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► **To cite this version:**

| Shalom Eliahou, Jean Fromentin. Gapsets of small multiplicity. 2019. hal-02059925v2

**HAL Id: hal-02059925**

**<https://hal.science/hal-02059925v2>**

Preprint submitted on 8 Mar 2019

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# GAPSETS OF SMALL MULTIPLICITY

S. ELIAHOU AND J. FROMENTIN

ABSTRACT. A *gapset* is the complement of a numerical semigroup in  $\mathbb{N}$ . In this paper, we characterize all gapsets of multiplicity  $m \leq 4$ . As a corollary, we provide a new simpler proof that the number of gapsets of genus  $g$  and fixed multiplicity  $m \leq 4$  is a nondecreasing function of  $g$ .

*Key words and phrases.* Numerical semigroups, genus, Kunz coordinates, gapset filtrations.

## 1. INTRODUCTION

Denote  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$  and  $\mathbb{N}_+ = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \dots\}$ . For  $a, b \in \mathbb{Z}$ , let  $[a, b] = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$  and  $[a, \infty[ = \{z \in \mathbb{Z} \mid a \leq z\}$  denote the integer intervals they span. A *numerical semigroup* is a subset  $S \subseteq \mathbb{N}$  containing 0, stable under addition and with finite complement in  $\mathbb{N}$ . Equivalently, it is a subset  $S \subseteq \mathbb{N}$  of the form  $S = \langle a_1, \dots, a_n \rangle = \mathbb{N}a_1 + \dots + \mathbb{N}a_n$  for some globally coprime positive integers  $a_1, \dots, a_n$ .

For a numerical semigroup  $S \subseteq \mathbb{N}$ , its *gaps* are the elements of  $\mathbb{N} \setminus S$ , its *genus* is  $g = |\mathbb{N} \setminus S|$ , its *multiplicity* is  $m = \min S \setminus \{0\}$ , its *Frobenius number* is  $f = \max \mathbb{Z} \setminus S$ , its *conductor* is  $c = f + 1$ , and its *embedding dimension*, usually denoted  $e$ , is the least number of generators of  $S$ , *i.e.* the least  $n$  such that  $S = \langle a_1, \dots, a_n \rangle$ . Note that the conductor  $c$  of  $S$  satisfies  $c + \mathbb{N} \subseteq S$ , and is minimal with respect to this property since  $c - 1 = f \notin S$ .

Given  $g \geq 0$ , the number  $n_g$  of numerical semigroups of genus  $g$  is finite, as easily seen. The values of  $n_g$  for  $g = 0, \dots, 15$  are as follows:

$$1, 1, 2, 4, 7, 12, 23, 39, 67, 118, 204, 343, 592, 1001, 1693, 2857.$$

In 2006, Maria Bras-Amorós made some remarkable conjectures concerning the growth of  $n_g$ . In particular, she conjectured [1] that

$$(1) \quad n_g \geq n_{g-1} + n_{g-2}$$

for all  $g \geq 2$ . This conjecture is widely open. Indeed, even the weaker inequality

$$(2) \quad n_g \geq n_{g-1}$$

whose validity has been settled by Alex Zhai [24] for all sufficiently large  $g$ , remains to be proved for all  $g \geq 1$ .

In that same paper, Zhai showed that ‘most’ numerical semigroups  $S$  satisfy  $c \leq 3m$ , where  $c$  and  $m$  are the conductor and multiplicity of  $S$ , respectively. For a more precise statement, let us denote by  $n'_g$  the number of numerical semigroups of genus  $g$  satisfying  $c \leq 3m$ . The values of  $n'_g$  for  $g = 0, \dots, 15$  are as follows:

$$1, 1, 2, 4, 6, 11, 20, 33, 57, 99, 168, 287, 487, 824, 1395, 2351.$$

Zhai showed then that  $\lim_{g \rightarrow \infty} n'_g/n_g = 1$ , as had been earlier conjectured by Yufei Zhao [25]. In that sense, numerical semigroups satisfying  $c \leq 3m$  may be considered as *generic*.

Recently, the strong conjecture (1) has been established for generic numerical semigroups. Here is the precise statement, first announced at the IMNS 2018 conference in Cortona [12].

**Theorem 1.1** ([8], Theorem 6.4). *The inequalities*

$$n'_{g-1} + n'_{g-2} + n'_{g-3} \geq n'_g \geq n'_{g-1} + n'_{g-2},$$

hold for all  $g \geq 3$ .

The proof of this result essentially rests on the notion of *gapset filtrations*, a new flexible framework to investigate numerical semigroups introduced in [8]. More details are given in Section 2 since, here also, gapsets filtrations are at the core of the present results.

**Notation 1.2.** *Let  $g \geq 0, m \geq 1$  be two integers. We denote by  $\Gamma_{g,m}$  the finite set of all numerical semigroups of genus  $g$  and multiplicity  $m$ , and by  $n_{g,m} = |\Gamma_{g,m}|$  its cardinality.*

Since, for a numerical semigroup  $S$  of multiplicity  $m$  and genus  $g$ , the integers  $1, \dots, m-1$  belong to the complement  $\mathbb{N} \setminus S$ , the relation  $g \geq m-1$  holds. Thus  $n_{g,m} = 0$  for  $m \geq g+2$ , and so we have

$$n_g = \sum_{m=1}^{g+1} n_{g,m}.$$

The first values of  $n_{g,m}$  for  $g \geq 0$  and small fixed  $m$  are given below.

$g$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
$m=1$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...
$m=2$	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	...
$m=3$	0	0	1	2	2	2	3	3	3	4	4	4	5	5	5	...
$m=4$	0	0	0	1	3	4	6	7	9	11	13	15	18	20	23	...
$m=5$	0	0	0	0	1	4	7	10	13	16	22	24	32	35	43	...
$m=6$	0	0	0	0	0	1	5	11	17	27	37	49	66	85	106	...

For instance, the unique numerical semigroup of multiplicity 1 is  $\mathbb{N}$ . Nathan Kaplan proposed the following conjecture in [13], a refinement of the conjectured inequality (2).

**Conjecture 1.3.** *Let  $m \geq 2$ . Then*

$$(3) \quad n_{g,m} \geq n_{g-1,m}$$

for all  $g \geq 1$ .

On the other hand, still for  $m \geq 2$  fixed, there is no hope a stronger inequality such as (1) may hold for the  $n_{g,m}$ , as the reader can check by looking at the rows of the above table.

Conjecture 1.3 is trivial for  $m = 2$  since  $n_{g,2} = 1$  for all  $g \geq 1$ , and has been settled for  $m = 3, 4, 5$  in 2018 by Pedro A. García-Sánchez, Daniel Marín-Aragón and Aureliano M. Robles-Pérez [11]. For that, they used a linear integer software to count the number of integral points of the associated Kunz polytope. With it, they first achieved formulas for  $n_{g,m}$  for  $m = 3, 4, 5$ , and then proved them to be increasing using a computer algebra system. The conjecture remains open for  $m \geq 6$ .

