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ABSTRACT. We consider *a linear phonon Boltzmann equation* with a reflecting/transmitting/absorbing interface. This equation appears as the Boltzmann-Grad limit for the energy density function of a harmonic chain of oscillators with inter-particle stochastic scattering in the presence of a heat bath at temperature T in contact with one oscillator at the origin. We prove that under the diffusive scaling the solutions of the phonon equation tend to the solution $\rho(t,y)$ of a heat equation with the boundary condition $\rho(t,0)\equiv T$.

1. INTRODUCTION

We consider *a linear phonon Boltzmann equation* in contact with a heat bath at the origin. This equation describes the evolution, after a proper kinetic limit, of the phonon energy in a chain of harmonic oscillators with random scattering of velocities, where one oscillator is in contact with a heat bath at temperature T.

We denoting by $W(t,y,k)$ the energy density of phonons at time $t\geq 0$, with respect to their position $y\in\mathbb{R}$ and frequency variable $k\in\mathbb{T}$ - the one dimensional circle, understood as the interval $[-1/2,1/2]$ with identified endpoints. The heat bath creates an interface localized at $y=0$. Outside the interface the density satisfies

$$\partial_t W(t,y,k)+\bar{\omega}'(k)\partial_y W(t,y,k)=\gamma_0 LW(t,y,k),\qquad (t,y,k)\in\mathbb{R}_+\times\mathbb{R}_*\times\mathbb{T}_*, \tag{1.1}$$

Here

$$\mathbb{R}_+:=(0,+\infty),\quad \mathbb{R}_*:=\mathbb{R}\setminus\{0\},\quad \mathbb{T}_*:=\mathbb{T}\setminus\{0\},$$

The parameter $\gamma_0>0$ represents the phonon scattering rate. The scattering operator L, acting only on the k-variable, is given by

$$LF(k):=\int_{\mathbb{T}} R(k,k')\left[F\left(k'\right)-F\left(k\right)\right]dk',\quad k\in\mathbb{T}$$

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for F belongs to $B_b(\mathbb{T})$ - the set of bounded measurable, real valued functions. Here

$$\bar{\omega}'(k) = \frac{\omega'(k)}{2\pi}, \quad k \in \mathbb{T}, \tag{1.2}$$

where $\omega : \mathbb{T} \to [0, +\infty)$ is *the dispersion relation* of the harmonic chain.

The interface conditions that describe the interaction of a phonon with a thermostat (placed at $y = 0$), at temperature $T > 0$, are given as follows:

- the outgoing densities are given in terms of the incoming ones as

$$\begin{aligned} W(t, 0^+, k) &= p_-(k)W(t, 0^+, -k) + p_+(k)W(t, 0^-, k) + T\mathfrak{g}(k), \quad \text{for } 0 < k \leq 1/2, \\ W(t, 0^-, k) &= p_-(k)W(t, 0^-, -k) + p_+(k)W(t, 0^+, k) + T\mathfrak{g}(k), \quad \text{for } -1/2 < k < 0. \end{aligned} \tag{1.3}$$

where $p_-, p_+, \mathfrak{g} : \mathbb{T} \to (0, 1]$ are continuous and

$$p_+(k) + p_-(k) + \mathfrak{g}(k) = 1. \tag{1.4}$$

In other words, $p_-(k)$ and $p_+(k)$ are the reflection and transmission coefficients across the interface, respectively. They correspond to the probabilities of the phonon being reflected, or transmitted, by the interface. The quantity $T\mathfrak{g}(k)$ is the phonon production rate by the thermostat as well as the absorption rate of the frequency k phonon by the interface. The parameter $T > 0$ is the heat bath temperature. This equation has been obtained in [7], without the heat bath, as the Boltzmann-Grad limit of the energy density function for a microscopic model of a heat conductor consisting of a one dimensional chain of harmonic oscillators, with inter-particle scattering conserving the energy and volume. In the presence of the thermostat, but with no scattering (the case $\gamma_0 = 0$), the limit has been proved [13]. It is believed that the limit also holds in case of the presence of scattering, i.e. when $\gamma_0 > 0$.

We are interested in the asymptotics of the solutions to (1.1) under the diffusive scaling, i.e. the limit, as $\epsilon \to 0$, for $W^\varepsilon(t, y, k) = W(t/\varepsilon^2, y/\varepsilon, k)$, i.e. we consider the equation

$$\begin{aligned} &\partial_t W^\varepsilon(t, y, k) + \frac{1}{\varepsilon}\,\bar{\omega}'(k)\partial_y W^\varepsilon(t, y, k) = \frac{\gamma}{\varepsilon^2}\int_{\mathbb{T}} R(k, k')\left[W^\varepsilon\left(t, y, k'\right) - W^\varepsilon\left(t, y, k\right)\right]\, dk', \qquad y \neq 0, \\ &W^\varepsilon(0, y, k) = W_0(y, k), \end{aligned} \tag{1.5}$$

with the interface conditions (1.3). Let $R(k) = \int R(k, k')dk'$. In our main result, see Theorem 2.2 below, we prove that under the assumption

$$\int_{\mathbb{T}} \frac{\omega'(k)^2}{R(k)} dk < +\infty, \tag{1.6}$$

and some other technical hypotheses, formulated in Sections 2.2 and 2.3 below, for any $G \in C_0^\infty(\mathbb{R}\times\mathbb{T})$ -compactly supported, smooth function - we have

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}\times\mathbb{T}} W^\varepsilon(t,y,k)G(y,k)dydk = \int_{\mathbb{R}\times\mathbb{T}}\rho(t,y)G(y,k)dydk, \tag{1.7}$$

where

$$\partial_t\rho(t,y) = D\partial_y^2\rho(t,y), \quad (t,y)\in\mathbb{R}_+\times\mathbb{R}_*, \tag{1.8}$$

$$\rho(t,0^\pm)\equiv T \tag{1.9}$$

$$\rho(0,y) = \rho_0(y) := \int_{\mathbb{T}} W_0(y,k)dk.$$

The diffusion coefficient is given by

$$D = \frac{1}{\gamma}\int \omega'(k)(-L)^{-1}\omega'(k)\,dk \tag{1.10}$$

that is finite by the assumption (1.6) and the properties of $R(k,k')$ made in section 2.2.

