(V)$ *such that* $\gamma(h) = 1$ *and for all* $t \ge 0$,

$$\gamma M_t = \mathrm{e}^{\lambda t}\gamma.$$

Moreover, there exists $C > 0$ *such that for all* $t \ge 0$ *and* $\mu \in \mathcal{M}_+(V)$,

$$\left\|\frac{\gamma(\cdot\,\psi)}{\gamma(\psi)} - \frac{\mu M_t(\cdot\,\psi)}{\mu M_t(\psi)}\right\|_{\mathcal{M}(1+\kappa V_0)} \le C\left(\frac{\mu(V)}{\mu(\psi)} + \frac{\theta}{\beta-\alpha}\right)\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}. \tag{2.10}$$

Using Lemma 2.6 and (A0) and (A1), we have $\mathrm{e}^{\lambda\tau}\gamma(V) = \gamma M_\tau V \le (\alpha+\theta)\gamma(V)$ and $\mathrm{e}^{\lambda\tau}\gamma(\psi) = \gamma M_\tau\psi \ge \beta\gamma(\psi)$. It yields the following estimate of the eigenelement

$$\frac{\log(\beta)}{\tau} \le \lambda \le \frac{\log(\alpha+\theta)}{\tau}. \tag{2.11}$$

We can now provide quantitative estimates for the exponential convergence to the profile given by these eigenelements.

Theorem 2.7. *Under Assumption* **A**, *there exists* $C > 0$ *such that for all* $\mu \in \mathcal{M}_+(V)$ *and* $t \geq 0$ *we have*

$$\left\|\mu M_t - \gamma \mathrm{e}^{\lambda t}\mu(h)\right\|_{\mathcal{M}(V)} \leq C\frac{\mu(V)}{\mu(\psi)}\mathrm{e}^{-\sigma t}\min\left\{\mu M_t\psi, \mu(V)\mathrm{e}^{\lambda t}\right\}, \tag{2.12}$$

where

$$\sigma = \frac{-\log\rho}{\mathfrak{p}\tau} > 0.$$

Remark 2.8. *The rate* σ *of exponential convergence is explicit in terms of the constants in Assumption* **A**. *Notice that* σ *is not equal to the spectral gap* ω *in Theorem 1.1, which is obtained by applying Theorem 2.7 with* $(V, h/\|h\|_{\mathcal{B}(V)})$ *after having checked that this couple satisfies Assumption* **A** *with other constants, see forthcoming Section 3.5. We recall that these spectral gaps and multiplicative constants are explicit in the proofs.*

Finally, the renormalisation of the semigroup by its mass $M_t\mathbf{1}$ may also be relevant for applications. Let us mention in particular the study of the convergence of the conditional probability to quasi-stationary distribution and the study of the typical trait in a structured branching process, see respectively Section 4.1 and *e.g.* [7, 50, 51]. For the sake of convenience, we use the notation $\mathcal{P}(V)$ for the set of probability measures which belong to $\mathcal{M}(V)$.

Corollary 2.9. *If Assumption* **A** *holds and* $\inf_{\mathcal{X}} V > 0$, *there exist* $C > 0$ *and* $\pi \in \mathcal{P}(V)$ *such that for every* $\mu \in \mathcal{M}_+(V)$ *and* $t \geq 0$,

$$\left\|\frac{\mu M_t}{\mu M_t\mathbf{1}} - \pi\right\|_{TV} \leq C\frac{\mu(V)}{\mu(h)}\mathrm{e}^{-\omega t}. \tag{2.13}$$

2.4. **Sufficient conditions: drift on the generator and irreducibility.** Assumptions (A0)-(A1)-(A2) can be checked conveniently through conditions on the generator $\mathcal{L}$ of the semigroup $(M_t)_{t\geq 0}$. We give here such sufficient conditions by adopting a weak but practical definition of the generator, which can be seen as a mild formulation of $\mathcal{L} = \partial_t M_{t|_{t=0}}$, similarly as in [39]. For $F, G \in \mathcal{B}(V)$ we say that

$$\mathcal{L}F = G$$

if for all $x \in \mathcal{X}$ the function $s \mapsto M_sG(x)$ is locally integrable, and for all $t \geq 0$

$$M_tF = F + \int_0^t M_sG\,ds.$$

In general for $F \in \mathcal{B}(V)$, there may not exist $G \in \mathcal{B}(V)$ such that $\mathcal{L}F = G$, meaning that F is not in the domain of $\mathcal{L}$. Therefore we relax the definition by saying that

$$\mathcal{L}F \leq G, \qquad \text{resp.} \quad \mathcal{L}F \geq G,$$

if for all $t \geq 0$

$$M_tF - F \leq \int_0^t M_sG\,ds, \qquad \text{resp.} \quad M_tF - F \geq \int_0^t M_sG\,ds.$$

We can now state the drift conditions on $\mathcal{L}$ guaranteeing the validity of Assumptions (A0)-(A1)-(A2). For convenience, we use the shorthand $\varphi \approx \psi$ to mean that $\psi \lesssim \varphi \lesssim \psi$.

Proposition 2.10. *Let* $V, \psi, \varphi : \mathcal{X} \to (0, \infty)$ *such that* $\psi \leq V$ *and* $\varphi \approx \psi$. *Assume that there exist constants* $a < b$ *and* $\zeta \geq 0$, $\xi \in \mathbb{R}$ *such that*

$$\mathcal{L}V \leq aV + \zeta\psi, \qquad \mathcal{L}\psi \geq b\psi, \qquad \mathcal{L}\varphi \leq \xi\varphi.$$

Then, for any $\tau > 0$, *there exists* $R > 0$ *such that* (V, ψ) *satisfies (A0)-(A1)-(A2) with* $K = \{V \leq R\,\psi\}$.

This result will be useful for the applications in Section 4. We provide now a sufficient condition for (A3)-(A4).

Proposition 2.11. *Let K be a finite subset of $\mathcal{X}$ and assume that there exists $\tau > 0$ such that for any $x, y \in K$,*

$$\delta_x M_\tau(\{y\}) > 0.$$

Then (A3)-(A4) are satisfied for any positive function $\psi \in \mathcal{B}(V)$.

This sufficient condition is relevant for the study of irreducible processes on discrete spaces. We refer to Section 4.1 for an application to the convergence to quasi-stationary distribution of birth and death processes. As a motivation, let us also mention the study of the first moment semigroup of discrete branching processes in continuous time and more generally of the exponential of denumerable non-negative matrices, for which irreducibility is generally easy to check.

We end this part by noting that Propositions 2.10 and 2.11 provide explicit constants for Assumptions (A0)-(A1)-(A2) and (A3)-(A4), see the proofs in Section 3.6.

3. Proofs

3.1. **Conservative semigroups: proof of Theorem 2.1.** For any function $V : \mathcal{X} \to [0, \infty)$, let us define the distance dist_V on $\mathcal{X}$ by

$$\mathrm{dist}_V(x, y) = \begin{cases} 0 & x = y \\ 2 + V(x) + V(y) & x \neq y \end{cases}.$$

We also introduce a semi-norm on measurable functions $f : \mathcal{X} \to \mathbb{R}$ defined by

$$\|f\|_{\mathrm{Lip}(V)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\mathrm{dist}_V(x, y)},$$

and the associated (Wasserstein) metric on $\mathcal{P}(\mathcal{X})$ given by

$$\mathrm{dist}_V(\mu_1, \mu_2) = \sup_{\|f\|_{\mathrm{Lip}(V)} \leq 1} \int_{\mathcal{X}} f(x)(\mu_1 - \mu_2)(dx).$$

We know from [40, Lemma 2.1] that for any couple of probability measures μ_1 and μ_2

$$\mathrm{dist}_V(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{\mathcal{M}(1+V)}.$$

The proof of Theorem 2.1 given below is a direct adaptation of the proof of Theorems 1.3 and 3.1 in [40].

Proof of Theorem 2.1. Let f be a test function such that $\|f\|_{\mathrm{Lip}(\kappa V_0)} \leq 1$ and take $x \neq y$. Fix $\mathfrak{a}' \in (\mathfrak{a} + 2\mathfrak{c}/\mathfrak{R}, 1)$ and $\mathfrak{b}' \in (0, \mathfrak{b})$, and set $\kappa = \mathfrak{b}'/\mathfrak{c}$ and $\mathfrak{y} = \max\{1 - \mathfrak{b} + \mathfrak{b}', (2 + \kappa\mathfrak{R}\mathfrak{a}')/(2 + \kappa\mathfrak{R})\}$. Considering successively the cases $V_1(x) + V_1(y) \geq \mathfrak{R}$ and $V_1(x) + V_1(y) \leq \mathfrak{R}$ as in [40, Proof of Theorem 3.1], and using (2.1) and (2.2), we obtain

$$|Pf(x) - Pf(y)| \leq \mathfrak{y}\, \mathrm{dist}_{\kappa V_1}(x, y),$$

so that

$$\|Pf\|_{\mathrm{Lip}(\kappa V_1)} \leq \mathfrak{y}.$$

Finally, for any probability measure μ_1 and μ_2,

$$\begin{aligned}\mathrm{dist}_{\kappa V_0}(\mu_1 P, \mu_2 P) &= \sup_{\|f\|_{\mathrm{Lip}(\kappa V_0)\le 1}} \int_{\mathcal{X}} Pf(x)(\mu_1-\mu_2)(dx) \\ &\le \sup_{\|Pf\|_{\mathrm{Lip}(\kappa V_1)}\le \mathfrak{y}} \int_{\mathcal{X}} Pf(x)(\mu_1-\mu_2)(dx) = \mathfrak{y}\,\mathrm{dist}_{\kappa V_1}(\mu_1,\mu_2)\end{aligned}$$

and the proof is complete. □

3.2. **Preliminary inequalities.** We first give useful inequalities which are directly deduced from iterations of inequalities in Assumption **A**. For all $t \ge 0$, let us define the following operator

$$\widehat{M}_t : f \mapsto M_t(\mathbf{1}_{K^c} f)$$

and for convenience, we introduce the following constants

$$\Theta = \frac{\theta}{\beta - \alpha}, \quad R = \sup_K \frac{V}{\psi},$$

which are well-defined and finite under Assumption **A**.

Lemma 3.1. *For all $k \ge 0$, we have*

i)
$$\widehat{M}_\tau^k M_\tau V \le \alpha^k M_\tau V,$$

ii) for all $\mu \in \mathcal{M}_+(V)$,
$$\frac{\mu M_{k\tau} V}{\mu M_{k\tau}\psi} \le \mathfrak{a}^k \frac{\mu(V)}{\mu(\psi)} + \Theta,$$

iii) for all $x \in K$ and $n \ge k$,
$$M_{n\tau}\psi(x) \le (\alpha(R+\Theta)+\theta)^k M_{(n-k)\tau}\psi(x),$$

iv) for all $x \in K$, and $f \in \mathcal{B}_+(V/\psi)$,
$$M_{(k+1)\tau}(f\psi)(x) \ge c_{k+1}\nu(f) M_{(k+1)\tau}\psi(x),$$

where
$$c_{k+1} = c^{k+1}\left(\frac{\beta}{\alpha(R+\Theta)+\theta}\right)^k. \tag{3.1}$$

Remark 3.2. *We observe that $(c_k)_{k\ge 1}$ is a decreasing geometric sequence. Indeed (A0), (A1) and (A2) ensure that on K*

$$\beta\psi \le M_\tau\psi \le M_\tau V \le (\alpha R+\theta)\psi,$$

so that $\beta \le (\alpha R+\theta) < (\alpha(R+\Theta)+\theta)$. Together with the fact that $c<1$, we get that $(c_k)_{k\ge 1}$ is a geometric progression with common ratio smaller than one.

Points *i*) and *ii*) of Lemma 3.1 are sharp inequalities , while *iv*) extends Assumption (A3) for any time.

Proof. Using (A1) we readily have $\mathbf{1}_{K^c} M_\tau V \leq \alpha V$ and $i)$ follows by induction.
Composing respectively (A1) and (A2) with $M_{k\tau}$ yields

$$M_{(k+1)\tau} V \leq \alpha M_{k\tau} V + \theta M_{k\tau} \psi; \quad M_{(k+1)\tau} \psi \geq \beta M_{k\tau} \psi.$$

Combining these inequalities gives

$$\frac{M_{(k+1)\tau} V}{M_{(k+1)\tau} \psi} \leq \mathfrak{a} \frac{M_{k\tau} V}{M_{k\tau} \psi} + \frac{\theta}{\beta}$$

and $ii)$ follows by induction recalling that $\mathfrak{a} < 1$.
By definition of R we immediately deduce from $ii)$ that for any $x \in K$,

$$\frac{M_{k\tau} V(x)}{M_{k\tau} \psi(x)} \leq R + \Theta.$$

Combining this inequality with

$$M_{n\tau} \psi \leq M_{(n-1)\tau} M_\tau V \leq M_{(n-1)\tau} (\alpha V + \theta \psi),$$

coming from (A1) and $\psi \leq V$, yields for $x \in K$,

$$M_{n\tau} \psi(x) \leq (\alpha(R + \Theta) + \theta) M_{(n-1)\tau} \psi(x).$$

The proof of $iii)$ is completed by induction.

Finally, let $x \in K$. We have

$$\begin{aligned} \frac{M_{(n+1)\tau}(f\psi)(x)}{M_{(n+1)\tau}\psi(x)} &= \frac{M_\tau(M_{n\tau}(f\psi))(x)}{M_{(n+1)\tau}\psi(x)} \\ &\geq c\nu\left(\frac{M_{n\tau}(f\psi)}{\psi}\right) \frac{M_\tau \psi(x)}{M_{(n+1)\tau}\psi(x)}, \end{aligned}$$

using (A3) with the function $M_{n\tau}(f\psi)/\psi$. Besides,

$$\begin{aligned} \nu\left(\frac{M_{n\tau}(f\psi)}{\psi}\right) &= \nu\left(\frac{M_\tau(M_{(n-1)\tau}(f\psi))}{\psi}\right) \\ &\geq c\nu\left(\nu\left(\frac{M_{(n-1)\tau}(f\psi)}{\psi}\right)\frac{M_\tau\psi}{\psi}\right) \geq c\beta\nu\left(\frac{M_{(n-1)\tau}(f\psi)}{\psi}\right), \end{aligned}$$

using again (A3) and (A2). Iterating the last inequality and plugging it in the previous one, we obtain

$$\frac{M_{(n+1)\tau}(f\psi)(x)}{M_{(n+1)\tau}\psi(x)} \geq c^{n+1}\beta^n \frac{M_\tau\psi(x)}{M_{(n+1)\tau}\psi(x)}\nu(f).$$

Moreover, for $x \in K$, we have

$$M_{(n+1)\tau}\psi(x) \leq (\alpha(R+\Theta)+\theta)^n M_\tau\psi(x)$$

using $iii)$. We obtain for all $x \in K$:

$$\frac{M_{(n+1)\tau}(f\psi)(x)}{M_{(n+1)\tau}\psi(x)} \geq c^{n+1}\left(\frac{\beta}{\alpha(R+\Theta)+\theta}\right)^n \nu(f),$$

which completes the proof. □

3.3. **Contraction of $P^{(t)}$: proofs of Lemmas 2.2 and 2.3 and Proposition 2.4.** The following statement proves that $(V_k)_{k\geq 0}$ is a family of Lyapunov functions for the sequence of operators $(P^{(n\tau)}_{k\tau,(k+1)\tau})_{0\leq k\leq n-1}$.

Lemma 3.3. *For all $k \geq 0$ and $n \geq m \geq k$, we have*

$$P^{(n\tau)}_{k\tau,m\tau} V_{n-m} \leq \mathfrak{a}^{m-k} V_{n-k} + \frac{\theta}{c\beta} \sum_{j=k}^{m-1} \mathfrak{a}^{m-j} P^{(n\tau)}_{k\tau,j\tau}\left(\mathbf{1}_K\right).$$

Proof. By definition of V_k in (2.6), we have, for $0 \leq k \leq n$,

$$P^{(n\tau)}_{(k-1)\tau,k\tau} V_{n-k} = \frac{M_\tau\left(V_{n-k} M_{(n-k)\tau}\psi\right)}{M_{(n-k+1)\tau}\psi} = \nu\left(\frac{M_{(n-k)\tau}\psi}{\psi}\right)\frac{M_\tau V}{M_{(n-k+1)\tau}\psi}$$

Using (A1) and (A2), we have $M_\tau V \leq \alpha V + \theta\psi \mathbf{1}_K$ and $M_{(n-k)\tau}\psi \leq M_{(n-k+1)\tau}\psi/\beta$. We obtain from the definitions of $\mathfrak{a}$ and V_{n-k+1} that

$$P^{(n\tau)}_{(k-1)\tau,k\tau} V_{n-k} \leq \mathfrak{a} V_{n-k+1} + \nu\left(\frac{M_{(n-k)\tau}\psi}{\psi}\right)\frac{\theta\psi\mathbf{1}_K}{M_{(n-k+1)\tau}\psi}.$$

Besides, combining (A2) and (A3) with $f = M_{(n-k)\tau}\psi/\psi$, we get

$$\nu\left(\frac{M_{(n-k)\tau}\psi}{\psi}\right)\frac{\psi\,\mathbf{1}_K}{M_{(n-k+1)\tau}\psi} \leq \frac{\mathbf{1}_K}{c\beta}.$$

The last two inequalities yield

$$P^{(n\tau)}_{(k-1)\tau,k\tau} V_{n-k} \leq \mathfrak{a} V_{n-k+1} + \frac{\theta\,\mathbf{1}_K}{c\beta}$$

and the conclusion follows by iteration using that $P^{(n\tau)}_{k\tau,m\tau} V_{n-m} = P^{(n\tau)}_{k\tau,(k+1)\tau} \cdots P^{(n\tau)}_{(m-1)\tau,m\tau} V_{n-m}$. □

Proof of Lemma 2.2. Using that $P^{(n\tau)}_{k\tau,(j-1)\tau}\left(\mathbf{1}_K\right) \leq 1$ and $\mathfrak{a} < 1$, it is a direct consequence of Lemma 3.3. □

Using (A3) and (A4) and following [20, 7], we prove a Doeblin condition (2.2) on the set K for the auxiliary semigroup $P^{(t)}$. However, K is not in general a sublevel set of V_k. This situation is reminiscent of [40, Assumption 3] and we adapt here their arguments. For that purpose, we need a lower bound for the Lyapunov functions $(V_k)_{k\geq 0}$, which is stated in the next lemma.

