Genus theory and ϵ-conjectures on p-class groups

Georges Gras

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Abstract. We suspect that the "genus part" of the class number of a number field K may be an obstruction for an "easy proof" of the classical p-rank ε-conjecture for p-class groups and, a fortiori, for a proof of the "strong ε-conjecture": $\#(\mathcal{C}\ell_K \otimes \mathbb{Z}_p) \ll_{d,p,\varepsilon} (\sqrt{D_K})^{\varepsilon}$ for all K of degree d. We analyze the weight of genus theory in this inequality by means of an infinite family of degree p cyclic fields with many ramified primes, then we prove the p-rank ε-conjecture: $\#(\mathcal{C}\ell_K \otimes \mathbb{F}_p) \ll_{d,p,\varepsilon} (\sqrt{D_K})^{\varepsilon}$, for $d = p$ and the family of degree p cyclic extensions (Theorem 2.5) then sketch the case of arbitrary base fields. The possible obstruction for the strong form, in the degree p cyclic case, is the order of magnitude of the set of "exceptional" p-classes given by a well-known non-predictible algorithm, but controled thanks to recent density results due to Koymans–Pagano. Then we compare the ε-conjectures with some p-adic conjectures, of Brauer–Siegel type, about the torsion group $\mathcal{T}_K$ of the Galois group of the maximal abelian p-ramified pro-p-extension of totally real number fields K. We give numerical computations with the corresponding PARI/GP programs.

Contents

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1. Introduction

For any number field K, we denote by $\mathcal{C}\ell_K$ the class group of K in the restricted sense and by $\mathcal{C}\ell_K \otimes \mathbb{Z}_p$ its p-class group, for any prime number p; to avoid any ambiguity, we shall write $\mathcal{C}\ell_K \otimes \mathbb{F}_p$ for the "p-torsion group", often denoted $\mathcal{C}\ell_K[p]$ in most papers, only giving the p-rank $\mathrm{rk}_p(\mathcal{C}\ell_K)$ of $\mathcal{C}\ell_K$. One knows the following classical result (weak form of theorems of Brauer, Brauer–Siegel, Tsfasman–Vladŭţ–Zykin [41, 44]):

For all $\varepsilon > 0$ there exists a constant $C_{d,\varepsilon}$ such that $\#\mathcal{C}\ell_K \le C_{d,\varepsilon} \cdot (\sqrt{D_K}\,)^{1+\varepsilon}$, where d is the degree of K and D_K the absolute value of its discriminant.

But it is clear that, arithmetically, the behaviour of the p-Sylow subgroups of $\mathcal{C}\ell_K$ depends, in conflicting manners (regarding p), on many parameters (signature, ramification, prime divisors of d, action of Galois groups, etc.).

After the Cohen–Lenstra–Martinet, Adam–Malle, Delaunay–Jouhet, Gerth, Koymans–Pagano,... heuristics, conjectures, or density statements, on the order and structure of $\mathcal{C}\ell_K \otimes \mathbb{Z}_p$ [4, 5, 1, 30, 8, 15, 28], many authors study and prove inequalities of the form $\#(\mathcal{C}\ell_K \otimes \mathbb{F}_p) \le C_{d,p,\varepsilon} \cdot (\sqrt{D_K}\,)^{c+\varepsilon}$, with positive constant $c < 1$ as small as possible (e.g., under GRH, the inequality $\#(\mathcal{C}\ell_K \otimes \mathbb{F}_p) \le C_{d,p,\varepsilon} \cdot (\sqrt{D_K}\,)^{1-\frac{1}{p\,(d-1)}+\varepsilon}$ [10, Proposition 1]; see also [14, § 1.1] for more examples and comments). The various links between this ε-conjecture and the above classical heuristics (or results) are described in [33, § 1.1, Theorem 1.2, Remark 3.3].

For a general history upon today about such inequalities, we refer to some recent papers of the bibliography (e.g., [9, 10, 14, 33, 43]) in which the reader can have a more complete list of recent contributions.

For short, we shall call "p-rank ε-conjecture" the case:

$$\#(\mathcal{C}\ell_K \otimes \mathbb{F}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^{\varepsilon},$$

and "strong ε-conjecture" the case:

$$\#(\mathcal{C}\ell_K \otimes \mathbb{Z}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^{\varepsilon}.$$

These kind of results do not separate the case of totally real fields, for which we know that the class number is rather small, regarding the case of non totally real fields (see the tables in Washington's book [42] and papers by Schoof as [36, 38], among many contributions; however, some real fields may have exceptional large class numbers regarding D_K [6, 7]). Moreover, by nature, the results that we have quoted deal with upper bounds of the p-rank $\mathrm{rk}_p(\mathcal{C}\ell_K)$ and precisely we shall see that genus theory gives, when it applies, large and maximal p-ranks and possibly unbounded orders, which probably makes harder proofs by classical complex analytic way.

Remark 1.1. Nevertheless, many arguments and computations are in favor of the strong ε-conjecture $\#(\mathcal{C}\ell_K \otimes \mathbb{Z}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^{\varepsilon}$, except possibly for subfamilies of density zero, even if such a conjecture is "a conjecture at infinity for fixed p" because, as we shall see in numerical calculations, the constants $C_{d,p,\varepsilon}$ are enormous, especially for $d = p$, so that, for "usual fields", the inequalities are trivial up to some huge values of the discriminant. So it will be difficult to give convincing computations "at infinity".

In this context, we shall only prove the case of the p-rank ε-conjecture for degree p cyclic fields K (Theorem 2.5):

Main Theorem. *Denote by $\mathcal{C}\ell_K$ the class group of any number field K. Let $p \geq 2$ be a prime number; then the p-rank ε-conjecture saying that: $\#(\mathcal{C}\ell_K \otimes \mathbb{F}_p) \ll_{d,p,\varepsilon} (\sqrt{D_K})^{\varepsilon}$ for all K of degree d, is true for the subfamily of all cyclic extension $F/\mathbb{Q}$ of degree $d = p$.*

For these fields, with Galois group $G =: \langle\sigma\rangle$, we shall use the exact sequence:

$$1 \to (\mathcal{C}\ell_K \otimes \mathbb{Z}_p)^G \longrightarrow \mathcal{C}\ell_K \otimes \mathbb{Z}_p \longrightarrow (\mathcal{C}\ell_K \otimes \mathbb{Z}_p)^{1-\sigma} \to 1$$

and consider the "non-genus part" $(\mathcal{C}\ell_K \otimes \mathbb{Z}_p)^{1-\sigma}$ (or set of "exceptional p-classes") as a random object for which densities results are known (see Remark 2.2).

The case $d = p = 2$ is known from Gauss genera theory, but its generalization is not obvious since for $p = 2$ the p-rank is canonical (given by Chevalley's formula (1)) while for $p > 2$ it depends on the algorithm which determines the complete structure of $\mathcal{C}\ell_K \otimes \mathbb{Z}_p$; indeed, for $d = p = 2$ and any class γ such that $\gamma^2 = 1$, we can write $\gamma^{\sigma-1} = \gamma^{\sigma+1-2} = \mathrm{N}_{K/\mathbb{Q}}(\gamma)\,\gamma^{-2} = 1$, which does not work for $d = p > 2$.

To our knowledge, for $d = p > 2$, only the case of cyclic cubic fields is proved [10, Corollary 1, case (3)] for the 3-rank ε-conjecture.

For CM fields, we shall try to put the "minus part" of $\mathcal{C}\ell_K \otimes \mathbb{Z}_p$ in "duality" with the "plus part" of the torsion group $\mathcal{T}_K$ of the Galois group of the maximal abelian p-ramified pro-p-extension of K for which, on the contrary, we know that there is no complete strong ε-conjecture because of some explicit families of density zero.

This suggests that, for any given p, the strong ε-conjecture may be true "for almost all field" in a sense to be specified.

2. A possible obstruction due to exceptional p-classes

We shall illustrate the above comments with a family of fields with optimal p-ranks, and give numerical illustrations with PARI/GP [32] programs.

2.1. Definition of the family $(F_{N,p})_{N\geq 1}$. We consider a fixed prime $p \geq 2$ and the sequence of all odd prime numbers ℓ_k, totally split in $\mathbb{Q}(\mu_p)$, whence such that $\ell_k \equiv 1 \pmod p$. For $p > 2$, let $F_{N,p}$ be any cyclic extension of degree p, of global conductor $f_{N,p} = \prod_{k=1}^{N} \ell_i$, for any $N \geq 1$. For $p = 2$ we shall consider $F_{N,2} := \mathbb{Q}(\sqrt{\pm\, \ell_1 \cdots \ell_N}) = \mathbb{Q}(\sqrt{\pm\, 3 \cdot 5 \cdots \ell_N})$ with $\pm$ such that 2 be unramified, and consider the restricted sense for class groups, which is more canonical for our purpose. We may consider degree p cyclic extensions ramified at p without any modification of the forthcoming reasonings, except an useless complexity of redaction (this modifies the above conductors up to the constant factor p^2 or 8).

The discriminant $D_{N,p}$ of $F_{N,p}$ is the product of the conductors associated to each nontrivial character of $G_{N,p} := \mathrm{Gal}(F_{N,p}/\mathbb{Q})$, thus $D_{N,p} = f_{N,p}^{p-1}$. There are $(p-1)^{N-1}$ such fields, all contained in $\mathbb{Q}(\mu_{\ell_1 \cdots \ell_N})$.

The Chevalley formula [3] gives the number of ambiguous classes (i.e., invariant by $G_{N,p}$) which is equal to the genus number for cyclic fields:

$$\#\big(\mathcal{C}\ell_{F_{N,p}} \otimes \mathbb{Z}_p\big)^{G_{N,p}} = p^{N-1}. \tag{1}$$

It is obvious that the group of ambiguous classes is p-elementary of p-rank $N-1 \leq \mathrm{rk}_p(\mathcal{C}\ell_{F_{N,p}})$ and generated by the ramified primes $\mathfrak{l}_i \mid \ell_i$, $1 \leq i \leq N$, with a (non-trivial) relation $\prod_{i=1}^{N} \mathfrak{l}_i^{n_i} = (a)$, $a \in F_{N,p}^{\times}$, due to the classical Kummer theory over $\mathbb{Q}(\mu_p)$ giving $F_{N,p}(\mu_p) = \mathbb{Q}(\mu_p)(\sqrt[p]{\alpha})$, $\alpha \in \mathbb{Q}(\mu_p)^{\times}$ depending of canonical Gauss sums.

2.2. About the exceptional p-classes. We know that $\#(\mathcal{C}\ell_{F_{N,p}} \otimes \mathbb{F}_p)$ and $\#(\mathcal{C}\ell_{F_{N,p}} \otimes \mathbb{Z}_p)$, both depend of a non-predictible algorithm that we recall below; this shows that the so-called Brumer–Silverman–Duke–Zhang–Ellenberg–Venkatesh ε-conjecture on the p-ranks is not so different from the strong ε-conjecture, in a logical point of view, except that we shall see that

the p-rank is an $O(N)$ contrary to $\#(\mathcal{C}\ell_{F_{N,p}} \otimes \mathbb{Z}_p)$ whose order of magnitude is unknown.

Definition 2.1. Let $\delta(N) \geq 0$ be such that $\#(\mathcal{C}\ell_{F_{N,p}} \otimes \mathbb{F}_p) = p^{N-1+\delta(N)}$ and let $\Delta(N) \geq \delta(N)$ be such that $\#(\mathcal{C}\ell_{F_{N,p}} \otimes \mathbb{Z}_p) = p^{N-1+\Delta(N)}$.

In other words, $p^{\Delta(N)} = \#(\mathcal{C}\ell_{F_{N,p}} \otimes \mathbb{Z}_p)^{1-\sigma}$ measures what we shall call the set of "exceptional p-classes" (i.e., non-invariant) obtained via the classical filtration of $\mathcal{C}\ell_{F_{N,p}} \otimes \mathbb{Z}_p$ (a general theoretical and numerical approach was given in [19, 20] and [39] after the historical papers of Inaba [25], Redei–Reichardt [35], Fröhlich [11, 12] and others; for a wide generalization to ray class groups and a survey, see [18]). Then $N - 1 + \delta(N)$ is the p-rank.

Remark 2.2. During the writing of this paper we have been informed, by Peter Koymans and Carlo Pagano, of their work [28] proving that the p-class groups of cyclic degree p fields have, under GRH, the (explicit) distribution conjectured by Frank Gerth III [15] and generalizing results of Jack Klys by means of methods developed by Etienne Fouvry–Jürgen Klüners, Alexander Smith and others.

These results prove that the strong ε-conjecture for degree p cyclic fields K is true (in the meaning that this occurs with probability 1), except possibly for very sparse families of fields of density zero. One deduces this from [28, Theorem 1.1] by proving that for any map $h : \mathbb{R} \to \mathbb{R}$, such that $h(X) \to \infty$ as $X \to \infty$, we have the following density property about the sets of exceptional p-classes of the p-class groups:

$$\lim_{X \to \infty} \left[\frac{\#\{K :\ \sqrt{D_K} < X \ \ \&\ \ \#\big(\mathcal{C}\ell(K) \otimes \mathbb{Z}_p\big)^{1-\sigma} > h(\sqrt{D_K})\,\}}{\#\{K :\ \sqrt{D_K} < X\}} \right] = 0.$$

The possible infinite "bad families" of degree p cyclic fields K are such that for some fixed ε and for any $C > 0$:

$$\#\big(\mathcal{C}\ell(K) \otimes \mathbb{Z}_p\big)^{1-\sigma} > C\ \frac{\big(\sqrt{D_K}\big)^{\varepsilon}}{p^{N-1}} =: C\, h_\varepsilon(\sqrt{D_K}),$$

where N is the number of ramified primes of K; we shall verify Section 2.3 that these functions h_ε fulfill the condition at infinity.