Our purpose in this paper is to give new proofs of Conjecture 1.3 for  $m = 3$  and  $m = 4$  by constructing explicit injections

$$\Gamma_{g,3} \rightarrow \Gamma_{g+1,3} \quad \text{and} \quad \Gamma_{g,4} \rightarrow \Gamma_{g+1,4}$$

for  $g \geq 0$ , thereby establishing the desired inequalities  $n_{g+1,3} \geq n_{g,3}$  and  $n_{g+1,4} \geq n_{g,4}$ . Thus, our proofs are computer-free and do not rest on counting formulas for  $n_{g,3}$  and  $n_{g,4}$ . These injections were first announced in [8].

## 2. GAPSET FILTRATIONS

The content of this section is mostly taken from [8].

**Definition 2.1.** *Let  $n \in \mathbb{N}_+$ . An additive decomposition of  $n$  is any expression of the form  $n = a + b$  with  $a, b \in \mathbb{N}_+$ . We refer to the positive integers  $a, b$  as the summands of this decomposition.*

**Definition 2.2.** *A gapset is a finite set  $G \subset \mathbb{N}_+$  satisfying the following property: for all  $z \in G$ , if  $z = x + y$  with  $x, y \in \mathbb{N}_+$ , then  $x \in G$  or  $y \in G$ . That is, for any additive decomposition of  $z \in G$ , at least one of its summands belongs to  $G$ .*

Notice the similarity of this definition with that of a prime ideal  $P$  in a ring  $R$ , where for any  $z \in P$ , any decomposition  $z = xy$  with  $x, y \in R$  implies  $x \in P$  or  $y \in P$ .

**Remark 2.3.** *It follows from the definition that a gapset  $G$  is nothing else than the set of gaps of a numerical semigroup  $S$ , where  $S = \mathbb{N} \setminus G$ .*

**Definition 2.4.** *We naturally extend the definitions of multiplicity, Frobenius number, conductor and genus of a gapset  $G$  as being those of the corresponding numerical semigroup  $S = \mathbb{N} \setminus G$ , respectively.*

More directly, for a gapset  $G$ , these notions may be described as follows:

- the multiplicity of  $G$  is the smallest integer  $m \geq 1$  such that  $m \notin G$ ;
- the Frobenius number of  $G$  is  $\max(G)$  if  $G \neq \emptyset$ , and  $-1$  otherwise;
- the conductor of  $G$  is  $1 + \max(G)$  if  $G \neq \emptyset$ , and  $0$  otherwise;
- the genus of  $G$  is  $g(G) = \text{card}(G)$ .

**Example 2.5.** *The set  $G = \{1, 2, 3, 4, 6, 7, 11\}$  is a gapset. For instance, for each additive decomposition of 11, namely*

$$1 + 10, \quad 2 + 9, \quad 3 + 8, \quad 4 + 7, \quad 5 + 6,$$

*at least one of the two summands belongs to  $G$ . Let  $S = \mathbb{N} \setminus G = \{0, 5, 8, 9, 10\} \cup [12, +\infty[$ . Then  $S = \langle 5, 8, 9 \rangle$  as easily seen, whence  $S$  is indeed a numerical semigroup. The multiplicity, conductor, Frobenius number, genus and embedding dimension of  $G$  and  $S$  are  $m = 5$ ,  $c = 11$ ,  $f = 12$ ,  $g = 7$  and  $e = 3$ , respectively.*

### 2.1. The canonical partition.

**Lemma 2.6.** *Let  $G$  be a gapset of multiplicity  $m$ . Then*

$$\begin{aligned} [1, m-1] &\subseteq G, \\ G \cap m\mathbb{N} &= \emptyset. \end{aligned}$$

*Proof.* By definition of the multiplicity,  $G$  contains  $[1, m-1]$  but not  $m$ . Let  $a \geq 2$  be an integer. The formula  $am = m + (a-1)m$  and induction on  $a$  imply that  $am \notin G$ .  $\square$

This motivates the following notation and definition.

**Notation 2.7.** *Let  $G$  be a gapset of multiplicity  $m$ . We denote  $G_0 = [1, m-1]$  and, more generally,*

$$(4) \quad G_i = G \cap [im + 1, (i+1)m - 1] \quad \text{for all } i \geq 0.$$

**Definition 2.8.** *Let  $G$  be a gapset of multiplicity  $m$  and conductor  $c$ . The depth of  $G$  is the integer  $q = \lceil c/m \rceil$ .*

**Proposition 2.9.** *Let  $G$  be a gapset of multiplicity  $m$  and depth  $q$ . Let  $G_i$  be defined as in (4). Then*

$$(5) \quad G = G_0 \sqcup G_1 \sqcup \cdots \sqcup G_{q-1}$$

*and  $G_{q-1} \neq \emptyset$ . Moreover  $G_{i+1} \subseteq m + G_i$  for all  $i \geq 0$ .*

*Proof.* As  $G \cap m\mathbb{N} = \emptyset$ , it follows that  $G$  is the disjoint union of the  $G_i$  for  $i \geq 0$ . Let  $c$  be the conductor of  $G$ . Then  $G \subseteq [1, c - 1]$ . Since  $(q - 1)m < c \leq qm$  by definition of  $q$ , it follows that  $G_i = \emptyset$  for  $i \geq q$ , whence (5). Let  $f = c - 1$ . Since  $f \in G$ ,  $(q - 1)m \leq f < qm$  and  $f \not\equiv 0 \pmod{m}$ , it follows that  $f \in G_{q-1}$ .

It remains to show that  $G_{i+1} \subseteq m + G_i$  for all  $i \geq 0$ . Let  $x \in G_{i+1}$ . Since  $G_{i+1} \subseteq [(i + 1)m + 1, (i + 2)m - 1]$ , we have

$$x - m \in [im + 1, (i + 1)m - 1].$$

Now  $x - m \in G$  since  $x = m + (x - m)$  and  $m \notin G$ . So  $x - m \in G_i$ .  $\square$

**Definition 2.10.** *Let  $G$  be a gapset. The canonical partition of  $G$  is the partition  $G = G_0 \sqcup G_1 \sqcup \cdots \sqcup G_{q-1}$  given by Proposition 2.9.*

**Remark 2.11.** *The multiplicity  $m$ , genus  $g$  and depth  $q$  of a gapset  $G$  may be read off from its canonical partition  $G = \sqcup_i G_i$  as follows :*

$$\begin{aligned} m &= \max(G_0) + 1, \\ g &= \sum_i |G_i|, \\ q &= \text{the number of parts of the partition.} \end{aligned}$$

**2.2. Gapset filtrations.** Let  $G \subset \mathbb{N}_+$  be a gapset. Let  $G = G_0 \sqcup G_1 \sqcup \cdots \sqcup G_{q-1}$  be its canonical partition. For all  $0 \leq i \leq q - 1$ , denote

$$(6) \quad F_i = -im + G_i.$$

Then  $F_{i+1} \subseteq F_i$  for all  $i$ , as follows from the inclusion  $G_{i+1} \subseteq m + G_i$  stated in Proposition 2.9. This gives rise to the following definition.