Lemma 3.4. *For every $n \geq 0$, we have*

$$d_1 M_{(n+1)\tau}\psi \leq \nu\left(\frac{M_{n\tau}\psi}{\psi}\right) M_\tau V \qquad \text{and} \qquad V_n \geq d_2,$$

with

$$d_1 = \left(1 - \mathfrak{a}\right)d, \qquad d_2 = \frac{\beta - \alpha}{\alpha + \theta} d.$$

Proof. First, using (A4),

$$\begin{aligned} d\, M_{(n+1)\tau}\psi &= d\, M_\tau \left[\mathbf{1}_K M_{n\tau}\psi + \mathbf{1}_{K^c} M_{n\tau}\psi\right] \\ &\le M_\tau \left[\nu\left(\frac{M_{n\tau}\psi}{\psi}\right)\mathbf{1}_K\,\psi + d\,\mathbf{1}_{K^c} M_{n\tau}\psi\right] \\ &\le \nu\left(\frac{M_{n\tau}\psi}{\psi}\right) M_\tau\psi + d\,\widehat{M}_\tau M_{n\tau}\psi. \end{aligned}$$

Then, by iteration, using (A2) and $\psi \le V$,

$$d\, M_{(n+1)\tau}\psi \le \nu\left(\frac{M_{n\tau}\psi}{\psi}\right)\sum_{j=0}^{n}\beta^{-j}\widehat{M}_\tau^j M_\tau\psi \le \nu\left(\frac{M_{n\tau}\psi}{\psi}\right)\sum_{j=0}^{n}\beta^{-j}\widehat{M}_\tau^j M_\tau V.$$

Hence by Lemma 3.1 *i*)

$$d\, M_{(n+1)\tau}\psi \le \frac{1}{1-\mathfrak{a}}\nu\left(\frac{M_{n\tau}\psi}{\psi}\right) M_\tau V$$

and the first identity is proved. From the definition of V_n we deduce

$$V_n \ge d_1 \frac{M_{(n+1)\tau}\psi}{M_{n\tau}\psi}\frac{V}{M_\tau V} \ge d_1\frac{\beta}{\alpha+\theta} = d_2$$

by using successively (A2), (A1), and $\psi \le V$. □

We now prove the Doeblin-type condition (2.2) for the auxiliary semigroup.

Proof of Lemma 2.3. First, we introduce the measure ν_i defined for all integer $i \ge 0$ by

$$\nu_i(f) = \nu\left(f\frac{M_{i\tau}\psi}{\psi}\right).$$

For any $x \in K$, $j \le k \le n$, we have using Lemma 3.1 *iv*) with the function $fM_{(n-k)\tau}\psi/\psi$,

$$P^{(n\tau)}_{(j-1)\tau,k\tau} f(x) = \frac{M_{(k-j+1)\tau}\left(fM_{(n-k)\tau}\psi\right)(x)}{M_{(n-j+1)\tau}\psi(x)} \ge c_{k-j+1}\ \nu_{n-k}\left(f\right)\frac{M_{(k-j+1)\tau}\psi(x)}{M_{(n-j+1)\tau}\psi(x)}.$$

for any nonnegative measurable function f. Then, Lemma 3.4 and (A2) yield

$$P^{(n\tau)}_{(j-1)\tau,k\tau} f(x) \ge d_1 c_{k-j+1}\frac{\nu_{n-k}\left(f\right)}{\nu_{n-j}(\mathbf{1})}\frac{M_{(k-j+1)\tau}\psi(x)}{M_\tau V(x)} \ge d_1 c_{k-j+1}\beta^{k-j}\frac{\nu_{n-k}\left(f\right)}{\nu_{n-j}(\mathbf{1})}\frac{M_\tau\psi(x)}{M_\tau V(x)}.$$

Recalling from (A1) and (A2) that for $x \in K$,

$$\frac{M_\tau\psi(x)}{M_\tau V(x)} \ge \frac{\beta}{\alpha R+\theta},$$

and from Lemma 3.1 *iii*) and $\nu(K) = 1$ that

$$\frac{\nu_{n-k}(\mathbf{1})}{\nu_{n-j}(\mathbf{1})} \ge \frac{1}{(\alpha(R+\Theta)+\theta)^{k-j}},$$

we get for $x \in K$,

$$P^{(n\tau)}_{(j-1)\tau,k\tau} f(x) \ge \alpha_{k-j}\frac{\nu_{n-k}\left(f\right)}{\nu_{n-k}(\mathbf{1})}, \tag{3.2}$$

where

$$\alpha_i = d_1 c^{i+1}\frac{\beta}{\alpha R+\theta}r^i, \qquad r = \left(\frac{\beta}{\alpha(R+\Theta)+\theta}\right)^2.$$

The previous bound holds only on K. We prove now that the semigroup charges K at an intermediate time and derive the expected lower bound. More precisely, setting

$$\omega_i = \frac{\mathfrak{a}^i}{\alpha_i} \quad \text{and} \quad S_\ell = \sum_{j=1}^{\ell} \omega_{\ell-j} = \frac{\alpha R+\theta}{dc(\beta-\alpha)} \cdot \sum_{j=1}^{\ell} \Big(\frac{\mathfrak{a}}{cr}\Big)^j,$$

we obtain for $k \le n-1$ and $1 \le \ell \le n-k$,

$$\begin{aligned} P^{(n\tau)}_{k\tau,(k+\ell)\tau} f &= \frac{1}{S_\ell} \sum_{j=k+1}^{k+\ell} \omega_{k+\ell-j} P^{(n\tau)}_{k\tau,(j-1)\tau} P^{(n\tau)}_{(j-1)\tau,(k+\ell)\tau}(f) \\ &\ge \frac{1}{S_\ell} \sum_{j=k+1}^{k+\ell} \omega_{k+\ell-j} P^{(n\tau)}_{k\tau,(j-1)\tau}(\mathbf{1}_K P^{(n\tau)}_{(j-1)\tau,(k+\ell)\tau} f) \ge B^{(\ell)}_{k,n} \frac{\nu_{n-k-\ell}(f)}{\nu_{n-k-\ell}(\mathbf{1})}, \end{aligned}$$

where the last inequality comes from (3.2) and

$$B^{(\ell)}_{k,n} = \frac{1}{S_\ell} \sum_{j=k+1}^{k+\ell} \mathfrak{a}^{k+\ell-j} P^{(n\tau)}_{k\tau,(j-1)\tau}(\mathbf{1}_K).$$

To conclude, we need to find a positive lower bound for $B^{(\ell)}_{k,n}$ which does not depend on k or n. For that purpose, we first observe that the second bound of Lemma 3.4 ensures that $P^{(n\tau)}_{k\tau,(k+\ell)\tau} V_{n-k-\ell} \ge d_2$. Using now Lemma 3.3 yields

$$\sum_{j=k+1}^{k+\ell} \mathfrak{a}^{k+\ell-j} P^{(n\tau)}_{k\tau,(j-1)\tau} \mathbf{1}_K \ge c\beta \frac{d_2 - \mathfrak{a}^\ell V_{n-k}}{\theta},$$

for $n \ge k+\ell$. For $x \in \{V_{n-k} \le \mathfrak{R}\}$ and $\ell = \mathfrak{p}$ defined in (2.7), we get

$$B^{(\mathfrak{p})}_{k,n} \ge c\beta \frac{d_2}{2\theta S_{\mathfrak{p}}} = \frac{c^2\beta^3 d_1^2}{2\theta(\alpha+\theta)(\alpha R+\theta)} \frac{1}{\sum_{j=1}^{\mathfrak{p}} (\mathfrak{a}/cr)^j}$$

which ends the proof. □

Proof of Proposition 2.4. Let n and k be two integers such that $0 \le k \le n-\mathfrak{p}$, and consider $\mathfrak{R} > 2\mathfrak{c}/(1-\mathfrak{a})$. According to Lemmas 2.2 and 2.3, the conservative operator $P^{(n\tau)}_{k\tau,(k+\mathfrak{p})\tau}$ satisfies condition (2.1) with the functions $V_{n-k-\mathfrak{p}}$ and V_{n-k} and condition (2.2) with the probability measure $\nu_{k,n}$. Applying Theorem 2.1 then yields the result. □

3.4. **Quantitative estimates: proofs of Lemmas 2.5 and 2.6 and Theorem 2.7.** The proof of Theorem 2.7 is split into several lemmas. We introduce the following constant

$$C_0 = \sup_{s \le \mathfrak{p}\tau} \max\left\{ \left\| \frac{M_s V}{V} \right\|_\infty, \left\| \frac{\psi}{M_s \psi} \right\|_\infty \right\}, \tag{3.3}$$

which is finite under Assumption (A0). We also consider for every $\mu \in \mathcal{M}_+(V)$ the family of operators $(Q^\mu_t)_{t\ge 0}$ defined by

$$Q^\mu_t f = \frac{\mu M_t(\psi f)}{\mu M_t(\psi)}$$

for $f \in \mathcal{B}(V_0)$. Fixing the measure μ, the operator $f \mapsto Q^\mu_t f$ is linear. Observe that $Q^{\delta_x}_t = \delta_x P^{(t)}_{0,t}$ so that Proposition 2.4 implies contraction inequalities for $\delta_x \mapsto Q^{\delta_x}_{n\mathfrak{p}\tau}$. Notice that $\mu \mapsto Q^\mu_{n\mathfrak{p}\tau}$ is non-linear and forthcoming Lemma 3.7 extends the contraction to general space of measures.

Then, in Lemma 3.8, we extend the inequalities to continuous time by a simple discretization argument. Finally, we prove the existence of the eigenvector and eigenmeasure, respectively stated in Lemma 2.5 and Lemma 2.6. The section ends with the proof of Theorem 2.7.

Let us first provide a useful upper bound for V_k. For that purpose, we set

$$p = \left\lfloor \frac{\log\left(2(1+\theta/\alpha)(\Theta+R)\right)}{\log\left(1/\mathfrak{a}\right)} \right\rfloor + 1, \qquad C_1 = \frac{2(\alpha(R+\Theta)+\theta)^{p+1}}{cc_{p-1}\beta^{p+1}}. \tag{3.4}$$

where $(c_k)_{k\geq 0}$ is defined in (3.1).

Lemma 3.5. *For all positive measure μ such that*

$$\frac{\mu(V)}{\mu(\psi)} \leq \Theta + R, \tag{3.5}$$

we have for all $k \geq p$,

$$\nu\left(\frac{M_{k\tau}\psi}{\psi}\right) \leq C_1 \frac{\mu M_{k\tau}\psi}{\mu(\psi)}. \tag{3.6}$$

The idea is the following: condition (3.5) ensures the existence of a time p at which the semigroup charges K. Then, (A3) yields (3.6). It will be needed in this form in the sequel, but could be extended to more general right hand sides in (3.5).

Proof. Recalling that $\widehat{M}_\tau = M_\tau\left(\mathbf{1}_{K^c}\,\cdot\,\right)$ and using that for all $g \in \mathcal{B}(V)$, $M_{(k+1)\tau}g = M_\tau(\mathbf{1}_K M_{k\tau}g) + \widehat{M}_\tau(M_{k\tau}g)$, we obtain by induction

$$M_{k\tau}g = \widehat{M}^k g + \sum_{j=1}^{k} \widehat{M}_\tau^{k-j} M_\tau\left(\mathbf{1}_K M_{(j-1)\tau}g\right).$$

Let $g = \psi\mathbf{1}_K$. Using Lemma 3.1 *iv)* with $f = \mathbf{1}_K$ and that $\nu(K) = 1$, we have

$$\begin{aligned}
M_{k\tau}\left(\mathbf{1}_K\psi\right) &\geq \sum_{j=1}^{k} \widehat{M}_\tau^{k-j} M_\tau\left(\mathbf{1}_K M_{(j-1)\tau}(\mathbf{1}_K\psi)\right) \\
&\geq \sum_{j=1}^{k} c_{j-1}\widehat{M}_\tau^{k-j} M_\tau\left(\mathbf{1}_K M_{(j-1)\tau}\psi\right) \\
&\geq c_{k-1}\sum_{j=1}^{k}\left(\widehat{M}_\tau^{k-j}M_{j\tau}\psi - \widehat{M}_\tau^{k-j}M_\tau\left(\mathbf{1}_{K^c}M_{(j-1)\tau}\psi\right)\right) \\
&= c_{k-1}\sum_{j=1}^{k}\left(\widehat{M}_\tau^{k-j}M_{j\tau}\psi - \widehat{M}_\tau^{k-j+1}M_{(j-1)\tau}\psi\right) \\
&= c_{k-1}\left(M_{k\tau}\psi - \widehat{M}_\tau^{k}\psi\right),
\end{aligned}$$

with the convention that $c_0 = 1$. Then, using (A2) and the fact that $\widehat{M}_\tau^k\psi \leq \widehat{M}_\tau^{k-1}M_\tau V$ together with Lemma 3.1 *i)* yields

$$M_{k\tau}\left(\mathbf{1}_K\psi\right) \geq c_{k-1}\left(\beta^k\psi - \alpha^{k-1}M_\tau V\right).$$

Next, (A1) and the fact that $V \geq \psi$ yield

$$M_{k\tau}\left(\mathbf{1}_K \psi\right) \geq c_{k-1}\beta^k \left(\psi - \mathfrak{a}^k \left(1+\theta/\alpha\right) V\right). \tag{3.7}$$

Using the definition (3.4) of p and the fact that $\mathfrak{a} \leq 1$ ensure that $\mathfrak{a}^p 2\left(1+\theta/\alpha\right)(\Theta + R) \leq 1$, and (3.5) yields

$$\mathfrak{a}^p \left(1+\theta/\alpha\right)\mu(V) \leq \mu(\psi)/2.$$

Then, (3.7) becomes

$$\mu M_{p\tau}(\mathbf{1}_K \psi) \geq c_{p-1}\beta^p \mu(\psi)/2. \tag{3.8}$$

Using

$$\mu M_{n\tau}\psi \geq \mu M_{p\tau}(\mathbf{1}_K M_{(n-p)\tau}\psi) = \mu M_{p\tau}(\mathbf{1}_K M_\tau((M_{(n-p-1)\tau}\psi/\psi)\psi))$$

for $n \geq p$ and successively (A3) with $f = M_{(n-p-1)\tau}\psi/\psi$, (A2) and (3.8), we get

$$\mu M_{n\tau}\psi \geq cc_{p-1}\beta^{p+1}\frac{\mu(\psi)}{2}\nu\left(\frac{M_{(n-p-1)\tau}\psi}{\psi}\right).$$

Finally, combining this estimate with Lemma 3.1 *iii*) ensures that

$$\nu\left(\frac{M_{n\tau}\psi}{\psi}\right) \leq (\alpha(R+\Theta)+\theta)^{p+1}\nu\left(\frac{M_{(n-p-1)\tau}\psi}{\psi}\right) \leq C_1\frac{\mu M_{n\tau}\psi}{\mu(\psi)},$$

which ends the proof. □

Lemma 3.6. *For all positive measure μ such that*

$$\frac{\mu(V)}{\mu(\psi)} \leq \Theta + R,$$

we have for all $s \geq p\tau$,

$$\nu\left(\frac{M_s\psi}{\psi}\right) \leq C_1'\frac{\mu M_s\psi}{\mu(\psi)},$$

where $C_1' = C_0^2 C_1 R$.

Proof. Let $u = s - \lfloor s/\tau \rfloor \tau$. First, by definition of C_0 in (3.3), we have

$$M_u\psi \geq C_0^{-1}\psi. \tag{3.9}$$

Moreover, for $x \in K$, using that $\psi < V$ and the definition of R, we get

$$M_u\psi(x) \leq M_u V(x) \leq C_0 V(x) \leq C_0 R\psi(x). \tag{3.10}$$

Then, using successively (3.10) combined with the fact that $\nu(K) = 1$, Lemma 3.5, and (3.9), we get

$$\nu\left(\frac{M_s\psi}{\psi}\right) \leq C_0 R\nu\left(\frac{M_{\lfloor s/\tau\rfloor\tau}\psi}{\psi}\right) \leq C_0 R C_1\frac{\mu M_{\lfloor s/\tau\rfloor\tau}\psi}{\mu(\psi)} \leq C_1'\frac{\mu M_s\psi}{\mu(\psi)},$$

which ends the proof. □

In the next lemma, we generalize Proposition 2.4 to the families $(Q^\mu_{n\mathfrak{p}\tau})_{n\geq 0}$. Recall that $\mathfrak{p}$ is defined in Lemma 2.3, κ and $\mathfrak{y}$ are defined in Proposition 2.4 and ρ is defined in (2.8).

Lemma 3.7. *For all measures $\mu_1, \mu_2 \in \mathcal{M}_+(V/\psi)$ and all $n \geq 0$ we have*

$$\left\|Q_{n\mathfrak{p}\tau}^{\mu_1} - Q_{n\mathfrak{p}\tau}^{\mu_2}\right\|_{\mathcal{M}(1+\kappa V_0)} \leq C_2 \rho^n \left(\frac{\mu_1(V)}{\mu_1(\psi)} + \frac{\mu_2(V)}{\mu_2(\psi)}\right), \tag{3.11}$$

where

$$C_2 = \max\left\{2\mathfrak{a}^{-\mathfrak{p}} + \kappa C_1\left(1 + 2\Theta\mathfrak{a}^{-\mathfrak{p}}\right), 2\left(1+\kappa\Theta\right)\mathfrak{a}^{-(p+\mathfrak{p})} + \kappa\right\} \tag{3.12}$$

with C_1, p defined in (3.4).