But the Theorem 2.5 shall prove (unconditionally) the p-rank ε-conjecture.

2.2.1. The algorithm. Recall briefly the computation of $\delta(N)$ and $\Delta(N)$ in the purpose of some probabilistic considerations.

Let F be *any* degree p cyclic extension of $\mathbb{Q}$ with $N \geq 1$ ramified primes, put $M := \mathcal{C}\ell_F \otimes \mathbb{Z}_p$, and let σ be a generator of $G := \mathrm{Gal}(F/\mathbb{Q}) \simeq \mathbb{Z}/p\mathbb{Z}$.

Let I_F be the group of ideals of F, prime to p, and let $M_i =: c\ell_F(\mathcal{I}_i)$, $i \geq 0$, $\mathcal{I}_i \subset I_F$, defined inductively by $M_0 := 1$, and:

$$M_{i+1}/M_i := (M/M_i)^G, \text{ for } 0 \leq i \leq m-1,$$

where $m \geq 1$ is the least integer i such that $M_i = M$ (i.e., such that $M_{i+1} = M_i$). Then $M_i = \{c \in M,\ c^{(1-\sigma)^i} = 1\}$ for all $i \geq 0$ and $M_1 = M^G$.

The sequence $\#(M_{i+1}/M_i)$, $0 \leq i \leq m$, decreases to 1 and is bounded by $\#M_1$ due to the injective maps $M_{i+1}/M_i \hookrightarrow M_i/M_{i-1} \hookrightarrow \cdots \hookrightarrow M_2/M_1 \hookrightarrow M_1$ defined by the operation of $1-\sigma$.

We have, for the fields F, the formulas [18, Corollary 3.7], for all $i \geq 0$:

$$\#\big(M_{i+1}/M_i\big) = \frac{p^{N-1}}{(\Lambda_i : \Lambda_i \cap \mathrm{N}_{F/\mathbb{Q}}(F^\times))} \ \ \&\ \ \Lambda_i := \{x \in \mathbb{Q}^\times,\ (x) \in \mathrm{N}_{F/\mathbb{Q}}(\mathcal{I}_i)\},$$

with $\mathcal{I}_0 = 1$ and $\Lambda_0 = 1$.

The progression of the algorithm depends on the $x \in \Lambda_i$, $(x) = \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{A})$, $\mathfrak{A} \in \mathcal{I}_i$, such that $x = \mathrm{N}_{F/\mathbb{Q}}(y)$, $y \in F^\times$, giving the equation:

$$(y) = \mathfrak{A} \cdot \mathfrak{B}^{1-\sigma},$$

in which the solutions $\mathfrak{B}$ (non-predictible) become new elements to be added to $\mathcal{I}_i$ to built $\mathcal{I}_{i+1} \supseteq \mathcal{I}_i$, then $\Lambda_{i+1} \supseteq \Lambda_i$, and so on.

Put, for all $i \geq 0$, $\#\big(M_{i+1}/M_i\big) := p^{N-1-t_i^N}$, $t_i^N \geq 0$, where $p^{t_i^N} \mid p^{N-1}$ is the index $(\Lambda_i : \Lambda_i \cap \mathrm{N}_{F/\mathbb{Q}}(F^\times))$ (since $x \in \Lambda_i$ is norm of an ideal, it is locally a norm everywhere, except perhaps at the N ramified primes, but the product formula for Hasse's normic symbols gives the above divisibility). We have:

$$t_0^N = 0 \leq t_1^N \leq \cdots \leq t_m^N = N-1.$$

Then $\#M = \#(\mathcal{C}\ell_F \otimes \mathbb{Z}_p) = \prod\limits_{i \geq 0} \#\big(M_{i+1}/M_i\big) = p^{m\,(N-1) - \sum_{i?=1}^{m} t_i^N}$.

We have the following result that we shall use to test the ε-conjecture.

Lemma 2.3 ([18, Lemma 4.2])**.** *Let $F/\mathbb{Q}$ be a degree p cyclic extension with $N \geq 1$ ramified primes. Then* $\mathrm{rk}_p(\mathcal{C}\ell_F) = (p-1)\cdot(N-1) - \sum_{i=1}^{p-2} t_i^N$. *Whence this yields* $\delta(N) = (p-2)\cdot(N-1) - \sum_{i=1}^{p-2} t_i^N$.

We see that the p-rank may vary in the interval $[N-1, (p-1)\,(N-1)]$ and is always $O(N)$ as $N \to \infty$ which shall be of a great importance; the p-rank is equal to $N-1$, for all $F/\mathbb{Q}$ of degree p, if and only if $p = 2$ as we have seen. In the same manner, the p^r-ranks are given by the expressions:

$$\sum_{i=(r-1)\,(p-1)}^{r\,(p-1)-1} \big(N-1-t_i^N\big) = (p-1)\cdot(N-1) - \sum_{i=(r-1)\,(p-1)}^{r\,(p-1)-1} t_i^N.$$

It is clear that the normic indices depend on the $\mathbb{F}_p$-ranks of $N \times N$-matrices of suitable Hilbert symbols [20, Ch. VI, § 2] generalizing, for instance, Rédei's matrices for the computation of the 4-ranks of a quadratic field.

But we emphasize the fact that if some heuristics on the $\mathbb{F}_p$-rank of these matrices are natural, *the number of steps of the algorithm* (i.e., m) only depends on distribution results (see Remark 2.2).

2.2.2. Examples of maximal rank. In fact, for $p > 2$, the maximal rank, equal to $(p-1)\cdot(N-1)$, is very rare. We remembered that in our thesis [20, p. 39], we gave the first example of cyclic cubic field, for which $\mathrm{rk}_3(\mathcal{C}\ell) = 2\,(N-1)$; this example has conductor $f = 9\cdot 577\cdot 757\cdot 991$ (the program below shows that the property is fulfilled for the eight fields of conductor f):

```
f=3895721091 N=4 P=x^3-1298573697*x+18010351461592  Cl=[9,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x-16034355152657  Cl=[3,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x+15707980296811  Cl=[3,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x-10132337699792  Cl=[3,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x-8835062576489   Cl=[3,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x-7829966535011   Cl=[9,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x+912031593193    Cl=[3,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x-221623244288    Cl=[9,3,3,3,3,3]
```

We publish, for any further examples, a program computing all the cyclic cubic fields such that there are exceptional 3-classes; so in most cases the 3-rank is N (instead of $N-1$) and many examples have a non-trivial 9-rank.

```
{bf=7;Bf=10^6;for(f=bf,Bf,e=valuation(f,3);if(e!=0 & e!=2,next);F=f/3^e;
if(Mod(F,3)!=1||core(F)!=F,next);F=factor(F);Div=component(F,1);
d=component(matsize(F),1);for(j=1,d-1,D=component(Div,j);
if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);
A=4*f-27*b^2;if(issquare(A,&a)==1,if(e==0,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);if(e==2,if(Mod(a,9)==3,a=-a);
P=x^3-f/3*x-f*a/27);N=omega(f);L=List;K=bnfinit(P,1);C8=component(K,8);
h=component(component(C8,1),1);Cl=component(component(C8,1),2);
dim=component(matsize(Cl),2);rho=0;for(i=1,dim,z=component(Cl,i);
v=valuation(z,3);if(v>=1,rho=rho+1;listput(L,3^v)));if(rho>=N,
print(" f=",f," N=",N," P=",P," Cl=",L)))))}

f=657 N=2 P=x^3-219*x-1241          Cl=[3,3]
f=657 N=2 P=x^3-219*x+730           Cl=[3,3]
(...)
f=3913 N=3 P=x^3+x^2-1304*x+17681   Cl=[3,3,3]
f=3913 N=3 P=x^3+x^2-1304*x-17536   Cl=[3,3,3]
(...)
f=4711 N=2 P=x^3+x^2-1570*x+19193   Cl=[9,3]
f=4711 N=2 P=x^3+x^2-1570*x-13784   Cl=[9,3]
f=4921 N=3 P=x^3+x^2-1640*x+23876   Cl=[3,3,3]
f=4921 N=3 P=x^3+x^2-1640*x-729     Cl=[3,3,3]
(...)
f=5383 N=2 P=x^3+x^2-1794*x+17744   Cl=[9,3]
f=5383 N=2 P=x^3+x^2-1794*x-9171    Cl=[9,3]
 (..)
f=15561 N=4 P=x^3-5187*x+141778     Cl=[9,3,3,3]
```

Remark 2.4. The Galois action of $G = \mathrm{Gal}(F/\mathbb{Q}) \simeq \mathbb{Z}/p\mathbb{Z}$ on the non-p-part of the class group gives rise, for all prime $q \mid \#\mathcal{C}\ell_F$, to:

$$\mathcal{C}\ell_F \otimes \mathbb{Z}_q \simeq \mathbb{Z}[\mu_p]\Big/\prod_{\mathfrak{q}\mid q}\mathfrak{q}^{n_\mathfrak{q}},\ n_\mathfrak{q} \geq 0,$$

whose $\mathbb{F}_q$-dimension is a multiple of the residue degree of q in $\mathbb{Q}(\mu_p)/\mathbb{Q}$, which explains the rarity of such divisibilities for large residue degrees.

2.3. Estimation of $C_{p,p,\varepsilon}$ for the fields $F_{N,p}$. As we have explained, we do not consider degree p cyclic extensions $F/\mathbb{Q}$ ramified at p. This shall modify the forthcoming computations by some $O(1)$ without any consequence on the statements since p is fixed in all the sequel.

We need a lower bound of the kth prime number $\ell_k \equiv 1 \pmod p$ to get an estimation of $f_{N,p}$.

We thank Gérald Tenenbaum for valuable indications for the good formula, from [40, Notes on Chapitre I, § 4.6], due to a result of Montgomery–Vaughan giving, for the kth prime number $\ell_k \equiv 1 \pmod p$:

$$\ell_k > \tfrac{p-1}{2} \cdot k \cdot \log\big(\tfrac{\ell_k}{p}\big), \text{ for all } k \geq 1. \tag{2}$$

Indeed, if $\pi(x;1,p) := \#\{\ell \leq x;\ \ell \equiv 1 \pmod p\}$ then:

$$\pi(x;1,p) \leq \frac{2\,x}{(p-1)\log\big(\frac{x}{p}\big)};$$

whence the result taking $x = \ell_k$ since $\pi(\ell_k;1,p) = k$.

We intend to test, for the family $(F_{N,p})_{N\geq 1}$ of discriminants $D_{N,p} := D_{F_{N,p}}$, the strong ε-conjecture, that is to say:

$$\#(\mathcal{C}\ell_{F_{N,p}} \otimes \mathbb{Z}_p) =: p^{N-1+\Delta(N)} \leq C_{p,p,\varepsilon} \cdot (\sqrt{D_{N,p}}\,)^{\varepsilon}, \tag{3}$$

where $\Delta(N) = (m-1)\cdot(N-1) - \sum_{i=1}^{m-1} t_i^N \geq 0$, related to the set of exceptional p-classes, has only a probabilistic value depending on m and the t_i^N.

It will be easy, from the forthcoming calculations, to test the p-rank ε-conjecture, $\#(\mathcal{C}\ell_{F_{N,p}} \otimes \mathbb{F}_p) \leq C_{p,p,\varepsilon} \cdot (\sqrt{D_{N,p}}\,)^{\varepsilon}$, but considering instead the weaker inequality:

$$p^{N-1+\delta(N)} = p^{(p-1)\,(N-1) - \sum_{i=1}^{p-2} t_i^N} \leq C_{p,p,\varepsilon} \cdot (\sqrt{D_{N,p}}\,)^{\varepsilon},$$

even in the less favorable case $t_i^N = 0$, for $1 \leq i \leq p-2$ and replacing for all N, $D_{N,p}$ by a lower bound $D'_{N,p}$ (in other words the existence of an inequality $p^{(p-1)\,(N-1)} \leq C'_{p,p,\varepsilon} \cdot (\sqrt{D'_{N,p}}\,)^{\varepsilon}$ proves the p-rank ε-conjecture for all degree p cyclic fields). We fix p and put $D_{N,p} =: D_N =: f_N^{p-1}$. The strong form is equivalent to prove that $\dfrac{p^{N-1+\Delta(N)}}{(\sqrt{D_N}\,)^{\varepsilon}}$ is bounded as $N \to \infty$, whence $(N-1+\Delta(N))\log(p) - \varepsilon \cdot \frac{p-1}{2} \sum_{k=1}^{N} \log(\ell_k) < \infty,$ as $N \to \infty$.