The result implies that only the absorption and the creation of phonons at the interface matter in this diffusive scale. Phonons that are reflected or transmitted will come back to the interface, due to scattering, and eventually get absorbed in a shorter time scale.

The diffusive limit has been considered, without the presence of interface, in [5, 11, 15]. It has been shown there that, if (1.6) is in force, then the solutions of the initial problem (1.5) converge, as in (1.7), to $\rho(t,y)$ - the solution of the Cauchy problem for the heat equation (1.8). When the condition (1.6) is violated a superdiffusive scaling may be required and the limit could be a fractional diffusion. This case has been also considered in [6, 8, 11, 15].

The case of the diffusive limit of the solution of a kinetic equation with an absorbing boundary has been considered in e.g. [14, 9, 10, 4, 3]. The diffusive limit with some other boundary conditions has also been discussed in the review paper [2], see the references contained therein. A related result for the radiative transport equation with some reflection/transmission condition has been obtained in [1] for the steady state, giving rise to different boundary conditions. We are not aware of a similar result in the dynamical case, as considered in the present paper.

The result for a fractional diffusive limit with the interface condition is a subject of the paper [12].

2. Preliminaries and the statement of the main result

2.1. **Weak solution of the kinetic equation with an interface.** In what follows we denote $\mathbb{R}_- := (-\infty, 0)$. Consider $\widetilde{W}(t,y,k) := W(t,y,k) - T$. It satisfies the equation (1.1) with the

interface given by

$$\widetilde{W}(t,0^+,k) = p_-(k)\widetilde{W}(t,0^+,-k) + p_+(k)\widetilde{W}(t,0^-,k)), \text{ for } 0 < k \le 1/2, \tag{2.1}$$

and

$$\widetilde{W}(t,0^-,k) = p_-(k)\widetilde{W}(t,0^-,-k) + p_+(k)\widetilde{W}(t,0^+,k), \quad \text{for } -1/2 < k < 0. \tag{2.2}$$

Definition 2.1. *A function $\widetilde{W} : \bar{\mathbb{R}}_+ \times \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ is called a (weak) solution to equation (1.1) with the interface conditions (2.1) and (2.2), provided that it belongs to $L^2_{\mathrm{loc}}(\mathbb{R}_+, L^2(\mathbb{R}\times\mathbb{T}))$, its restrictions $\widetilde{W}_\iota$ to $\mathbb{R}_+ \times \mathbb{R}_\iota \times \mathbb{T}$, $\iota \in \{-,+\}$ extend to continuous functions on $\mathbb{R}_+ \times \bar{\mathbb{R}}_\iota \times \mathbb{T}$ that satisfy (2.1) and (2.2), and*

$$\begin{aligned} 0 = &\int_0^{+\infty} \int_{\mathbb{R}\times\mathbb{T}} \widetilde{W}(t,y,k) \left[\partial_t\varphi(t,y,k) + \bar{\omega}'(k)\partial_y\varphi(t,y,k) + \gamma L\varphi(t,y,k)\right] dtdydk \\ &+ \int_{\mathbb{R}\times\mathbb{T}} W_0(y,k)\varphi(0,y,k)dydk \end{aligned} \tag{2.3}$$

for any test function $\varphi \in C_0^\infty(\bar{\mathbb{R}}_+ \times \mathbb{R}_* \times \mathbb{T})$,

2.2. **Assumption about the dispersion relation and the scattering kernel.** We assume that $\omega(\cdot)$ is even, belongs to $C^\infty(\mathbb{T}\setminus\{0\})$, i.e. it smooth outside $k = 0$. Furthermore we assume that $\omega(\cdot)$ is unimodal, that implies that $k\omega'(k) \ge 0$ for $k \in (-1/2, 1/2)$.

We assume that the scattering kernel is symmetric

$$R(k,k') = R(k',k), \tag{2.4}$$

positive, except for 0 frequency, i.e.

$$R(k,k') > 0, \quad (k,k') \in \mathbb{T}^2_* \tag{2.5}$$

and the total scattering kernel

$$R(k) := \int_{\mathbb{T}} R(k,k')dk' \tag{2.6}$$

is such that the stochastic kernel

$$p(k,k') := \frac{R(k,k')}{R(k)} \in C^\infty(\mathbb{T}^2). \tag{2.7}$$

In addition we assume that (1.6) is in force.

Example. Suppose that

$$R(k) \sim R_0|\sin(\pi k)|^\beta, \quad |k| \ll 1$$

for some $\beta \ge 0$ and $R_0 > 0$ and

$$\omega'(k) \sim 2\omega_0' \operatorname{sign}(k)\,|\sin(\pi k)|^\kappa, \quad |k| \ll 1 \tag{2.8}$$

for some $\kappa \geq 0$. Then (1.6) holds, provided that

$$0 \leq \beta < 1 + 2\kappa. \tag{2.9}$$

2.3. **About the reflection, transmission and absorption coefficients.** In [13] the coefficient $p_\pm(k)$ and $\mathfrak{g}(k)$ are obtained from the microscopic dynamics and depends on the dispersion relation as follows.