Proof. Fix $\mu_1, \mu_2 \in \mathcal{M}_+(V/\psi)$, $f \in \mathcal{B}(V/\psi)$ with $\|f\|_{\mathcal{B}(1+\kappa V_0)} \leq 1$ and an integer $n \geq 0$. Set

$$m = \left\lfloor \frac{\log\left(\frac{\mu_1(V)}{\mu_1(\psi)} + \frac{\mu_2(V)}{\mu_2(\psi)}\right)}{\mathfrak{p}\log\left(1/\mathfrak{a}\right)} \right\rfloor + 1, \tag{3.13}$$

and for convenience,

$$\mathfrak{n} = \mathfrak{p}\tau n, \quad \mathfrak{m} = \mathfrak{p}\tau m.$$

By definition of the auxiliary semigroup in (2.5), we have

$$\begin{aligned}
&\mu_1 M_{\mathfrak{n}}(f\psi)\,\mu_2 M_{\mathfrak{n}}\psi - \mu_2 M_{\mathfrak{n}}(f\psi)\,\mu_1 M_{\mathfrak{n}}\psi \\
&\quad = \int_{\mathcal{X}^2} M_{\mathfrak{n}}\psi(x)\; M_{\mathfrak{n}}\psi(y)\; \left(\frac{M_{\mathfrak{n}}(f\psi)(x)}{M_{\mathfrak{n}}\psi(x)} - \frac{M_{\mathfrak{n}}(f\psi)(y)}{M_{\mathfrak{n}}\psi(y)}\right) \mu_1(dx)\mu_2(dy) \\
&\quad \leq \int_{\mathcal{X}^2} M_{\mathfrak{n}}\psi(x)\, M_{\mathfrak{n}}\psi(y)\, \left\|\delta_x P_{0,\mathfrak{n}}^{(\mathfrak{n})} - \delta_y P_{0,\mathfrak{n}}^{(\mathfrak{n})}\right\|_{\mathcal{M}(1+\kappa V_0)} \mu_1(dx)\mu_2(dy).
\end{aligned}$$

Using Proposition 2.4, we get for $n \geq m$,

$$\begin{aligned}
&\mu_1 M_{\mathfrak{n}}(f\psi)\,\mu_2 M_{\mathfrak{n}}\psi - \mu_2 M_{\mathfrak{n}}(f\psi)\,\mu_1 M_{\mathfrak{n}}\psi \\
&\quad \leq \mathfrak{y}^{n-m} \int_{\mathcal{X}^2} M_{\mathfrak{n}}\psi(x)\, M_{\mathfrak{n}}\psi(y)\, \left\|\delta_x P_{0,\mathfrak{m}}^{(\mathfrak{n})} - \delta_y P_{0,\mathfrak{m}}^{(\mathfrak{n})}\right\|_{\mathcal{M}(1+\kappa V_{(n-m)\mathfrak{p}})} \mu_1(dx)\mu_2(dy).
\end{aligned} \tag{3.14}$$

Using the definition of the norm on $\mathcal{M}(1+\kappa V_{(n-m)\mathfrak{p}})$ and the definition of $V_{(n-m)\mathfrak{p}}$ in (2.6), we obtain

$$\begin{aligned}
\left\|\delta_x P_{0,\mathfrak{m}}^{(\mathfrak{n})} - \delta_y P_{0,\mathfrak{m}}^{(\mathfrak{n})}\right\|_{\mathcal{M}(1+\kappa V_{(n-m)\mathfrak{p}})} &\leq \int_{\mathcal{X}} (1 + \kappa V_{(n-m)\mathfrak{p}}(z)) \left|\delta_x P_{0,\mathfrak{m}}^{(\mathfrak{n})} - \delta_y P_{0,\mathfrak{m}}^{(\mathfrak{n})}\right|(dz) \\
&\leq 2 + \kappa\nu\left(\frac{M_{\mathfrak{n}-\mathfrak{m}}\psi}{\psi}\right)\left(\frac{M_{\mathfrak{m}}V(x)}{M_{\mathfrak{n}}\psi(x)} + \frac{M_{\mathfrak{m}}V(y)}{M_{\mathfrak{n}}\psi(y)}\right).
\end{aligned}$$

Combining this inequality with (3.14) and $\rho \geq \mathfrak{y}$, we get

$$\begin{aligned}
&Q_{\mathfrak{n}}^{\mu_1} f - Q_{\mathfrak{n}}^{\mu_2} f \\
&\quad = \frac{\mu_1 M_{\mathfrak{n}}(f\psi)\,\mu_2 M_{\mathfrak{n}}\psi - \mu_2 M_{\mathfrak{n}}(f\psi)\,\mu_1 M_{\mathfrak{n}}\psi}{\mu_1 M_{\mathfrak{n}}\psi.\mu_2 M_{\mathfrak{n}}\psi} \\
&\quad \leq \rho^n\left(2\rho^{-m} + \kappa\rho^{-m}\nu\left(\frac{M_{\mathfrak{n}-\mathfrak{m}}\psi}{\psi}\right)\left(\frac{\mu_1 M_{\mathfrak{m}}V}{\mu_1 M_{\mathfrak{n}}\psi} + \frac{\mu_2 M_{\mathfrak{m}}V}{\mu_2 M_{\mathfrak{n}}\psi}\right)\right).
\end{aligned} \tag{3.15}$$

We now bound each term of the right-hand side. First, using that $\mathfrak{a}^{\mathfrak{p}} \leq \rho$ and (3.13), we have

$$\mathfrak{a}^{\mathfrak{p}} \leq \rho^m \left(\frac{\mu_1(V)}{\mu_1(\psi)} + \frac{\mu_2(V)}{\mu_2(\psi)}\right). \tag{3.16}$$

Second, Lemma 3.1 *ii*) ensures that for $\mu \in \{\mu_1, \mu_2\}$,

$$\frac{\mu M_{\mathfrak{m}}V}{\mu M_{\mathfrak{m}}\psi} \leq \mathfrak{a}^{m\mathfrak{p}}\frac{\mu(V)}{\mu(\psi)} + \Theta. \tag{3.17}$$

Besides (3.13) also guarantees that for $\mu \in \{\mu_1, \mu_2\}$.

$$\mathfrak{a}^{m\mathfrak{p}} \frac{\mu(V)}{\mu(\psi)} \leq 1.$$

It means that that the positive measure $\mu M_{\mathfrak{m}}$ satisfies inequality (3.5), since $1 \leq R \leq R + \Theta$. Then, Lemma 3.5 applied to $\mu M_{\mathfrak{m}}$ with $k = (n-m)\mathfrak{p}$ yields for all $n \geq m + p/\mathfrak{p}$,

$$\nu\left(\frac{M_{\mathfrak{n}-\mathfrak{m}}\psi}{\psi}\right)\frac{\mu M_{\mathfrak{m}}V}{\mu M_{\mathfrak{n}}\psi} \leq C_1 \frac{\mu M_{\mathfrak{m}} M_{\mathfrak{n}-\mathfrak{m}}\psi}{\mu M_{\mathfrak{m}}\psi}\frac{\mu M_{\mathfrak{m}}V}{\mu M_{\mathfrak{n}}\psi} \leq C_1 \frac{\mu M_{\mathfrak{m}}V}{\mu M_{\mathfrak{m}}\psi}.$$

Finally, using again (3.17) and (3.16), we get

$$\begin{aligned}\nu\left(\frac{M_{\mathfrak{n}-\mathfrak{m}}\psi}{\psi}\right)&\left(\frac{\mu_1 M_{\mathfrak{m}}V}{\mu_1 M_{\mathfrak{n}}\psi} + \frac{\mu_2 M_{\mathfrak{m}}V}{\mu_2 M_{\mathfrak{n}}\psi}\right)\\ &\leq C_1\left(\frac{\mu_1 M_{\mathfrak{m}}V}{\mu_1 M_{\mathfrak{m}}\psi} + \frac{\mu_2 M_{\mathfrak{m}}V}{\mu_2 M_{\mathfrak{m}}\psi}\right) \leq C_1(1+2\Theta\mathfrak{a}^{-\mathfrak{p}})\rho^m\left(\frac{\mu_1(V)}{\mu_1(\psi)} + \frac{\mu_2(V)}{\mu_2(\psi)}\right).\end{aligned}$$

Plugging the last inequality in (3.15) ensures that for all $n \geq m + p/\mathfrak{p}$,

$$Q^{\mu_1}_{\mathfrak{n}} f - Q^{\mu_2}_{\mathfrak{n}} f \leq (2\mathfrak{a}^{-\mathfrak{p}} + \kappa C_1(1+2\Theta\mathfrak{a}^{-\mathfrak{p}}))\left(\frac{\mu_1(V)}{\mu_1(\psi)} + \frac{\mu_2(V)}{\mu_2(\psi)}\right)\rho^n.$$

To conclude, it remains to show that (3.11) also holds for $n \leq m + p/\mathfrak{p}$. We have

$$\|Q^{\mu_1}_{\mathfrak{n}} - Q^{\mu_2}_{\mathfrak{n}}\|_{\mathcal{M}(1+\kappa V_0)} \leq \|Q^{\mu_1}_{\mathfrak{n}}\|_{\mathcal{M}(1+\kappa V_0)} + \|Q^{\mu_2}_{\mathfrak{n}}\|_{\mathcal{M}(1+\kappa V_0)} \leq 2 + \kappa\frac{\mu_1 M_{\mathfrak{n}}V}{\mu_1 M_{\mathfrak{n}}\psi} + \kappa\frac{\mu_2 M_{\mathfrak{n}}V}{\mu_2 M_{\mathfrak{n}}\psi}.$$

Using again (3.17), we have for $\mu \in \{\mu_1, \mu_2\}$

$$\frac{\mu M_{\mathfrak{n}}V}{\mu M_{\mathfrak{n}}\psi} \leq \mathfrak{a}^{n\mathfrak{p}}\frac{\mu(V)}{\mu(\psi)} + \Theta \leq \rho^n \frac{\mu(V)}{\mu(\psi)} + \Theta,$$

so that

$$\|Q^{\mu_1}_{\mathfrak{n}} - Q^{\mu_2}_{\mathfrak{n}}\|_{\mathcal{M}(1+\kappa V_0)} \leq 2\,(1+\kappa\Theta) + \kappa\rho^n\left(\frac{\mu_1(V)}{\mu_1(\psi)} + \frac{\mu_2(V)}{\mu_2(\psi)}\right).$$

Finally, $\rho \geq \mathfrak{a}^{\mathfrak{p}}$ and $n \leq m + p/\mathfrak{p}$ and (3.13) yield

$$1 \leq \rho^n \mathfrak{a}^{-(p+m\mathfrak{p})} = \rho^n \mathfrak{a}^{-(p+\mathfrak{p})}\mathfrak{a}^{-(m-1)\mathfrak{p}} \leq \rho^n \mathfrak{a}^{-(p+\mathfrak{p})}\left(\frac{\mu_1(V)}{\mu_1(\psi)} + \frac{\mu_2(V)}{\mu_2(\psi)}\right),$$

and we get

$$\|Q^{\mu_1}_{\mathfrak{n}} - Q^{\mu_2}_{\mathfrak{n}}\|_{\mathcal{M}(1+\kappa V_0)} \leq \rho^n\left(2(1+\kappa\Theta)\mathfrak{a}^{-(p+\mathfrak{p})} + \kappa\right)\left(\frac{\mu_1(V)}{\mu_1(\psi)} + \frac{\mu_2(V)}{\mu_2(\psi)}\right),$$

for all $n \leq m + p/\mathfrak{p}$, which ends the proof. □

We now extend the previous lemma to continuous time.

Lemma 3.8. *For all $\mu_1, \mu_2 \in \mathcal{M}_+(V/\psi)$ and all $t \geq 0$,*

$$\|Q^{\mu_1}_t - Q^{\mu_2}_t\|_{\mathcal{M}(1+\kappa V_0)} \leq C_2'\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\left(\frac{\mu_1(V)}{\mu_1(\psi)} + \frac{\mu_2(V)}{\mu_2(\psi)}\right),$$

where $C_2' = C_0^2 C_2$ and C_0, C_2 have been defined in (3.3) and (3.12) respectively.

Proof. Let $n = \lfloor t/\mathfrak{p}\tau \rfloor$ and $\delta = t - n\mathfrak{p}\tau \in [0, \mathfrak{p}\tau)$. According to Lemma 3.7, we have

$$\begin{aligned}\|Q_t^{\mu_1} - Q_t^{\mu_2}\|_{\mathcal{M}(1+\kappa V_0)} &= \left\| \frac{\mu_1 M_\delta M_{n\mathfrak{p}\tau}(f\psi)}{\mu_1 M_\delta M_{n\mathfrak{p}\tau}\psi} - \frac{\mu_2 M_\delta M_{n\mathfrak{p}\tau}(f\psi)}{\mu_2 M_\delta M_{n\mathfrak{p}\tau}\psi} \right\|_{\mathcal{M}(1+\kappa V_0)} \\ &\leq C_2 \rho^n \left(\frac{\mu_1 M_\delta(V)}{\mu_1 M_\delta(\psi)} + \frac{\mu_2 M_\delta(V)}{\mu_2 M_\delta(\psi)} \right).\end{aligned}$$

Using the definition (3.3) of C_0 for the last term ends the proof. □

We have now all the ingredients to prove the existence of the eigenelements and the associated estimates. We start with the right eigenfunction and preliminary estimates.

Proof of Lemma 2.5. We define

$$\eta(\cdot) = \nu(\cdot/\psi)$$

and, for $t \geq 0$,

$$h_t = \frac{M_t\psi}{\nu\left(M_t\psi/\psi\right)} = \frac{M_t\psi}{\eta M_t\psi}.$$

The proof is divided into four steps. We begin by giving preliminary estimates on $M_t\psi$. Next, we show that h_t converges in $\mathcal{B}(V^2/\psi)$ as $t \to \infty$. Then, we establish that its limit h is an eigenvector. Finally, we give a lower bound for h.

Recalling the definition of C_0 in (3.3) and using the first part of Lemma 3.4 and (A1), we obtain for any $t \geq 0$,

$$M_t\psi(x) \leq C_0 M_{(\lfloor t/\tau \rfloor + 1)\tau}\psi(x) \leq C_0 d_1^{-1}(\alpha + \theta)(\eta M_{\lfloor t/\tau \rfloor \tau}\psi)V(x). \tag{3.18}$$

Using now $M_{\lfloor t/\tau \rfloor \tau}\psi \leq C_0 M_t\psi$, we get

$$M_t\psi \leq C_0^2 d_1^{-1}(\alpha + \theta)(\eta M_t\psi)V. \tag{3.19}$$

Since $\|M_t\psi/\psi\|_{\mathcal{B}(1+\kappa V_0)} = \sup_{\mathcal{X}} M_t\psi/(\psi + \kappa V)$, we get for any $t \geq 0$,

$$\left\| \frac{M_t\psi}{\psi} \right\|_{\mathcal{B}(1+\kappa V_0)} \leq C_0^2 d_1^{-1}(\alpha + \theta)(\eta M_t\psi) \sup_{\mathcal{X}} \frac{V}{\psi + \kappa V} \leq \frac{C_0^2(\alpha + \theta)}{d_1 \kappa} \eta M_t\psi. \tag{3.20}$$

Using again the definition of C_0, we get

$$\eta M_{t+s}\psi \geq C_0^{-1}(\eta M_{\lfloor t/\tau \rfloor \tau})(M_s\psi).$$

Besides, from Lemma 3.1 (ii) and the fact that $V \leq R\psi$ on K

$$\frac{\eta M_{\lfloor t/\tau \rfloor \tau} V}{\eta M_{\lfloor t/\tau \rfloor \tau}\psi} \leq \mathfrak{a}^{\lfloor t/\tau \rfloor}\eta(V) + \Theta \leq R + \Theta,$$

so $\mu = \eta M_{\lfloor t/\tau \rfloor \tau}$ verifies (3.5). Lemma 3.6 applied to μ gives for all $s \geq p\tau$,

$$(\eta M_{\lfloor t/\tau \rfloor \tau})(M_s\psi) \geq {C_1'}^{-1}(\eta M_{\lfloor t/\tau \rfloor \tau}\psi)(\eta M_s\psi).$$

Putting the two last estimates together, we get for all $\lfloor s/\tau \rfloor \geq p$

$$\eta M_{t+s}\psi \geq (C_0 C_1')^{-1}(\eta M_{\lfloor t/\tau \rfloor \tau}\psi)\, \eta M_s\psi. \tag{3.21}$$

We can now proceed to the second step: the convergence of $(h_t)_{t \geq 0}$. Let $\mu \in \mathcal{M}_+(V)$. We use that

$$\mu(h_{t+s}) = \frac{\mu M_{t+s}\psi / \mu M_t\psi}{\eta M_{t+s}\psi / \eta M_t\psi} \mu(h_t)$$

to obtain that

$$|\mu(h_{t+s})-\mu(h_t)| \le \left|\frac{\mu M_{t+s}\psi}{\mu M_t\psi}-\frac{\eta M_{t+s}\psi}{\eta M_t\psi}\right| \frac{1}{\eta M_{t+s}\psi/\eta M_t\psi}\mu(h_t)$$
$$= \left|Q_t^\mu\left(\frac{M_s\psi}{\psi}\right)-Q_t^\eta\left(\frac{M_s\psi}{\psi}\right)\right| \frac{\mu M_t\psi}{\eta M_{t+s}\psi}.$$

Then, Lemma 3.8 yields

$$|\mu(h_{t+s})-\mu(h_t)| \le C_2'\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\left\|\frac{M_s\psi}{\psi}\right\|_{\mathcal{B}(1+\kappa V_0)}\left(\frac{\mu(V)}{\mu(\psi)}+\nu\left(\frac{V}{\psi}\right)\right)\frac{\mu M_t\psi}{\eta M_{t+s}\psi}. \tag{3.22}$$

Plugging (3.18), (3.20) and (3.21) into (3.22) yields for all $\lfloor s/\tau\rfloor \ge p$

$$|\mu(h_{t+s})-\mu(h_t)| \le C_0^4C_1'C_2'\frac{(\alpha+\theta)^2}{d_1^2\kappa}\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\left(\frac{\mu(V)}{\mu(\psi)}+\nu\left(\frac{V}{\psi}\right)\right)\mu(V).$$

Recalling that $\psi \le V$, $V \le R\psi$ on K and $\nu(K^c)=0$, we get for all $\lfloor s/\tau\rfloor \ge p$

$$|\mu(h_{t+s})-\mu(h_t)| \le C_3\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\frac{\mu(V)^2}{\mu(\psi)}, \tag{3.23}$$

with $C_3 = C_0^4C_1'C_2'\frac{(\alpha+\theta)^2}{d_1^2\kappa}(1+R)$. Taking $\mu=\delta_x$ we deduce

$$\|h_{t+s}-h_t\|_{\mathcal{B}(V^2/\psi)} \le C_3\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}.$$

By Cauchy criterion, this ensures that h_t converges as $t\to\infty$ in $\mathcal{B}(V^2/\psi)$ to a limit denoted by h. Moreover from Lemma 3.4, we have that $d_2 \le V_n = V/h_{n\tau}$ and letting $n\to\infty$ yields

$$h \le d_2^{-1}V. \tag{3.24}$$

Letting $s\to\infty$ in (3.23) implies (2.9).

We now move on to the third step and check that h is an eigenfunction. For all $x\in\mathcal{X}$, we have

$$\eta M_t\left(\frac{M_s\psi}{\eta M_s\psi}\right)\frac{M_{t+s}\psi(x)}{\eta M_{t+s}\psi} = \delta_x M_t\left(\frac{M_s\psi}{\eta M_s\psi}\right).$$

Letting $s\to+\infty$ in this identity and using boundedness condition from (3.19), we get

$$(\eta M_t h)\,.\,h(x) = M_t h(x).$$

Hence h is an eigenvector of M_t associated to the eigenvalue $\eta M_t h$. Moreover,

$$\eta M_{t+s}h = \eta M_t(M_s h) = (\eta M_s h)\,.\,(\eta M_t h),$$

and $t\mapsto \eta M_t h$ is locally bounded, since from (3.24) and Assumption (A0) we have that $Mh \lesssim MV \lesssim V$ on $[0,T]\times\mathcal{X}$ for any $T>0$. Then, there exists $\lambda\in\mathbb{R}$ such that for all $t\ge 0$, $\eta M_t h = \mathrm{e}^{\lambda t}\eta(h) = \mathrm{e}^{\lambda t}$, since $\eta(h)=1$, and this completes the proof.