We then have, replacing ℓ_k by a lower bound ℓ'_k, to compute, using (1), (2), (3), the following quantity:

$$\begin{aligned} X(N) &:= (N-1+\Delta(N))\log(p) - \varepsilon\cdot\tfrac{p-1}{2}\sum_{k=1}^{N}\log(\ell'_k) \\ &= (N-1+\Delta(N))\log(p) - \varepsilon\cdot\tfrac{p-1}{2}\sum_{k=1}^{N}\log\big(\tfrac{p-1}{2}\big) + \log(k) + \log_2\big(\tfrac{\ell_k}{p}\big). \end{aligned}$$

We verify that $\sum_{k=1}^{N}\log_2\big(\frac{\ell_k}{p}\big)$ can be neglected, subject to adding -1 to the sum, and consider, instead:

$$\begin{aligned} X(N) &= (N-1+\Delta(N))\log(p) - \varepsilon\cdot\tfrac{p-1}{2}\Big[-1+\sum_{k=1}^{N}\big[\log\big(\tfrac{p-1}{2}\big)+\log(k)\big]\Big] \\ &= (N-1+\Delta(N))\log(p) - \varepsilon\cdot\tfrac{p-1}{2}\big[-1+N\cdot\log\big(\tfrac{p-1}{2}\big)+\log(N!)\big]. \end{aligned}$$

The expression of $N!$ leads to $\log(N!) = N\log(N) - N + \frac{1}{2}\log(N) + O(1)$, whence, with $\gamma_p := \log\big(\frac{p-1}{2}\big) - 1$:

$$\begin{aligned} X(N) =& (N-1+\Delta(N))\log(p) \\ &- \varepsilon\cdot\tfrac{p-1}{2}\big[N\log(N) + N\cdot\gamma_p + \tfrac{1}{2}\log(N) + O(1)\big]. \end{aligned}$$

Now we write $X(N)$ under the form:

$$\begin{aligned} X(N) &= N\log(p) + \Delta(N)\log(p) - \varepsilon\cdot\tfrac{p-1}{2}N\log(N) \\ &\quad - \log(p) - \varepsilon\cdot\tfrac{p-1}{2}N\cdot\gamma_p - \varepsilon\cdot\tfrac{p-1}{4}\log(N) - \varepsilon\cdot O(1) \\ &= N\cdot\Big[-\varepsilon\cdot\tfrac{p-1}{2}\log(N) + \tfrac{\Delta(N)}{N}\log(p) \\ &\qquad + \log(p) - \tfrac{\log(p)}{N} - \varepsilon\cdot O(1)\Big] \\ &= N\cdot\Big[-\varepsilon\cdot\tfrac{p-1}{2}\log(N) + \tfrac{\Delta(N)}{N}\log(p) \\ &\qquad + (1-N^{-1})\log(p) - \varepsilon\cdot O(1)\Big]. \end{aligned} \tag{4}$$

Replacing $\Delta(N)$ by the maximal value $(p-2)(N-1)$ of $\delta(N)$, the dominant term $-\varepsilon\cdot\frac{p-1}{2}\log(N)$ ensures the existence of a positive constant C_ε since $\frac{(p-1)(N-1)}{N}\log(p) = O(1)$ giving:

$$X_0(N) = -\varepsilon\,\tfrac{p-1}{2}\,N\log(N) + N\big[(p-1)\log(p) - \varepsilon\,O(1) - o(1)\big]. \tag{5}$$

It is easy to verify that $X_0(N)$, as function of N, admits, for an $N_0 \gg 0$, a computable maximum, only depending on p and ε (e.g., for $p=7$, $\varepsilon = 0.1$, $N_0 \approx 2935394\cdot 10^{10}$, $X_0(N_0) \approx 88\cdot 10^{14}$).

This proves the p-rank ε-conjecture for the family $(F_{N,p})_{N\geq 1}$, even assuming always a maximal p-rank $(p-1)(N-1)$. Whence we can state:

Theorem 2.5. *Let $p \geq 2$ be a given prime number.*

(i) The p-rank ε-conjecture on the existence, for all $\varepsilon > 0$, of a constant $C_{d,p,\varepsilon}$ such that $\#(\mathcal{C}\ell_K \otimes \mathbb{F}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^{\varepsilon}$ for all K of degree d, is true for the subfamily of all cyclic extension $F/\mathbb{Q}$ of degree $d = p$.

(ii) For any cyclic extension $F/\mathbb{Q}$ of degree p, let $\Delta(N) \geq 0$ be defined by $\#(\mathcal{C}\ell_F \otimes \mathbb{Z}_p) = p^{N-1+\Delta(N)}$, where N is the number of ramified primes. Then the strong ε-conjecture, $\#(\mathcal{C}\ell_K \otimes \mathbb{Z}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^{\varepsilon}$, is true for the family of degree p cyclic extensions under the condition $\Delta(N) \ll N \log(N)$.

Proof. Consider arbitrary prime numbers $\ell_{n_i} \equiv 1 \pmod p$, $1 \leq i \leq N$, and a field K, cyclic of degree p, of conductor $\prod_{i=1}^{N} \ell_{n_i}$. Thus $D_K \geq D_{F_N}$ and the maximal p-rank of $\mathcal{C}\ell_K$ is still $(p-1)(N-1)$. We have seen that if we put $\delta(N) = (p-2)(N-1)$ in (4), giving (5), the main term ensures the existence of the constant $C_{p,p,\varepsilon}$. We omit the details when p^2 (or 8) divides the conductor. □

Very probably, considering the Remark 2.2, $\Delta(N)$ is almost all the time of order much less than $N \log(N)$ since p^r-ranks are in practice very rare for $r \geq 2$ as well as a maximal p-rank equal to $(p-1)(N-1)$ (see § 2.2.2); but the dominant terms in (4) being $\Delta(N)\log(p) - \varepsilon \cdot \frac{p-1}{2} N \log(N)$, this may give some trouble for the proof of an universal strong ε-conjecture assuming for instance $\Delta(N) = O(1)\log(N)$, or more, for infinite families of degree p cyclic fields, even of density zero.

2.4. A lower bound for $C_{p,p,\varepsilon}$. This part is not essential for our purpose, but it will show that a lower bound of $C_{p,p,\varepsilon}$ is of the same order of magnitude as for the upper bound deduced from (5). We shall use the following property which may be justified from results given in [31]. For all $k \geq 1$, the kth prime $\ell_k \equiv 1 \pmod p$ fulfills, for some constants c_p only depending on p, the inequality $\ell_k < c_p\,((p-1)\,k) \cdot \log((p-1)\,k) < c_p\,((p-1)\,k)^2$. Then:

$$\begin{aligned}(N-1+\delta(N))\log(p) \;&\leq\; \log(C_\varepsilon) + \varepsilon\,\tfrac{p-1}{2}\textstyle\sum_{k=1}^{N}\log(\ell_k)\\ &\leq\; \log(C_\varepsilon) + \varepsilon\,\tfrac{p-1}{2}\textstyle\sum_{k=1}^{N}\big[\log(c_p) + 2\log(p-1) + 2\log(k)\big],\\ &\leq\; \log(C_\varepsilon) + \varepsilon\,\tfrac{p-1}{2}\cdot\big[N\,\gamma'_p + 2\log(N!)\big]\end{aligned}$$

where $\gamma'_p := \log(c_p) + 2\log(p-1) > 0$. Whence, from the value of $\log(N!)$:

$$(N-1+\delta(N))\log(p) \leq \log(C_\varepsilon) + \varepsilon\tfrac{p-1}{2}\big[2N\log(N) + N\,\gamma''_p + \log(N) + O(1)\big],$$

with $\gamma''_p = \gamma'_p - 2 > 0$. Thus:

$$\begin{aligned}\log(C_\varepsilon) &\geq (N-1+\delta(N))\log(p) - \varepsilon\,(p-1)\,N\log(N)\\ &\qquad - \varepsilon\,\tfrac{p-1}{2}N\,\gamma''_p - \varepsilon\,\tfrac{p-1}{2}\log(N) - \varepsilon\,O(1)\\ &\geq N\cdot\Big[-\varepsilon\,(p-1)\log(N) + \tfrac{\delta(N)}{N}\log(p) + O(1)\Big].\end{aligned} \tag{6}$$

Remarks 2.6. (i) As soon as we replace ε by $c+\varepsilon$, $0<c<1$, as it is done in much papers giving general proofs for $\#(\mathcal{C}\ell_K \otimes \mathbb{F}_p) \leq C_{d,p,c,\varepsilon} \cdot (\sqrt{D_K}\,)^{c+\varepsilon}$, the above computations becomes, replacing $\Delta(N)$ by $\delta(N)$:

$$\log(C_{p,p,c,\varepsilon}) \leq N \cdot \Big[-(c+\varepsilon) \cdot \tfrac{p-1}{2} \log(N) + \tfrac{\delta(N)}{N} \log(p) + O(1) \Big]$$
$$\log(C_{p,p,c,\varepsilon}) \geq N \cdot \Big[-(c+\varepsilon) \cdot (p-1) \log(N) + \tfrac{\delta(N)}{N} \log(p) + O(1) \Big].$$

For $\varepsilon \to 0$, the formulas (5) and (6) give a limit value $C_{p,c}$ of $C_{p,p,c,\varepsilon}$ such that $\log(C_{p,c}) \approx (N-1+\delta(N)) \log(p) - c\,\frac{p-1}{2}\,N \log(N) - N\,O(1)$, rapidely negative as N increases, giving for degree p cyclic fields, an obvious proof of the "p-rank $(c+\varepsilon)$-property".

(ii) If we replace $\mathbb{Q}$ by a number field k and the fields $F_{N,p}$ by the degree p cyclic extensions $F_{k,N,p}$ of k with N ramified prime ideals of k, the details of computations are more complicate, but the results and comments are similar since the p-rank of $\mathcal{C}\ell_{F_{k,N,p}}$ is still equivalent to $O(N)$ because of the exact sequence of genus theory and the inequalities [16, Theorem IV.4.5.1]:

$$N - c_{k,p} \leq \mathrm{rk}_p\big(\mathcal{C}\ell^G_{F_{k,N,p}}\big) \leq N + c'_{k,p},$$

where the constants $c_{k,p}$, $c'_{k,p}$ depend on the p-classes and units of k. Then, the general algorithm computing the p-rank via the filtration $(M_i)_{i\geq 0}$ is identical up to similar modifications (see [18, § 4.4]) and shall give the p-rank ε-conjecture for the relative degree p cyclic case without too much difficulties. The case of abelian p-extensions may be accessible from the p^n-cyclic cases, with some effort...

The proof of the strong ε-conjecture, for the degree p cyclic extensions of k, remains, theoretically, open, but is clearly not a folk conjecture for such very particular real fields because of the possible generalization of the density results of [28] and the conclusion, about the possible existence of pathological families of density zero, is still relevant. Meanwhile a particular study of the algorithm giving $\Delta(N)$, independently of any density results, should be a crucial step for many questions in number theory.

3. PARI/GP programs computing $\mathcal{C}\ell_{F_{N,p}}$

3.1. Structure of some $\mathcal{C}\ell_{F_{N,p}}$. The following numerical results show that exceptional p-classes (indicated by *) are not excessively frequent for these fields. We examine the cases $p=3,5,7$, then $p=2$ (one must precise p and the conductor $f=\ell_1\cdots\ell_N$, $\ell_i \equiv 1 \pmod p$), in the following program giving all the $(p-1)^{N-1}$ fields of conductor f):

```
{p=3;f=7*13*19*31*37;V=polsubcyclo(f,p);d=matsize(V);d=component(d,2);
for(k=1,d,P=component(V,k);if(nfdisc(P)!=f^(p-1),next);K=bnfinit(P,1);
C8=component(K,8);C81=component(C8,1);h=component(C81,1);
Cl=component(C81,2);print("p=",p," f=",f," P=",P," Cl=",Cl))}

p=3  f=1983163=7*13*19*31*37
```

```
P=x^3+x^2-661054*x+ 49725976    Cl=[6,6,3,3]
P=x^3+x^2-661054*x+198463201    Cl=[3,3,3,3]
P=x^3+x^2-661054*x+97321888     Cl=[39,3,3,3,3]*
P=x^3+x^2-661054*x-186270421    Cl=[3,3,3,3]
P=x^3+x^2-661054*x-79179619     Cl=[6,6,3,3]
P=x^3+x^2-661054*x-188253584    Cl=[3,3,3,3,3]*
P=x^3+x^2-661054*x+138968311    Cl=[3,3,3,3]
P=x^3+x^2-661054*x-146607161    Cl=[3,3,3,3,3]*
P=x^3+x^2-661054*x-158506139    Cl=[3,3,3,3]
P=x^3+x^2-661054*x-140657672    Cl=[6,6,3,3]
P=x^3+x^2-661054*x+186564223    Cl=[3,3,3,3]
P=x^3+x^2-661054*x+81456584     Cl=[3,3,3,3]
P=x^3+x^2-661054*x+206395853    Cl=[3,3,3,3]
P=x^3+x^2-661054*x-206102051    Cl=[12,12,3,3]
P=x^3+x^2-661054*x+2130064      Cl=[3,3,3,3,3]*
P=x^3+x^2-661054*x+27911183     Cl=[3,3,3,3,3]*

p=3  f=85276009=7*13*19*31*37*43
P=x^3-x^2-28425336*x+58104545836    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+56995957719    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-7472705085     Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+10264704787    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+51623569152    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+30901498965    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-50622365639    Cl=[6,6,3,3,3]
P=x^3-x^2-28425336*x-29132811371    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-46614393216    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-33226059803    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-56506410260    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-12248161589    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+23994142236    Cl=[6,6,3,3,3]
P=x^3-x^2-28425336*x+52220501215    Cl=[3,3,3,3,3,3]*
P=x^3-x^2-28425336*x+11799672949    Cl=[6,6,3,3,3]
P=x^3-x^2-28425336*x-57529722368    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-21287418543    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-47040773261    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+36273887532    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-53436473936    Cl=[6,6,3,3,3,3]*
P=x^3-x^2-28425336*x+40964068027    Cl=[21,3,3,3,3]
P=x^3-x^2-28425336*x+45824800540    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+56313749647    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+13078813084    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+39940755919    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-37830964289    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+57848717809    Cl=[6,6,3,3,3]
P=x^3-x^2-28425336*x-9775157328     Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-2526696563     Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-58126654431    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x+28428494704    Cl=[3,3,3,3,3]
P=x^3-x^2-28425336*x-51986781783    Cl=[3,3,3,3,3]

p=5  f=13981=11*31*41
```

```
P=x^5+x^4-5592*x^3+46417*x^2+4301003*x-26664769     Cl=[5,5]
P=x^5+x^4-5592*x^3+46417*x^2+2623283*x-26664769     Cl=[5,5]
P=x^5+x^4-5592*x^3-261165*x^2-4479065*x-26832541    Cl=[55,5,5]*
P=x^5+x^4-5592*x^3+32436*x^2+1518784*x+1814528      Cl=[5,5]
P=x^5+x^4-5592*x^3-205241*x^2-2074333*x-6028813     Cl=[5,5,5]*
P=x^5+x^4-5592*x^3-135336*x^2+847696*x+1143440      Cl=[5,5]
P=x^5+x^4-5592*x^3-205241*x^2-1878599*x+234675      Cl=[5,5]
P=x^5+x^4-5592*x^3+74379*x^2+3993421*x-7035445      Cl=[5,5]
P=x^5+x^4-5592*x^3+158265*x^2-424575*x-12851541     Cl=[5,5]
P=x^5+x^4-5592*x^3+130303*x^2+442247*x+346523       Cl=[5,5]
P=x^5+x^4-5592*x^3+46417*x^2+1644613*x-16878069     Cl=[5,5]
P=x^5+x^4-5592*x^3+214189*x^2-2493763*x+5715227     Cl=[5,5]
P=x^5+x^4-5592*x^3+74379*x^2+3238447*x-28174717     Cl=[5,5]
P=x^5+x^4-5592*x^3+32436*x^2+5992704*x-2659392      Cl=[5,5,5]*
P=x^5+x^4-5592*x^3-37469*x^2+4384889*x-11397517     Cl=[5,5]
P=x^5+x^4-5592*x^3-9507*x^2+7432747*x+1185383       Cl=[5,5]
```