**Definition 2.12.** *Let  $G \subset \mathbb{N}_+$  be a gapset of multiplicity  $m$  and depth  $q$ . The gapset filtration associated to  $G$  is the finite sequence*

$$(F_0, F_1, \dots, F_{q-1}) = (G_0, -m + G_1, \dots, -(q - 1)m + G_{q-1}),$$

i.e. with  $F_i$  defined as in (6) for all  $i$ . Thus, as seen above, we have

$$(7) \quad F_0 = [1, m - 1] \supseteq F_1 \supseteq \cdots \supseteq F_{q-1}.$$

We define the *multiplicity*, *Frobenius number*, *conductor* and *genus* of a gapset filtration  $F = (F_0, \dots, F_{q-1})$  from those of the corresponding gapset  $G$ , namely:

- the multiplicity of  $F$  is  $1 + \max(F_0)$  if  $F_0 \neq \emptyset$  and 0 otherwise;
- the Frobenius number of  $F$  is  $qm + \max(F_{q-1})$  if  $F_0 \neq \emptyset$  and  $-1$  otherwise;
- the conductor of  $F$  is  $1 + qm + \max(F_{q-1})$  if  $F_0 \neq \emptyset$  and 0 otherwise;
- the genus of  $F$  is  $\text{card}(F_0) + \cdots + \text{card}(F_{q-1})$ .

**Example 2.13.** Consider the gapset  $G = \{1, 2, 3, 4, 6, 7, 11\}$  of Example 2.5. Its multiplicity is  $m = 5$ , and its canonical partition is given by  $G_0 = \{1, 2, 3, 4\}$ ,  $G_1 = \{6, 7\}$  and  $G_2 = \{11\}$ . Thus, its associated filtration is  $F = (\{1, 2, 3, 4\}, \{1, 2\}, \{1\})$ .

**Definition 2.14.** For integers  $g \geq 1, m \geq 1$ , we denote by  $\mathcal{F}(g, m)$  the set of all gapset filtrations of genus  $g$  and multiplicity  $m$ .

Note that any given gapset filtration  $F = (F_0, \dots, F_{q-1})$  corresponds to a unique gapset  $G$ , since (6) is equivalent to

$$(8) \quad G_i = im + F_i.$$

In particular, there is a straightforward bijection between gapsets  $G$  and gapset filtrations  $F$ , which naturally preserves the multiplicity, Frobenius number, conductor and genus. Here is a direct consequence.

**Proposition 2.15.** For any integers  $g \geq 1, m \geq 1$ , we have

$$n_{g,m} = |\mathcal{F}(g, m)|.$$

*Proof.* Straightforward from the above discussion.  $\square$

This result allows us to study properties of the sequence  $g \mapsto n_{g,m}$  in the setting of gapset filtrations of multiplicity  $m$ . In particular, in order to establish its growth, it suffices to exhibit injections from  $\mathcal{F}(g, m)$  to  $\mathcal{F}(g+1, m)$ . This is what we achieve in subsequent sections for  $m = 3$  and  $m = 4$ .

We start with the separate case  $m = 3$ , which can be treated in a straightforward way and which points to a general strategy for larger values of  $m$ . Then, following those clues, we introduce some general tools, and we end up applying them to the case  $m = 4$ .

### 3. THE CASE $m = 3$

Any filtration  $(F_0, \dots, F_t)$  such that

$$\{1, 2\} = F_0 \supseteq F_1 \supseteq \dots \supseteq F_t \neq \emptyset$$

is of one of the two possible forms below, with the terms on the left standing as a compact notation:

$$(12)^r(1)^s = \underbrace{(\{1, 2\}, \dots, \{1, 2\})}_r, \underbrace{(\{1\}, \dots, \{1\})}_s,$$

$$(12)^r(2)^s = \underbrace{(\{1, 2\}, \dots, \{1, 2\})}_r, \underbrace{(\{2\}, \dots, \{2\})}_s,$$

both with  $r \geq 1$  since  $F_0 = \{1, 2\}$ , and  $s \geq 0$ . We now characterize those filtrations which are gapset filtrations of multiplicity 3.

**Theorem 3.1.** *The gapset filtrations of multiplicity  $m = 3$  are exactly the following ones:*

$$\begin{aligned} (12)^r(2)^s & \text{ with } 0 \leq s \leq r, \\ (12)^r(1)^s & \text{ with } 0 \leq s \leq r + 1, \end{aligned}$$

both with  $r \geq 1$ .

Note that  $g = 2r + s$  in both cases, since the genus of a gapset filtration  $F = (F_0, \dots, F_{q-1})$  is given by the sum of the  $|F_i|$ .

*Proof.* We start with the second case.

Case  $F = (12)^r(2)^s$ . Then

$$\begin{aligned} F_0 &= \dots = F_{r-1} = \{1, 2\}, \\ F_r &= \dots = F_{r+s-1} = \{2\}. \end{aligned}$$

Using (8) with  $m = 3$ , namely  $G_i = 3i + F_i$  for all  $i$ , let

$$(9) \quad G = G_0 \cup \dots \cup G_{r+s-1}$$

be the corresponding finite set. By construction,  $F$  is a gapset filtration if and only if  $G$  a gapset. So, when is it the case that  $G$  is a gapset? We now proceed to answer this question.

**Step 1.** The set  $G$  given by (9) has the following properties:

$$\begin{aligned} 3\mathbb{N} \cap G &= \emptyset \\ 3i + 1 \in G &\iff i \leq r - 1 \\ 3i + 2 \in G &\iff i \leq r + s - 1. \end{aligned}$$

Indeed, this directly follows from the definition  $G_i = 3i + F_i$  and (9).

**Step 2.** For  $i \in \mathbb{N}$ , any additive decomposition  $3i + 1 = a + b$  is of the form

$$(a, b) = (3x + 1, 3(i - x)) \text{ or } (3y + 2, 3(i - 1 - y) + 2)$$

for some integers  $0 \leq x \leq i - 1$  or  $0 \leq y \leq i - 1$ . Similarly, any additive decomposition  $3i + 2 = a + b$  is of the form

$$(a, b) = (3x + 2, 3(i - x)) \text{ or } (3y + 1, 3(i - y) + 1)$$

for some integers  $0 \leq x \leq i - 1$  or  $0 \leq y \leq i$ .

**Step 3.** Let  $3i + 1 \in G$ , i.e. with  $i \leq r - 1$  according to Step 1. We now show that for any additive decomposition  $3i + 1 = a + b$ , either  $a$  or  $b$  belongs to  $G$ . Using Step 1, if  $(a, b) = (3x + 1, 3(i - x))$ , then  $a \in G$  since  $x \leq i$  and we are done. Similarly, if  $(a, b) = (3y + 2, 3(i - 1 - y) + 2)$ , then  $a \in G$  since  $y \leq i \leq r - 1 \leq r + s - 1$  and we are done again.