Let us proceed to the last step and show that h is lower bounded. Combining the fact that

$$M_{n\tau}\psi \ge M_{k\tau}(\mathbf{1}_K M_{(n-k)\tau}\psi) = M_{k\tau}(\mathbf{1}_K M_\tau((M_{(n-k-1)\tau}\psi/\psi)\psi))$$

for all $k\le n$ with (A3) and (A2), we get

$$M_{n\tau}\psi \ge c\beta(\eta M_{(n-k-1)\tau}\psi)M_{k\tau}(\mathbf{1}_K\psi).$$

Recalling (3.7) and dividing by $\eta M_{n\tau}\psi$, we obtain

$$\frac{M_{n\tau}\psi}{\eta M_{n\tau}\psi} \ge c\beta c_{k-1}\left(\beta^k\psi-\alpha^{k-1}(\alpha+\theta)V\right)\frac{\eta M_{(n-k-1)\tau}\psi}{\eta M_{n\tau}\psi}.$$

Let $n \to \infty$, the left-hand side converges to h and using from Lemma 3.1 iii) that

$$\frac{\eta M_{(n-k-1)\tau}\psi}{\eta M_{n\tau}\psi} \geq \frac{1}{(\alpha(R+\Theta)+\theta)^{k+1}}$$

and recalling the expression of c_k in (3.1), we obtain

$$h \geq c\,(cr)^k \left(\psi - \mathfrak{a}^{k-1}\frac{\alpha+\theta}{\beta}V\right).$$

Considering now

$$k = k(x) = \left\lfloor \frac{\log\left(\frac{\psi(x)}{V(x)}\frac{\beta}{2(\alpha+\theta)}\right)}{\log(\mathfrak{a})} \right\rfloor + 2,$$

and recalling that $r = \beta^2/(\alpha(R+\Theta)+\theta)^2$ ensures that

$$\psi - \mathfrak{a}^{k-1}\frac{\alpha+\theta}{\beta}V \geq \psi/2.$$

We get

$$h \geq c_1' \left(\frac{\psi}{V}\right)^q \psi,$$

with $c_1' = c\,(cr)^{2+\log\left(\frac{\beta}{2(\alpha+\theta)}\right)/\log(\mathfrak{a})}/2 > 0$ and $q = \log(cr)/\log(\mathfrak{a}) > 0$. □

Remark 3.9. *Notice that the eigenfunction h built in this proof satisfies $\nu(h/\psi) = 1$ and the constants in $(V/\psi)^q\psi \lesssim h \lesssim V$ depend on this normalization. If we normalize h such that $\|h\|_{\mathcal{B}(V)} = 1$ as in Theorem 1.1 we get $c_1 d_2\,(\psi/V)^q\,\psi \leq h \leq V$.*

We consider now the left eigenelement.

Proof of Lemma 2.6. Let us use again $\eta = \nu(\cdot/\psi)$. Applying Lemma 3.8 to $\mu_1 = \eta$ and $\mu_2 = \eta M_s$, we get for $t, s \geq 0$,

$$\left\|Q_{t+s}^\eta - Q_t^\eta\right\|_{\mathcal{M}(1+\kappa V_0)} \leq C_2' \rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\left(\nu\left(\frac{V}{\psi}\right) + \frac{\eta M_s V}{\eta M_s \psi}\right).$$

Then, using Lemma 3.1 *ii*), $V \leq R\psi$ on K, $\nu(K) = 1$ and the definition of C_0 in (3.3), we have

$$\left\|Q_{t+s}^\eta - Q_t^\eta\right\|_{\mathcal{M}(1+\kappa V_0)} \leq C_2' \rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\left(R + C_0^2\mathfrak{a}^{\lfloor s/\tau\rfloor}R + \Theta\right).$$

Therefore, the sequence of probabilities $(Q_t^\eta)_{t\geq 0}$ satisfies the Cauchy criterion in $\mathcal{M}(1+\kappa V_0)$ and it then converges to a probability measure $\pi \in \mathcal{M}(1+\kappa V_0)$. Similarly, applying Lemma 3.8 to $\mu_1 = \mu$ and $\mu_2 = \eta M_s$, we also have

$$\left\|Q_{t+s}^\eta - Q_t^\mu\right\|_{\mathcal{M}(1+\kappa V_0)} \leq C_2' \rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\left(\frac{\mu(V)}{\mu(\psi)} + C_0^2\mathfrak{a}^{\lfloor s/\tau\rfloor}R + \Theta\right)$$

for any $\mu \in \mathcal{M}(1+\kappa V_0)$. Letting s tend to infinity yields

$$\|\pi - Q_t^\mu\|_{\mathcal{M}(1+\kappa V_0)} \leq C_2'\left(\frac{\mu(V)}{\mu(\psi)} + \Theta\right)\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}. \tag{3.25}$$

Now, we have $\pi(h/\psi) \lesssim \pi(V/\psi) = \pi(V_0) < +\infty$ and we can then define $\gamma \in \mathcal{M}(\psi + \kappa V)$ by

$$\gamma(f) = \frac{\pi(f/\psi)}{\pi(h/\psi)}$$

for $f \in \mathcal{B}(\psi + \kappa V) = \mathcal{B}(V)$. Observe that $\gamma(h) = 1$. Next,

$$Q^\eta_{t+s}(f/\psi) = Q^\eta_t(M_s f/\psi)\frac{\eta M_t\psi}{\eta M_{t+s}\psi}. \tag{3.26}$$

Applying (2.9) to $\mu = \eta M_s$ and $\mu = \eta$,

$$\frac{\eta M_t\psi}{\eta M_{t+s}\psi} \xrightarrow[t\to+\infty]{} \mathrm{e}^{-\lambda s}.$$

Then, letting $t \to \infty$ in (3.26), we obtain

$$\pi(f/\psi) = \pi(M_s f/\psi)\mathrm{e}^{-\lambda s},$$

which ensures that γ is an eigenvector. Adding that $\pi(f) = \gamma(f\psi)/\gamma(\psi)$ since π is a probability measure, (2.10) follows from (3.25). □

Proof of Theorem 2.7. Using that

$$\|\pi - Q^\mu_t\|_{\mathcal{M}(1+\kappa V_0)} = \sup_{f\in\mathcal{B}(1+\kappa V_0)}\left|\frac{\gamma(f\psi)}{\gamma(\psi)} - \frac{\mu M_t(f\psi)}{\mu M_t\psi}\right| = \left\|\frac{\gamma}{\gamma(\psi)} - \frac{\mu M_t}{\mu M_t\psi}\right\|_{\mathcal{M}(\psi+\kappa V)}$$

and multiplying (3.25) by $\mu M_t\psi$, we get

$$\left\|\mu M_t\psi\frac{\gamma}{\gamma(\psi)} - \mu M_t\right\|_{\mathcal{M}(\psi+\kappa V)} \le C_2'\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\left(\frac{\mu(V)}{\mu(\psi)} + \Theta\right)\mu M_t\psi. \tag{3.27}$$

Moreover, $h \in \mathcal{M}(\psi + \kappa V)$ since $h \lesssim V$. As $\gamma(h) = 1$, the previous inequality applied to the eigenfunction h yields

$$\left|\frac{\mu M_t\psi}{\gamma(\psi)} - \mu(h)\mathrm{e}^{\lambda t}\right| \le C_2'\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\left(\frac{\mu(V)}{\mu(\psi)} + \Theta\right)\mu M_t\psi.$$

Then, recalling that $\psi \le V$, we have

$$\begin{aligned}\left\|\mu M_t\psi\frac{\gamma}{\gamma(\psi)} - \gamma\mathrm{e}^{\lambda t}\mu(h)\right\|_{\mathcal{M}(\psi+\kappa V)} &= \left|\frac{\mu M_t\psi}{\gamma(\psi)} - \mathrm{e}^{\lambda t}\mu(h)\right|\|\gamma\|_{\mathcal{M}(\psi+\kappa V)} \\ &\le C_2'\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\left(\frac{\mu(V)}{\mu(\psi)} + \Theta\right)\mu M_t\psi \times (1+\kappa)\gamma(V).\end{aligned} \tag{3.28}$$

Combining (3.27) and (3.28), by triangular inequality, we get

$$\left\|\mu M_t - \gamma\mathrm{e}^{\lambda t}\mu(h)\right\|_{\mathcal{M}(\psi+\kappa V)} \le C_2'\rho^{\lfloor t/\mathfrak{p}\tau\rfloor}\left(\frac{\mu(V)}{\mu(\psi)} + \Theta\right)\mu M_t\psi\,(1 + (1+\kappa)\gamma(V))\,.$$

This gives the first part of (2.12). Finally, by integration of (3.18)

$$\mu M_t\psi \le C_0 d_1^{-1}(\alpha+\theta)\nu(M_{\lfloor t/\tau\rfloor\tau}\psi/\psi)\,\mu(V).$$

Adding that $\gamma(V)/\gamma(\psi) \le \Theta$ according to Lemma 3.1 *ii)* and $\gamma M_{\lfloor t/\tau\rfloor\tau} = \mathrm{e}^{\lambda\lfloor t/\tau\rfloor\tau}\gamma$ from Lemma 2.6, Lemma 3.5 applied to $\mu = \gamma$ yields

$$\nu(M_{\lfloor t/\tau\rfloor\tau}\psi/\psi) \le C_1\mathrm{e}^{\lambda\lfloor t/\tau\rfloor\tau}$$

for $t \ge p\tau$ and we obtain

$$\mu M_t\psi \le C_0C_1 d_1^{-1}(\alpha+\theta)\mathrm{e}^{|\lambda|\tau}\mu(V)\mathrm{e}^{\lambda t}.$$

It proves (2.12) for $t \ge p\tau$ with

$$C = C_2'(1+\Theta)\,(1 + (1+\kappa)\gamma(V))\max(1, C_0C_1d_1^{-1}(\alpha+\theta)\mathrm{e}^{|\lambda|\tau}).$$

The fact that (2.12) holds for some constant C also for $t \le p\tau$ is a consequence of (A0). □

Proof of Corollary 2.9. For convenience and without loss of generality, we assume that $V \geq \mathbf{1}$. Then, $\gamma(\mathbf{1}) < \infty$. Next, if $\gamma(\mathbf{1}) = 0$, then $\gamma(\mathcal{X}) = 0$. In this case, $\gamma = 0$, which is absurd because $\gamma(\psi) > 0$ and $\psi > 0$. Therefore, $\gamma(\mathbf{1}) > 0$.

We set $\pi(\cdot) = \gamma(\cdot)/\gamma(\mathbf{1})$ and we have by triangular inequality

$$\begin{aligned}\left\| \frac{\mu M_t}{\mu M_t \mathbf{1}} - \pi \right\|_{\mathrm{TV}} &\leq \left\| \frac{\mu M_t}{\mu M_t \mathbf{1}} - \pi \right\|_{\mathcal{M}(V)} \\ &= \frac{\mathrm{e}^{\lambda t}}{\mu M_t \mathbf{1}} \left\| \mathrm{e}^{-\lambda t} \mu M_t - \pi \mathrm{e}^{-\lambda t} \mu M_t \mathbf{1} \right\|_{\mathcal{M}(V)} \\ &\leq \frac{\mathrm{e}^{\lambda t}}{\mu M_t \mathbf{1}} \left[\left\| \mathrm{e}^{-\lambda t} \mu M_t - \gamma \mu(h) \right\|_{\mathcal{M}(V)} + \left| \gamma(\mathbf{1}) \mu(h) - \mathrm{e}^{-\lambda t} \mu M_t \mathbf{1} \right| \pi(V) \right].\end{aligned}$$

From (1.2), we can deal with the first term of the right-hand side and

$$\left\| \mathrm{e}^{-\lambda t} \mu M_t - \gamma \mu(h) \right\|_{\mathcal{M}(V)} \leq C \mu(V) \mathrm{e}^{-\omega t}.$$

Using this estimate with $V \geq \mathbf{1}$, we can also control the second term and

$$\left| \gamma(\mathbf{1}) \mu(h) - \mathrm{e}^{-\lambda t} \mu M_t \mathbf{1} \right| \leq C \mu(V) \mathrm{e}^{-\omega t}. \tag{3.29}$$

Combining the three last estimates yields

$$\left\| \frac{\mu M_t}{\mu M_t \mathbf{1}} - \pi \right\|_{\mathcal{M}(V)} \leq C \frac{\mathrm{e}^{\lambda t}}{\mu M_t \mathbf{1}} \mu(V) \mathrm{e}^{-\omega t} (1 + \pi(V)). \tag{3.30}$$

Now on the first hand, Equation (3.29) also gives

$$\mathrm{e}^{-\lambda t} \mu M_t \mathbf{1} \geq \gamma(\mathbf{1}) \mu(h) - C \mu(V) \mathrm{e}^{-\omega t}$$

and for any $t \geq t(\mu) = \frac{1}{\omega} \log \left(\frac{2C}{\gamma(\mathbf{1})} \frac{\mu(V)}{\mu(h)} \right)$, we have

$$\mathrm{e}^{-\lambda t} \mu M_t \mathbf{1} \geq \mu(h) \gamma(\mathbf{1})/2. \tag{3.31}$$

Plugging (3.31) in (3.30) yields (2.13) when $t \geq t(\mu)$. Otherwise,

$$\left\| \frac{\mu M_t}{\mu M_t \mathbf{1}} - \pi \right\|_{\mathrm{TV}} \leq 1 \leq \mathrm{e}^{-\omega t} \mathrm{e}^{\omega t(\mu)} \leq C \frac{\mu(V)}{\mu(h)} \mathrm{e}^{-\omega t},$$

which ends the proof. □

3.5. **Proof of Theorem 1.1.**

Proof of Theorem 1.1 (i). We assume that Assumption **A** is satisfied by (V, ψ) for a set K and constants $\alpha, \beta, \theta, c, d$ and a probability ν. Then, from Lemmas 2.5 and 2.6 and Theorem 2.7, there exist eigenelements (γ, h, λ) such that (2.12) and

$$\beta \leq \mathrm{e}^{\lambda \tau} \leq \alpha + \theta, \qquad c_1 d_2 (\psi/V)^q \psi \leq h \leq V.$$

We check now that (V, h) satisfies also Assumption **A** with the same set K and constant α as (V, ψ) but other constants β', θ', c', d' and an other probability measure ν'.
The fact (V, ψ) verifies (A0) and that $M_t h = \mathrm{e}^{\lambda t} h$ for any $t \geq 0$ ensure that (V, h) satisfies (A0) too. Moreover h satisfies (A2) with $\beta' = \exp(\lambda \tau) \geq \beta > \alpha$. Recalling that $R = \sup_K V/\psi < \infty$, we also have

$$\sup_K \frac{\psi}{h} \leq \frac{R^q}{c_1 d_2}.$$

Adding that (V,ψ) satisfies (A1) with constants α,θ yields

$$M_\tau V \le \alpha V + \theta' \mathbf{1}_K h, \quad \text{with } \theta' = \theta \frac{R^q}{c_1 d_2},$$

which gives (A1) for (V,h). We use now (A3) and (A2) for (V,ψ) and get for $x \in K$

$$\delta_x M_\tau(fh) \ge c\nu\left(\frac{fh}{\psi}\right)\delta_x M_\tau \psi \ge c\beta\nu\left(\frac{fh}{\psi}\right)\psi(x).$$

Using again that $M_\tau h = \mathrm{e}^{\lambda\tau}h$, we obtain for $x \in K$,

$$\delta_x M_\tau(fh) \ge c'\nu'(f)\delta_x M_\tau h,$$

with

$$\nu' = \frac{\nu(\cdot\,\frac{h}{\psi})}{\nu(\frac{h}{\psi})}, \quad c' = c\beta \mathrm{e}^{-\lambda\tau}\nu\left(\frac{h}{\psi}\right)\inf_K \frac{\psi}{h} \ge \frac{\beta}{\alpha+\theta}\frac{c_1 d_2}{R^{q+1}} > 0.$$

Finally, (A4) is satisfied since

$$\sup_{x\in K}\frac{M_{n\tau}h(x)}{h(x)} = \mathrm{e}^{\lambda n\tau} = \nu'\left(\frac{M_{n\tau}h}{h}\right).$$

Then, Theorem 2.7 applied to M with functions (V,h), yields (1.2) since $\mu M_t h = \mathrm{e}^{\lambda t}h$. Adding that uniqueness is a direct consequence of $\omega > 0$ ends the proof of Theorem 1.1 (i). □

Proof of Theorem 1.1 (ii). Assume that there exist a positive measurable function V, a triplet $(\gamma,h,\lambda) \in \mathcal{M}_+(V)\times\mathcal{B}_+(V)\times\mathbb{R}$, and constants $C,\omega > 0$ such that (1.1) and (1.2) hold. Without loss of generality we can suppose that $\|h\|_{\mathcal{B}(V)} = \gamma(h) = 1$. It remains to check that (V,h) satisfies Assumption **A**.

Fix $R > \gamma(V)$ and $\tau > 0$ such that

$$\mathrm{e}^{-\omega\tau/2}C\left(R+\gamma(V)\right) < 1 - \frac{\gamma(V)}{R}. \tag{3.32}$$

It ensures that

$$\alpha := \mathrm{e}^{\lambda\tau}\left(C\mathrm{e}^{-\omega\tau} + \frac{\gamma(V)}{R}\right) < \beta := \mathrm{e}^{\lambda\tau}.$$

By (1.1), $Mh \gtrsim h$ and $M_\tau h \ge \beta h$ so that (A0), (A2) and (A4) are satisfied by h with $d = 1$ and for any probability measure ν.