3.2. Examples of exeptional p-classes. If one drops the invariant classes from the class group, this gives a very small component (often equal to 1); for instance we list some fields of the above tables with exceptional p-classes and/or nontrivial non-p-parts of the class group (for $p = 3, 5, 7, 11$):

```
p=3
f=1983163  P=x^3+x^2-661054*x+97321888 Cl=[39,3,3,3,3]=[13]x[3,3,3,3,3]*
f=1983163  P=x^3+x^2-661054*x-206102051Cl=[12,12,3,3]=[4,4]x[3,3,3,3]
f=85276009 P=x^3-x^2-28425336*x-53436473936
                                       Cl=[6,6,3,3,3,3]=[2,2]x[3,3,3,3,3,3]*
p=5
f=13981    P=x^5+x^4-5592*x^3-261165*x^2-4479065*x-26832541
                                       Cl=[55,5,5]=[11]x[5,5,5]*
f=13981    P=x^5+x^4-5592*x^3-205241*x^2-2074333*x-6028813
                                       Cl=[5, 5, 5]*
f=13981    P=x^5+x^4-5592*x^3+32436*x^2+5992704*x-2659392
                                       Cl=[5, 5, 5]*
p=7
f=88537   P=x^7+x^6-37944*x^5-1134719*x^4+324123632*x^3
+13095064100*x^2-393352790753*x-8536744545007
                                       Cl=[301,7]=[43]x[7,7]
f=10004681   P=x^7-x^6-4287720*x^5-1266715121*x^4+5127549957760*x^3
+2650344024068794*x^2-951078919604894529*x-488606218944681147667
                                       Cl=[791,7,7]=[113]x[7,7,7]
f=10004681   P=x^7-x^6-4287720*x^5+3455494311*x^4+176273349584*x^3
-471685834336278*x^2-36087798097778993*x+6487901368894795147
                                       Cl=[7,7,7,7]*
f=10004681   P=x^7-x^6-4287720*x^5+2985274304*x^4+1830007100160*x^3
-1788009977372784*x^2+233770322355404864*x-5599630780142239232
                                       Cl=[14,14,14]=[2,2,2]x[7,7,7]
f=10004681   P=x^7-x^6-4287720*x^5+6276814353*x^4-4047542893720*x^3
+1360785664233294*x^2-232635292693132049*x+15953699891990750023
                                       Cl=[7,7,7,7]*
f=10004681   P=x^7-x^6-4287720*x^5-6169008811*x^4-3865457699520*x^3
-1247559831026016*x^2-202130636944756129*x-12969698603184144677
```

```
                                        Cl=[7,7,7,7]*
f=10004681    P=x^7-x^6-4287720*x^5+1224450448*x^4+5079527488960*x^3
-2161274540764336*x^2-16771617132875296 64*x+812143879436422435328
                                        Cl=[7,7,7,7]*
f=10004681    P=x^7-x^6-4287720*x^5-1306733845*x^4+4849019638720*x^3
+2420349994235592*x^2-659732951886568641*x-207964718993797238079
                                        Cl=[7,7,7,7]*
f=10004681    P=x^7-x^6-4287720*x^5+5756570941*x^4-3107102879720*x^3
+760318637129246*x^2-74394888056758073*x+1594979915105904419
                                        Cl=[7,7,7,7]
f=10004681    P=x^7-x^6-4287720*x^5-1746939809*x^4+2392910471944*x^3
+1648023037138232*x^2+241022177190387487*x-9669121620934915453
                                        Cl=[7,7,7,7]*
f=10004681    P=x^7-x^6-4287720*x^5-4188081973*x^4-284662313448*x^3
+706702291060440*x^2+90226184532822239*x-3998693323498243787
                                        Cl=[7,7,7,7]*
f=10004681    P=x^7-x^6-4287720*x^5-2227164497*x^4+1569805356712*x^3
+369664737597974*x^2-166416491455189217*x+3813962316737479895
                                        Cl=[7,7,7,7]*
f=10004681    P=x^7-x^6-4287720*x^5-3307670045*x^4+1296557509240*x^3
+1587033521303494*x^2+232482552302284071*x-10617355312468435915
                                        Cl=[203,7,7]=[203]x[7,7,7]
f=10004681    P=x^7-x^6-4287720*x^5+1654651731*x^4+4187910318240*x^3
-1965195178959414*x^2-1012216801097102473*x+451605062388713519719
                                        Cl=[14,14,14]=[2,2,2]x[7,7,7]
f=10004681    P=x^7-x^6-4287720*x^5+1544600240*x^4+4818925558272*x^3
-2343209264562096*x^2-1609606088655340480*x+886992887186768275456
                                        Cl=[203,7,7]=[203]x[7,7,7]
f=10004681    P=x^7-x^6-4287720*x^5-1306733845*x^4+4427622475000*x^3
+1798213808544282*x^2-1313378138040048441*x-613859706212867719587
                                        Cl=[301,7,7]=[301]x[7,7,7]
p=11
f=137149    P=x^11+x^10-62340*x^9-2099173*x^8+998038116*x^7+30321726924*x^6
-5078707527329*x^5+3275334221180*x^4+8066546096404148*x^3
-214723422858644515*x^2+1902599837479513519*x-4121588229203611219
                                        Cl=[253,11]=[23]x[11,11]
f=137149    P=x^11+x^10-62340*x^9-727683*x^8+1217887963*x^7+33409088063*x^6
-7760886906947*x^5-350751766601032*x^4+3398347545513222*x^3
+236507399684756272*x^2+2593585988882665302*x+8529384350363670191
                                        Cl=[979,11]=[89]x[11,11]
f=137149    P=x^11+x^10-62340*x^9-1550577*x^8+1265615815*x^7+66889353347*x^6
-7866931610939*x^5-712972865698216*x^4-12900860936489076*x^3
+1432769815949223 36*x^2+1596818122984871186*x+3178173588229813309
                                        Cl=[737,11]=[67]x[11,11]
f=137149    P=x^11+x^10-62340*x^9-7310835*x^8+70636578*x^7+43296296622*x^6
+1378934348258*x^5-28471672310749*x^4-944763467217249*x^3
+12433374265353417*x^2+31266667418235948*x-324164722946199831
                                        Cl=[253,11]=[23]x[11,11]
f=137149    P=x^11+x^10-62340*x^9-727683*x^8+1153290784*x^7+18055120364*x^6
-7130815486262*x^5-134781314432095*x^4+13777568483843493*x^3
+310799275320778321*x^2+883774494827373474*x-5728549445587601897
                                        Cl=[253,11]=[23]x[11,11]
```

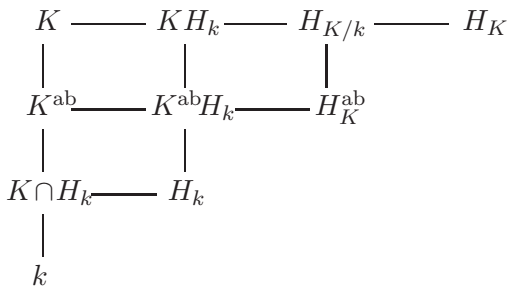
For $p = 2$, $F_{N,2}$ may be real or complex; as we know, the 2-rank is always $N - 1$ and any exceptional classes give non-trivial 4-ranks. The PARI/GP instruction bnfnarrow allows the 2-structure in the restricted sense:

```
{m=1;for(N=2,100,el=prime(N);m=(-1)^((el-1)/2)*el*m;P=x^2-m;
K=bnfinit(P,1);L=bnfnarrow(K);Cl=component(L,2);print("m=",m," Cl=",Cl))}
m=-15  Cl=[2]
m=+105  Cl=[2,2]
m=-1155  Cl=[2,2,2]
m=-15015  Cl=[12,2,2,2]
m=-255255  Cl=[16,2,2,2,2]*
m=+4849845  Cl=[4,2,2,2,2,2]*
m=-111546435  Cl=[42,2,2,2,2,2,2]
m=-3234846615  Cl=[308,2,2,2,2,2,2,2]*
m=+100280245065  Cl=[2,2,2,2,2,2,2,2,2]
m=+3710369067405  Cl=[34,2,2,2,2,2,2,2,2,2]
m=+152125131763605  Cl=[2,2,2,2,2,2,2,2,2,2,2]
m=-6541380665835015  Cl=[28284,2,2,2,2,2,2,2,2,2,2,2]*
m=+307444891294245705  Cl=[14,2,2,2,2,2,2,2,2,2,2,2,2]
m=+16294579238595022365  Cl=[2,2,2,2,2,2,2,2,2,2,2,2,2,2]
m=-961380175077106319535  Cl=[1210734,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
m=-58644190679703485491635  Cl=[1622526,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
m=+39291607755401335279395 45 Cl=[2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
m=-27897041506334948048370 7695
                  Cl=[83911452,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]*
m=-20364840299624512075310661735
                  Cl=[362626834,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
m=+16088223836703364539495422 77065
                  Cl=[4,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]*
m=-13353225784463792567781200 8996395
                  Cl=[2322692420,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]*
```

4. Genus theory – Abelian p-ramification theory

4.1. Reminders on Genus theory. For wide information on genus theory, see for example [12, 13, 16, 27, 29, 34].

4.1.1. Genus field and genus number $g_{K/k}$. Introduce the genus field according to an extension K/k (we shall take $k = \mathbb{Q}$ in the sequel) which is the maximal subextension $H_{K/k}$ of H_K (in the restricted sense) equal to the compositum of K with an abelian extension of k:



Thus $H_{K/k}$ is for instance equal to the compositum of K with $H_K^{\rm ab}$ (the maximal abelian subextension of H_K/k), according to the diagram above, where $K^{\rm ab}$ is the maximal abelian subextension of K/k. The genus number is $g_{K/k} := [H_{K/k} : KH_k]$. When K/k is cyclic, the genus number $g_{K/k}$ is equal to the number of invariant classes by $\mathrm{Gal}(K/k)$ given by Chevalley's formula (1). In the general Galois case, we have the similar expression $g_{K/k} = \dfrac{\#\mathcal{C}\ell_k \cdot \prod_{\mathfrak{l}} e_{\mathfrak{l}}^{\rm ab}}{[K^{\rm ab} : k] \cdot (E_k^{\rm pos} : E_k^{\rm pos} \cap \mathcal{N}_{K/k})}$, where $E_k^{\rm pos}$ is the group of *totally positive* units of k, $e_{\mathfrak{l}}^{\rm ab}$ the index of ramification of $\mathfrak{l}$ in $K^{\rm ab}/k$ and $\mathcal{N}_{K/k}$ the group of local norms in K/k. A general formula does exist for non-Galois fields (e.g.,[16, Theorem IV.4.2 & Corollaries]).