Let $\gamma > 0$ (the thermostat strength) and

$$\tilde{g}(\lambda) := \left(1 + \gamma \int_{\mathbb{T}} \frac{\lambda dk}{\lambda^2 + \omega^2(k)}\right)^{-1}, \quad \operatorname{Re}\lambda > 0. \tag{2.10}$$

It turns out, see [13], that

$$|\tilde{g}(\lambda)| \leq 1, \quad \lambda \in \mathbb{C}_+ := [\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0]. \tag{2.11}$$

The function $\tilde{g}(\cdot)$ is analytic on $\mathbb{C}_+$. By Fatou's theorem we know that

$$\nu(k) := \lim_{\epsilon \to 0+} \tilde{g}(\epsilon - i\omega(k)) \tag{2.12}$$

exists a.e. in $\mathbb{T}$ and in any $L^p(\mathbb{T})$ sense for $p \in [1, \infty)$. Denote

$$\begin{aligned} \mathfrak{g}(k) &:= \frac{\gamma|\nu(k)|^2}{|\bar{\omega}'(k)|^2}, \qquad \mathfrak{P}(k) := \frac{\gamma\nu(k)}{2|\bar{\omega}'(k)|} \\ p_+(k) &:= |1 - \mathfrak{P}(k)|^2 \\ p_-(k) &:= |\mathfrak{P}(k)|^2 . \end{aligned} \tag{2.13}$$

It has been shown in Section 10 of [13] that

$$\operatorname{Re}\nu(k) = \left(1 + \frac{\gamma}{2|\bar{\omega}'(k)|}\right)|\nu(k)|^2. \tag{2.14}$$

This identity implies in particular that (1.4) is in force.

2.4. **Scaled kinetic equations and the formulation of the main result.** Consider W^ε the solution of a rescaled kinetic equations (1.5). Our main result can be stated as follows.

Theorem 2.2. *Suppose that $W_0(y,k) = T + \widetilde{W}_0(y,k)$, where $\widetilde{W}_0 \in L^2(\mathbb{R}\times\mathbb{T})$. Under the assumptions made about the scattering kernel $R(\cdot,\cdot)$ and dispersion relation $\omega(\cdot)$, for any test function $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{T})$ we have*

$$\lim_{\epsilon \to 0} \int_0^{+\infty} dt \int_{\mathbb{R}\times\mathbb{T}} W^\varepsilon(t,y,k)\varphi(t,y,k)dydk = \int_0^{+\infty} dt \int_{\mathbb{R}\times\mathbb{T}} \rho(t,y)\varphi(t,y,k)dydk, \tag{2.15}$$

where $\rho(\cdot,\cdot)$ is the solution of the heat equation

$$\partial_t \rho(t,y) = D\partial_y^2 \rho(t,y), \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}_*,$$

(2.16)

$$\rho(0,y) = \rho_0(y) := \int_{\mathbb{T}} W_0(y,k)dk, \quad \rho(t,0) \equiv T,\ t>0.$$

Here, the coefficient $D>0$ is given by (4.6) *below.*

Defining $\widetilde{W^\varepsilon} = W^\varepsilon - T$, one can easily see that it also satisfies (1.5) with

$$\widetilde{W^\varepsilon}(t,0^+,k) = p_-(k)\widetilde{W^\varepsilon}(t,0^+,-k) + p_+(k)\widetilde{W^\varepsilon}(t,0^-,k), \text{ for } 0 \le k \le 1/2, \tag{2.17}$$

and

$$\widetilde{W^\varepsilon}(t,0^-,k) = p_-(k)\widetilde{W^\varepsilon}(t,0^-,-k) + p_+(k)\widetilde{W^\varepsilon}(t,0^+,k), \quad \text{for } -1/2 \le k \le 0. \tag{2.18}$$

The initial condition $\widetilde{W^\varepsilon}(0,y,k) = \widetilde{W}_0(y,k) := W_0(y,k) - T$ belongs to $L^2(\mathbb{R}\times\mathbb{T})$. This means that it is enough to prove Theorem 2.2 for $T=0$. This proof is presented in Section 4.

3. Some auxiliaries

3.1. **Some functional spaces.** Let H^1_+ be the Hilbert obtained as the completion of the Schwartz class $\mathcal{S}(\mathbb{R}\times\mathbb{T})$ in the norm

$$\|\varphi\|^2_{H^1_+} := \|\varphi\|^2_{L^2(\mathbb{R}\times\mathbb{T})} + \int_0^{+\infty}\int_{\mathbb{T}} |\omega'(k)|[\partial_y\varphi(y,k)]^2 dydk$$

Similarly, we introduce H^1_-.

Let $\mathcal{H}$ be the Hilbert space obtained as the completion of $\mathcal{S}(\mathbb{R}\times\mathbb{T})$ in the norm

$$\|\varphi\|^2_{\mathcal{H}} := \|\varphi\|^2_{H^1_-} + \|\varphi\|^2_{H^1_+}. \tag{3.1}$$

Let also

$$\|\varphi\|^2_{\mathcal{H}_0} := \int_{\mathbb{T}} |\omega'(k)|\mathfrak{g}(k)\left\{[\varphi(0^+,k)]^2 + [\varphi(0^-,k)]^2\right\}dk + \int_{\mathbb{R}\times\mathbb{T}^2}[\varphi(y,k') - \varphi(y,k)]^2 dydkdk'$$