By (1.2), we have for all $x \in \mathcal{X}$

$$\mathrm{e}^{-\lambda t}M_t V(x) - h(x)\gamma(V) \le CV(x)\mathrm{e}^{-\omega t}.$$

We define $K = \{x \in \mathcal{X}, V(x) \le Rh(x)\}$, which is not empty since $\|h\|_{\mathcal{B}(V)} = 1$ and $R > \gamma(V) \ge \gamma(h) = 1$. Writing $\theta = \gamma(V)\mathrm{e}^{\lambda\tau}$ and using $h(x) = \mathbf{1}_{K^c}h(x)/V(x).V(x) + \mathbf{1}_K h(x)$, we get

$$M_\tau V(x) \le \alpha V(x) + \mathbf{1}_K \theta h(x)$$

for all $x \in \mathcal{X}$. Therefore, (A0) and (A1) hold for (V,h) and it remains to prove (A3).
We define the probability measure π by

$$\pi = \gamma(\cdot\, h)$$

and we use the Hahn-Jordan decomposition of the following family of signed measure indexed by $x \in \mathcal{X}$,

$$\nu^x = \frac{\delta_x M_{\tau/2}(\cdot\, h)}{\mathrm{e}^{\lambda\tau/2} h(x)} - \pi = \nu_+^x - \nu_-^x.$$

As $h \le V$, Equation (1.2) with $t = \tau/2$ and $\mu = \delta_x$ yields

$$\nu_\pm^x(\mathbf{1}) \le \nu_\pm^x(V/h) \le \|\nu^x\|_{\mathcal{M}(V/h)} = \left\| \frac{\delta_x M_{\tau/2}}{\mathrm{e}^{\lambda\tau/2} h(x)} - \gamma \right\|_{\mathcal{M}(V)} \le C \frac{V(x)}{h(x)} \mathrm{e}^{-\omega\tau/2}. \tag{3.33}$$

For every $f \in \mathcal{B}_+(V/h)$ and $x \in \mathcal{X}$ we have

$$\frac{\delta_x M_\tau(hf)}{\mathrm{e}^{\lambda\tau} h(x)} = \frac{\delta_x M_{\tau/2}}{\mathrm{e}^{\lambda\tau/2} h(x)} \left(\frac{M_{\tau/2}(hf)}{\mathrm{e}^{\lambda\tau/2} h} h \right) \ge (\pi - \nu_-^x) \left(\frac{M_{\tau/2}(hf)}{\mathrm{e}^{\lambda\tau/2} h} \right). \tag{3.34}$$

Next,

$$\pi \left(\frac{M_{\tau/2}(hf)}{\mathrm{e}^{\lambda\tau/2} h} \right) = \frac{\gamma M_{\tau/2}(hf)}{\mathrm{e}^{\lambda\tau/2}} = \frac{\mathrm{e}^{\lambda\tau/2} \gamma(hf)}{\mathrm{e}^{\lambda\tau/2}} = \pi(f) \tag{3.35}$$

and

$$\nu_-^x \left(\frac{M_{\tau/2}(hf)}{\mathrm{e}^{\lambda\tau/2} h} \right) = \int_{\mathcal{X}} \frac{\delta_y M_{\tau/2}(hf)}{\mathrm{e}^{\lambda\tau/2} h(y)} \nu_-^x(dy) \le \int_{\mathcal{X}} \left(\pi(f) + \nu_+^y(f) \right) \nu_-^x(dy). \tag{3.36}$$

Combining (3.34) with (3.35) and (3.36), we get

$$\frac{\delta_x M_\tau(hf)}{\mathrm{e}^{\lambda\tau} h(x)} \ge \pi(f)(1 - \nu_-^x(\mathbf{1})) - \int_{\mathcal{X}} \nu_+^y(f) \nu_-^x(dy).$$

The minimality property of the Hahn-Jordan decomposition entails that $\nu_-^x \le \pi$, and (3.33) ensures that $\nu_-^x(\mathbf{1}) \le CR\,\mathrm{e}^{-\omega\tau/2}$ when $x \in K$. We deduce that for all $x \in K$

$$\frac{\delta_x M_\tau(hf)}{\mathrm{e}^{\lambda\tau} h(x)} \ge \pi(f)(1 - CR\,\mathrm{e}^{-\omega\tau/2}) - \int \nu_+^y(f) \pi(dy) =: \eta(f).$$

Point (A3) then holds with

$$\nu = \frac{\eta_+(\cdot\, \mathbf{1}_K)}{\eta_+(\mathbf{1}_K)} \qquad \text{and} \qquad c = \eta_+(\mathbf{1}_K).$$

The positivity of the constant c is guaranteed by (3.32). Indeed, $\eta_+(\mathbf{1}_K) \ge \eta(\mathbf{1}_K) = \eta(\mathbf{1}) - \eta(\mathbf{1}_{K^c})$ while using (3.33),

$$\begin{aligned} \eta(\mathbf{1}) &\ge 1 - CR\,\mathrm{e}^{-\omega\tau/2} - \int_{\mathcal{X}} \nu_+^y(\mathbf{1}) \pi(dy) \\ &\ge 1 - CR\,\mathrm{e}^{-\omega\tau/2} - C\pi(V/\psi)\mathrm{e}^{-\omega\tau/2} = 1 - C\mathrm{e}^{-\omega\tau/2}(R + \gamma(V)) \end{aligned}$$

and

$$\eta(\mathbf{1}_{K^c}) \le \eta\left(\frac{V}{R\psi}\right) \le \pi\left(\frac{V}{R\psi}\right) = \frac{\gamma(V)}{R}.$$

It ends the proof. □

3.6. Drift condition and irreducibility: proofs of Propositions 2.10 and 2.11.

Proof of Proposition 2.10. Let $C > 0$ be such that

$$C^{-1}\psi \le \varphi \le C\psi.$$

By assumptions $\mathcal{L}\psi \ge b\psi$ and $\mathcal{L}\varphi \le \xi\varphi$ so that we have for all $t \ge 0$

$$M_t\psi \ge \psi + b\int_0^t M_s\psi\, ds \qquad \text{and} \qquad M_t\varphi \le \varphi + \xi\int_0^t M_s\varphi\, ds,$$

which yields by Grönwall's lemma

$$M_t\psi \ge \mathrm{e}^{bt}\psi \qquad \text{and} \qquad M_t\psi \le CM_t\varphi \le C\mathrm{e}^{\xi t}\varphi \le C^2\mathrm{e}^{\xi t}\psi.$$

Similarly, setting $\phi = V - \frac{\zeta}{b-a}\psi$, we have

$$\mathcal{L}\phi \le aV + \zeta\psi - \frac{b\,\zeta}{b-a}\psi = a\phi,$$

so by Grönwall's lemma $M_t\phi \le \mathrm{e}^{at}\phi$ and

$$M_tV \le \mathrm{e}^{at}\Big(V - \frac{\zeta}{b-a}\psi\Big) + \frac{\zeta}{b-a}M_t\psi \le \mathrm{e}^{at}V + \frac{C^2\zeta}{b-a}\mathrm{e}^{\xi t}\psi.$$

Since $\psi \le V$, Assumption (A0) is satisfied for any K sublevel set of V/ψ. Now fix $\tau > 0$,

$$R > \frac{C^2\zeta}{b-a}\,\frac{\mathrm{e}^{\xi\tau}}{\mathrm{e}^{b\tau} - \mathrm{e}^{a\tau}} > 0$$

and define $K = \{x \in \mathcal{X},\ V(x) \le R\,\psi(x)\}$. Adding that $\psi \le V/R$ on K^c we get

$$M_\tau V \le \Big(\mathrm{e}^{a\tau} + \frac{C^2\zeta}{(b-a)R}\mathrm{e}^{\xi\tau}\Big)V + \frac{C^2\zeta}{b-a}\mathrm{e}^{\xi\tau}\mathbf{1}_K\psi$$

and by definition of R

$$\mathrm{e}^{a\tau} + \frac{C^2\zeta}{(b-a)R}\mathrm{e}^{\xi\tau} < \mathrm{e}^{b\tau}.$$

So Assumptions (A1)-(A2) are verified for $K = \{V \le R\,\psi\}$, $R > \frac{C^2\zeta}{b-a}\frac{\mathrm{e}^{\xi\tau}}{\mathrm{e}^{b\tau}-\mathrm{e}^{a\tau}}$, with the constants

$$\alpha = \mathrm{e}^{a\tau} + \frac{C^2\zeta}{(b-a)R}\mathrm{e}^{\xi\tau}, \qquad \beta = \mathrm{e}^{b\tau} \qquad \text{and} \qquad \theta = \frac{C^2\zeta}{b-a}\mathrm{e}^{\xi\tau}.$$

It ends the proof. □

Proof of Proposition 2.11. Let $\psi : \mathcal{X} \to (0,\infty)$ and define $\nu = (\#K)^{-1}\sum_{x\in K}\delta_x$ the uniform measure on K, where $\#K$ stands for the cardinal of K. We have for all $f \ge 0$ and $x, y \in K$,

$$\delta_x M_\tau(f\psi) \ge \delta_x M_\tau(\{y\})f(y)\psi(y) \ge cf(y)M_\tau\psi(x).$$

where

$$c = \min_{x,y\in K}\frac{\psi(y)\delta_x M_\tau(\{y\})}{M_\tau\psi(x)} > 0$$

using the irreducibility condition $\delta_x M_\tau(\{y\}) > 0$. Integrating with respect to ν shows that (A3) holds and Assumption (A4) is trivially satisfied with $d = 1/\#K$. □

4. Applications

4.1. **Convergence to quasi-stationary distribution.** Let $(X_t)_{t\geq 0}$ be a càd-làg Markov process on the state space $\mathcal{X}\cup\{\partial\}$, where $\mathcal{X}$ is measurable space and ∂ is an absorbing state. In this section, we apply the results to the (non-conservative) semigroup defined by

$$M_t f(x) = \mathbb{E}_x\left[f(X_t)1_{X_t\neq\partial}\right],$$

where $x\in\mathcal{X}$. The semigroup M is defined for any measurable bounded functions f on $\mathcal{X}$. We consider a positive function V and assume that for any $t>0$, there exists $C_t>0$ such that for any $x\in\mathcal{X}$, $\mathbb{E}_x[V(X_t)]\leq C_t V(x)$. This allows to extend the definition above and ensures that the semigroup M acts on $\mathcal{B}(V)$ and that we can use the framework of Section 2.
A quasi-stationary distribution (QSD) is a probability law π on $\mathcal{X}$ such that

$$\forall t\geq 0,\ \mathbb{P}_\pi(X_t\in\cdot\ |X_t\neq\partial)=\pi(\cdot).$$

Theorem 2.7 and Corollary 2.9 directly give existence and uniqueness of a QSD and quantitative estimates for the convergence. We state them below using the total variation norm for finite signed measures

$$\|\mu\|_{\mathrm{TV}}=\|\mu\|_{\mathcal{M}(\mathbf{1})}=|\mu|(\mathcal{X})=\sup_{\|f\|_\infty\leq 1}|\mu(f)|.$$

We recall that $\mathcal{P}(V)$ stands for the set of probability measures which integrate V.

Theorem 4.1. *Assume that $(M_t)_{t\geq 0}$ satisfies Assumption $\boldsymbol{A}$ with $\inf_{\mathcal{X}} V>0$. Then, there exist a unique quasi-stationary distribution $\pi\in\mathcal{P}(V)$, and $\lambda_0>0, h\in\mathcal{B}_+(V)$, $C,w>0$ such that for all $\mu\in\mathcal{P}(V)$ and $t\geq 0$*

$$\left\|\mathrm{e}^{\lambda_0 t}\mathbb{P}_\mu(X_t\in\cdot)-\mu(h)\pi\right\|_{TV}\leq C\mu(V)\mathrm{e}^{-\omega t},$$

and

$$\|\mathbb{P}_\mu(X_t\in\cdot\ |\ X_t\neq\partial)-\pi\|_{TV}\leq C\frac{\mu(V)}{\mu(h)}\mathrm{e}^{-\omega t}.$$

It extends recent known results, see in particular [21] for various interesting examples and discussions below for comparisons of statements.

As an application, we consider the simple but interesting case of a continuous time random walk on integers, with jumps $+1$ and -1, absorbed in 0. We obtain new and optimal results for the exponential convergence to quasi-stationary distribution. Let us consider the Markov process X whose transition rates and generator are given by the linear operator

$$\mathscr{L}f(n)=b_n(f(n+1)-f(n))+d_n(f(n-1)-f(n)),$$

which is defined for any $n\in\mathbb{N}$ and $f:\mathbb{N}\to\mathbb{R}$ with

$$b_i=b>0,\quad d_i=d>0\quad\text{for any } i\geq 2,\qquad b_1,d_1>0,\qquad b_0=d_0=0.$$

This process is a birth and death process which follows a simple random walk before reaching 1. If $d\geq b$, this process is almost surely absorbed in 0. The convergence in law of such processes conditionally on non-absorption has been studied in many works [69, 68, 72, 52, 1, 38, 49, 75, 44]. The necessary and sufficient condition for $\xi-$positive recurrence of birth and death processes is known from the work of Van Doorn [69]. More precisely here, the fact that there exists $\lambda>0$

such that for any $x > 0$ and $i > 0$, $\mathrm{e}^{-\lambda t}\mathbb{P}_x(X_t = i)$ converges to a positive finite limit as $t \to \infty$ is given by the following condition

$$(\mathbf{H}) \qquad \Delta := (\sqrt{b} - \sqrt{d})^2 + b_1\left(\sqrt{d/b} - 1\right) - d_1 > 0.$$

We notice that $b = d$ is excluded by condition (**H**) and indeed in this case $t \mapsto \mathbb{P}(X_t \neq 0)$ decreases polynomially. Similarly, the case $b_1 = b$ and $d_1 = d$ is excluded and there is an additional linear term in the exponential decrease of $\mathbb{P}_x(X_t = i)$.
Moreover we know from [68] that condition (**H**) ensures that $\mathbb{P}_x(X_t \in \cdot | X_t \neq 0)$ converges to the unique quasi-stationary distribution π for any $x > 0$. To the best of our knowledge, under Assumption (**H**), the speed of convergence of $\mathrm{e}^{-\lambda t}\mathbb{P}_x(X_t = i)$ or $\mathbb{P}_x(X_t = i | X_t \neq 0)$ and the extension of the convergence to infinite support masses were unknown, see *e.g.* [68, page 695]. For a subset of parameters satisfying **(H)**, [72] obtains the convergence to quasi-stationary distributions for non-compactly supported initial measures μ such that $\mu(V) < \infty$. Using the same Lyapunov functions as in [72] or those defined below, results in [21] allow to get exponential convergence for a subset of parameters satisfying **(H)**. Our approach allows to relax these conditions. We obtain below quantitative exponential estimates for the full range of parameters given by (**H**), allowing also non-compactly supported initial measures.

More precisely, we set

$$\mathcal{X} = \mathbb{N} \setminus \{0\} = \{1, \dots\}, \qquad V : n \mapsto \sqrt{d/b}^{\,n}, \qquad \psi : n \mapsto \eta^n,$$

for $n \in \mathcal{X}$, where $\eta = \sqrt{d/b} - \Delta/2b_1 \in (0, \sqrt{d/b})$.

Corollary 4.2. *Under Assumption* (**H**)*, there exists a unique quasi-stationary distribution* $\pi \in \mathcal{P}(V)$*, and* $\lambda_0 > 0, h \in \mathcal{B}_+(V)$ *and* $C, w > 0$ *such that for all* $\mu \in \mathcal{P}(V)$ *and* $t \geq 0$*,*

$$\left\|\mathrm{e}^{\lambda_0 t}\mathbb{P}_\mu(X_t \in \cdot) - \mu(h)\pi\right\|_{TV} \leq C\mu(V)\mathrm{e}^{-\omega t}$$

and

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid X_t \neq 0) - \pi\|_{TV} \leq C\frac{\mu(V)}{\mu(h)}\mathrm{e}^{-\omega t}.$$

Note that the constants above can be explicitly derived from Lemma 2.5. We also recall that these estimates hold for non-compactly supported initial laws and that V and ψ are not eigenelements. As perspectives, we expect that such statement can be generalized to birth and death processes where b_n, d_n are constant outside some compact set of $\mathbb{N}$. Finally, we hope that the proof will help to also study the non-exponential decrease of the non-absorption probability, in particular for random walks, corresponding to $b = b_1, d = d_1$.