**Step 4.** Let  $3i + 2 \in G$ , i.e. with  $i \leq r + s - 1$ . Let  $3i + 2 = a + b$  be any additive decomposition. If  $(a, b) = (3x + 2, 3(i - x))$ , then  $a \in G$  since  $x \leq i$  and we are done. Assume now  $(a, b) = (3y + 1, 3(i - y) + 1)$  with  $0 \leq y \leq i$ . Then  $a, b \notin G$  if and only if  $y, i - y \geq r$ . This is only possible if  $i \geq 2r$  and, since  $i \leq r + s - 1$  by hypothesis, the latter is equivalent to  $s - 1 \geq r$ . In particular, if  $s \leq r$ , then either  $a$  or  $b$  belongs to  $G$ . In summary, we have

$$(12)^r(2)^s \text{ is a gapset filtration} \iff G \text{ is a gapset} \iff s \leq r,$$

as desired.

Case  $F = (12)^r(1)^s$ . The arguments are similar to those of the previous case. Here, to start with, we have

$$\begin{aligned} F_0 &= \cdots = F_{r-1} = \{1, 2\}, \\ F_r &= \cdots = F_{r+s-1} = \{1\}. \end{aligned}$$

The corresponding set  $G$  defined by  $G_i = 3i + F_i$  for all  $i$  and (8) has the following properties:

$$\begin{aligned} 3\mathbb{N} \cap G &= \emptyset \\ 3i + 1 \in G &\iff i \leq r + s - 1 \\ 3i + 2 \in G &\iff i \leq r - 1. \end{aligned}$$

Analogously to Step 3 above, it is easy to see that for any additive decomposition  $a + b = 3i + 2$  where  $3i + 2 \in G$ , then either  $a$  or  $b$  belongs to  $G$ .

On the other hand, let  $3i + 1 \in G$ . Then, analogously to Step 4 above, we find that there exists an additive decomposition  $3i + 1 = a + b$  with  $a, b \notin G$  if and only if  $s \geq r + 2$ . The details, using Step 2 and the above properties of  $G$ , are straightforward and left to the reader. Therefore,  $G$  is a gapset if and only if  $s \leq r + 1$ , as claimed. This concludes the proof of the proposition.  $\square$

Here is a straightforward consequence of the above characterization and the main result of this section.

**Corollary 3.2.** *For all  $g \geq 0$ , there is a natural injection*

$$\mathcal{F}(g, 3) \longrightarrow \mathcal{F}(g + 1, 3).$$

*In particular, we have  $n_{g+1,3} \geq n_{g,3}$  for all  $g \geq 0$ .*

*Proof.* Since  $\mathcal{F}(g, 3) = \emptyset$  for  $g \leq 1$ , the statement holds in this case. Assume now  $g \geq 2$ . For  $F = (F_0, \dots, F_{q-1}) \in \mathcal{F}(g, 3)$ , let us denote by  $f_1(F)$  the insertion of a 1 in  $F$  at the unique possible position to get

a new nonincreasing sequence of subsets of  $[1, 2]$ . That is, for  $r, s \geq 1$ , we define

$$\begin{aligned} (12)^r &\xrightarrow{f_1} (12)^r(1) \\ (12)^r(1)^s &\xrightarrow{f_1} (12)^r(1)^{s+1} \\ (12)^r(2)^s &\xrightarrow{f_1} (12)^{r+1}(2)^{s-1}. \end{aligned}$$

When is it the case that  $f_1(F)$  is still a *gapset* filtration, of course automatically of genus  $g + 1$ ? In other words, when do we have that  $f_1(F)$  belongs  $\mathcal{F}(g + 1, 3)$ ? Theorem 3.1 easily provides the following answer.

- If  $F = (12)^r(2)^s \in \mathcal{F}(g, 3)$ , then  $f_1(F) \in \mathcal{F}(g + 1, 3)$  for all  $r, s$ .
- If  $F = (12)^r(1)^s \in \mathcal{F}(g, 3)$ , then  $f_1(F) \in \mathcal{F}(g + 1, 3)$  if and only if  $s \leq r$ .

Recall that  $g = 2r + s$  in both cases. In particular, the only case where  $F \in \mathcal{F}(g, 3)$  but  $f_1(F) \notin \mathcal{F}(g + 1, 3)$  is for  $F = (12)^r(1)^s$  with  $s = r + 1$ , i.e. for  $F = (12)^r(1)^{r+1} \in \mathcal{F}(g, 3)$  where  $g = 3r + 1$ .

Consequently,  $f_1$  provides a well-defined map

$$f_1: \mathcal{F}(g, 3) \longrightarrow \mathcal{F}(g + 1, 3),$$

obviously injective by construction, whenever  $g \not\equiv 1 \pmod{3}$ .

Similarly, for  $F \in \mathcal{F}(g, 3)$ , denote by  $f_2(F)$  the insertion of a 2 in  $F$  where it makes sense. That is, for  $r, s \geq 1$ , define

$$\begin{aligned} (12)^r &\xrightarrow{f_2} (12)^r(2) \\ (12)^r(1)^s &\xrightarrow{f_2} (12)^{r+1}(1)^{s-1} \\ (12)^r(2)^s &\xrightarrow{f_2} (12)^r(2)^{s+1}. \end{aligned}$$

By Theorem 3.1 again, we have

- If  $F = (12)^r(2)^s \in \mathcal{F}(g, 3)$ , then  $f_2(F) \in \mathcal{F}(g + 1, 3)$  if and only if  $s \leq r - 1$ .
- If  $F = (12)^r(1)^s \in \mathcal{F}(g, 3)$ , then  $f_2(F) \in \mathcal{F}(g + 1, 3)$  for all  $r, s \geq 1$ .

In particular, the only case where  $F \in \mathcal{F}(g, 3)$  but  $f_2(F) \notin \mathcal{F}(g + 1, 3)$  is for  $F = (12)^r(2)^r \in \mathcal{F}(g, 3)$  with  $g = 3r$ . Therefore,  $f_2$  provides a well-defined injective map

$$f_2: \mathcal{F}(g, 3) \longrightarrow \mathcal{F}(g + 1, 3)$$

whenever  $g \not\equiv 0 \pmod{3}$ .

Summarizing, we end up with a well-defined injective map

$$f: \mathcal{F}(g, 3) \longrightarrow \mathcal{F}(g + 1, 3)$$

defined by  $f = f_1$  if  $g \equiv 0, 2 \pmod{3}$ , and  $f = f_2$  otherwise.  $\square$

#### 4. SOME MORE GENERAL TOOLS

In order to facilitate discussing gapsets and gapset filtrations, and gather more tools to treat more cases, it is useful to consider somewhat more general subsets of  $\mathbb{N}_+$ .