3.2. **Apriori bounds.** Computing the time derivative we have

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{W^\varepsilon}(t)\|^2_{L^2} = -\frac{\gamma}{2\varepsilon^2}\int_{-\infty}^{\infty} dy\mathcal{D}(\widetilde{W^\varepsilon}(t,y,\cdot)) - \frac{1}{2\varepsilon}\int_{\mathbb{T}} \bar{\omega}'(k)\left[\widetilde{W^\varepsilon}(t,0^-,k)^2 - \widetilde{W^\varepsilon}(t,0^+,k)^2\right]dk, \tag{3.2}$$

with

$$\mathcal{D}(f) := \int_{\mathbb{T}^2} R(k,k')[f(k) - f(k')]^2 dkdk'. \tag{3.3}$$

Taking into account (2.17) and (2.18) we obtain

$$\begin{aligned}
&\int_{\mathbb{T}} \bar\omega'(k)\left\{[\widetilde{W^\varepsilon}(t,0^-,k)]^2-[\widetilde{W^\varepsilon}(t,0^+,k)]^2\right\}dk\\
&=\int_0^{1/2}\bar\omega'(k)\left\{[\widetilde{W^\varepsilon}(t,0^-,k)]^2-\left[p_-(k)\widetilde{W^\varepsilon}(t,0^+,-k)+p_+(k)\widetilde{W^\varepsilon}(t,0^-,k)\right]^2\right\}dk\\
&+\int_{-1/2}^0\bar\omega'(k)\left\{\left[p_-(k)\widetilde{W^\varepsilon}(t,0^-,-k)+p_+(k)\widetilde{W^\varepsilon}(t,0^+,k)\right]^2-[\widetilde{W^\varepsilon}(t,0^+,k)]^2\right\}dk.
\end{aligned}$$

After straightforward calculations (recall that coefficients $p_\pm(k)$ are even, while $\bar\omega'(k)$ is odd) we conclude that the right hand side equals

$$\begin{aligned}
\int_0^{1/2}\bar\omega'(k)\Big\{&\left(\widetilde{W^\varepsilon}(t,0^-,k)^2+\widetilde{W^\varepsilon}(t,0^+,-k)^2\right)\left(1-p_+^2(k)-p_-^2(k)\right)\\
&-4p_-(k)p_+(k)\widetilde{W^\varepsilon}(t,0^+,-k)\widetilde{W^\varepsilon}(t,0^-,k)\Big\}dk.
\end{aligned}$$

Since $p_+(k)+p_-(k)\le 1$ we have $1-p_+^2(k)-p_-^2(k)\ge 0$. In addition,

$$\det\begin{bmatrix} 1-p_+^2(k)-p_-^2(k) & -2p_-(k)p_+(k)\\ -2p_-(k)p_+(k) & 1-p_+^2(k)-p_-^2(k)\end{bmatrix}=\left[1-(p_+(k)+p_-(k))^2\right]\left[1-(p_+(k)-p_-(k))^2\right].$$

Using (1.4) we conclude that the quadratic form

$$(x,y)\mapsto\left(1-p_+^2(k)-p_-^2(k)\right)(x^2+y^2)-4p_-(k)p_+(k)xy$$

is positive definite as long as $p_+(k)+p_-(k)<1$. The eigenvalues of the form can be determined from the equation

$$0=\left[1-\lambda-(p_+(k)+p_-(k))^2\right]\left[1-\lambda-(p_+(k)-p_-(k))^2\right],$$

which yields

$$\lambda_+:=1-(p_+(k)-p_-(k))^2,\quad \lambda_-:=1-(p_+(k)+p_-(k))^2$$

and $\lambda_+>\lambda_-$. Note that

$$2\mathfrak{g}(k)\ge\lambda_-=\mathfrak{g}(k)\left[1+p_+(k)+p_-(k)\right]\ge\mathfrak{g}(k).$$

Equality (3.2) allows us to obtain the following apriori bounds

$$\begin{aligned}
&\|\widetilde{W^\varepsilon}(t)\|^2_{L^2(\mathbb{R}\times\mathbb{T})}\le\|\widetilde{W}_0\|^2_{L^2(\mathbb{R}\times\mathbb{T})},\\
&\int_0^t ds\int_{\mathbb{R}}\mathcal{D}(\widetilde{W^\varepsilon}(s,y,\cdot))dy\le\frac{\varepsilon^2}{\gamma}\|\widetilde{W}_0\|^2_{L^2(\mathbb{R}\times\mathbb{T})},\\
&\int_0^t ds\int_0^{1/2}\bar\omega'(k)\mathfrak{g}(k)dk\ \left(\widetilde{W^\varepsilon}(s,0^-,k)^2+\widetilde{W^\varepsilon}(s,0^+,-k)^2\right)\le\varepsilon\|\widetilde{W}_0\|^2_{L^2(\mathbb{R}\times\mathbb{T})}.
\end{aligned}\tag{3.4}$$

By (2.17) and (2.18) we obtain that

$$\begin{aligned}\widetilde{W^\varepsilon}(s,0^+,k)^2 &\le \widetilde{W^\varepsilon}(s,0^-,k)^2+\widetilde{W^\varepsilon}(s,0^+,-k)^2, \quad k\in(0,1/2)\\ \widetilde{W^\varepsilon}(s,0^-,k)^2 &\le \widetilde{W^\varepsilon}(s,0^-,-k)^2+\widetilde{W^\varepsilon}(s,0^+,k)^2, \quad k\in(-1/2,0).\end{aligned} \tag{3.5}$$

Then using the unimodality of $\omega(k)$ it follows that

$$\int_0^t ds\int_{\mathbb{T}} dk|\omega'(k)|\mathfrak{g}(k)\ \left(\widetilde{W^\varepsilon}(s,0^-,k)^2+\widetilde{W^\varepsilon}(s,0^+,k)^2\right)\le 2\varepsilon\|\widetilde{W}_0\|^2_{L^2(\mathbb{R}\times\mathbb{T})}. \tag{3.6}$$