Proof of Corollary 4.2. For $u \geq 1$, let $\varphi_u : n \mapsto u^n$ for $n \geq 1$ and $\varphi_u(0) = 0$. We have

$$\mathscr{L}\varphi_u(n) = \lambda_u(n)\varphi_u(n),$$

for any $n \in \mathbb{N}$, where

$$\lambda_u(n) = \lambda_u = b(u-1) + d\,(1/u - 1) \quad (n \geq 2), \qquad \lambda_u(1) = b_1(u-1) - d_1.$$

We set

$$a = \inf_{u>0} \lambda_u = \lambda_{\sqrt{d/b}} = -(\sqrt{d} - \sqrt{b})^2, \qquad \zeta = \Delta\frac{V(1)}{\psi(1)}.$$

Note that from (**H**), $\zeta > 0$. Then, setting $V(0) = \psi(0) = 0$, $V = \varphi_{\sqrt{d/b}}$ on $\mathbb{N} = \mathcal{X} \cup \{0\}$ and

$$\mathscr{L}V = aV + \zeta\mathbf{1}_{\{n=1\}}\psi \leq aV + \zeta\psi \tag{4.1}$$

on $\mathbb{N}$. Moreover $\psi = \varphi_\eta$ and

$$b\psi \leq \mathscr{L}\psi \leq \xi\psi \tag{4.2}$$

on $\mathbb{N}$, where

$$b = \min(\lambda_\eta, \lambda_\eta(1)) = \min\left(\lambda_\eta, a + \frac{\Delta}{2}\right) > a = \inf_{u>0} \lambda_u, \qquad \xi = \max(\lambda_\eta, \lambda_\eta(1)).$$

Using now a classical localization argument, we check that the drift conditions (4.1)-(4.2) ensure that for any $n \geq 1$ and $t \geq 0$,

$$\mathbb{E}_n[V(X_t)] \leq V(x) + \int_0^t \mathbb{E}_n[(aV + \zeta\psi)(X_s)]ds, \tag{4.3}$$

$$\psi(x) + \int_0^t \mathbb{E}_n[b\psi(X_s)]ds \leq \mathbb{E}_x[\psi(X_t)] \leq \psi(x) + \int_0^t \mathbb{E}_n[\xi\psi(X_s)]ds. \tag{4.4}$$

Indeed, following [55], for $m \geq 1$, we let $T_m = \inf\{t > 0 : X_t \geq m\}$ and $(X_t^m)_{t\geq 0}$ be the Markov process defined by

$$X_t^m = X_t \mathbf{1}_{t<T_m}.$$

We extend functions V, ψ on $\mathbb{N}$ by setting $V(0) = \psi(0) = 0$. Using (4.1) and $V(0) \leq V(m)$, its strong generator $\mathscr{L}^m$ satisfies

$$\mathscr{L}^m V \leq aV + \zeta\psi \quad \text{and} \quad \mathscr{L}^m\psi \leq \xi\psi$$

on $O_m = \{0, 1, \dots m-1\}$,

$$\mathscr{L}^m\psi(m-1) = \mathscr{L}\psi(m-1) - b\psi(m) \geq b\psi(m-1) - b\psi(m) = b\psi(m-1) - b\eta\psi(m-1)$$

and $\mathscr{L}^m\psi \geq b\psi$ on O_{m-1}. First, using $\mathscr{L}^m V \leq (a+\zeta)V$ on O_m and $V(n) \to \infty$ as $n \to \infty$, [55, Theorem 2.1] ensures that $\lim_{m\to\infty} T_m = \infty$ and

$$\mathbb{E}_n[V(X_t)] \leq e^{(a+\zeta)t}V(n)$$

for every $n \in \mathbb{N}$. Second $\mathscr{L}^m\psi \leq \xi\psi$ on O_m and ψ is bounded on O_m. Using that X^m coincides with X on $[0, T_m)$, Fatou's lemma and Kolmogorov equation give

$$\begin{aligned}\mathbb{E}_n[\psi(X_t)] &\leq \liminf_{m\to\infty} \mathbb{E}_x[\psi(X_t^m)] \\ &= \psi(n) + \liminf_{m\to\infty} \mathbb{E}_n\left[\int_0^{t\wedge T_m} \mathscr{L}^m\psi(X_s)ds\right] \leq \psi(x) + \xi\int_0^t \mathbb{E}_n[\psi(X_s)]ds.\end{aligned}$$

Moreover $\psi(X_t^m) = \mathbf{1}_{t\leq T_m}\psi(X_t) \leq \psi(X_t)$ and $X_t^m \to X_t$ as $m \to \infty$. Using $\mathscr{L}^m\psi \geq b\psi - b\eta\psi\mathbf{1}_{m-1}$ on O_m and bounded convergence twice yields $\mathbb{E}(\int_0^t \psi(X_s)1_{X_s=m-1}ds) \to 0$ as $m \to \infty$ and

$$\begin{aligned}\mathbb{E}_n[\psi(X_t)] &= \lim_{m\to\infty} \mathbb{E}_x[\psi(X_t^m)] \\ &= \psi(n) + \lim_{m\to\infty} \mathbb{E}_n\left[\int_0^{t\wedge T_m} \mathscr{L}^m\psi(X_s)ds\right] \geq \psi(x) + b\int_0^t \mathbb{E}_n[\psi(X_s)]ds.\end{aligned}$$

Using Fatou's lemma as above for V ends the proof of (4.3)-(4.4). Considering the generator $\mathcal{L}$ of the semigroup $M_t f(x) = \mathbb{E}[f(X_t)1_{X_t\neq 0}]$ defined for $x \in \mathcal{X}$ and $f \in \mathcal{B}(V)$ and recalling the definition of Section 2.4, these inequalities ensure that

$$\mathcal{L}V \leq aV + \zeta\psi, \qquad \mathcal{L}\psi \geq b\psi, \qquad \mathcal{L}\psi \leq \xi\psi.$$

Finally, the fact that $b_i, d_i > 0$ for $i \geq 1$ ensures $\delta_i M_t(\{j\}) > 0$ for any $i, j \in \mathcal{X}$ and $t > 0$ by irreducibility. Then combining Propositions 2.10 and 2.11 ensures that Assumption **A** holds for M with the functions (V, ψ). Applying then Theorem 4.1 ends the proof. □

4.2. **The growth-fragmentation equation.** In this section we apply our general result to the so-called growth-fragmentation partial differential equation

$$\partial_t u_t(x) + \partial_x u_t(x) + B(x)u_t(x) = \int_0^1 B\Big(\frac{x}{z}\Big) u_t\Big(\frac{x}{z}\Big) \frac{\wp(\mathrm{d}z)}{z}, \qquad t, x > 0. \tag{4.5}$$

This nonlocal partial differential equation is complemented with the zero flux boundary condition $u_t(0) = 0$ for all $t > 0$ and an initial data $u_0 = \mu$. This equation appears in the modeling of various physical or biological phenomena [53, 63, 3, 67] as well as in telecommunication. The unknown $u_t(x)$ represents the concentration at time t of some "particles" with "size" $x > 0$, which can be for instance the size of a cell [28, 41], the length of a fibrillar polymer [33], the window size in data transmission over the Internet [19, 8], or the time elapsed since the last discharge of a neuron [61, 16]. Each particle grows with speed 1, and splits with rate $B(x)$ to produce smaller particles of sizes zx with $0 < z < 1$ distributed with respect to the fragmentation kernel $\wp$.

We assume that $B : (0, \infty) \to [0, \infty)$ is a continuously differentiable increasing function and $\wp$ is a positive measure on $[0, 1]$ for which there exist $z_0 \in (0, 1)$, $\epsilon \in [0, z_0]$ and $c_0 > 0$ such that

$$\wp(dz) \geq \frac{c_0}{\epsilon} \mathbf{1}_{[z_0-\epsilon, z_0]}(z)dz \quad \text{if } \epsilon > 0 \qquad \text{or} \qquad \wp \geq c_0 \delta_{z_0} \quad \text{if } \epsilon = 0. \tag{4.6}$$

For any $r \in \mathbb{R}$ we denote by $\wp_r \in [0, +\infty]$ the moment of order r of $\wp$

$$\wp_r = \int_0^1 z^r \wp(dz).$$

Notice that Assumption (4.6) implies that $r \mapsto \wp_r$ is strictly decreasing. The mass conservation during the fragmentation process leads to impose

$$\wp_1 = 1.$$

The zero order moment $\wp_0$ represents the mean number of fragments. The conditions above ensure that $\wp_0 > 1$ and as a consequence the growth-fragmentation equation we consider is not conservative. The conservative form where $\wp_1 = 1$ is replaced by $\wp_0 = 1$ also appears in some situations [19, 8, 59, 61, 16]. In this case, the eigenelements are given by $h(x) = 1$, $\lambda = 0$, and the classical theory of the conservative Harris' theorem applies [15]. Here we are interested in the more challenging case of a non-conservative fragmentation kernel.

We can associate to Equation (4.5) a semigroup $(M_t)_{t \geq 0}$. We only give here the definition of this semigroup as well as its main properties which are useful to verify Assumption **A**, and we refer to the appendix Section 5 for the proofs. For any $f : (0, \infty) \to \mathbb{R}$ measurable and locally bounded, we define the family $(M_t f)_{t \geq 0}$ as the unique solution to the equation

$$M_t f(x) = f(x + t)\mathrm{e}^{-\int_0^t B(x+s)ds} + \int_0^t \mathrm{e}^{-\int_0^s B(x+s')ds'} B(x + s) \int_0^1 M_{t-s} f(z(x + s))\wp(dz)\, ds.$$

This semigroup is positive and preserves $C^1(0, \infty)$. More precisely if $f \in C^1(0, \infty)$ then the function $(t, x) \mapsto M_t f(x)$ is continuously differentiable on $[0, \infty) \times (0, \infty)$ and satisfies

$$\partial_t M_t f(x) = \mathscr{L} M_t f(x) = M_t \mathscr{L} f(x)$$

where the infinitesimal generator $\mathscr{L} : C^1(0,\infty) \to C^0(0,\infty)$ is defined by

$$\mathscr{L}f(x) = f'(x) + B(x)\left[\int_0^1 f(zx)\wp(dz) - f(x)\right].$$

To apply our main result to the semigroup $(M_t)_{t\geq 0}$, we choose a real number $k > 1$ and the Lyapunov function

$$V(x) = 1 + x^k.$$

The space $\mathcal{B}(V)$ is invariant under $(M_t)_{t\geq 0}$ and for any $\mu \in \mathcal{M}(V)$ we can define by duality $\mu M_t \in \mathcal{M}(V)$. The family $(\mu M_t)_{t\geq 0}$ is then the solution to Equation (4.5) with initial data μ, in a weak sense made precise in the appendix Section 5.

An important phenomenon in the long time behavior of the growth-fragmentation equation is the property of asynchronous exponential growth [73]. This property refers to a separation of the variables t and x when time t becomes large: the size repartition of the population stabilizes and the total mass grows exponentially in time. It is a typical example of application of our main result, Theorem 1.1. This question attracted a lot of attention in the last decades, see *e.g.* [2, 4, 11, 25, 28, 36, 37, 45, 48, 56, 58, 64, 66, 74], references therein, and discussion below for more details about this literature. As far as we know, these works assume that the fragmentation rate has at most a polynomial growth. In our statement below, we relax this condition and we do not assume any upper bound on the division rate. We obtain thus the existence of the Perron eigentriplet for super-polynomial fragmentation rates. Second, an explicit spectral gap was known only in the case of a constant division rate [64, 48, 58, 74]. Our method allows to get it for much more general fragmentation rate. Finally, it guarantees exponential convergence for measure solutions while only convergence under strong assumptions on the coefficients [25] and without specific rate was known before.

Theorem 4.3. *Under the above assumptions, there exists of a unique triplet* $(\gamma, h, \lambda) \in \mathcal{M}_+(V)\times \mathcal{B}_+(V) \times \mathbb{R}$ *of eigenelements of* M *with* $\gamma(h) = \|h\|_{\mathcal{B}(V)} = 1$, *i.e. satisfying for all* $t \geq 0$

$$\gamma M_t = \mathrm{e}^{\lambda t}\gamma \qquad \text{and} \qquad M_t h = \mathrm{e}^{\lambda t} h.$$

Moreover there exist constants $C, \omega > 0$ *such that for all* $\mu \in \mathcal{M}(V)$ *and all* $t \geq 0$,

$$\left\|\mathrm{e}^{-\lambda t}\mu M_t - \mu(h)\gamma\right\|_{\mathcal{M}(V)} \leq C\mathrm{e}^{-\omega t}\|\mu\|_{\mathcal{M}(V)}. \tag{4.7}$$

Remark 4.4. *The eigenfunction* h *is continuously differentiable on* $(0,\infty)$ *and satisfies*

$$\mathscr{L}h = \lambda h \qquad \text{and} \qquad (1+x)^{1-q(k-1)} \lesssim h \lesssim (1+x)^k \text{ with } q > 0.$$

The eigenmeasure γ *satisfies, for any* $f \in C_c^1(0,\infty)$,

$$\gamma(\mathscr{L}f) = \lambda\gamma(f).$$

Notice that we cannot expect the convergence (4.7) to hold true in $\mathcal{M}(h)$ in general. It is proven that it is wrong when B is bounded for instance [10].

Before proving Theorem 4.3, let us make a brief review of the large and still growing literature on the asynchronous exponential growth of the growth-fragmentation equation and situate our result in this literature.
The first results have been obtained for a compact state space, namely a bounded subinterval of $(0,\infty)$, by Heijmans, Diekmann and Thieme [28]. They proved an exponential convergence in the case of equal mitosis by adopting a semigroup approach and using a spectral result obtained by Heijmans in [45]. The same kind of method has then been used in [37, 66, 4], still for a

bounded state space. It is worth noticing that in [4] a h-transform is performed with the right eigenfunction to define a stochastic semigroup (see also [22] for a similar renormalization).
The first study on the whole state space $(0,\infty)$ is due to Perthame and Ryzhik [64] for the equal mitosis. Similarly to the results mentioned above, the strategy is to first solve the Perron eigenvalue problem and then prove the (exponential) convergence. This strategy also applies for more general fragmentation kernels. The eigenvalue problem has been solved in [56, 31] for general coefficients. Then, the *General Relative Entropy* technique developed by Michel, Mischler and Perthame [57] (see also [25] for an extension to measure solutions) guarantees the convergence, but without specifying any decay rate. The question of obtaining an exponential rate of convergence once the eigentriplet is known has been treated in several works after [64], by means of functional inequalities [2, 36], semigroup methods [11], or a combination of both [18, 17]. As we already mentioned, a Krein-Rutman theorem with exponential convergence is proposed in [58] so that the whole problem is treated at once. Our result is then closer to this approach, but does not rely on spectral analysis. More recently, the problem was also revisited and solved by stochastic techniques by Bertoin and Watson [13, 12], relying on a Feynman-Kac formula. Convergence at exponential speed was also proved by Marguet [51] in a time-inhomogeneous framework by means of ergodicity techniques.

The end of the section is devoted to the proof of Theorem 4.3. We prove that the drift and Doeblin conditions are satisfied with the functions (V,ψ) given by

$$V(x) = 1 + x^k, \qquad \psi(x) = \frac{1}{2}(1+x),$$

where we recall that $k > 1$ and observe that $\psi \le V$. Using Section 2.4, we obtain that Assumption **A** holds and conclude thanks to Theorem 1.1.

Let $x_0 \ge 0$ and $\underline{B} > 0$ such that for all $x \ge x_0$,

$$B(x) \ge \underline{B}.$$

Now define

$$t_0 = \frac{1 + z_0 + (1+\epsilon)x_0}{1 - z_0} + \frac{1}{2}, \qquad t_1 = \frac{1 - z_0}{2z_0}, \qquad \tau = t_0 + t_1, \tag{4.8}$$

and for all integer $n \ge 0$,

$$y_n = \Big(\frac{1+z_0}{2z_0}\Big)^n + x_0.$$

Lemma 4.5. *(i) Setting $\varphi(x) = 1 - \sqrt{x} + x$, we have $\psi \le \varphi \le 2\psi$ and there exist $\zeta > 0$ and $a < b < \xi$ such that*

$$\mathcal{L}V \le aV + \zeta\psi, \qquad \mathcal{L}\psi \ge b\psi, \qquad \mathcal{L}\varphi \le \xi\varphi,$$

where $\mathcal{L}$ is the generator of M in the sense defined in Section 2.4.
(ii) For all $n \ge 0$, all $x \in [0, y_n]$, and all $f : (0,\infty) \to [0,\infty)$ locally bounded we have

$$M_\tau f(x) \ge \mathrm{e}^{-\tau B(y_n+\tau)} \frac{(c_0\underline{B})^{n+1}}{1-z_0} \frac{t_1^n}{n!}\, \nu(f)$$

where ν is the probability measure defined by

$$\nu(f) = \int_{z_0(y_0+\tau)}^{z_0(y_0+\tau)+1} f(y)\, dy.$$

(iii) For all $\eta > 0$ there exists $c_\eta > 0$ such that for all $t, x \ge 0$ and $y \in [\eta x, x]$

$$c_\eta \le \frac{M_t\psi(y)}{M_t\psi(x)} \le 1.$$

(iv) For all $n \geq 0$, there exists $d > 0$ such that

$$d\frac{M_t\psi(x)}{\psi(x)} \leq \frac{M_t\psi(y)}{\psi(y)}$$

for all $t \geq 0$, $x \in [0, y_n]$ and $y \in [z_0(y_0+\tau), z_0(y_0+\tau)+1] = \operatorname{supp}\nu$.

Proof of Lemma 4.5 (i). Since the identity $\partial_t M_t = M_t\mathscr{L}$ is valid for all C^1 functions and the semigroup M is positive, we only need to prove that $\mathscr{L}V \leq aV + \zeta\psi$, $\mathscr{L}\psi \geq b\psi$, $\mathscr{L}\varphi \leq \xi\varphi$. First,

$$\mathscr{L}x^r = rx^{r-1} + (\wp_r - 1)B(x)x^r$$

for any $r \geq 0$. We deduce that

$$2\,\mathscr{L}\psi(x) = 1 + (\wp_0 - 1)B(x) \geq 0,$$

so that $b = 0$ suits. For φ we have

$$\mathscr{L}\varphi(x) = 1 - \frac{1}{2\sqrt{x}} + (\wp_0 - 1)B(x) - (\wp_{\frac{1}{2}} - 1)B(x)\sqrt{x}.$$

Since $x \mapsto (\wp_0 - 1) - (\wp_{\frac{1}{2}} - 1)\sqrt{x}$ is negative for $x > \left(\frac{\wp_0 - 1}{\wp_{1/2} - 1}\right)^2$ and B is increasing we deduce

$$\mathscr{L}\varphi(x) \leq 1 + (\wp_0 - 1)B\left(\left(\frac{\wp_0 - 1}{\wp_{\frac{1}{2}} - 1}\right)^2\right) =: \frac{\xi}{2} \leq \xi\varphi(x).$$

For V we have

$$\mathscr{L}V(x) = kx^{k-1} + (\wp_0 - 1)B(x) + (\wp_k - 1)B(x)x^k = \underbrace{\left[\left((\wp_k - 1) + (\wp_0 - 1)x^{-k}\right)B(x) + \frac{k}{x}\right]}_{\to\, l := (\wp_k - 1)\lim_{x\to+\infty} B(x) \text{ when } x\to+\infty} x^k.$$

Since $\wp_k < 1$ and B is increasing, the limit l belongs to $[-\infty, 0)$ and we can find $x_1 > 0$ such that for all $x \geq x_1$

$$\mathscr{L}V(x) \leq ax^k = aV(x) - a,$$

where $a = \max\{l/2, -1\} < 0$. For all $x \in [0, x_1]$ we have

$$\mathscr{L}V(x) \leq kx_1^{k-1} + (\wp_0 - 1)B(x_1)$$

and finally setting $\zeta = 2(kx_1^{k-1} + (\wp_0 - 1)B(x_1) - a)$, we get that for all $x \geq 0$

$$\mathscr{L}V(x) \leq aV(x) + \frac{\zeta}{2} \leq aV(x) + \zeta\psi(x).$$

It ends the proof of (i). □

Before proving (ii), let us briefly comment on the definition of t_0, t_1 and y_n. The time t_1 and the sequence y_n are chosen in such a way that

$$y_0 > 0, \quad y_0 \geq x_0, \qquad \lim_{n\to\infty} y_n = +\infty, \qquad \text{and} \qquad z_0(y_{n+1} + t_1) \leq y_n.$$

The choice of the value of t_0 appears in the proof of the case $n = 0$ and the definition of ν.

Since τ is independent of n and $y_n \to +\infty$ when $n \to \infty$ we can find R and n large enough so that $\operatorname{supp}\nu \subset K \subset [0, y_n]$, where $K = \{x,\ V(x) \leq R\psi(x)\}$, and thus (ii) guarantees that Assumption (A3) is satisfied with time τ on K. More precisely it suffices to take R and n large enough so that

$$\frac{1 + (z_0(y_0+\tau)+1)^k}{1 + z_0(y_0+\tau) + 1} \leq \frac{R}{2} \leq \frac{1 + y_n^k}{1 + y_n}. \tag{4.9}$$

Proof of Lemma 4.5 (ii). Let $f \geq 0$. We prove by induction on n that for all $x \in [0, X_n]$ and all $t \in [0, t_1]$ we have

$$M_{t_0+t} f(x) \geq \mathrm{e}^{-(t_0+t)B(y_n+\tau)} \frac{(c_0 \underline{B})^{n+1}}{1-z_0} \frac{t^n}{n!} \nu(f), \tag{4.10}$$

which yields the desired result by taking $t = t_1$.