##### 4.1. On $m$ -extensions and $m$ -filtrations.

**Definition 4.1.** *Let  $m \in \mathbb{N}_+$ . An  $m$ -extension is a finite set  $A \subset \mathbb{N}_+$  containing  $[1, m - 1]$  and admitting a partition*

$$(10) \quad A = A_0 \sqcup A_1 \sqcup \cdots \sqcup A_t$$

for some  $t \geq 0$ , where  $A_0 = [1, m - 1]$  and  $A_{i+1} \subseteq m + A_i$  for all  $i \geq 0$ .

In particular, an  $m$ -extension  $A$  satisfies  $A \cap m\mathbb{N} = \emptyset$ . Moreover, the above conditions on the  $A_i$  imply

$$(11) \quad A_i = A \cap [im + 1, (i + 1)m - 1]$$

for all  $i \geq 0$ , whence the  $A_i$  are *uniquely determined* by  $A$ .

**Remark 4.2.** *Every gapset of multiplicity  $m$  is an  $m$ -extension. This follows from Proposition 2.9.*

Closely linked is the notion of  $m$ -filtration.

**Definition 4.3.** *Let  $m \in \mathbb{N}_+$ . An  $m$ -filtration is a finite sequence  $F = (F_0, F_1, \dots, F_t)$  of nonincreasing subsets of  $\mathbb{N}_+$  such that*

$$F_0 = [1, m - 1] \supseteq F_1 \supseteq \cdots \supseteq F_t.$$

The genus  $g$  of  $F$  is defined as  $g = \sum_{i=0}^t |F_i|$ .

For  $m \in \mathbb{N}_+$ , there is a straightforward bijection between  $m$ -extensions and  $m$ -partitions.

**Proposition 4.4.** *Let  $A = A_0 \sqcup A_1 \sqcup \cdots \sqcup A_t$  be an  $m$ -extension. Set  $F_i = -im + A_i$  for all  $i$ . Then  $(F_0, F_1, \dots, F_t)$  is an  $m$ -filtration. Conversely, let  $(F_0, F_1, \dots, F_t)$  be an  $m$ -filtration. Set  $A_i = im + F_i$  for all  $i$ , and let*

$$A = \bigsqcup_{i=0}^t A_i = \bigsqcup_{i=0}^t (im + F_i).$$

Then  $A$  is an  $m$ -extension.

*Proof.* We have  $F_i = -im + A_i$  if and only if  $A_i = im + F_i$ .  $\square$

**Notation 4.5.** If  $A$  is an  $m$ -extension, we denote by  $F = \varphi(A)$  the  $m$ -filtration associated to it by Proposition 4.4. Conversely, if  $F$  is an  $m$ -filtration, we denote by  $A = \tau(F)$  its associated  $m$ -extension.

By Proposition 4.4, the maps  $\varphi$  and  $\tau$  are inverse to each other.

## 4.2. Gapset filtrations revisited.

**Definition 4.6.** Let  $G \subset \mathbb{N}_+$  be a gapset of multiplicity  $m$ . The gapset filtration associated to  $G$  is the  $m$ -filtration  $F = \varphi(G)$ .

By Remark 4.2, every gapset  $G$  of multiplicity  $m$  is an  $m$ -extension, whence  $\varphi(G)$  is well-defined.

Concretely, let  $G$  be a gapset of multiplicity  $m$  and depth  $q$ . As in (4), let  $G_i = G \cap [im+1, (i+1)m-1]$  for all  $i \geq 0$ , so that  $G_0 = [1, m-1]$  and

$$G = G_0 \sqcup \cdots \sqcup G_{q-1}.$$

The associated  $m$ -filtration  $F = \varphi(G)$  is then given by  $F = (F_0, \dots, F_{q-1})$  where  $F_i = -im + G_i$  for all  $i \geq 0$ .

It follows from Remark 2.11 and the equality  $|F_i| = |G_i|$  for all  $i$ , that the genus of  $F$  is equal to  $|F_0| + \cdots + |F_{q-1}|$  and that its depth is equal to the number of nonzero  $F_i$ .

**4.3. A compact representation.** In this section, we use permutations of  $[1, m-1]$  and exponent vectors to represent  $m$ -filtrations in a compact way. We denote by  $\mathfrak{S}_{m-1}$  the symmetric group on  $[1, m-1]$ .

**Proposition 4.7.** Let  $F = (F_0, \dots, F_t)$  be an  $m$ -filtration. Then there exists a permutation  $\sigma \in \mathfrak{S}_{m-1}$  and exponents  $e_0, \dots, e_{m-2} \in \mathbb{N}$  such that

$$F = (\underbrace{F'_0, \dots, F'_0}_{e_0}, \underbrace{F'_1, \dots, F'_1}_{e_1}, \dots, \underbrace{F'_{m-2}, \dots, F'_{m-2}}_{e_{m-2}}),$$

where  $F'_0 = [1, m-1]$  and  $F'_i = F'_{i-1} \setminus \{\sigma(i)\}$  for  $1 \leq i \leq m-2$ . In particular, we have  $|F'_i| = m-1-i$  for all  $0 \leq i \leq m-2$ .

*Proof.* By hypothesis, we have

$$[1, m-1] = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_t.$$

Equalities may occur in this chain. Removing repetitions, let

$$[1, m-1] = H_0 \supsetneq H_1 \supsetneq \cdots \supsetneq H_s$$

denote the underlying descending chain, i.e. with

$$\{F_0, F_1, \dots, F_t\} = \{H_0, H_1, \dots, H_s\}$$

and  $H_i \neq H_j$  for all  $i \neq j$ . Each  $H_i$  comes with some repetition frequency  $\mu_i \geq 1$  in  $\{F_0, F_1, \dots, F_t\}$ . Thus, we have

$$F = \underbrace{(H_0, \dots, H_0)}_{\mu_0}, \underbrace{(H_1, \dots, H_1)}_{\mu_1}, \dots, \underbrace{(H_s, \dots, H_s)}_{\mu_s}.$$

Now, between each consecutive pair  $H_{i-1} \supseteq H_i$ , we insert some maximal descending chain of subsets  $H'_{i,j}$ , i.e.

$$H_{i-1} = H'_{i,0} \supseteq H'_{i,1} \supseteq \dots \supseteq H'_{i,k_i} = H_i,$$

where  $k_i = |H_{i-1}| - |H_i|$ . Thus  $|H'_{i,j}| = |H'_{i-1}| - j$  for all  $0 \leq j \leq k_i$ .