3.3. **Uniform continuity at $y=0$.** Suppose that $y>0$. Let

$$V_\varepsilon(t,y,k):=\int_0^t \widetilde{W^\varepsilon}(s,y,k)ds. \tag{3.7}$$

Since $\widetilde{W^\varepsilon}(s,y,k)$ satisfies (1.5) we can write

$$\varepsilon\left[\widetilde{W^\varepsilon}(t,y,k)-\widetilde{W^\varepsilon}(0,y,k)\right]+\ \bar\omega'(k)\partial_y V_\varepsilon(t,y,k)=F_\epsilon(t,y,k), \tag{3.8}$$

with

$$F_\epsilon(t,y,k):=\frac{\gamma}{\varepsilon}\int_{\mathbb{T}} R(k,k')\left[V_\varepsilon\left(t,y,k'\right)-V_\varepsilon\left(t,y,k\right)\right]\ dk',\qquad y\neq 0.$$

Hence, using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|F_\epsilon(t,\cdot)\|^2_{L^2(\mathbb{R}\times\mathbb{T})}&=\left(\frac{\gamma}{\varepsilon}\right)^2\int_{\mathbb{R}\times\mathbb{T}}dydk\left\{\int_0^t ds\int_{\mathbb{T}}R(k,k')\left[\widetilde{W^\varepsilon}(s,y,k')-\widetilde{W^\varepsilon}(s,y,k)\right]\ dk'\right\}^2\\ &\le\left(\frac{\gamma}{\varepsilon}\right)^2\int_{\mathbb{R}\times\mathbb{T}}dydk\left\{\int_0^t ds\int_{\mathbb{T}}R(k,k')dk'\right\}\left\{\int_0^t ds\int_{\mathbb{T}}R(k,k')\left[\widetilde{W^\varepsilon}(s,y,k')-\widetilde{W^\varepsilon}(s,y,k)\right]^2\ dk'\right\}\\ &\le t\|R\|_\infty\left(\frac{\gamma}{\varepsilon}\right)^2\int_0^t ds\int_{\mathbb{R}}\mathcal{D}(\widetilde{W^\varepsilon}(s,y,\cdot))dy.\end{aligned}$$

Using the second estimate of (3.4) we conclude that for each $t_0>0$

$$\sup_{\epsilon\in(0,1]}\sup_{t\in[0,t_0]}\|F_\epsilon(t,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{T})}\le\gamma t_0\|R\|_\infty\|\widetilde{W}_0\|^2_{L^2(\mathbb{R}\times\mathbb{T})}<+\infty. \tag{3.9}$$

From (3.8) we conclude that

$$\partial_y V_\varepsilon(t,y,k)=\frac{\tilde F_\epsilon(t,y,k)}{\omega'(k)},\qquad y,\bar\omega'(k)\neq 0, \tag{3.10}$$

where

$$\tilde F_\epsilon(t,y,k)=F_\epsilon(t,y,k)-\varepsilon\left[\widetilde{W^\varepsilon}(t,y,k)-\widetilde{W^\varepsilon}(0,y,k)\right].$$

From (3.9) and the first estimate of (3.4) we conclude

$$\sup_{\epsilon\in(0,1]}\sup_{t\in[0,T]}\|\tilde F_\epsilon(t,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{T})}=:\tilde F_*(T)<+\infty.$$

We have

$$\|V_\varepsilon(t,\cdot)\|_{H^1_\pm}\le\|\tilde F_\epsilon(t,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{T})}\quad t\ge 0. \tag{3.11}$$

Since $\dot V_\epsilon(t)=\widetilde{W}_\epsilon(t)$, from the first estimate of (3.4) we conclude that for any $t_0>0$

$$\sup_{\epsilon\in(0,1]}\left\|\dot V_\varepsilon\right\|_{L^\infty([0,t_0];L^2(\mathbb{R}\times\mathbb{T}))}<+\infty.$$

From (3.11) we get also (cf (3.1))

$$\sup_{\epsilon\in(0,1]}\|V_\varepsilon\|_{L^\infty([0,t_0];\mathcal{H})}<+\infty.$$

Summarizing, we have shown the following.

Proposition 3.1. *For any $t_0>0$*

$$C(t_0):=\sup_{\epsilon\in(0,1]}\left(\|V_\varepsilon\|_{L^\infty([0,t_0];\mathcal{H})}+\left\|\dot V_\varepsilon\right\|_{L^\infty([0,t_0];L^2(\mathbb{R}\times\mathbb{T}))}\right)<+\infty \tag{3.12}$$

and

$$\lim_{\varepsilon\to0+}\|V_\varepsilon\|_{L^\infty([0,t_0];\mathcal{H}_0)}=0.$$

Denote by $W_0^{1,\infty}([0,t_0];L^2(\mathbb{R}\times\mathbb{T}))$ the completion of the space of smooth functions $f:[0,t_0]\to L^2(\mathbb{R}\times\mathbb{T})$ satisfying $f(0)=0$, with respect to the norm

$$\|f\|_{W_0^{1,\infty}([0,t_0];L^2(\mathbb{R}\times\mathbb{T}))}:=\sup_{t\in[0,t_0]}\|\dot f\|_{L^2(\mathbb{R}\times\mathbb{T})}.$$

As a consequence of the above proposition we immediately conclude the following.