We start with the case $n = 0$. The Duhamel formula

$$M_t f(x) = f(x+t)\mathrm{e}^{-\int_0^t B(x+s)ds} + \int_0^t \mathrm{e}^{-\int_0^s B(x+s')ds'} B(x+s) \int_0^1 M_{t-s} f(z(x+s))\wp(dz)\, ds$$

ensures, using the positivity of M_t and the growth of B, that for all $t, x \geq 0$

$$M_t f(x) \geq \mathrm{e}^{-tB(x+t)} \int_0^t B(x+s) \int_0^1 f(z(x+s)+t-s)\wp(dz)\, ds.$$

Thus for $t \geq x_0$ we have for all $x \geq 0$, using Assumption (4.6) for the last inequality,

$$\begin{aligned} M_t f(x) &\geq \mathrm{e}^{-tB(x+t)} \underline{B} \int_{x_0}^t \int_0^1 f(z(x+s)+t-s)\wp(dz)\, ds \\ &\geq \mathrm{e}^{-tB(x+t)} \underline{B} \int_0^1 \int_{z(x+t)}^{z(x+x_0)+t-x_0} f(y)\, dy\, \frac{\wp(dz)}{1-z} \\ &\geq \mathrm{e}^{-tB(x+t)} \underline{B} \frac{c_0}{1-z_0} \int_{z_0(x+t)}^{(z_0-\epsilon)(x+x_0)+t-x_0} f(y)\, dy. \end{aligned}$$

We deduce that for $t \in [t_0, t_0+t_1]$ and $x \in [0, X_0]$

$$M_t f(x) \geq \mathrm{e}^{-tB(x_0+\tau)} \underline{B} \frac{c_0}{1-z_0} \int_{z_0(x_0+t_0+t_1)}^{(z_0-\epsilon)x_0+t_0-x_0} f(y)\, dy.$$

The time t_0 has been defined in such a way that $(z_0-\epsilon)x_0 + t_0 - x_0 = z_0(x_0+t_0+t_1)+1$ so

$$\int_{z_0(x_0+t_0+t_1)}^{(z_0-\epsilon)x_0+t_0-x_0} f(y)\, dy = \nu(f)$$

and this finishes the proof of the case $n = 0$.

Assume now that (4.10) is valid for n and let's check it for $n+1$. By the Duhamel formula, using that $y_n \geq x_0$ and $z_0(X_{n+1}+t_1) \leq y_n$, we have for $x \in [x_n, x_{n+1}]$ and $t \in [0, t_1]$

$$\begin{aligned} M_{t_0+t} f(x) &\geq \int_0^t \mathrm{e}^{-\int_0^s B(x+s')ds'} B(x+s) \int_0^1 M_{t_0+t-s} f(z(x+s))\wp(dz)\, ds \\ &\geq \underline{B} \int_0^t \mathrm{e}^{-sB(x_{n+1}+t_1)} \int_0^{z_0} M_{t_0+t-s} f(z(x+s))\wp(dz)\, ds \\ &\geq \underline{B}^{n+2} \frac{c_0^{n+1}}{1-z_0} \nu(f) \int_0^t \mathrm{e}^{-sB(x_{n+1}+t_1)} \mathrm{e}^{-(t_0+t-s)B(x_n+\tau)} \frac{(t-s)^n}{n!} \int_0^{z_0} \wp(dz)\, ds \\ &\geq \mathrm{e}^{-(t_0+t)B(y_{n+1}+\tau)} \underline{B}^{n+2} \frac{c_0^{n+2}}{1-z_0} \frac{t^{n+1}}{(n+1)!} \nu(f) \end{aligned}$$

and the proof is complete. □

We now turn to the proof of *(iii)*, which uses the monotonicity results proved in Lemma 5.4, see the appendix Section 5.

Proof of Lemma 4.5 (iii). The second inequality readily follows from Lemma 5.4 (*ii*). For the first one, we start with a technical result on $\wp$. Due to the assumption we made on $\wp$, if we set $z_1 > \max(z_0, 1 - c_0(z_0 - \epsilon/2))$, we have

$$\varrho := \int_{z_1}^1 \wp(dz) \le \frac{1}{z_1}\Big(1 - \int_0^{z_1} z\,\wp(dz)\Big) \le \frac{1 - c_0(z_0 - \epsilon/2)}{z_1} < 1.$$

Using Lemma 5.4 (*ii*) and (*iii*), we deduce that for all $t \ge s \ge 0$ and all $x > 0$

$$\int_0^1 M_{t-s}\psi(z(x+s))\,\wp(dz) \le \int_0^1 M_t\psi(zx)\,\wp(dz) \le \wp_0 M_t\psi(z_1 x) + \varrho M_t\psi(x).$$

Now from the Duhamel formula we get, using that $t \mapsto t\,\mathrm{e}^{-\int_0^t B(s)\,ds}$ is bounded on $[0,\infty)$,

$$\begin{aligned} M_t\psi(x) &= \psi(x+t)\mathrm{e}^{-\int_0^t B(x+s)\,ds} + \int_0^t \mathrm{e}^{-\int_0^s B(x+s')\,ds'} B(x+s) \int_0^1 M_{t-s}\psi(z(x+s))\,\wp(dz)ds \\ &\le (1+x+t)\mathrm{e}^{-\int_0^t B(s)\,ds} + \Big(1 - \mathrm{e}^{-\int_0^t B(x+s)\,ds}\Big)\Big(\wp_0 M_t\psi(z_1 x) + \varrho M_t\psi(x)\Big) \\ &\le C_0\psi(x) + \wp_0 M_t\psi(z_1 x) + \varrho M_t\psi(x). \end{aligned}$$

Choosing an integer n such that $z_1^n \le \eta$ we obtain

$$M_t\psi(x) \le \frac{C_0}{1-\varrho}\sum_{k=0}^{n-1}\Big(\frac{\wp_0}{1-\varrho}\Big)^k \psi(x) + \Big(\frac{\wp_0}{1-\varrho}\Big)^n M_t\psi(\eta x) = C_1\psi(x) + C_2\,M_t\psi(\eta x)$$

and since $M_t\psi(\eta x) \ge \psi(\eta x) \ge \eta\psi(x)$ we obtain for any $y \in [\eta x, x]$,

$$\frac{M_t\psi(y)}{M_t\psi(x)} \ge \frac{M_t\psi(\eta x)}{C_1\psi(x) + C_2 M_t\psi(\eta x)} \ge \frac{\eta}{C_1 + C_2\eta},$$

which ends the proof. □

Proof of Lemma 4.5 (iv). We apply (iii) with

$$\eta = \frac{z_0(y_0+\tau)}{y_n}$$

and we obtain that for all $x \in [0, y_n]$ and $y \in [z_0(y_0+\tau), z_0(y_0+\tau)+1]$

$$\frac{M_t\psi(y)}{\psi(y)} \ge \frac{c_\eta}{z_0(y_0+\tau)+2}\frac{M_t\psi(x)}{\psi(x)}.$$

□

We are now in position to prove Theorem 4.3.

Proof of Theorem 4.3 and Remark 4.4. Fix τ defined in (4.8). In Lemma 4.5 (i) we have verified the assumptions of Proposition 2.10, so we can find a real $R > 0$ and an integer $n \ge 0$ large enough so that (4.9) and Assumptions (A0)-(A1)-(A2) are satisfied with $K = \{V \le R\psi\}$. Then, points (ii) and (iv) in Lemma 4.5 ensure that Assumptions (A3) and (A4) are also satisfied. So Assumption **A** is verified for (V,ψ) and by virtue of Theorem 1.1 inequality (4.7) is proved, as well as the bounds on h in Remark 4.4. It remains to check that h is continuously differentiable and that h and γ satisfy the eigenvalue equations $\mathscr{L}h = \lambda h$ and $\gamma\mathscr{L} = \lambda\gamma$. By definition of h, the Duhamel formula gives

$$h(x)\mathrm{e}^{\lambda t} = h(x+t)\mathrm{e}^{-\int_0^t B(x+s)ds} + \int_0^t \mathrm{e}^{-\int_0^s B(x+s')ds'} B(x+s)\int_0^1 \mathrm{e}^{\lambda(t-s)} h(z(x+s))\wp(dz)\,ds$$

and we deduce that for any $x>0$ the function $t\mapsto h(t+x)$ is continuous and then continuously differentiable. Moreover, we have the identity $\partial_t M_t h = M_t \mathscr{L} h$ and since $M_t h = \mathrm{e}^{\lambda t} h$ we deduce

$$\mathscr{L} h = \lambda h.$$

For the equation on γ, we start from Proposition 5.3 which ensures that for any $f\in C^1_c(0,\infty)$

$$\mathrm{e}^{\lambda t}\gamma(f) = (\gamma M_t)(f) = \gamma(f) + \int_0^t (\gamma M_s)(\mathscr{L} f)\, ds = \gamma(f) + \frac{\mathrm{e}^{\lambda t}-1}{\lambda}\gamma(\mathscr{L} f).$$

Differentiating with respect to t yields the result. □

4.3. **Comments and a few perspectives.** First and as for Harris conservative semigroup [40], our proof relies on a (quantitative) contraction method for the discrete time semigroup $(M_{n\tau})_{n\geq 0} = (M_\tau^n)_{n\geq 0}$. It has been extended easily to the continuous setting. We can thus actually state analogous results for a discrete time semigroup $(M^n)_{n\in\mathbb{N}}$ by making the following assumption for a couple of positive functions (V,ψ).

Assumption B. *There exist some integers $\tau, T>0$, real numbers $\beta>\alpha>0$, $\theta\geq 0$, $(c,d)\in(0,1]^2$, some set $K\subset\mathcal{X}$ and some probability measure ν on $\mathcal{X}$ supported by K such that*

(B0) $\psi\leq V$ *on* $\mathcal{X}$ *and* $V\lesssim\psi$ *on* K; $M^k V\lesssim V$ *and* $M^k\psi\gtrsim\psi$ *for* $k\leq T$ *on* $\mathcal{X}$,

(B1) $M^\tau V\leq \alpha V+\theta\mathbf{1}_K\psi$,

(B2) $M^\tau\psi\geq\beta\psi$,

(B3) For all $x\in K$ *and* $f\in\mathcal{B}_+(V/\psi)$,

$$M^\tau(f\psi)(x)\geq c\,\nu(f)\,M^\tau\psi(x),$$

(B4) For any integer n,

$$d\sup_{x\in K}\frac{M^{n\tau}\psi(x)}{\psi(x)}\leq\nu\left(\frac{M^{n\tau}\psi}{\psi}\right).$$

The counterpart of Theorem 1.1 becomes

Theorem 4.6. *(i) Let (V,ψ) be a couple of measurable functions from $\mathcal{X}$ to $(0,\infty)$ which satisfies Assumption* **B**. *Then, there exists a unique triplet $(\gamma,h,\lambda)\in\mathcal{M}_+(V)\times\mathcal{B}_+(V)\times\mathbb{R}$ of eigenelements of M with $\gamma(h)=\|h\|_{\mathcal{B}(V)}=1$, i.e. satisfying*

$$\gamma M=\lambda\gamma \qquad \text{and} \qquad Mh=\lambda h. \tag{4.11}$$

Moreover, there exists $C>0$ and $\rho\in(0,1)$ such that for all $n\geq 0$ and $\mu\in\mathcal{M}(V)$,

$$\left\|\lambda^{-n}\mu M^n-\mu(h)\gamma\right\|_{\mathcal{M}(V)}\leq C\,\|\mu\|_{\mathcal{M}(V)}\,\rho^{-n}. \tag{4.12}$$

(ii) Assume that there exist a positive measurable function V, a triplet $(\gamma,h,\lambda)\in\mathcal{M}_+(V)\times\mathcal{B}_+(V)\times\mathbb{R}$, and constants $C,\rho>0$ such that (4.11) and (4.12) hold. Then, the couple $(V,h/\|h\|_{\mathcal{B}(V)})$ satisfies Assumption **B**.

In addition, we recover the various bounds on the eigenevector as in Lemma 2.5 or on the eigenvalue as in (2.11). We also recover that if $\inf_{\mathcal{X}} V>0$ then there exists $C>0$ and $\pi\in\mathcal{P}(V)$ such that for all $\mu\in\mathcal{P}(V)$ and $n\geq 0$,

$$\left\|\frac{\mu M^n}{\mu M^n\mathbf{1}}-\pi\right\|_{\mathrm{TV}}\leq C\frac{\mu(V)}{\mu(h)}\rho^n. \tag{4.13}$$

As a consequence, Assumption **B** gives sufficient conditions to have the existence, uniqueness and convergence to a quasi-stationary distribution for a Markov chain $(X_n)_{n\geq 0}$. Moreover, the convergence of the Q-process, the description of the domain of attraction and the bounds on the extinction times can be then obtained by usual procedure, see *e.g.* [21, 71].

Second, for the sake of simplicity, we have not allowed ψ to vanish in this paper. This excludes reducible structures. To illustrate this fact, let us consider the case where $\mathcal{X} = \{1, 2\}$ similar to [9, Example 3.5] where

$$M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

for $a, b, c > 0$. In any case, Theorem 4.6 with $\mathcal{X} = \{2\}$ implies that if $\mu(\{1\}) = 0$ then $c^{-n}\mu M^n$ tends to δ_2. Of course, here it is an equality, but it can be easily generalized into a reducible example on two subspaces. Now if $\mu(\{1\}) > 0$ then one can use Theorem 4.6 only when $c > a$. Indeed, in this case, one can choose ψ to be the right eigenvector:

$$\psi = (b, c - a) = b\mathbf{1}_{\{1\}} + (c - a)\mathbf{1}_{\{2\}}$$

and show that, up to a renormalisation, μM^n converges to δ_2. When $c \leq a$, one can not use Theorem 4.6 and indeed the conclusion is wrong : there is no positive right eigenvector. However, allowing ψ to vanish enables to treat the case $c < a$, as in [21, Section 6]. Focusing on initial measures μ such that $\mu(\psi) > 0$, a large part of our results actually holds when $\psi \geq 0$. Indeed (A0) and (A2) give that if $\mu(\psi) > 0$ then $\mu M_t \psi > 0$ for every $t \geq 0$.
Up to our knowledge, the critical case $a = b$ remains a challenging issue. In this case, there is no spectral gap. Nevertheless, we believe that our approach could be extended to this case by allowing n-dependent constant $d = d_n$ in (A4) to vary with time as in [9, Assumption $(\mathbf{H_4})$].

Finally, we recall that the results obtained here rely on a contraction method. Several extensions to the non-homogeneous setting are expected, in the same vein as [7] for time inhomogeneous linear PDEs. One can now relax the "coming down from infinity" property imposed by the generalized Doeblin condition of [7]. Similarly, let us recall that the expectation of a branching process yields the first moment semigroup, which usually drives the extinction of the process (criticality) and provides its deterministic renormalization (Kesten Stigum theorem). The method of this paper should provide a powerful tool to analyse the first moment semigroup of a branching process with infinite number of types, including in varying environment, see [5, 6, 51] for some motivations in population dynamics and queuing systems. We also mention that time inhomogeneity provides a natural point of view to deal with non-linearity in large population approximations of systems with interaction. These points should be partially addressed in forthcoming works.

5. Appendix: The growth-fragmentation semigroup

We give here the details of the construction of the growth-fragmentation semigroup and prove its basic properties. For a function $f \in \mathcal{B}_{loc}(0, \infty)$, *i.e.* measurable and locally bounded on $(0, \infty)$, we define the family $(M_t f)_{t\geq 0} \subset \mathcal{B}_{loc}(0, \infty)$ through the Duhamel formula

$$M_t f(x) = f(x + t)\mathrm{e}^{-\int_0^t B(x+s)ds} + \int_0^t \mathrm{e}^{-\int_0^s B(x+s')ds'} B(x + s) \int_0^1 M_{t-s} f(z(x + s))\wp(dz)\, ds.$$

We first prove that this indeed defines uniquely the family $(M_t f)_{t\geq 0}$. Then, we verify that the associated family $(M_t)_{t\geq 0}$ is a semigroup of linear operators, which provides the unique solution to the growth-fragmentation (4.5) on the space $\mathcal{M}(V)$ with $V(x) = 1 + x^k$, $k > 1$. Finally we

provide some useful monotonicity properties for this semigroup, which are consequences of the monotonicity assumption on B.

Lemma 5.1. *For any $f \in \mathcal{B}_{loc}(0,\infty)$ there exists a unique $\bar f \in \mathcal{B}_{loc}([0,\infty)\times(0,\infty))$ such that for all $t \geq 0$ and $x > 0$*

$$\bar f(t,x) = f(x+t)\mathrm{e}^{-\int_0^t B(x+s)ds} + \int_0^t \mathrm{e}^{-\int_0^s B(x+s')ds'} B(x+s) \int_0^1 \bar f(t-s, z(x+s))\wp(dz)\, ds.$$

Moreover if f is nonnegative/continuous/continuously differentiable, then so does $\bar f$. In the latter case $\bar f$ satisfies the partial differential equation

$$\partial_t \bar f(t,x) = \mathscr{L}\bar f(t,x) = \partial_x \bar f(t,x) + B(x)\Big[\int_0^1 \bar f(t,zx)\wp(dz) - \bar f(t,x)\Big].$$

Proof. Let $f \in \mathcal{B}_{loc}(0,\infty)$ and define on $\mathcal{B}_{loc}([0,\infty)\times(0,\infty))$ the mapping Γ by

$$\Gamma g(t,x) = f(x+t)\mathrm{e}^{-\int_0^t B(x+s)ds} + \int_0^t \mathrm{e}^{-\int_0^s B(x+s')ds'} B(x+s) \int_0^1 g(t-s, z(x+s))\wp(dz)\, ds.$$

Now for $T, A > 0$ define the set $\Omega_{T,A} = \{(t,x) \in [0,T]\times(0,\infty),\ x+t < A\}$ and denote by $\mathcal{B}_b(\Omega_{T,A})$ the Banach space of bounded measurable functions on $\Omega_{T,A}$, endowed with the supremum norm $\|\cdot\|_\infty$. Clearly Γ induces a mapping $\mathcal{B}(\Omega_{T,A}) \to \mathcal{B}(\Omega_{T,A})$, still denoted by Γ. To build a fixed point of Γ in $\mathcal{B}_{loc}([0,\infty)\times(0,\infty))$ we prove that it admits a unique fixed point in any $\mathcal{B}_b(\Omega_{A,A})$.