We end up with a maximal descending chain of subsets

$$F' = [1, m-1] = F'_0 \supseteq F'_1 \supseteq \dots \supseteq F'_{m-2},$$

where each term has one less element than the preceding one, i.e. where  $|F'_j| = |F'_{j-1}| - 1$  for all  $1 \leq j \leq m-2$ . By construction, we have

$$\{F_0, F_1, \dots, F_t\} = \{H_0, H_1, \dots, H_s\} \subseteq \{F'_0, F'_1, \dots, F'_{m-2}\},$$

and each  $F'_i$  arises with some frequency  $e_i \geq 0$  in  $\{F_0, F_1, \dots, F_t\}$ . Thus

$$F = \underbrace{(F'_0, \dots, F'_0)}_{e_0}, \underbrace{(F'_1, \dots, F'_1)}_{e_1}, \dots, \underbrace{(F'_{m-2}, \dots, F'_{m-2})}_{e_{m-2}}.$$

Finally, since each  $F'_i$  is obtained by removing one distinct element from  $F'_{i-1}$  for  $1 \leq i \leq m-2$ , there is a permutation  $\sigma$  of  $[1, m-1]$  such that

$$F'_i = F'_{i-1} \setminus \{\sigma(i)\}$$

for  $1 \leq i \leq m-2$ . □

**Notation 4.8.** Given  $\sigma \in \mathfrak{S}_{m-1}$  and  $e = (e_0, \dots, e_{m-2}) \in \mathbb{N}^{m-1}$  such that  $e_0 \geq 1$ , we denote by  $F(\sigma, e)$  the  $m$ -filtration

$$F = \underbrace{(F'_0, \dots, F'_0)}_{e_0}, \underbrace{(F'_1, \dots, F'_1)}_{e_1}, \dots, \underbrace{(F'_{m-2}, \dots, F'_{m-2})}_{e_{m-2}}$$

where  $F'_i = F'_{i-1} \setminus \{\sigma(i)\}$  for  $1 \leq i \leq m-2$ .

**Example 4.9.** Consider the 5-filtration  $F = (\{1, 2, 3, 4\}, \{1, 2\}, \{1\})$  of Example 2.13. Let  $\sigma = (3, 4, 2, 1) \in \mathfrak{S}_4$  and  $e = (1, 0, 1, 1)$ . Then  $F = F(\sigma, e)$  as readily checked. Note that we also have  $F = F(\sigma', e)$  where  $\sigma' = (4, 3, 2, 1)$ .

One important question is: when is the  $m$ -filtration  $F = F(\sigma, e)$  a gapset filtration? The next section provides an answer.

#### 4.4. Complementing an $m$ -extension.

**Notation 4.10.** Let  $F = F(\sigma, e)$  be an  $m$ -filtration, where  $\sigma \in \mathfrak{S}_{m-1}$  and  $e = (e_0, \dots, e_{m-2}) \in \mathbb{N}^{m-1}$  with  $e_0 \geq 1$ . We denote by  $G = G(\sigma, e)$  the corresponding  $m$ -extension, i.e.  $G = \tau(F)$  using Notation 4.5.

Here is how to determine the set complement in  $\mathbb{N}$  of the  $m$ -extension  $G = G(\sigma, e)$ .

**Proposition 4.11.** Let  $F = F(\sigma, e)$  be an  $m$ -filtration, where  $\sigma \in \mathfrak{S}_{m-1}$  and  $e = (e_0, \dots, e_{m-2}) \in \mathbb{N}^{m-1}$  with  $e_0 \geq 1$ . Let  $G = G(\sigma, e)$  be the corresponding  $m$ -extension, i.e.  $G = \tau(F)$ . Then

$$(12) \quad \mathbb{N} \setminus G = \bigsqcup_{i=0}^{m-1} \sigma(i) + m(e_0 + \dots + e_{i-1} + \mathbb{N}),$$

with the conventions  $\sigma(0) = 0$  and  $e_0 + \dots + e_{i-1} = 0$  for  $i = 0$ .

*Proof.* For  $0 \leq i \leq m-1$ , denote  $F_i = [1, m-1] \setminus \{\sigma(0), \dots, \sigma(i)\}$ . Thus  $F_0 = [1, m-1]$ ,  $F_1 = [1, m-1] \setminus \{\sigma(1)\}$ ,  $F_2 = [1, m-1] \setminus \{\sigma(1), \sigma(2)\}$  and so on. By definition of  $F = F(\sigma, e)$ , we have

$$F = \underbrace{(F_0, \dots, F_0)}_{e_0}, \underbrace{(F_1, \dots, F_1)}_{e_1}, \dots, \underbrace{(F_{m-2}, \dots, F_{m-2})}_{e_{m-2}}.$$

Let  $G = \tau(F)$ . For  $k \in [0, m-1]$ , set  $G^{(k)} = \{x \in G \mid x \equiv k \pmod{m}\}$ .

Then  $G = \bigsqcup_{k=0}^{m-1} G^{(k)}$ . Since  $G$  is an  $m$ -extension, we have  $G \cap m\mathbb{N} = \emptyset$ ,

i.e.  $G^{(0)} = \emptyset$ . We now proceed to determine  $G^{(k)}$  for  $k \geq 1$ . Since  $\sigma$  is a permutation of  $[1, m-1]$ , there exists  $i \in [1, m-1]$  such that  $k = \sigma(i)$ . We claim that

$$(13) \quad G^{(k)} = G^{(\sigma(i))} = \sigma(i) + m[0, e_0 + \dots + e_{i-1} - 1].$$

Indeed by construction, for all  $r \geq 0$  we have

$$(14) \quad \sigma(i) \in F_r \Leftrightarrow r \leq i - 1.$$

Now, by definition of the map  $\tau$ , we have

$$(15) \quad G = \bigsqcup_{l=0}^{m-2} \left( \bigsqcup_{j=e_0+\dots+e_{l-1}}^{e_0+\dots+e_l-1} (jm + F_l) \right).$$

It follows from (14) and (15) that

$$\sigma(i) + jm \in G \Leftrightarrow j < e_0 + \dots + e_{i-1}$$

for all  $j \geq 0$ . This proves (13). Taking the complement in  $\mathbb{N}$ , it follows that

$$\sigma(i) + jm \in \mathbb{N} \setminus G \Leftrightarrow j \geq e_0 + \cdots + e_{i-1}.$$

This proves (12).  $\square$

**Notation 4.12.** Given  $\sigma \in \mathfrak{S}_{m-1}$  and  $e = (e_0, \dots, e_{m-2}) \in \mathbb{N}^{m-1}$  with  $e_0 \geq 1$ , we denote

$$S(\sigma, e) = \bigsqcup_{i=0}^{m-1} \sigma(i) + m(e_0 + \cdots + e_{i-1} + \mathbb{N}).$$

Thus, the above proposition amounts to the statement

$$\mathbb{N} = G(\sigma, e) \sqcup S(\sigma, e)$$

for all  $\sigma \in \mathfrak{S}_{m-1}$  and  $e = (e_0, \dots, e_{m-2}) \in \mathbb{N}^{m-1}$  with  $e_0 \geq 1$ .

This yields the following way to construct all gapsets of given multiplicity  $m \geq 3$ .