Corollary 3.2. *The family $(V_\varepsilon(\cdot))_{\epsilon\in(0,1]}$ is bounded in $W_0^{1,\infty}([0,t_0];L^2(\mathbb{R}\times\mathbb{T}))\cap L^\infty([0,t_0];\mathcal{H})$ for aby $t_0>0$. Any $\star$-weak limit point $V(\cdot)$ of $V_\varepsilon(\cdot)$, as $\varepsilon\to0+$, satisfies the following:*

1) $V(t,y,k)\equiv\bar V(t,y):=\displaystyle\int_{\mathbb{T}}V(t,y,k)dk$ *for* $(t,y,k)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{T}$)
2) *the mapping* $\mathbb{R}_+\times\mathbb{R}_\iota\ni(t,y)\mapsto\bar V(t,y)$ *extends to a mapping from* $C(\bar{\mathbb{R}}_+\times\bar{\mathbb{R}}_\iota)$, $\iota\in\{-,+\}$,
3) $V(t,0^\pm)=0$ *for each* $t>0$,
4) $V(0,y)\equiv0$, $y\in\mathbb{R}$.

4. Proof of Theorem 2.2

Thanks to the above estimates, we conclude that the solutions $\widetilde{W^\varepsilon}(\cdot)$ are $\star$-weakly sequentially compact in $L^\infty\left([0,+\infty);L^2_w(\mathbb{R}\times\mathbb{T})\right)$, where $L^2_w(\mathbb{R}\times\mathbb{T})$ denotes $L^2(\mathbb{R}\times\mathbb{T})$ with the weak topology. Suppose that $\bar W(t,y,k)$ is a limiting point for some subsequence $(\widetilde{W^{\varepsilon_n}}(s,y,k))$, where $\varepsilon_n\to0$. For convenience sake we shall denote the subsequence by $(\widetilde{W^\varepsilon}(s,y,k))$. Thanks to (3.4) for each $t>0$ we have (cf (3.3))

$$\lim_{\epsilon\to0}\int_0^t ds\int_{\mathbb{R}}\mathcal{D}\left(\widetilde{W^\varepsilon}(s,y,\cdot)\right)dy=0 \tag{4.1}$$

thus $\bar W(t,y,k)\equiv\rho(t,y)$, for a.e. $(t,y,k)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{T}$.

Lemma 4.1. *Equation*

$$-LX_1 = \bar{\omega}' \tag{4.2}$$

has a unique solution such that

$$\int_{\mathbb{T}} X_1(k)R(k)dk = 0$$

and

$$\int_{\mathbb{T}} X_1^2(k)R(k)dk < +\infty. \tag{4.3}$$

Proof. Let μ be a Borel probability measure on $\mathbb{T}$ given by

$$\mu(dk) = \frac{R(k)}{\bar{R}}dk,$$

where

$$\bar{R} := \int_{\mathbb{T}} R(k)dk.$$

We can reformulate (4.2) as

$$X_1 - PX_1 = \frac{\bar{\omega}'}{R}, \tag{4.4}$$

where, by virtue of (1.6), the right hand side belongs to $L^2(\mu)$ and $P : L^2(\mu) \to L^2(\mu)$ is a symmetric operator on $L^2(\mu)$ given by

$$PF(k) := \int_{\mathbb{T}} p(k,k')F(k')dk', \quad F \in L^2(\mu).$$

The operator is a compact contraction and, since

$$\int_{\mathbb{T}} F(k)(I-P)F(k)F(k)\mu(dk) = \mathcal{D}(F)$$

we conclude that 1 is a simple eigenvalue, with the respective eigenspace spanned on the eigenvector $F_0 \equiv 1$. Thus the conclusion of the lemma follows, as $\bar{\omega}'/R \perp F_0$. □

Proposition 4.2. *For any function $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}_*)$ we have*

$$\int_0^{+\infty} \int_{\mathbb{R}} \rho(t,y)\left[\partial_t\varphi(t,y) + D\partial_{yy}^2\varphi(t,y)\right]dtdy = 0. \tag{4.5}$$

with

$$0 < D := \frac{1}{\gamma}\int_{\mathbb{T}} \bar{\omega}'(k)X_1(k)dk < +\infty. \tag{4.6}$$

Proof. The claim made in (4.6) follows immediately from the fact that $X_1 \neq 0$ and $D = \mathcal{D}(\chi)$.

To prove (4.5) we apply a version of the perturbed test function technique. Let $\varphi_\varepsilon(t,y,k)$ be determined by

$$\varphi_\varepsilon(t,y,k) = \varphi(t,y) + \varepsilon\chi_1(t,y,k) + \varepsilon^2\chi_2(t,y,k), \qquad y \neq 0. \tag{4.7}$$

where $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}_*)$ and $\chi_j(t,y,k)$, $j = 1,2$ are yet to be determined.

We can write

$$\begin{aligned}
0 &= \int_0^{+\infty}\int_{\mathbb{R}\times\mathbb{T}} \partial_t\left[\widetilde{W^\varepsilon}(t,y,k)\varphi_\varepsilon(t,y,k)\right] dtdydk \\
&= \int_0^{+\infty}\int_{\mathbb{R}\times\mathbb{T}} \left[\widetilde{W^\varepsilon}(t,y,k)\partial_t\varphi_\varepsilon(t,y,k) + \partial_t\widetilde{W^\varepsilon}(t,y,k)\varphi_\varepsilon(t,y,k)\right] dtdydk \\
&= \int_0^{+\infty}\int_{\mathbb{R}\times\mathbb{T}} \left[\widetilde{W^\varepsilon}(t,y,k)\partial_t\varphi_\varepsilon(t,y,k) - \frac{1}{\varepsilon}\,\bar\omega'(k)\partial_y\widetilde{W^\varepsilon}(t,y,k)\varphi_\varepsilon(t,y,k) + \frac{\gamma}{\varepsilon^2}L\widetilde{W^\varepsilon}(t,y,k)\varphi_\varepsilon(t,y,k)\right] dtdydk.
\end{aligned} \tag{4.8}$$