Let $A > 0$ and $T < 1/(\wp_0 B(A))$. For any $g_1, g_2 \in \mathcal{B}_b(\Omega_{T,A})$ we have

$$\|\Gamma g_1 - \Gamma g_2\|_\infty \leq \wp_0 T B(A)\|g_1 - g_2\|_\infty$$

and Γ is a contraction. The Banach fixed point theorem then guarantees the existence of a unique fixed point $g_{T,A}$ of Γ in $\mathcal{B}_b(\Omega_{T,A})$. The same argument on $\Omega_{T,A-T}$ with f being replaced by $g_{T,A}(T,\cdot)$ ensures that $g_{T,A}$ can be extended into a unique fixed point $g_{2T,A}$ of Γ on $\Omega_{2T,A}$. Iterating the procedure we finally get a unique fixed point g_A of Γ in $\mathcal{B}(\Omega_{A,A})$.

For $A' > A > 0$ we have $g_{A'|\Omega_A} = g_A$ by uniqueness of the fixed point in $\mathcal{B}_b(\Omega_A)$, and we can define $\bar f$ by setting $\bar f_{|\Omega_A} = g_A$ for any $A > 0$. Clearly the function $\bar f$ thus defined is the unique fixed point of Γ in $\mathcal{B}_{loc}([0,\infty)\times(0,\infty))$. Since Γ preserves the closed cone of nonnegative functions if f is nonnegative, the fixed point $\bar f$ necessarily belongs to this cone when f is nonnegative. Similarly, the space $C([0,\infty)\times(0,\infty))$ of continuous functions being a closed subspace of $\mathcal{B}_{loc}([0,\infty)\times(0,\infty))$, the fixed point $\bar f$ is continuous when f is so.

Consider now that f is continuously differentiable on $(0,\infty)$. The space $C^1([0,\infty)\times(0,\infty))$ is not closed in $\mathcal{B}_{loc}([0,\infty)\times(0,\infty))$ for the norm $\|\cdot\|_\infty$. For proving the continuous differentiability of $\bar f$ we repeat the fixed point argument in

$$\{g \in C^1(\Omega_{T,A}),\ g(0,\cdot) = f\}$$

endowed with the norm

$$\|g\|_{C^1} = \|g\|_\infty + \|\partial_t g\|_\infty + \|\partial_x g\|_\infty.$$

Differentiating Γg with respect to t we get

$$\partial_t \Gamma g(t,x) = \mathscr{L} f(x+t)\mathrm{e}^{-\int_0^t B(x+s)ds} + \int_0^t \mathrm{e}^{-\int_0^s B(x+s')ds'} B(x+s) \int_0^1 \partial_t g(t-s, z(x+s))\wp(dz)\, ds$$

and differentiating the alternative formulation

$$\Gamma g(t,x) = f(x+t)\mathrm{e}^{-\int_x^{x+t} B(y)dy} + \int_x^{x+t} \mathrm{e}^{-\int_x^y B(y')dy'} B(y) \int_0^1 g(t+x-y, zy)\wp(dz)\, dy$$

with respect to x we obtain

$$\begin{aligned}
\partial_x \Gamma g(t,x) &= \mathscr{L} f(t,x)\mathrm{e}^{-\int_x^{x+t} B(y)dy} + B(x)\Big(f(x+t)\mathrm{e}^{-\int_x^{x+t} B(y)dy} - \int_0^1 g(t,zx)\wp(dz)\Big) \\
&\quad + B(x)\int_x^{x+t} \mathrm{e}^{-\int_x^y B(y')dy'} B(y) \int_0^1 g(t+x-y,zy)\wp(dz)\, dy \\
&\quad + \int_x^{x+t} \mathrm{e}^{-\int_x^y B(y')dy'} B(y) \int_0^1 \partial_t g(t+x-y,zy)\wp(dz)\, dy \\
&= \Big[\mathscr{L} f(t,x) + B(x)f(x+t) - B(x)\int_0^1 f(zx)\wp(dz)\Big]\mathrm{e}^{-\int_x^{x+t} B(y)dy} \\
&\quad + \int_x^{x+t} \mathrm{e}^{-\int_x^y B(y')dy'} \big(B(y) - B(x)\big) \int_0^1 \partial_t g(t+x-y,zy)\wp(dz)\, dy.
\end{aligned}$$

On the one hand using the first expression of $\partial_x \Gamma g(t,x)$ above we deduce that for $g_1, g_2 \in C^1(\Omega_{T,A})$ such that $g_1(0,\cdot) = g_2(0,\cdot) = f$ we have

$$\|\Gamma g_1 - \Gamma g_2\|_{C^1} \le \wp_0 TB(A)\|g_1 - g_2\|_\infty + 2\wp_0 TB(A)\|\partial_t g_1 - \partial_t g_2\|_\infty \le 2TB(A)\|g_1 - g_2\|_{C^1}.$$

Thus Γ is a contraction for $T < 1/(2\wp_0 TB(A))$ and this guarantees that the fixed point $\bar{f}$ necessarily belongs to $C^1([0,\infty)\times(0,\infty))$. On the other hand using the second expression of $\partial_x \Gamma g(t,x)$ we have

$$\partial_t \Gamma g(t,x) - \partial_x \Gamma g(t,x) = B(x)\Big[\int_0^1 g(t,zx)\wp(dz) - \Gamma g(t,x)\Big]$$

and accordingly the fixed point satisfies $\partial_t \bar{f} = \mathscr{L}\bar{f}$. □

With Lemma 5.1 at hand we can define for any $t \ge 0$ the mapping M_t on $\mathcal{B}_{loc}(0,\infty)$ by setting

$$M_t f(x) = \bar{f}(t,x).$$

Proposition 5.2. *The family $(M_t)_{t\ge 0}$ defined above is a positive semigroup of linear operators on $\mathcal{B}_{loc}(0,\infty)$. If $f \in C^1(0,\infty)$ then the function $(t,x) \mapsto M_t f(x)$ is continuously differentiable and satisfies*

$$\partial_t M_t f(x) = \mathscr{L} M_t f(x) = M_t \mathscr{L} f(x).$$

Additionally for any $k > 1$ the space $\mathcal{B}(V)$ with $V(x) = 1 + x^k$ is invariant under $(M_t)_{t\ge 0}$, and for all $t \ge 0$ the restriction of M_t to $\mathcal{B}(V)$ is a bounded operator.

Proof. The linearity and the semigroup property readily follow from the uniqueness of the fixed point in Lemma 5.1. The positivity and the stability of $C^1(0,\infty)$ are direct consequences of Lemma 5.1, as well as the relation $\partial_t M_t f = \mathscr{L} M_t f$. For getting the second one $\partial_t M_t f = M_t \mathscr{L} f$, it suffices to remark from the computation of $\partial_t \Gamma g$ in the proof of Lemma 5.1 that $\partial_t M_t f$ is the unique fixed point of Γ with initial data $\mathscr{L} f$. For the invariance of $\mathcal{B}(V)$ we compute

$$\mathscr{L} V(x) = 1 + kx^{k-1} + (\wp_0 - 1)B(x) + (\wp_k - 1)B(x)x^k$$

which is bounded on $(0,\infty)$ since $(\wp_0-1)B(x)+(\wp_k-1)B(x)x^k\le 0$ when $x\ge\left(\frac{\wp_0-1}{1-\wp_k}\right)^{\frac{1}{k}}$. We deduce that there exists $C>0$ such that $\mathscr{L}V\le CV$ and since $V\in C^1(0,\infty)$ we get

$$M_tV(x)=V(x)+\int_0^t M_s(\mathscr{L}V)(x)\,ds\le \mathrm{e}^{Ct}V(x).$$

Finally by positivity of M_t we have for any $f\in\mathcal{B}(V)$

$$|M_tf|\le M_t|f|\le \|f\|_{\mathcal{B}(V)}M_tV\le \mathrm{e}^{Ct}\|f\|_{\mathcal{B}(V)}V$$

which yields

$$\|M_tf\|_{\mathcal{B}(V)}\le \mathrm{e}^{Ct}\|f\|_{\mathcal{B}(V)}.$$

□

Now we define, for $t\ge 0$ and $\mu\in\mathcal{M}_+(V)$, the positive measure μM_t by setting for any measurable set $A\subset(0,\infty)$

$$(\mu M_t)(A)=\mu(M_t\mathbf{1}_A).$$

The axioms of a positive measure are satisfied. Clearly $(\mu M_t)(\emptyset)=0$ and $(\mu M_t)(A\cup B)=(\mu M_t)(A)+(\mu M_t)(B)$ when A and B are two disjoint measurable sets. If $(A_n)_{n\ge0}$ is an increasing sequence of measurable sets then by positivity of M_t the sequence $(M_t\mathbf{1}_{A_n})_{n\ge0}$ is an increasing sequence of measurable functions bounded by M_tV. Passing to the limit in the Duhamel formula we deduce from the uniqueness of the solution that the pointwise limit of $(M_t\mathbf{1}_{A_n})_{n\ge0}$ is $M_t\mathbf{1}_A$, where $A=\bigcup_{n\ge0}A_n$. We conclude by dominated or monotone convergence theorem that $(\mu M_t)(A)=\lim_{n\to\infty}(\mu M_t)(A_n)$. By construction we have $(\mu M_t)(f)=\mu(M_tf)$ for any positive measurable function f, and since $\mathcal{B}(V)$ is invariant under M_t the measure μM_t belongs to $\mathcal{M}_+(V)$. Then, for $\mu\in\mathcal{M}(V)$ we define $\mu M_t\in\mathcal{M}(V)$ as the equivalence class of (μ_+M_t,μ_-M_t).

Proposition 5.3. *The family $(M_t)_{t\ge0}$ defined above is a positive semigroup of bounded linear operators on $\mathcal{M}(V)$. Moreover for any $\mu\in\mathcal{M}(V)$ the family $(\mu M_t)_{t\ge0}$ is solution to Equation (4.5) in the sense that for all $f\in C^1_c(0,\infty)$ and all $t\ge0$*

$$(\mu M_t)(f)=\mu(f)+\int_0^t(\mu M_s)(\mathscr{L}f)\,ds.$$

Proof. Let $\mu\in\mathcal{M}(V)$ and $f\in C^1_c(0,\infty)$. From Proposition 5.2 we know that $\partial_tM_tf=M_t\mathscr{L}f$ which gives by integration in time

$$M_tf(x)=f(x)+\int_0^tM_s\mathscr{L}f(x)\,ds=f(x)+\int_0^tM_s(f'-Bf)(x)\,ds+\int_0^tM_s\mathcal{F}f(x)\,ds$$

for all $x\in(0,\infty)$, where we have set

$$\mathcal{F}f(x)=B(x)\int_0^1 f(zx)\wp(dz).$$

Since $f'-Bf\in\mathcal{B}(V)$ we have $|M_s(f'-Bf)|\le\|f'-Bf\|_{\mathcal{B}(V)}\,\mathrm{e}^{Cs}V$ and Fubini's theorem ensures that

$$\mu\Big(\int_0^tM_s(f'-Bf)\,ds\Big)=\int_0^t(\mu M_s)(f'-Bf)\,ds.$$

The last term deserves a bit more attention since $\mathcal{F}f$ can be not bounded by V. Consider $g\in C^1_c(0,\infty)$ such that $g\ge|f|$. By positivity of M_s and $\mathcal{F}$ we have $|M_s\mathcal{F}f|\le M_s\mathcal{F}|f|\le M_s\mathcal{F}g$

and since $g \in C_c^1(0,\infty)$

$$\mu_\pm\Big(\int_0^t M_s\mathcal{F}g\,ds\Big) = \mu_\pm\Big(M_t g - g - \int_0^t M_s(g' - Bg)\,ds\Big) < +\infty.$$

This guarantees that $(s,x) \mapsto M_s\mathcal{F}f(x)$ is $(ds\times\mu)$-integrable and Fubini's theorem yields

$$\mu\Big(\int_0^t M_s\mathcal{F}f\,ds\Big) = \int_0^t (\mu M_s)(\mathcal{F}f)\,ds,$$

which ends the proof. □

We end this appendix by giving some monotonicity results on $(M_t)_{t\geq 0}$, which are useful for verifying (A4) in Section 4.2. They are valid under the monotonicity assumption we made on the fragmentation rate B.

Lemma 5.4. *(i) For any $x>0$, $t\mapsto M_t\psi(x)$ is increasing.*
(ii) For any $t\geq 0$, $x\mapsto M_t\psi(x)$ is increasing.
(iii) For any $T>0$, $z\in[0,1]$, and $x\geq 0$, $t\mapsto M_t\psi\big(z(x+T-t)\big)$ is increasing on $[0,T]$.

Proof. The point (i) readily follows from $\partial_t M_t\psi = M_t(\mathscr{L}\psi)$, since M_t is positive and

$$2\,\mathscr{L}\psi(x) = 1 + (\wp_0 - 1)B(x) \geq 0.$$

Let us prove (ii). Define $f(t,x) = \partial_x M_t\psi(x)$ which satisfies

$$\begin{aligned}\partial_t f(t,x) = \partial_x f(t,x) - B(x)f(t,x) + B(x)\int_0^1 f(t,zx)z\wp(dz)\\ - B'(x)M_t\psi(x) + B'(x)\int_0^1 M_t\psi(zx)\,\wp(dz).\end{aligned}$$

Since $\partial_t M_t\psi(x) = \mathscr{L}M_t\psi(x)$, $\partial_t M_t\psi(x)\geq 0$, and $B'\geq 0$, we have

$$-B'(x)M_t\psi(x) + B'(x)\int_0^1 M_t\psi(zx)\,\wp(dz) = \frac{B'(x)}{B(x)}\big(\partial_t M_t\psi(x) - \partial_x M_t\psi(x)\big) \geq -\frac{B'(x)}{B(x)}f(t,x)$$

and as a consequence

$$\partial_t f(t,x) \geq \mathcal{A}f(t,x) := \partial_x f(t,x) - \left(B(x) + \frac{B'(x)}{B(x)}\right)f(t,x) + B(x)\int_0^1 f(t,zx)z\wp(dz).$$

Similarly to $\mathscr{L}$ the operator $\mathcal{A}$ generates a positive semigroup $(U_t)_{t\geq 0}$. It is a standard result that it enjoys the following maximum principle

$$\partial_t f(t,x) \geq \mathcal{A}f(t,x) \qquad \Longrightarrow \qquad f(t,x)\geq U_t f_0(x)$$

where $f_0 = f(0,\cdot)$. Since $f(0,x) = \psi'(x) = \frac{1}{2} \geq 0$ we deduce from the positivity of U_t that $f(t,x)\geq 0$ for all $t,x>0$, and this finishes the proof of (ii).

We turn to the proof of (iii). The case $z=0$ corresponds to (ii) and we consider now $z\in(0,1]$. Setting $f(t,x) = M_t\psi\big(z(x+T-t)\big)$ we have using (ii)

$$\partial_x f(t,x) = z\,\partial_x M_t\psi\big(z(x+T-t)\big) \geq 0$$

and

$$\begin{aligned}\partial_t f(t,x) &= (\partial_t M_t\psi)\big(z(x+T-t)\big) - z\,(\partial_x M_t\psi)\big(z(x+T-t)\big)\\ &= (1-z)\,\partial_x M_t\psi\big(z(x+T-t)\big) - B\big(z(x+T-t)\big)M_t\psi\big(z(x+T-t)\big)\\ &\qquad + B\big(z(x+T-t)\big)\int_0^1 M_t\psi\big(z'z(x+T-t)\big)\,\wp(dz')\\ &= \frac{1-z}{z}\,\partial_x f(t,x) - B\big(z(x+T-t)\big)f(t,x)\\ &\qquad + B\big(z(x+T-t)\big)\int_0^1 f\big(t,z'x-(1-z')(T-t)\big)\,\wp(dz'). \qquad (5.1)\end{aligned}$$

Now define $g(t,x)=\partial_t f(t,x)$ and differentiate the above equation with respect to t to get

$$\begin{aligned}\partial_t g(t,x) &= \frac{1-z}{z}\,\partial_x g(t,x) - B\big(z(x+T-t)\big)g(t,x)\\ &\qquad + B\big(z(x+T-t)\big)\int_0^1 g\big(t,z'x-(1-z')(T-t)\big)\,\wp(dz')\\ &\quad + zB'\big(z(x+T-t)\big)f(t,x) - zB'\big(z(x+T-t)\big)\int_0^1 f\big(t,z'x-(1-z')(T-t)\big)\,\wp(dz')\\ &\quad + B\big(z(x+T-t)\big)\int_0^1 (1-z')\partial_x f\big(t,z'x-(1-z')(T-t)\big)\,\wp(dz').\end{aligned}$$

Using again (5.1) we get

$$\begin{aligned}\partial_t g(t,x) &= \frac{1-z}{z}\,\partial_x g(t,x) - B\big(z(x+T-t)\big)g(t,x)\\ &\qquad + B\big(z(x+T-t)\big)\int_0^1 g\big(t,z'x-(1-z')(T-t)\big)\,\wp(dz')\\ &\quad + z\frac{B'}{B}\big(z(x+T-t)\big)\Big(\frac{1-z}{z}\partial_x f(t,x) - g(t,x)\Big)\\ &\quad + B\big(z(x+T-t)\big)\int_0^1 (1-z')\partial_x f\big(t,z'x-(1-z')(T-t)\big)\,\wp(dz')\end{aligned}$$

and using the positivity of $\partial_x f$, B and B' we finally obtain

$$\begin{aligned}\partial_t f(t,x) &\geq \frac{1-z}{z}\,\partial_x g(t,x) - \Big(B + z\frac{B'}{B}\Big)\big(z(x+T-t)\big)g(t,x)\\ &\qquad + B\big(z(x+T-t)\big)\int_0^1 g\big(t,z'x-(1-z')(T-t)\big)\,\wp(dz').\end{aligned}$$

Since $g(0,x) = \frac{1-z}{2} + \frac{\wp_0-1}{2}B\big(z(x+T)\big) \geq 0$ we deduce from the maximum principle that $g(t,x)\geq 0$. □

Acknowledgments

B.C. and V.B. and A.M. have received the support of the Chair "Modélisation Mathématique et Biodiversité" of VEOLIA-Ecole Polytechnique-MnHn-FX. The authors have been also supported by ANR projects, funded by the French Ministry of Research: B.C. by ANR MESA (ANR-18-CE40-006), V.B. by ANR ABIM (ANR-16-CE40-0001) and ANR CADENCE (ANR-16-CE32-0007), and A.M. by ANR MEMIP (ANR-16-CE33-0018).

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