4.1.2. Variants of the strong ε-conjecture. Since the p-genus group of K may be an obstruction to the strong ε-conjecture, we may consider, in the exact sequence $1 \to \mathcal{C}\ell'_K \otimes \mathbb{Z}_p \to \mathcal{C}\ell_K \otimes \mathbb{Z}_p \to \mathrm{Gal}(H_{K/\mathbb{Q}}/K) \otimes \mathbb{Z}_p \to 1$, the number $\#(\mathcal{C}\ell'_K \otimes \mathbb{Z}_p)$ (giving the number of exceptional p-classes instead of the whole p-class group) and propose the following form of the ε-conjecture:

Conjecture 4.1. *For a number field K, let H_K be its Hilbert's class field, $H_{K/\mathbb{Q}}$ its genus field and $\mathcal{C}\ell'_K := \mathrm{Gal}(H_K/H_{K/\mathbb{Q}})$. Let p be a prime number. For all $\varepsilon > 0$ there exists $C'_{d,p,\varepsilon}$ such that $\#(\mathcal{C}\ell'_K \otimes \mathbb{Z}_p) \le C'_{d,p,\varepsilon} \cdot (\sqrt{D_K}\,)^{\varepsilon}$ holds for all K of degree d, except possibly for sparse families of density zero.*

One may ask what happens for a "global" strong ε-conjecture on the form:

$$\#\mathcal{C}\ell_K \le \widehat{C}_{d,\varepsilon} \cdot (\sqrt{D_K}\,)^{\varepsilon}.$$

The paper [6] of Ryan Daileda recalls some results by Littlewood showing that (under GRH) there exist imaginary quadratic fields with arbitrary large discriminant for which $\#\mathcal{C}\ell_K \ge c \cdot \sqrt{D_K} \log_2(D_K)$, where c is an absolute constant. For real quadratic fields a result of Montgomery and Weinberger is that there exist real quadratic fields with arbitrary large discriminant whose class numbers satisfy $\#\mathcal{C}\ell_K \ge c \cdot \sqrt{D_K}\, \frac{\log_2(D_K)}{\log(D_K)}$. Analogous results are known for cyclic cubic fields and Daileda proves that there exists an absolute constant $c > 0$ so that there are totally real non-abelian cubic fields, with arbitrary large discriminant satisfying $\#\mathcal{C}\ell_K \ge c \cdot \sqrt{D_K} \left(\frac{\log_2(D_K)}{\log(D_K)}\right)^2$. All this has been generalized to CM number fields in [7].

So, if we consider an inequality of "strong global ε-conjecture" type:

$$\#\mathcal{C}\ell_K \le \widehat{C}_{d,\varepsilon} \cdot (\sqrt{D_K})^{\varepsilon}, \quad \text{for all } K \text{ of degree } d,$$

any *infinite family* $\mathcal{K}$ of fiields K such that $\#\mathcal{C}\ell_K \ge c \cdot \sqrt{D_K}$, for a constant $c > 0$, independent of $K \in \mathcal{K}$, yields:

$$\log(\widehat{C}_{d,\varepsilon}) \ge \log(c) + (1 - \varepsilon) \cdot \log(\sqrt{D_K}\,),$$

which is absurd. In other words, there is in general no strong global ε-conjecture; nevertheless the question of the strong form we have considered

(for p fixed and for $\mathcal{C}\ell_K$ or $\mathcal{C}\ell'_K$):

$$\#(\mathcal{C}\ell_K \otimes \mathbb{Z}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K}\,)^{\varepsilon},$$

depends on the finiteness of fields K such that $\#(\mathcal{C}\ell_K \otimes \mathbb{Z}_p) \geq c_p \cdot \sqrt{D_K}$. For instance, in the quadratic case and $p > 2$, this should give some p-class groups for which the p-rank and/or the exponent tend to infinity with D_K; even in the imaginary case, this may occur only for very sparse families.

4.1.3. Computation of some successive local maxima. The following program, for imaginary quadratic fields, allows a study of this question by computing the successive local maxima of $C'_{2,\varepsilon}$, in C, the discriminant and the class number obtained for each local maximum, in D and h, respectively:

```
{eps=0.05;Cm=0;bD=2;BD=10^9;for(D=bD,BD,e=valuation(D,2);M=D/2^e;
if(core(M)!=M,next);if((e==1||e>3)||(e==0&Mod(M,4)!=-1)||(e==2&Mod(M,4)!=1),
next);h=qfbclassno(-D);N=omega(D);C=h/(2^(N-1)*(sqrt(D)^eps));
if(C>Cm,Cm=C;print("D=",-D," h=",h," C=",C)))}
```

As far as the program has run, it slows down between $\mathsf{C} = 6 \cdot 10^4$ and $\mathsf{C} = 7 \cdot 10^4$, but no conclusion is possible as expected from the above. But the most spectacular fact, that we have discover, is that, for each local maximum C, the corresponding discriminant is prime (whence h odd) whatever ε, as shown by this short excerpt:

```
 D           h       C
-3           1       0.97290843486946871070224166894116640 7
-23          3       2.77381861789069460660685132125197163
-47          5       4.54116788512456422032574050922901447 9
-71          7       6.29240375129760563573361906287211578 5
-167         11      9.67887259926842956029905432916082159 7
-191         13     11.40033250135200530420041581651016836 7
-239         15     13.08070982213482245667961267913645681 9
-311         19     16.46018042090937533079809708596767676 3
-431         21     18.04501980216218208216159247749867928 6
-479         25     21.42553232035947469017824818477988697 9
(...)
-1118314391  77395  45972.57253910331355222092339789302205 5
-1130984399  77697  46138.96360776461282728294161333397435 5
-1139075159  78141  46394.35648869968770045168123604396161 1
-1184068679  81267  48203.63680771683317432383382921075862 4
-1229647319  81419  48248.21489894054599858298640184169402 8
-1237871879  82171  48685.72927956435449720226626156908132 1
-1250370239  83503  49462.50565126222272704002972194038929 6
```

We have no counterexample in the selected interval $D \leq 2 \cdot 10^9$ and no serious explanation, but if we test the local maxima of $\frac{\#\mathcal{C}\ell_K}{(\sqrt{D_K}\,)^{\varepsilon}}$ instead of $\frac{2^{-(N-1)}\,\#\mathcal{C}\ell_K}{(\sqrt{D_K}\,)^{\varepsilon}}$, we have for instance the following normal behaviour:

```
D=[3, 1; 5, 1]          h=2     C=1.86907924178300163332887292329694026 12
D=Mat([23, 1])          h=3     C=2.77381861789069460660685132125197163 2
D=[3, 1; 13, 1]         h=4     C=3.64992025702981056321138523821870543 19
D=Mat([47, 1])          h=5     C=4.54116788512456422032574050922901447 95
```

```
D=Mat([71, 1])           h=7     C=6.292403751297605635733619062872115785 2
D=[5, 1; 19, 1]          h=8     C=7.139156409132720177137076795990845551 7
D=[7, 1; 17, 1]          h=10    C=8.873834525487285759850632858167064985 8
(...)
D=Mat([63839, 1])        h=423   C=320.78438186581184951419633278279394585
D=[23, 1; 2833, 1]       h=424   C=321.37826092893978747476207092089453868
D=[113, 1; 607, 1]       h=434   C=328.53606501556853121813362584226004750
D=[19, 1; 23, 1; 163, 1] h=440   C=332.76370450428389891215784283045533831
D=[41, 1; 1831, 1]       h=454   C=342.90123385517606193005461615124054801
D=[19, 1; 37, 1; 113, 1] h=468   C=352.97586311634473415004028909597003881
D=[43, 1; 1973, 1]       h=480   C=361.43179021472306392157190052394995551
D=Mat([88079, 1])        h=487   C=366.35924203761999218970143789845358049
```

In the same way, if we compute the successive maxima of the 3-class groups, we obtain a similar result:

```
{p=3;bD=1;BD=10^9;Cm=0;for(D=bD,BD,e=valuation(D,2);M=D/2^e;
if(core(M)!=M,next);if((e==1||e>3)||(e==0&Mod(M,4)!=-1)||(e==2&Mod(M,4)!=1),
next);h=qfbclassno(-D);hp=p^valuation(h,p);Cp=hp;if(Cp>Cm,Cm=Cp;
C=log(Cp)/log(sqrt(D));print("D=",-D," h=",h," hp=",hp," C=",C))}
D=-23          h=3     hp=3      C=0.70075861284442195481324
D=-199         h=9     hp=9      C=0.83019007976763598642971
D=-983         h=27    hp=27     C=0.95661698654993161545339
D=-3671        h=81    hp=81     C=1.07074359233325762042197
D=-29399       h=243   hp=243    C=1.06778367209896382287404
D=-178559      h=729   hp=729    C=1.09019287826209803280171
D=-2102999     h=2187  hp=2187   C=1.05643959875714455523718
D=-14868719    h=6561  hp=6561   C=1.06436822551851827813563
D=-98311919    h=19683 hh=19683  C=1.07451592116950263349372
```

Then, for $p = 2$:

```
{p=2;bD=1;BD=10^9;Cm=0;for(D=bD,BD,e=valuation(D,2);M=D/2^e;
if(core(M)!=M,next);if((e==1||e>3)||(e==0&Mod(M,4)!=-1)||(e==2&Mod(M,4)!=1),
next);h=qfbclassno(-D);hp=p^valuation(h,p);Cp=hp;if(Cp>Cm,Cm=Cp;
C=log(Cp)/log(sqrt(D));print("D=",-D," h=",h," hp=",hp," C=",C))}
D=-15          h=2      hp=2      C=0.51191604961963097877535535 7
D=-39          h=4      hp=4      C=0.75680143806748014932554416 2
D=-95          h=8      hp=8      C=0.91326208027946021270580184 6
D=-399         h=16     hp=16     C=0.92589967750355568293970045 0
D=-791         h=32     hp=32     C=1.03868759331275047494288787 0
D=-2519        h=64     hp=64     C=1.06207515934603303597607213 3
D=-10295       h=128    hp=128    C=1.05028965338218139897549157 6
D=-39431       h=256    hp=256    C=1.04800912247037747176961883 3
D=-132599      h=512    hp=512    C=1.05778376718171543436060171 7
D=-328319      h=1024   hp=1024   C=1.09142074599919442326097591 7
D=-1333631     h=2048   hp=2048   C=1.08124429773319866438847447 4
D=-4599839     h=4096   hp=4096   C=1.08434623636863164815987990 2
D=-18855359    h=8192   hp=8192   C=1.07578173625955568996506213 3
D=-63836951    h=16384  hp=16384  C=1.07991825466773727688253810 4
D=-266675639   h=32768  hp=32768  C=1.07179180171460729596093915 0
D=-966467519   h=65536  hp=65536  C=1.07209338875617949823722663 9
```

We shall examine elsewhere all these strange phenomena which seems valid for all p and suggest the existence of families for which the p-part of the class number has maximal values, so that $\dfrac{\log(\#(\mathcal{C}\ell_K \otimes \mathbb{Z}_p))}{\log(\sqrt{D_K}\,)} \to 1$ as $D_K \to \infty$.

4.1.4. Reciprocal study. To try to suggest the existence of analogous families giving huge p-class groups, a trick is to consider normic equations, in integers a, b, of the form:

$$a^2 + m\,b^2 = 4 \cdot q^{p^\rho},\ \ \gcd(a,b) \in \{1,2\},$$

where $q > 1$ is any fixed integer and ρ an exponent as large as possible; then when $a \geq 1$ increases, we deduce b and the square free integer m. This kind of experiment has shown, in [17, § 5.3], that there exist huge discriminants giving interesting p-adic invariants. Moreover the function:

$$C_{K,p} := \frac{\log(\#(\mathcal{C}\ell_K \otimes \mathbb{Z}_p))}{\log(\sqrt{D_K})},$$

giving (for any $\varepsilon > 0$):

$$\log(C_\varepsilon) \geq \big(C_{K,p} - \varepsilon\big) \cdot \log(\sqrt{D_K}),$$

may constitute an obstruction for the strong ε-conjecture as soon as:

$$\liminf_{K \in \mathcal{K}} C_{K,p} > 0$$

for an infinite subfamily $\mathcal{K}$ of the set of imaginary quadratic fields. But a priori, this does not affect the p-rank ε-conjecture.