Since the support in the y variable of the test function $\varphi_\epsilon(t,y,k)$ will turn out to be isolated from 0, the integration by parts yield the equation

$$0 = \int_0^{+\infty}\int_{\mathbb{R}\times\mathbb{T}} \widetilde{W^\varepsilon}(t,y,k)\left[\partial_t\varphi_\varepsilon(t,y,k) + \frac{1}{\varepsilon}\,\bar\omega'(k)\partial_y\varphi_\varepsilon(t,y,k) + \frac{\gamma}{\varepsilon^2}L\varphi_\varepsilon(t,y,k)\right] dtdydk. \tag{4.9}$$

Substituting from (4.7) we obtain that the term in the brackets has the form

$$\frac{1}{\epsilon}I + II + \epsilon III_\epsilon, \tag{4.10}$$

with

$$\begin{aligned}
I &:= \gamma L\chi_1 + \bar\omega'(k)\partial_y\varphi(t,y), \\
II &:= \gamma L\chi_2 + \bar\omega'(k)\partial_y\chi_1 + \partial_t\varphi(t,y), \\
III_\epsilon &:= \partial_t\chi_1(t,y,k) + \varepsilon\partial_t\chi_2(t,y,k) + \bar\omega'(k)\partial_y\chi_2.
\end{aligned} \tag{4.11}$$

We stipulate that

$$I = 0 \quad \text{and} \quad II = \partial_t\varphi(t,y) + D\partial^2_{yy}\varphi(t,y). \tag{4.12}$$

The first condition yields

$$\chi_1(t,y,k) = -\frac{1}{\gamma}\partial_y\varphi(t,y)L^{-1}\bar\omega'(k) = -\frac{1}{\gamma}\partial_y\varphi(t,y)X_1(k), \tag{4.13}$$

while the second implies that $\chi_2(t,y,k)$ is the solution of

$$\gamma L\chi_2(t,y,k) = D\partial^2_{yy}\varphi(t,y) - \bar\omega'\partial_y\chi_1(t,y,k). \tag{4.14}$$

Substituting from (4.13) we get

$$L\chi_2(t,y,k) = \frac{1}{\gamma}\left(D + \frac{1}{\gamma}\bar{\omega}' X_1\right)\partial^2_{yy}\varphi(t,y). \tag{4.15}$$

Since $(D + \bar{\omega}' X_1/\gamma)/R$ belongs to $L^2(\mu)$ and its orthogonal to constants, we can solve the equation

$$LX_2 = D + \frac{1}{\gamma}\bar{\omega}' X_1$$

using the same argument as in Lemma 4.1. Then,

$$\chi_2(t,y,k) = \frac{1}{\gamma}\partial^2_{yy}\varphi(t,y)X_2(k).$$

Clearly $III_\epsilon = O(1)$. Taking the limit in (4.9) we obtain (4.5), that ends the proof of the proposition.

□

Suppose that $\bar{V}(t,y)$ is a limiting point for some subsequence

$$V_{\varepsilon_n}(t,y,k) = \int_0^t \widetilde{W}_{\epsilon_n}(s,y,k)ds$$

where $\varepsilon_n \to 0$, in the sense described by Corollary 3.2. We can also assume that the respective sequence $(\widetilde{W}_\varepsilon(\cdot))$ $\star$-weakly converge to $\rho(t,y)$ in $L^\infty\left([0,+\infty); L^2_w(\mathbb{R}\times\mathbb{T})\right)$.

For convenience sake we shall denote the subsequences by $(V_\varepsilon(\cdot))$, $(\widetilde{W}_\varepsilon(\cdot))$, respectively. We have

$$\bar{V}(t,y) = \int_0^t \rho(s,y)ds, \quad \text{for all } t \geq 0, \text{ and a.e. } y. \tag{4.16}$$

Proposition 4.3. *For any function $\varphi \in C_0^\infty\left(\bar{\mathbb{R}}_+ \times \mathbb{R}_*\right)$ we have*

$$\int_0^{+\infty}\int_{\mathbb{R}} \left\{\bar{V}(t,y)\left[\varphi(t,y) + D\partial^2_{yy}\varphi(t,y)\right] - \varphi(t,y)\rho_0(y)\right\}dtdy = 0. \tag{4.17}$$

with D given by (4.6) *and ρ_0 defined in* (2.16).

Before showing the proof of the proposition we show how to finish, with its help, the proof of Theorem 2.2. According to Proposition 4.3 $\bar{V}(t,y)$ is a weak solution of the heat equation

$$\partial_t \bar{V}(t,y) = D\partial^2_{yy}\bar{V}(t,y) + \rho_0(y)$$

satisfying $\bar{V}(0,y) \equiv 0$. According to part 3) of Corollary 3.2 we also have $\bar{V}(t,0^\pm) \equiv 0$, $t \geq 0$. Hence,

$$\bar{V}(t,y) = \int_0^t \frac{1}{\sqrt{4\pi D s}}\int_{\mathbb{R}_\pm}\left\{\exp\left\{-\frac{(y-y')^2}{4Ds}\right\} - \exp\left\{-\frac{(y+y')^2}{4Ds}\right\}\right\}\rho_0(y')dy', \quad t, \pm y > 0.$$

This, combined with (4.16), implies that

$$\rho(t,y) = \frac{1}{\sqrt{4\pi D t}}\int_{\mathbb{R}_\pm}\left\{\exp\left\{-\frac{(y-y')^2}{4Dt}\right\} - \exp\left\{-\frac{(y+y')^2}{4Dt}\right\}\right\}\rho_0(y')dy', \quad t, \pm y > 0, \tag{4.18}$$

which satisfies the conclusion of Theorem 2.2 for $T = 0$. The only thing yet to be shown is the proof of Proposition 4.3.

