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# Proof that Bombieri's question about the abc-conjecture is Yes

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## Abstract

In [1] Enrico Bombieri, asked the question “For example, is it always true that

$$c \leq \prod_{p/abc} p^2 \text{ ?}.”$$

in present paper we prove that the answer is: Yes.

## 1 The series $\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s}$

We introduce the series

$$\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \tag{1}$$

where

$$\begin{aligned} R(n) &:= \text{the multiplicative radical of } n, \\ t &:= \text{a real positive variable,} \\ s &:= \text{a real variable, such that } s > t + 1. \end{aligned}$$

$\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s}$  converges because it is majorized by  $\sum_{n=1}^{\infty} \frac{1}{n^{s-t}}$ , since  $R(n)^t \leq n^s$  and the

latter series converges since  $s > t + 1$ . We denote the set

$$\{t > 0, s > t + 1\},$$

the convergence region of  $\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s}$ , by  $CR(s, t)$ , indicating its dependance on  $s$  and  $t$ .

## 2 The Euler product of $\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s}$

Since  $R(n)^t$  is also multiplicative if  $R(n)$  is, we have

$$\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} = \prod_p \left(1 + \frac{p^t}{p^s - 1}\right). \quad (2)$$

The product converges since it is majorized by

$$\prod_p \left(1 + \frac{p^s}{p^s - 1}\right) = \zeta(s)$$

taking logarithm of (2), we have

$$\lg \sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} = \sum_p \lg \left(1 + \frac{p^t}{p^s - 1}\right). \quad (3)$$

## 3 The functions $T(s, t)$ and $S(s, t)$

Differentiating (3), firstly, with respect to  $t$ , legitimate because the  $n$ -th prime is smaller than  $n$  [2], we have

$$\frac{\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg R(n)}{\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s}} = \sum_p \frac{p^t}{p^t + p^s - 1} \lg p, \quad (4)$$

which we denote by  $T(s, t)$ , and for short by  $T$ .

Secondly, differentiating (3) with respect to  $s$ , legitimate because of [2], we have

$$\frac{-\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg n}{\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s}} = \sum_p \frac{-\frac{p^s}{p^s - 1} \frac{p^t}{p^s - 1}}{1 + \frac{p^t}{p^s - 1}} \lg p$$

or

$$\frac{\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg n}{\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s}} = \sum_p \frac{p^s}{p^s - 1} \frac{p^t}{p^t + p^s - 1} \lg p, \quad (5)$$

which we denote by  $S(s, t)$  and, for short, by  $S$ .

#### 4 Inequalities for $T(s, t)$ and $S(s, t)$

Comparing the series for  $T(s, t)$  and  $S(s, t)$  we have on the one hand

$$T(s, t) < 2 S(s, t)$$

since  $\frac{p^s}{p^s - 1} < 2$ , and, on the other hand,

$$T(s, t) < S(s, t)$$

since  $1 < \frac{p^s}{p^s - 1}$ .

Combining, we get

$$T(s, t) < S(s, t) < 2 T(s, t),$$

or

$$1 < \frac{S(s, t)}{T(s, t)} < 2. \quad (6)$$

#### 5 The identity $\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg \frac{R(n)^s}{n^T} = 0$

By multiplying (4) by  $S \sum_{n=1}^{\infty} \frac{R(n)^t}{n^s}$ , and (5) by  $T \sum_{n=1}^{\infty} \frac{R(n)^t}{n^s}$ , we have, respectively,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg R(n) &= T \sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \\ \sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg n &= S \sum_{n=1}^{\infty} \frac{R(n)^t}{n^s}. \end{aligned}$$

From these we get

$$S \sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg R(n) = T \sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg n$$

or

$$\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg R(n)^S = \sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg n^T,$$

so that

$$\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg \frac{R(n)^S}{n^T} = 0. \quad (7)$$

## 6 Splitting $\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg \frac{R(n)^S}{n^T}$ in summands

Taking as point of reference of the split the ratio

$$\frac{S}{T} = \frac{S(s, t)}{T(s, t)},$$

(7) becomes

$$\sum_{n=1}^{\infty} \frac{R(n)^t}{n^s} \lg \frac{R(n)^S}{n^T} = \sum_{n \leq R(n)^{S/T}} \frac{R(n)^t}{n^s} \lg \frac{R(n)^S}{n^T} + \sum_{n > R(n)^{S/T}} \frac{R(n)^t}{n^s} \lg \frac{R(n)^S}{n^T}.$$

After evaluation of the first two terms, the left side of this is actually equal to

$$\frac{S-T}{2^{s-t}} \lg 2 + \sum_{n \geq 3}^{\infty} \frac{R(n)^t}{n^s} \lg \frac{R(n)^S}{n^T}.$$

Hence (7) becomes

$$\begin{aligned} \frac{S-T}{2^{s-t}} \lg 2 + \sum_{n \geq 3}^{\infty} \frac{R(n)^t}{n^s} \lg \frac{R(n)^S}{n^T} \\ = \sum_{n \leq R(n)^{S/T}} \frac{R(n)^t}{n^s} \lg \frac{R(n)^S}{n^T} + \sum_{n > R(n)^{S/T}} \frac{R(n)^t}{n^s} \lg \frac{R(n)^S}{n^T}. \end{aligned} \quad (8)$$

Since, however, the sets  $\{n \leq R(n)^{S/T}\}$  and  $\{n > R(n)^{S/T}\}$  are disjoint, it results

that the sum  $\sum_{n \leq R(n)^{S/T}} \frac{R(n)^t}{n^s} \lg \frac{R(n)^S}{n^T}$  contains all numbers  $n \geq 3$ , i.e. we have

$$n \leq R(n)^{S/T}. \quad (9)$$

## 7 Bombieri's "abc-question"

Because of (6) and (9), we deduce that for all  $c \geq 3$ , we have

$$c \leq R(c)^2,$$

and therefore, a fortiori

$$c \leq R(abc)^2$$

or

$$c \leq \prod_{p/abc} p^2. \tag{10}$$

That

$$R(abc) = \prod_{p/abc} p$$

results from the fundamental theorem of arithmetic [2], page 2.

Summarizing, we have shown that the answer to Bombieri's "abc-question" is "Yes".

## References

- [1] Enric Bombieri, *Forty Years of Effective Results in Diophantine Theory*, p. 206, in a Panorama of Number Theory, edited by Gisbert Wüstholz, Cambridge University Press, 2002.
- [2] G. H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, fourth edition, p. 253 subscript, Oxford at the Clarendon Press, 1960.