Of course, the right member of the normic equation being rapidely too large when ρ increases, the experimentation is very limited regarding PARI/GP possibilities. However, even for small values of ρ predicting, a priori, p-classes of order aroud p^ρ we obtain much large orders, and the following numerical results may be convincing enough about the infiniteness of such utmost examples.

```
{p=3;rho=4;q=2;Y=4*q^(p^rho);ba=1;Ba=sqrt(Y);H=1;for(a=ba,Ba,B=Y-a^2;
m=core(B);D=m;if(Mod(m,4)!=-1,D=4*m);b=component(core(B,1),2);
if(gcd(a,b)>2,next);h=qfbclassno(-D);vh=valuation(h,p);hp=p^vh;
if(hp>H,H=hp;Cp=log(hp)/log(sqrt(D));Hp=component(quadclassunit(-D),2);
d=component(matsize(Hp),2);L=List;for(k=1,d,c=component(Hp,k);
w=valuation(c,p);if(valuation(c,p)!=0,listput(L,p^w)));
print("D=",D," a=",a," b=",b," Cp=",Cp," hp=",hp," Hp=",L)))}
rho=4
D=9671406556917033397649407 a=1 b=1 Cp=0.152767 hp=81 Hp=[81]
D=197375644018714967298967  a=5 b=7 Cp=0.204814 hp=243 Hp=[243]
D=9671406556917033397648447 a=31 b=1 Cp=0.229151 hp=729 Hp=[729]
D=9671406556917033397648319 a=33 b=1 Cp=0.267343 hp=2187 Hp=[729,3]
D=9671406556917033397644647 a=69 b=1 Cp=0.305534 hp=6561 Hp=[243,27]
D=9671406556917033397435039 a=463 b=1 Cp=0.343726 hp=19683 Hp=[19683]
D=9671406556917033397373783 a=525 b=1 Cp=0.381918 hp=59049 Hp=[59049]
D=9671406556917033395993039 a=1287 b=1 Cp=0.420110 hp=177147 Hp=[177147]
D=9671406556917033372819119 a=4983 b=1 Cp=0.496494 hp=1594323 Hp=[531441,3]
```

```
D=96714065569170220180937 83 a=106675 b=1 Cp=0.534686 hp=4782969
                                                        Hp=[1594323,3]
{p=2;rho=6;q=2;Y=4*q^(p^rho);ba=1;Ba=sqrt(Y);H=1;for(a=ba,Ba,B=Y-a^2;
m=core(B);D=m;if(Mod(m,4)!=-1,D=4*m);b=component(core(B,1),2);
if(gcd(a,b)>2,next);h=qfbclassno(-D);vh=valuation(h,p);hp=p^vh;
if(hp>H,H=hp;Cp=log(hp)/log(sqrt(D));Hp=component(quadclassunit(-D),2);
d=component(matsize(Hp),2);L=List;for(k=1,d,c=component(Hp,k);
w=valuation(c,p);if(valuation(c,p)!=0,listput(L,p^w)));
print("D=",D," a=",a," b=",b," Cp=",Cp," hp=",hp," Hp=",L))}
rho=6
D=8198552921648689607 a=1 b=3 Cp=0.445646 hp=16384 Hp=[512,2,2,2,2,2]
D=18446744073709551615 a=2 b=2 Cp=0.468750 hp=32768 Hp=[512,4,2,2,2,2]
D=73786976294838206415 a=7 b=1 Cp=0.575757 hp=524288 Hp=[32768,4,2,2]
D=73786976294838175135 a=177 b=1 Cp=0.636363 hp=2097152 Hp=[8192,16,2,2,2,2]
D=73786976294831146815 a=2657 b=1 Cp=0.666666 hp=4194304
                                                    Hp=[16384,8,2,2,2,2,2]
D=8198552921599834167 a=20969 b=3 Cp=0.732133 hp=8388608
                                                    Hp=[65536,4,2,2,2,2,2]
D=73786976293564644495 a=35687 b=1 Cp=0.787878 hp=67108864
                                                    Hp=[2048,512,2,2,2,2,2,2]
D=73786976290585731943 a=65211 b=1 Cp=0.969696 hp=4294967296
                                                    Hp=[33554432,4,2,2,2,2,2]

{p=2;rho=6;q=3;Y=4*q^(p^rho);ba=1;Ba=sqrt(Y);H=1;for(a=ba,Ba,B=Y-a^2;
m=core(B);D=m;if(Mod(m,4)!=-1,D=4*m);b=component(core(B,1),2);
if(gcd(a,b)>2,next);h=qfbclassno(-D);vh=valuation(h,p);hp=p^vh;
if(hp>H,H=hp;Cp=log(hp)/log(sqrt(D));Hp=component(quadclassunit(-D),2);
d=component(matsize(Hp),2);L=List;for(k=1,d,c=component(Hp,k);
w=valuation(c,p);if(valuation(c,p)!=0,listput(L,p^w)));
print("D=",D," a=",a," b=",b," Cp=",Cp," hp=",hp," Hp=",L))}
rho=6
D=13734735281170049938631396357123 a=1 b=1 Cp=0.154682 hp=256 Hp=[64,2,2]
D=536513096920705075727788920 20 a=2 b=32 Cp=0.272429 hp=8192
                                                Hp=[64,4,4,2,2,2]
D=13734735281170049938631396357108 a=4 b=2 Cp=0.270694 hp=16384
                                                Hp=[64,8,2,2,2,2,2]
D=54938941124680199754525585428 3 a=7 b=5 Cp=0.303662 hp=32768
                                                Hp=[64,4,2,2,2,2,2,2,2]
D=38046358119584625868785031460 a=8 b=38 Cp=0.379179 hp=262144
                                                Hp=[512,8,2,2,2,2,2,2]
D=13734735281170049938631396327195  a=173 b=1 Cp=0.464048 hp=16777216
                                                Hp=[32768,8,4,2,2,2,2]
```

We note the exceptional case $D_K = 73786976290585731943$ with

$$\mathcal{C}\ell_K \otimes \mathbb{Z}_2 \simeq \mathbb{Z}/2^{25}\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^5,$$

giving the large value $C_{K,2} = 0.969696$.

One computes that the group $\mathcal{T}_K$ that we shall study in the next section is isomorphic to $\mathbb{Z}/2^2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^6$ which yields $[\widetilde{K} \cap H_K : K] = 2^{25}$, where $\widetilde{K}$ is the compositum of the $\mathbb{Z}_2$-extensions of K.

Many families are described by means of parametized radicals as the family of fields $K = \mathbb{Q}(\sqrt{k^2 - q^n})$ with any prime $q \neq 2$, $k^2 - q^n < 0$, in which $\mathcal{C}\ell_K$ has, under some conditions on the parameters, an element of order n (see [2, Theorem 3.1] and its bibliography); applied to $n = p^r$, we get, for $C_{K,p}$, the upper bound $\frac{O(1)\,r}{p^r} \to 0$ as $r \to \infty$, not sufficient to give "bad families". It is difficult to say if some of the above huge discriminants may be obtained with explicit parametrized expressions.

4.2. Reminders on p-ramification theory. We intend to give now some analogies with the torsion group $\mathcal{T}_K$ of the Galois group of the maximal abelian p-ramified (i.e., unramified outside p and ∞) pro-p-extension of K; this extension contains the p-Hilbert class field of K (in the ordinary sense) and the compositum of the $\mathbb{Z}_p$-extensions of K. This Galois group introduces the "normalized" p-adic regulator of K defined in [21, § 5].

Since in this section the non-p-part of the class group does not intervene, unless otherwise stated, we shall put, by abuse of notation, $\mathcal{C}\ell_K := \mathcal{C}\ell_K \otimes \mathbb{Z}_p$.

4.2.1. Structure of the p-torsion group $\mathcal{T}_K$. Let K be any number field and let $p \geq 2$ be a prime number; we denote by $\mathfrak{p} \mid p$ the prime ideals of K dividing p. Consider the group E_K of p-principal global units of K (i.e., units $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}\mid p} \mathfrak{p}}$). For each $\mathfrak{p} \mid p$, let $K_\mathfrak{p}$ be the $\mathfrak{p}$-completion of K and $\overline{\mathfrak{p}}$ the corresponding prime ideal of the ring of integers of $K_\mathfrak{p}$; then let:

$$U_K := \Big\{ u \in \bigoplus_{\mathfrak{p}\,\mid\, p} K_\mathfrak{p}^\times,\ \ u = 1 + x,\ \ x \in \bigoplus_{\mathfrak{p}\,\mid\, p} \overline{\mathfrak{p}} \Big\} \quad \& \quad W_K := \mathrm{tor}_{\mathbb{Z}_p}(U_K),$$

the $\mathbb{Z}_p$-module of principal local units at p and its torsion subgroup.

We consider the diagonal embedding $E_K \otimes \mathbb{Z}_p \longrightarrow U_K$ whose image is $\overline{E}_K$, the topological closure of E_K in U_K.

We assume in this paper that K satisfies the Leopoldt conjecture at p. Whence the following p-adic result ([21, Lemma 3.1, Corollary 3.2], [16, Lemma III.4.2.4], [26, Définition 2.11, Proposition 2.12]):

Lemma 4.2. *Let μ_K be the group of global roots of unity of p-power order of K. Under the Leopoldt conjecture for p in K, we have $\mathrm{tor}_{\mathbb{Z}_p}(\overline{E}_K) = \mu_K$ and the exact sequence (where* log *is the p-adic logarithm):*

$$1 \to W_K/\mu_K \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}\big(U_K/\overline{E}_K\big) \xrightarrow{\log} \mathrm{tor}_{\mathbb{Z}_p}\big(\log\big(U_K\big)/\log(\overline{E}_K)\big) \to 0.$$

Put $\mathcal{W}_K := W_K/\mu_K$ & $\mathcal{R}_K := \mathrm{tor}_{\mathbb{Z}_p}\big(\log(U_K)/\log(\overline{E}_K)\big)$. Then the above exact sequence becomes $1 \to \mathcal{W}_K \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}\big(U_K/\overline{E}_K\big) \xrightarrow{\log} \mathcal{R}_K \to 0$.

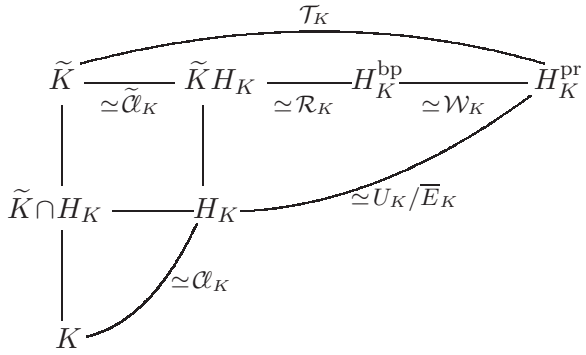
Let $\widetilde{K}$ be the compositum of the $\mathbb{Z}_p$-extensions, H_K the p-Hilbert class field and H_K^{pr} the maximal Abelian p-ramified pro-p-extension, of K. Then let H_K^{bp} be the Bertrandias–Payan field (compositum of the p-cyclic extensions of K embeddable in p-cyclic extensions of arbitrary large degree).

In the following diagram, class field theory yields:

$$\mathrm{Gal}(H_K^{\mathrm{pr}}/H_K) \simeq U_K/\overline{E}_K \text{ and } \mathrm{Gal}(H_K^{\mathrm{pr}}/H_K^{\mathrm{bp}}) \simeq \mathcal{W}_K.$$

We denote by $\widetilde{\mathcal{C}\ell}_K$ the subgroup of the p-class group $\mathcal{C}\ell_K$ corresponding to $\mathrm{Gal}(H_K/\widetilde{K}\cap H_K)$ by class field theory.

Then $\mathcal{R}_K$ is isomorphic to $\mathrm{Gal}(H_K^{\mathrm{bp}}/\widetilde{K}H_K)$:



We have $\#\mathcal{T}_K = \#\widetilde{\mathcal{C}\ell}_K \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K$ and the following inequalities:

$$\begin{aligned} \mathrm{rk}_p(\mathcal{T}_K) &\le \mathrm{rk}_p(\widetilde{\mathcal{C}\ell}_K) + \mathrm{rk}_p(\mathcal{R}_K) + \mathrm{rk}_p(\mathcal{W}_K) \\ &\le \mathrm{rk}_p(\mathcal{C}\ell_K) + r_1 + r_2 - 1 + \#S_K, \end{aligned} \tag{7}$$

where (r_1, r_2) is the signature of K and S_K the set of p-places of K. So, for a constant degree d, the p-rank ε-conjecture for the p-class groups implies the p-rank ε-conjecture for the torsion groups $\mathcal{T}_K$ and conversely since we have the other inequality:

$$\begin{aligned} \mathrm{rk}_p(\mathcal{C}\ell_K) &\le \mathrm{rk}_p(\widetilde{\mathcal{C}\ell}_K) + \mathrm{rk}_p(\mathrm{Gal}(\widetilde{K}\cap H_K/K)) \\ &\le \mathrm{rk}_p(\widetilde{\mathcal{C}\ell}_K) + r_2 + 1 \le \mathrm{rk}_p(\mathcal{T}_K) + r_2 + 1. \end{aligned} \tag{8}$$

For more precise rank formulas for $\mathcal{T}_K$, see [16, Corollary III.4.2.3] and the reflection theorem that we shall recall in § 4.4.

4.2.2. The p-adic Brauer–Siegel conjecture for $\mathcal{T}_K$. We have proposed in [17], for the totally real case, after extensive numerical computations, the following conjecture:

Conjecture 4.3. *Let $p \ge 2$ be prime and let d be a given degree. For any number field K (under Leopoldt's conjecture), let $\mathcal{T}_K$ be the torsion group of the Galois group of the maximal abelian p-ramified pro-p-extension of K. There exists a constant $\widetilde{C}_{d,p}$ such that:*

$$\#\mathcal{T}_K \le (\sqrt{D_K}\,)^{\widetilde{C}_{d,p}}, \text{ for all } K \text{ totally real of degree } d.$$

We put, for p fixed and for any totally real number field K:

$$\widetilde{\mathcal{C}}_{K,p} := \frac{\log(\#\mathcal{T}_K)}{\log(\sqrt{D_K})} \leq \widetilde{\mathcal{C}}_{d,p}. \tag{9}$$

In practice, $\widetilde{\mathcal{C}}_{K,p}$ may be much smaller than 1 (and it is often 0), except very sparse cases as that of $K = \mathbb{Q}(\sqrt{19})$ and $p = 13599893$, for which $\mathcal{T}_K = \mathcal{R}_K \simeq \mathbb{Z}/p\mathbb{Z}$, whence $\widetilde{\mathcal{C}}_{K,p} = \dfrac{\log(\#\mathcal{T}_K)}{\log(\sqrt{4\times 19})} = \dfrac{\log(13599893)}{\log(\sqrt{4\times 19})} = 7.5855.$ But $\mathcal{C}\ell_K = 1$.

4.2.3. Estimation of $\widetilde{\mathcal{C}}_{F_{N,p},p}$. Put $F := F_{N,p}$ and $\widetilde{\mathcal{C}}_{F_N,p} =: \widetilde{\mathcal{C}}_F$ for p fixed. Then, from the computations in Subsection 2.3, using for $\mathcal{T}_F$ the analog of Chevalley's formula given in [16, Theorem IV.3.3], we can put similarly $\#\mathcal{T}_F =: p^{N-r+\widetilde{\Delta}(N)}$, $\widetilde{\Delta}(N) \geq 0$, where $r \geq 0$ depends on p-adic properties of the ramified primes $\ell \neq p$, and we may estimate that, as $N \to \infty$:

$$\widetilde{\mathcal{C}}_F \approx \frac{(N - r + \widetilde{\Delta}(N))\log(p)}{\frac{p-1}{2}\left[N\log(N) + N\,\gamma_p + \frac{1}{2}\log(N) + O(1)\right]} \sim c_p \cdot \frac{1+o(1)}{\log(N)},$$

where $c_p = \frac{2\log(p)}{p-1}$ and assuming a small order of magnitude of $\widetilde{\Delta}(N)$.