Proof of Proposition 4.3. According to (3.8) we have

$$\partial_t V_\varepsilon(t,y,k) + \frac{1}{\varepsilon}\bar{\omega}'(k)\partial_y V_\varepsilon(t,y,k) = \frac{\gamma}{\epsilon^2} L V_\epsilon(t,y,k) + \widetilde{W^\varepsilon}(0,y,k) \tag{4.19}$$

and obviously $V_\epsilon(0,y,k) = 0$ a.e. Let $\varphi_\varepsilon(t,y,k)$ be given by (4.7). From (4.19) we can write

$$\begin{aligned}
0 &= \int_0^{+\infty}\int_{\mathbb{R}\times\mathbb{T}} \partial_t \left[V_\varepsilon(t,y,k)\varphi_\varepsilon(t,y,k)\right] dtdydk \\
&= \int_0^{+\infty}\int_{\mathbb{R}\times\mathbb{T}} \Big\{ \Big[V_\varepsilon(t,y,k)\partial_t\varphi_\varepsilon(t,y,k) - \frac{1}{\varepsilon}\,\bar{\omega}'(k)\partial_y V_\varepsilon(t,y,k)\varphi_\varepsilon(t,y,k) \\
&\quad + \frac{\gamma}{\varepsilon^2} L V_\varepsilon(t,y,k)\varphi_\varepsilon(t,y,k)\Big] + \widetilde{W^\varepsilon}(0,y,k)\varphi_\varepsilon(t,y,k)\Big\} dtdydk \\
&= \int_0^{+\infty}\int_{\mathbb{R}\times\mathbb{T}} \Big\{ V_\varepsilon(t,y,k)\Big[\partial_t\varphi_\varepsilon(t,y,k) + \frac{1}{\varepsilon}\,\bar{\omega}'(k)\partial_y\varphi_\varepsilon(t,y,k) \\
&\quad + \frac{\gamma}{\varepsilon^2} L\varphi_\varepsilon(t,y,k)\Big] + W_0(y,k)\varphi_\varepsilon(t,y,k)\Big\} dtdydk.
\end{aligned} \tag{4.20}$$

Substituting from (4.7) we obtain that the term in the square brackets has the form (4.10), with I, II and III_ϵ as given in (4.16). Taking the limit in (4.20) we obtain (4.7). □

References

[1] Bal, G., Ryzhik, L., *Diffusion approximation of radiative transfer problems with interfaces*, SIAM Journal on Applied Mathematics, Volume 60 Issue 6, May/June (2000) Pages 1887-1912

[2] C. Bardos, F. Golse, Y. Sone, *Half-Space Problems for the Boltzmann Equation: A Survey.* Journal of Statistical Physics, Vol. 124, Nos. 2-4, July/August 2006.

[3] C.Bardos, E. Bernard, F. Golse, R. Sentis. *The Diffusion Approximation for the Linear Boltzmann Equation with Vanishing Scattering Coefficient.* Communications in Math. Sciences, 2015, 13 (3), pp. 641-671.

[4] Bardos, C.; Santos, R.; Sentis, R., *Diffusion approximation and computation of the critical size*, Trans. Amer. Math. Soc. 284 (1984), no. 2, 617649.

[5] Basile, G., *From a kinetic equation to a diffusion under an anomalous scaling* Ann. Inst. H. Poincar Probab. Statist. **50**, no. 4, 1301-1322, (2014).

[6] Basile, G.; Bovier, A., *Convergence of a kinetic equation to a fractional diffusion equation*, Markov Process. Related Fields 16 (2010), no. 1, 15-44.

[7] G. Basile, S. Olla, H. Spohn, *Wigner functions and stochastically perturbed lattice dynamics*, Arch.Rat.Mech., Vol. 195, no. 1, 171-203, 2009.

[8] N. Ben Abdallah, A. Mellet and M. Puel, *Fractional diffusion limit for collisional kinetic equations: a Hilbert expansion approach.* Kinet. Relat. Models 4 (2011), no. 4, 873 - 900.

[9] Bensoussan, A., Lions, J. L., Papanicolaou, G.,*Boundary Layers and Homogenization of transport processes*, Publ. RIMS, Kyoto Univ.15 (1979), 53-157.

[10] Dautray, L., Lions, J. L., *Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 6*, Springer Verlag, Berlin (1993).

[11] Jara, M., T. Komorowski, S. Olla, *Limit theorems for additive functionals of a Markov chain*, Ann. of Appl. Prob. **19**, 2270-2300, (2009).

[12] T. Komorowski, S. Olla, L. Ryzhik, *Fractional diffusion limit for a kinetic equation with an interface - probabilistic approach*, preprint (2019).

[13] T. Komorowski, S. Olla, L. Ryzhik, H. Spohn, *High frequency limit for a chain of harmonic oscillators with a point Langevin thermostat*, 2018, arXiv:1806.02089

[14] Larsen, E. W., Keller, J. B., *Asymptotic solution of neutron transport problems for small mean free paths*, J. Math. Phys. 15, (1974), 75-81.

[15] A. Mellet, S. Mischler, C. Mouhot, *Fractional diffusion limit for collisional kinetic equations*, Archive for Rational Mechanics and Analysis, (2011), **199**, 2, 493-525.

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