**Proposition 4.13.** *Let  $m \geq 2$ . Every numerical semigroup  $S$  of multiplicity  $m$  is of the form  $S = S(\sigma, e)$  for some  $\sigma \in \mathfrak{S}_{m-1}$  and  $e = (e_0, \dots, e_{m-2}) \in \mathbb{N}^m$  with  $e_0 \geq 1$ .*

*Proof.* Let  $S$  be a numerical semigroup of multiplicity  $m$ . Let  $G = \mathbb{N} \setminus S$  and  $F = \varphi(G)$  be the associated gapset and gapset filtration, respectively. Then  $F$  is an  $m$ -filtration, whence by Proposition 4.7, it is of the form  $F = F(\sigma, e)$  for some  $\sigma$  and  $e$  of the desired type. Then  $G = \tau(F) = G(\sigma, e)$ , whence  $S = \mathbb{N} \setminus G = S(\sigma, e)$ .  $\square$

We now determine the conditions under which a set of the form  $S(\sigma, e)$  is a numerical semigroup.

**Theorem 4.14.** *Let  $m \geq 3$ . Let  $\sigma \in \mathfrak{S}_{m-1}$  and  $e = (e_0, \dots, e_{m-2}) \in \mathbb{N}^m$  with  $e_0 \geq 1$ . Then  $S(\sigma, e)$  is a numerical semigroup if and only if for all  $1 \leq i, j, k \leq m-1$  with  $i \leq j < k$ , we have*

$$e_j + \cdots + e_{k-1} \leq \begin{cases} e_0 + \cdots + e_{i-1} & \text{if } \sigma(i) + \sigma(j) = \sigma(k), \\ e_0 + \cdots + e_{i-1} + 1 & \text{if } \sigma(i) + \sigma(j) = \sigma(k) + m. \end{cases}$$

*Proof.* Denote  $S_0 = \mathbb{N}$  and  $S_i = \sigma(i) + m(e_0 + \cdots + e_{i-1} + \mathbb{N})$  for  $1 \leq i \leq m-1$ . Let  $S' = S(\sigma, e)$ . Then

$$S' = \bigsqcup_{i=0}^{m-1} S_i$$

by definition. We have  $0 \in S_0 \subset S'$ . The complement of  $S'$  in  $\mathbb{N}$  is finite, since  $\mathbb{N} \setminus S(\sigma, e) = G(\sigma, e)$ . It remains to prove that  $S'$  is stable under addition if and only if the stated inequalities are satisfied.

Let  $i, j$  be integers such that  $0 \leq i \leq j \leq m - 1$ . If  $i = 0$  then  $S_i + S_j = S_j + m\mathbb{N} = S_j$ . We now assume  $i \neq 0$ . There are three cases.

• **Case**  $\sigma(i) + \sigma(j) \leq m - 1$ . There exists  $k \in [1, m - 1]$  satisfying  $\sigma(k) = \sigma(i) + \sigma(j)$ . Then

$$\begin{aligned} S_i + S_j &= \sigma(i) + m(e_0 + \cdots + e_{i-1} + \mathbb{N}) + \sigma(j) + m(e_0 + \cdots + e_{j-1} + \mathbb{N}) \\ &= \sigma(k) + m(e_0 + \cdots + e_{i-1} + e_0 + \cdots + e_{j-1} + \mathbb{N}). \end{aligned}$$

Therefore  $S_i + S_j$  is contained in  $S'$  if and only if it is contained in  $S_k$ , and this occurs if and only if

$$e_0 + \cdots + e_{k-1} \leq e_0 + \cdots + e_{i-1} + e_0 + \cdots + e_{j-1}.$$

This condition is plainly satisfied if  $k < j$ , and is equivalent to

$$e_j + \cdots + e_{k-1} \leq e_0 + \cdots + e_{i-1}$$

if  $k > j$ .

• **Case**  $\sigma(i) + \sigma(j) \geq m + 1$ . There exists  $k \in [1, m - 1]$  satisfying  $\sigma(k) + m = \sigma(i) + \sigma(j)$ . Then

$$\begin{aligned} S_i + S_j &= \sigma(i) + m(e_0 + \cdots + e_{i-1} + \mathbb{N}) + \sigma(j) + m(e_0 + \cdots + e_{j-1} + \mathbb{N}) \\ &= \sigma(k) + m(e_0 + \cdots + e_{i-1} + e_0 + \cdots + e_{j-1} + 1 + \mathbb{N}). \end{aligned}$$

Again,  $S_i + S_j$  is contained in  $S'$  if and only if it is contained in  $S_k$ , and this occurs if and only if

$$e_0 + \cdots + e_{k-1} \leq e_0 + \cdots + e_{i-1} + e_0 + \cdots + e_{j-1} + 1.$$

This is plainly satisfied if  $j < k$ , and is equivalent to

$$e_j + \cdots + e_{k-1} \leq e_0 + \cdots + e_{i-1} + 1$$

otherwise.

• **Case**  $\sigma(i) + \sigma(j) = m$ . Then  $S_i + S_j \subseteq m\mathbb{N} = S_0 \subset S'$ . □

**Remark 4.15.** For a gapset filtration  $F = F(\sigma, e)$  of multiplicity  $m$ , there is a strong connection between its exponent vector  $e \in \mathbb{N}^{m-1}$  and the Kunz coordinates of the associated numerical semigroup  $S(\sigma, e)$ .

Indeed, let  $S$  be a numerical semigroup of multiplicity  $m$ . Recall that the Apéry set of  $S$  is  $\text{Ap}(S) = \{x \in S \mid x - m \notin S\}$ . By Lemma 1.4 of [18], we have  $\text{Ap}(S) = \{0 = w(0), w(1), \dots, w(m-1)\}$  where  $w(i)$  is the smallest element of  $S$  which is congruent to  $i$  modulo  $m$ . Hence for  $i \in [0, m-1]$  there exist  $k_i \in \mathbb{N}$  such that  $w(i) = i + mk_i$ . The integers  $k_1, \dots, k_{m-1}$  are the *Kunz coordinates* of  $S$ . From (12), we obtain that the smallest element of  $S(\sigma, e)$  which is congruent to  $\sigma(i)$  modulo  $m$  is  $\sigma(i) + m(e_0 + \cdots + e_{i-1})$ . Hence for all  $i \in [1, m-1]$ , we have

$$k_{\sigma(i)} = e_0 + \cdots + e_{i-1}.$$



4.5. **The insertions maps  $f_i$ .** Let  $m \geq 3$  and let  $F = (F_0, \dots, F_t)$  be an  $m$ -filtration, i.e. with

$$[1, m-1] = F_0 \supseteq F_1 \supseteq \dots \supseteq F_t.$$

Let  $g = \sum_{j=0}^t |F_j|$  be the genus of  $F$ . Given  $i \in [1, m-1]$ , we wish to insert  $i$  in  $F$  so as to end up with an  $m$ -filtration of genus  $g+1$ . There is only one way to do this, namely to insert  $i$  in the first  $F_j$  for which  $i \notin F_j$ . More formally, we define  $f_i(F)$  as follows:

- If  $i \in F_s \setminus F_{s+1}$  for some  $s \leq t-1$ , then  $f_i(F) = (F'_0, \dots, F'_t)$  where

$$F'_j = \begin{cases} F_j & \text{if } j \neq s+1, \\ F_{s+1} \sqcup \{i\} & \text{if } j = s+1. \end{cases}$$

- If  $i \in F_t$ , then  $f_i(F) = (F_0, \dots, F_t, F_{t+1})$  where  $F_{t+1} = \{i\}$ .