Give a program computing (in Cp) $\widetilde{\mathcal{C}}_F$; in the imaginary quadratic case for $p = 2$, the conjectural inequality implies $\#(\widetilde{\mathcal{C}}\ell_F) \leq (\sqrt{D_F})^{\widetilde{\mathcal{C}}_{2,2}}$ (indeed, $\#\mathcal{R}_F = 1$ and $\#\mathcal{T}_F = \#\widetilde{\mathcal{C}}\ell_F \cdot \mathcal{W}_F$, where $\mathcal{W}_F = 2$ (resp. 1) if $m \equiv \pm 1 \pmod 8$ (resp. if not)). So $\mathcal{C}\ell_F$ (in Clres) may be larger than $\#\mathcal{T}_F$ (in Tor):

```
{p=2;n=12;m=1;for(N=2,100,el=prime(N);m=(-1)^((el-1)/2)*el*m;P=x^2-m;
K=bnfinit(P,1);D=abs(m);Kpn=bnrinit(K,p^n);r=1;if(m<0,r=2);L=List;
Hpn=component(component(Kpn,5),2);e=component(matsize(Hpn),2);T=1;
for(k=1,e-r,c=component(Hpn,e-k+1);if(Mod(c,p)==0,q=p^valuation(c,p);
T=T*q;listinsert(L,q,1)));C8=component(K,8);C81=component(C8,1);
h=component(C81,1);Clord=component(C81,2);K=bnfnarrow(K);
Cl=component(K,2);print("m=",m," Clres=",Clres," Clord =",Clord);
print("Structure of T=",L);print("#Tor=",T," Cp=",log(T)/log(sqrt(D))))}

m=-15   Clres=[2]
Structure of Tor=[2]
#Tor=2              Cp=0.51191604961963097877535535772960454081
m=105   Clres=[2,2]=[2]xClord
Structure of Tor=[2,2]
#Tor=4              Cp=0.59574824743531323067786608868687642325
m=-1155   Clres=[2,2,2]
Structure of Tor=[2,2,2]
#Tor=8              Cp=0.58975726471501581115878339498474155345
m=-15015   Clres=[12,2,2,2]
Structure of Tor=[2,2,2,2]
#Tor=16             Cp=0.57661327808675875001115538902772596330
m=-255255   Clres=[16,2,2,2,2]
Structure of Tor=[2,2,2,2,2]
#Tor=32             Cp=0.55674390390043840097934284424073618196
```

```
m=4849845   Clres=[4,2,2,2,2,2]=[2]xClord
Structure of Tor=[4,2,2,2,2,2]
#Tor=128            Cp=0.630360676991495277038104950758385809 18
m=-111546435   Clres=[42,2,2,2,2,2,2]
Structure of Tor=[2,2,2,2,2,2,2]
#Tor=128            Cp=0.523695948021823169408235983903668654 09
m=-3234846615   Clres=[308,2,2,2,2,2,2]
Structure of Tor=[4,2,2,2,2,2,2,2]
#Tor=512            Cp=0.569781628298232806460495028873981090 11
m=100280245065   Clres=[2,2,2,2,2,2,2,2]=[2]xClord
Structure of Tor=[2,2,2,2,2,2,2,2]
#Tor=512            Cp=0.492540117753113272971871058158919797 76
m=3710369067405   Clres=[34,2,2,2,2,2,2,2,2]=[2]xClord
Structure of Tor=[2,2,2,2,2,2,2,2,2]
#Tor=1024           Cp=0.478987996043361052403494647510699891 67
m=152125131763605   Clres=[2,2,2,2,2,2,2,2,2,2]=[2]xClord
Structure of Tor=[4,2,2,2,2,2,2,2,2,2]
#Tor=4096           Cp=0.509421627329977331859912688182144590 06
m=-6541380665835015   Clres=[28284,2,2,2,2,2,2,2,2,2,2]
Structure of Tor=[2,2,2,2,2,2,2,2,2,2,2]
#Tor=4096           Cp=0.456807720414074067382427576978255794 15
m=307444891294245705   Clres=[14,2,2,2,2,2,2,2,2,2,2,2]=[2]xClord
Structure of Tor=[2,2,2,2,2,2,2,2,2,2,2,2]
#Tor=8192           Cp=0.447557413221261954433129893660824580 89
m=16294579238595022365   Clres=[2,2,2,2,2,2,2,2,2,2,2,2,2]=[2]xClord
Structure of Tor=[32,2,2,2,2,2,2,2,2,2,2,2,2]
#Tor=262144         Cp=0.564077425758339760131648911022717366 55
m=-961380175077106319535   Clres=[1210734,2,2,2,2,2,2,2,2,2,2,2,2,2]
Structure of Tor=[2,2,2,2,2,2,2,2,2,2,2,2,2,2]
#Tor=32768          Cp=0.430393413307650140329071017748555773 16
m=-58644190679703485491635   Clres=[1622526,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
Structure of Tor=[2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
#Tor=65536          Cp=0.423087871911122270687039920141079428 67
```

We do not know an algorithm computing the "exceptional elements" of $\mathcal{T}_F$ as for p-class groups.

The case $p = 3$ is similar and gives for instance for the 16 cyclic cubic fields of conductor $f = 7 \cdot 13 \cdot 19 \cdot 31 \cdot 37$:

```
P=x^3+x^2-661054*x+49725976       Cl=[3,3,3,3,3]
Structure of Tor=[9,3,3,3,3]
#Tor=729   Cp=0.45459180523024141673723101157712338880
P=x^3+x^2-661054*x+198463201      Cl=[6,6,3,3]
Structure of Tor=[9,3,3,3,3]
#Tor=729   Cp=0.45459180523024141673723101157712338880
P=x^3+x^2-661054*x+97321888       Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243   Cp=0.37882650435853451394769250964760282400
P=x^3+x^2-661054*x-186270421      Cl=[39,3,3,3,3]
Structure of Tor=[9,3,3,3,3]
#Tor=729   Cp=0.45459180523024141673723101157712338880
P=x^3+x^2-661054*x-79179619       Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3]
```

```
#Tor=81    Cp=0.303061203486827611158154007718082259 20
P=x^3+x^2-661054*x-188253584      Cl=[6,6,3,3]
Structure of Tor=[9,3,3,3,3]
#Tor=729   Cp=0.454591805230241416737231011577123388 80
P=x^3+x^2-661054*x+138968311      Cl=[3,3,3,3,3]
Structure of Tor=[3,3,3,3]
#Tor=81    Cp=0.303061203486827611158154007718082259 20
P=x^3+x^2-661054*x-146607161      Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243   Cp=0.378826504358534513947692509647602824 00
P=x^3+x^2-661054*x-158506139      Cl=[3,3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243   Cp=0.378826504358534513947692509647602824 00
P=x^3+x^2-661054*x-140657672      Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243   Cp=0.378826504358534513947692509647602824 00
P=x^3+x^2-661054*x+186564223      Cl=[6,6,3,3]
Structure of Tor=[3,3,3,3]
#Tor=81    Cp=0.303061203486827611158154007718082259 20
P=x^3+x^2-661054*x+81456584       Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243   Cp=0.378826504358534513947692509647602824 00
P=x^3+x^2-661054*x+206395853      Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3]
#Tor=81    Cp=0.303061203486827611158154007718082259 20
P=x^3+x^2-661054*x-206102051      Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3]
#Tor=81    Cp=0.303061203486827611158154007718082259 20
P=x^3+x^2-661054*x+2130064        Cl=[12,12,3,3]
Structure of Tor=[3,3,3,3,3,3]
#Tor=729   Cp=0.454591805230241416737231011577123388 80
P=x^3+x^2-661054*x+27911183       Cl=[3,3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243   Cp=0.378826504358534513947692509647602824 00
```

Now give some examples for $p = 2$ and $p = 3$ where the constant $\widetilde{\mathcal{C}}_F$ is rather large; the degree p cyclic extensions F are not necessary of the form F_N.

(i) $p = 2$, m divides $5 \cdot 7 \cdots 41 \cdot 43$.

```
m=5005 Clres=[2,2,2] Clord =[2,2]
Structure of Tor=[4,2,2]
#Tor=16      Cp=0.650980512550578216644892474705583907 64
m=1078282205 Clres=[4,2,2,2,2,2,2] Clord =[4,2,2,2,2,2]
Structure of Tor=[32,4,2,2,2,2,2]
#Tor=4096    Cp=0.799837695345799798526117004572367490 58
m=215656441 Clres=[2,2,2,2,2,2] Clord =[2,2,2,2,2]
Structure of Tor=[8,2,2,2,2,2]
#Tor=256     Cp=0.577947831957780515341525239621331806 36
m=436092044389001 Clres=[2,2,2,2,2,2,2,2,2,2] Clord =[2,2,2,2,2,2,2,2,2]
Structure of Tor=[32,2,2,2,2,2,2,2,2,2]
#Tor=16384   Cp=0.575757018693368765603504001037074965 94
m=46189 Clres=[2,2,2] Clord =[2,2]
Structure of Tor=[16,2,2]
```

```
#Tor=64      Cp=0.774430289654552790953875560521617456 44
m=221 Clres=[4] Clord =[2]
Structure of Tor=[16]
#Tor=16      Cp=1.027234218583384833339701021166259299 4
m=435656388001 Clres=[2,2,2,2,2,2,2] Clord =[2,2,2,2,2,2]
Structure of Tor=[8,2,2,2,2,2,2]
#Tor=512     Cp=0.465544535036782352293093962940364175 44
```

We note the case of $m = 221$ for which $\mathcal{C}\ell_F^{\rm res} \simeq \mathbb{Z}/4\mathbb{Z}$, $\mathcal{C}\ell_F^{\rm ord} \simeq \mathbb{Z}/2\mathbb{Z}$, but $\mathcal{T}_F \simeq \mathbb{Z}/16\mathbb{Z}$ due to the 2-adic regulator. This gives the exceptional value $\widetilde{\mathcal{C}}_{F,2} \approx 1.02723422$.

(ii) $p = 3$, f divides $13 \cdot 19 \cdot 31 \cdot 37 \cdot 43$.

```
f=10621   P=x^3+x^2-3540*x-60579    Cl=[3,3]
Structure of Tor=[9,9,3]
#Tor=243     Cp=0.336677613825041926919634840737486842 74
P=x^3+x^2-4060762*x-3150249179    Cl=[3,3,3,3]
Structure of Tor=[27,27,3,3,3]
#Tor=19683   Cp=0.606019704885075468455342713327476316 93
f=12182287   P=x^3+x^2-4060762*x+187697459    Cl=[9,9,3,3,3]
Structure of Tor=[9,9,3,3,3]
#Tor=2187    Cp=0.471348659355058697687488777032481579 83
f=12182287   P=x^3+x^2-4060762*x+2380509119    Cl=[3,3,3,3]
Structure of Tor=[9,3,3,3,3]
#Tor=729     Cp=0.404013136590050312303561808884984211 29
f=12182287   P=x^3+x^2-4060762*x-2309671376    Cl=[3,3,3,3,3]
Structure of Tor=[9,9,3,3,3]
#Tor=2187    Cp=0.471348659355058697687488777032481579 83
f=641173   P=x^3-x^2-213724*x-29968901    Cl=[6,6,3]
Structure of Tor=[9,3,3,3]
#Tor=243     Cp=0.336677613825041926919634840737486842 74
f=392977   P=x^3-x^2-130992*x+6826156    Cl=[3,3,3]
Structure of Tor=[9,3,3,3]
#Tor=243     Cp=0.336677613825041926919634840737486842 74
f=7657   P=x^3+x^2-2552*x+47360    Cl=[9,3,3]
Structure of Tor=[9,9,3]
#Tor=243     Cp=0.336677613825041926919634840737486842 74
```

4.3. *p*-adic Brauer–Siegel conjecture versus ε-conjectures. The p-adic Brauer–Siegel Conjecture 4.3 concerns more essentially the totally real case for the following reasons which are yet given by the rank inequalities (7) and (8).

4.3.1. Analysis by means of CM-fields. We have, with obvious notations and $p \neq 2$:

$$\mathcal{C}\ell_K = \mathcal{C}\ell_K^- \bigoplus \mathcal{C}\ell_K^+, \qquad \mathcal{T}_K = \mathcal{T}_K^- \bigoplus \mathcal{T}_K^+,$$
$$\mathcal{R}_K = \mathcal{R}_K^- \bigoplus \mathcal{R}_K^+, \qquad \mathcal{W}_K = \mathcal{W}_K^- \bigoplus \mathcal{W}_K^+;$$

but we have the following properties which explain the differences between real fields and non real ones (under the Leopoldt conjecture):

(i) $\#\mathcal{R}_K^- = 1$ since all the units of infinite order of K are real;

(ii) $\#\mathcal{C}\ell_K^- = \#\widetilde{\mathcal{C}\ell}_K^- \cdot \#\mathrm{Gal}(\widetilde{K} \cap H_K/K)^-$ where $\mathrm{Gal}(\widetilde{K} \cap H_K/K)^- \simeq \mathcal{C}\ell_K^-/\widetilde{\mathcal{C}\ell}_K^-$ may be large (but with bounded rank) since $\mathrm{Gal}(\widetilde{K}/K)^- \simeq \mathbb{Z}_p^{\frac{d}{2}}$ contrary to $\mathrm{Gal}(\widetilde{K}/K)^+ \simeq \mathbb{Z}_p$;

(iii) $\#\mathcal{T}_K^- = \#\widetilde{\mathcal{C}\ell}_K^- \cdot \#\mathcal{W}_K^-$ is essentially equal to $\#\widetilde{\mathcal{C}\ell}_K^-$ since $\mathcal{W}_K^-$ is most often trivial and does not intervene in estimations of class groups for d fixed;

(iv) $\#\mathcal{R}_K^+$ is the main p-adic invariant which may be nontrivial for much primes p, even if we have conjectured in [24] that it is trivial for all p large enough;

(v) $\#\mathcal{C}\ell_K^+$ is essentially equal to $\#\widetilde{\mathcal{C}\ell}_K^+$ since the part of the Hilbert class field, contained in the cyclotomic $\mathbb{Z}_p$-extension, is very limited;

(vi) $\#\mathcal{T}_K^+ = \#\widetilde{\mathcal{C}\ell}_K^+ \cdot \#\mathcal{R}_K^+ \cdot \#\mathcal{W}_K^+$ is thus essentially the product $\#\mathcal{C}\ell_K^+ \cdot \#\mathcal{R}_K^+$.