By construction, for all  $i \in [1, m-1]$ , we have that  $f_i(F)$  is an  $m$ -filtration of genus  $g+1$ .

One delicate question is the following. If  $F$  is a *gapset* filtration of multiplicity  $m$ , for which  $i \in [1, m-1]$  does it hold that  $f_i(F)$  remains a gapset filtration? This question was successfully addressed in Section 3 for  $m=3$ .

## 5. THE CASE $m=4$

We now use the above tools to characterize all gapset filtrations of multiplicity  $m=4$  and to derive a counting-free proof of the inequality  $n_{g+1,4} \geq n_{g,4}$  for all  $g \geq 0$ .

Let  $F$  be a gapset filtration of multiplicity  $m=4$ . By Proposition 4.7, there exists  $\sigma \in \mathfrak{S}_3$  and  $e = (a, b, c) \in \mathbb{N}^3$  with  $a \geq 1$  such that  $F = F(\sigma, e)$ . Moreover, Theorem 4.14 gives the exact conditions for  $S(\sigma, e)$  to be a numerical semigroup, i.e. for  $F(\sigma, e)$  to be a gapset filtration. This yields the following characterization, where the six elements of  $\mathfrak{S}_3$  are displayed in window notation.

**Theorem 5.1.** *The gapset filtrations of multiplicity  $m=4$  are exactly the filtrations  $F = F(\sigma, e)$  given in the table below, with  $\sigma \in \mathfrak{S}_3$  and  $e = (a, b, c) \in \mathbb{N}^3$  such that  $a \geq 1$  and subject to the conditions below:*

$\sigma \in \mathfrak{S}_3$	$F = F(\sigma, e)$	conditions on $a, b, c$
(1, 2, 3)	$(123)^a(23)^b(3)^c$	$b \leq a, c \leq a$
(1, 3, 2)	$(123)^a(23)^b(2)^c$	$b + c \leq a$
(2, 1, 3)	$(123)^a(13)^b(3)^c$	$c \leq a$
(2, 3, 1)	$(123)^a(13)^b(1)^c$	$c \leq a + 1$
(3, 1, 2)	$(123)^a(12)^b(2)^c$	$b + c \leq a + 1, c \leq a + b$
(3, 2, 1)	$(123)^a(12)^b(1)^c$	$b \leq a + 1, c \leq a + 1$

*Proof.* Consider for instance the case  $\sigma = (1, 3, 2)$ . We have  $\sigma(1) + \sigma(1) = \sigma(3)$  and  $\sigma(2) + \sigma(2) = \sigma(3) + m$ . Hence, by Theorem 4.14, the conditions on  $e = (a, b, c)$  for  $S = S(\sigma, e)$  to be a numerical semigroup, i.e. for  $F = F(\sigma, e)$  to be a gapset filtration, are exactly  $b + c \leq a$  and  $c \leq a + b + 1$ . Since the latter condition is implied by the former one, it may be ignored. We end up with the sole condition  $b + c \leq a$ , as stated in the table. The proof in the five other cases is again a straightforward application of Theorem 4.14 and is left to the reader.  $\square$

**Corollary 5.2.** *For all  $g \geq 0$ , there is an explicit injection*

$$\mathcal{F}(g, 4) \longrightarrow \mathcal{F}(g + 1, 4).$$

*In particular, we have  $n_{g+1,4} \geq n_{g,4}$  for all  $g \geq 0$ .*

*Proof.* The statement is trivial for  $g \leq 2$  since  $\mathcal{F}(g, 4) = \emptyset$  in this case. Assume now  $g \geq 3$ . Let  $F \in \mathcal{F}(g, 4)$  be a gapset filtration of genus  $g$ . Write  $F = F(\sigma, e)$  for some  $\sigma \in \mathfrak{S}_3$  and  $e = (a, b, c) \in \mathbb{N}^3$  with  $a \geq 1$ . For  $i = 1, 3$ , consider the 4-filtrations  $F' = f_1(F)$  and  $F'' = f_3(F)$  of genus  $g + 1$  obtained by the insertion maps  $f_1$  and  $f_3$ , respectively. Then  $F' = F(\sigma, e')$  where

$$e' = \begin{cases} (a + 1, b - 1, c) & \text{if } \sigma \in \{(1, 2, 3), (1, 3, 2)\}, \\ (a, b + 1, c - 1) & \text{if } \sigma \in \{(2, 1, 3), (3, 1, 2)\}, \\ (a, b, c + 1) & \text{if } \sigma \in \{(2, 3, 1), (3, 2, 1)\}. \end{cases}$$

It follows from (16) that  $F'$  fails to be a gapset filtration, i.e.  $F' \notin \mathcal{F}(g + 1, 4)$ , if and only if  $\sigma = (2, 3, 1)$  or  $(3, 2, 1)$  and  $e = (a, b, a + 1)$ . This corresponds to  $F$  being one of

$$(123)^a(13)^b(1)^{a+1} \text{ or } (123)^a(23)^b(1)^{a+1}.$$

Here  $g = 3a + 2b + a + 1 = 4a + 2b + 1$ , whence  $g$  is odd. In particular, if  $g \not\equiv 1 \pmod{2}$ , then  $F'$  is always a gapset filtration. We conclude that, whenever  $g$  is *even*, then  $f_1$  yields a well-defined injection

$$\mathcal{F}(g, 4) \longrightarrow \mathcal{F}(g + 1, 4).$$

Let us now turn to  $F'' = f_3(F)$ . Then  $F'' = F(\sigma, e'')$  where  $e''$  is easily described by a table similar to (16). Omitting details, it follows that  $F''$  fails to be a gapset filtration if and only if  $F$  is one of

$$(123)^a(13)^b(3)^a \text{ or } (123)^a(23)^b(3)^a.$$

In this case we have  $g = 3a + 2b + a = 4a + 2b$ , which is even. We conclude that whenever  $g$  is *odd*, then  $f_3$  yields a well-defined injection

$$\mathcal{F}(g, 4) \longrightarrow \mathcal{F}(g + 1, 4).$$

This concludes the proof of the corollary.  $\square$

**5.1. Concluding remark.** We have shown that for  $m = 3$  and  $4$ , an injection  $\mathcal{F}(g, m) \rightarrow \mathcal{F}(g + 1, m)$  is provided by one of the insertion maps  $f_i$ , where  $i \in [1, m - 1]$  depends on the class of  $g$  modulo  $3$  and  $2$ , respectively.

Unfortunately, for any given  $m \geq 5$ , this is no longer true in general. That is, one should not expect that for each  $g \geq 1$ , an injection  $\mathcal{F}(g, m) \rightarrow \mathcal{F}(g + 1, m)$  will be provided by just one of the insertion maps  $f_i$ . Constructing such injections for all  $m, g$  remains open at the time of writing.

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