So if we assume that $\#\mathcal{T}_K^+$ is controled, we may consider that $\#\mathcal{C}\ell_K^+$ is much less than $\#\mathcal{T}_K^+$ because of the regulator; then, since $\#\mathcal{T}_K^-$ is independent of any regulator, it is mesured by a divisor of $\#\mathcal{C}\ell_K^-$ (equal to $\#\widetilde{\mathcal{C}\ell}_K^-$) and by $\#\mathcal{W}_K^-$ controled, which explains (from the factor $[\widetilde{K} \cap H_K : K]$) a bigger order of magnitude of $\#\mathcal{C}\ell_K^-$ regarding $\#\mathcal{T}_K^-$ than $\#\mathcal{C}\ell_K^+$ regarding $\#\mathcal{T}_K^+$.

4.3.2. Computation of $\widetilde{\mathcal{C}}_{K,p}$ for imaginary quadratic fields. We shall illustrate the cases $p = 2$, then $p = 3$, in various intervals of negative discriminants to observe the local decreasing of the variable $\widetilde{\mathcal{C}}_{K,p}$ (9).

Note that for $p > 3$, the group $\mathcal{W}_K$ is trivial contrary to the cases $p = 2$ and 3 where $\mathcal{W}_K$ may be $\mathbb{Z}/p\mathbb{Z}$, which must probably be discarded in our considerations.

(a) Case $p = 2$. The case $p = 2$ is interesting because of the influence of exceptional classes and gives ($v_p(\#\mathcal{T}_K)$ in vptor, $\widetilde{\mathcal{C}}_{K,p}$ in Cp):

```
{p=2;bD=10^8;BD=2*10^8;Lp=log(p);vp=0;n=20;
for(D=bD,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3)||(e==0 & Mod(M,4)!=-1)||(e==2 & Mod(M,4)==-1),next);
m=D;if(e!=0,m=D/4);P=x^2+m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p)-(n-1);
if(vptor>vp,vp=vptor);if(vptor>=vp,Cp=vptor*Lp/log(sqrt(D));
print("D=",-D," m=",-m," vptor=",vptor," Cp=",Cp)))}

p=2, Interval [10^6, 2*10^6]
D=-1000011 m=-1000011    vptor=3   Cp=0.30102975598343592993344579 3
D=-1000020 m=-250005     vptor=3   Cp=0.30102955988341649589389941 88
D=-1000036 m=-250009     vptor=4   Cp=0.40137228168819760538120614 27
D=-1000132 m=-250033     vptor=5   Cp=0.50171186616102856876824493 15
(...)
D=-1003620 m=-250905     vptor=5   Cp=0.50158546915117461334327775 19
D=-1003940 m=-250985     vptor=6   Cp=0.60188867794243255322386671 09
```

```
D=-1005843 m=-1005843    vptor=7   Cp=0.702107244840955486811297669
D=-1007492 m=-251873     vptor=8   Cp=0.80231319118712060282768700 51
(...)
D=-1327972 m=-331993     vptor=8   Cp=0.78659665658319691092490074 96
D=-1345476 m=-336369     vptor=9   Cp=0.88410011254375912149444467 38
D=-1347524 m=-336881     vptor=10  Cp=0.98222759657812904087763114 5

p=2, Interval [10^7, 2*10^7]
D=-10000004 m=-2500001   vptor=3   Cp=0.25802570416574861895099915
(...)
D=-10000136 m=-2500034   vptor=3   Cp=0.25802549285588586651933610
D=-10000212 m=-2500053   vptor=4   Cp=0.34403382825873844184039068
D=-10000228 m=-2500057   vptor=5   Cp=0.43004224263528166242354902
(...)
D=-10001220 m=-2500305   vptor=5   Cp=0.43003959612044524303603555
D=-10001355 m=-10001355  vptor=7   Cp=0.60205493037545335612706 83
(...)
D=-10028164 m=-2507041   vptor=10  Cp=0.85993565199906305664334 45
D=-11423624 m=-2855906   vptor=10  Cp=0.85304154074284467596277 85
D=-11434244 m=-2858561   vptor=11  Cp=0.93829204458797711306639 80
D=-19227908 m=-4806977   vptor=11  Cp=0.90921495043360109692441 16

p=2, Interval [10^8, 2*10^8]
D=-100000011 m=-100000011 vptor=2   Cp=0.15051499693318294862950
D=-100000020 m=-25000005  vptor=3   Cp=0.22577249429669243057496 9
(...)
D=-100000072 m=-25000018  vptor=3   Cp=0.22577248792333196292489 1
D=-100000120 m=-25000030  vptor=4   Cp=0.30102997605364433827863 4
(...)
D=-100000228 m=-25000057  vptor=4   Cp=0.30102995840436347172636 5
D=-100000324 m=-25000081  vptor=6   Cp=0.45154491407419732789549 6
(...)
D=-100009811 m=-100009811  vptor=6  Cp=0.45154258866246136591601
D=-100009988 m=-25002497   vptor=7  Cp=0.52679963615921044837325 2
(...)
D=-100042692 m=-25010673   vptor=9  Cp=0.67730179636262193119943 7
D=-100120215 m=-100120215  vptor=11 Cp=0.827778498960280742930 3
D=-100703939 m=-100703939  vptor=11 Cp=0.827517363447336823439 3
D=-101091716 m=-25272929   vptor=13 Cp=0.977771142543422825517 17
D=-196241540 m=-49060385   vptor=13 Cp=0.943805281087295501440 90
```

One sees some influence of genus theory since, for $D = -101091716$, we have $\widetilde{\mathcal{C}}_{K,2} \approx 0.977771$ because of $\#\mathcal{T}_K = 2^{13}$, but to be put in relation with $\widetilde{\mathcal{C}}_{K,2} \approx 0.982227$ for $D = -1347524$, of the first interval, with $\#\mathcal{T}_K = 2^{10}$.

The structure of the class group given by PARI/GP is $[1024, 2, 2, 2]$.

Then consider the program computing the structure of $\mathcal{T}_K$ [23, Programme I, § 3.2] that we recall for the convenience of the reader (choose p, nt such that p^{nt} be a multiple of the exponent of $\mathcal{T}_K$, then the polynomial P):

```
{p=2;nt=32;P=x^2+101091716;K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);
S=component(component(Kpn,1),7);r=component(component(S,2),2)+1;
```

```
Hpn=component(component(Kpn,5),2);L=List;e=component(matsize(Hpn),2);
R=0;for(k=1,e-r,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;
listinsert(L,p^valuation(c,p),1)));print("Structure of T: ",L)}
```

Then we obtain $\mathcal{T}_K \simeq \mathbb{Z}/2^{10}\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The difference comes from $\mathcal{W}_K \simeq \mathbb{Z}/2\mathbb{Z}$ (since $-101091716 \equiv -1 \pmod{16}$) and $[\widetilde{K} \cap H_K : K] = 2$.

(b) Case $p = 3$.

```
{p=3;bD=10^6;BD=2*10^6;Lp=log(p);vp=0;n=8;
for(D=bD,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3)||(e==0 & Mod(M,4)!=-1)||(e==2 & Mod(M,4)==-1),next);
m=D;if(e!=0,m=D/4);P=x^2+m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p)-(n-1);
if(vptor>vp,vp=vptor);if(vptor>=vp,Cp=vptor*Lp/log(sqrt(D));
print("D=",-D," m=",-m," vptor=",vptor," Cp=",Cp)))}
p=3, Interval [10^6, 2*10^6]
D=-1000011 m=-1000011     vptor=1   Cp=0.15904029161166201314203485 20
D=-1000020 m=-250005      vptor=1   Cp=0.15904018800793628192781961 25
D=-1000043 m=-1000043     vptor=1   Cp=0.15903992324770874163046635 99
(...)
D=-1020548 m=-255137      vptor=3   Cp=0.47641985068062433717831070 10
D=-1021332 m=-255333      vptor=4   Cp=0.63519121307967637890964575 80
D=-1022687 m=-1022687     vptor=5   Cp=0.79391294390880033794338891 302
(...)
D=-1898859 m=-1898859     vptor=5   Cp=0.75992961400032134557533065 74
p=3, Interval [10^7, 10^7+10^6]
D=-10002927 m=-10002927   vptor=3   Cp=0.40895365007950900178024254
D=-10003224 m=-2500806    vptor=4   Cp=0.54527052902405852884991426
(...)
D=-10065279 m=-10065279   vptor=4   Cp=0.54506139909286008239956811
D=-10066440 m=-2516610    vptor=5   Cp=0.68132187532448910070906763
(...)
D=-14316744 m=-3579186    vptor=5   Cp=0.66675746065456780942779488
D=-14547531 m=-14547531   vptor=6   Cp=0.79933316809564910995167969
(...)
D=-19767512 m=-4941878    vptor=6   Cp=0.78474406738375920976115602
p=3, Interval [10^8, 10^8+10^7]
D=-100075971 m=-100075971  vptor=4   Cp=0.47710158545562108829625 7
D=-100080003 m=-100080003  vptor=5   Cp=0.59637567751709031081139 1
(...)
D=-100787315 m=-100787315  vptor=5   Cp=0.59614776775460508148221 6
D=-100867844 m=-25216961   vptor=6   Cp=0.71534631865650297347805 3
(...)
D=-117344127 m=-117344127  vptor=6   Cp=0.70952134255385508340293 0
D=-119846559 m=-119846559  vptor=7   Cp=0.82683589044322150833198 5
(...)
D=-135140024 m=-33785006   vptor=7   Cp=0.82153179482816497011618 6
D=-136159455 m=-136159455  vptor=8   Cp=0.93851674579229036761487 3
```

For $D_K = -136159455$, we get $\mathcal{W}_K \simeq \mathbb{Z}/3\mathbb{Z}$, $\mathcal{T}_K \simeq \mathbb{Z}/3^7\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathcal{C}\ell_K \simeq \mathbb{Z}/3^6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ (using the instruction quadclassunit(-136159455)).

4.3.3. Conclusion. Consider a prime p and a fixed degree d. Assume, tentatively, a strong ε-conjecture for the groups $\mathcal{T}_K$ in the case of totaly real number fields K of degree d, which implies a strong ε-conjecture for the p-class groups. We then have the existence, for all $\varepsilon > 0$, of a constant $\widetilde{C}_{d,p,\varepsilon}$ such that:

$$\log(\#\mathcal{T}_K) \leq \log(\widetilde{C}_{d,p,\varepsilon}) + \varepsilon \cdot \log\big(\sqrt{D_K}\,\big);$$

then introduce the function $\widetilde{\mathcal{C}}_{K,p}$:

$$\widetilde{\mathcal{C}}_{K,p} := \frac{\log(\#\mathcal{T}_K)}{\log\big(\sqrt{D_K}\,\big)} \leq \frac{\log(\widetilde{C}_{d,p,\varepsilon})}{\log\big(\sqrt{D_K}\,\big)} + \varepsilon. \tag{10}$$

So, when $D_K \to \infty$, we get $\widetilde{\mathcal{C}}_{K,p} = \varepsilon + o(1)$. But in [17, § 5.3], we have proved that there exist explicit infinite families of real quadratic fields for which $\widetilde{\mathcal{C}}_{K,p} \approx 1$ (contradiction).

We obtain, generalizing to arbitrary degrees, the following heuristic:

There is no absolute strong ε-conjecture for the $\mathcal{T}_K$ groups of totally real number fields K and $\mathcal{T}_K$ is essentially governed by the normalized p-adic regulator $\mathcal{R}_K$. Nevertheless, as for p-class groups, one may conjecture that the exceptions to the strong ε-conjecture are due to sparse subfamilies of density zero.

Recall that, from (7) and (8), the p-rank ε-conjecture for the $\mathcal{T}_K$ does exist if and only if the p-rank ε-conjecture does exist for the p-class groups $\mathcal{C}\ell_K$.

4.4. Reflection theorem and p-rank ε-conjectures. Another justification of the above comments is to recall the reflection theorem [22] which exchanges, roughly speaking, "imaginary components" of p-class groups $\mathcal{C}\ell$ with "real components" of p-torsion groups $\mathcal{T}$, and conversely, subject to consider fields K containing the group μ_p of pth roots of unity.

In full generality, the following result precises (7) and (8) when $\mu_p \subset K$:

Proposition 4.4 ([16, Theorem III.4.2.2])**.** *Let K be a number field containing μ_p and fulfilling Leopoldt's conjecture at p. Then we have the rank formula (reflection theorem)*[1] $\mathrm{rk}_p(\mathcal{T}_K^{\mathrm{ord}}) = \mathrm{rk}_p(\mathcal{C}\ell_K^{S\,\mathrm{res}}) + \#S - 1$, *where S is the set of p-places of K and $\mathcal{C}\ell_K^{S\,\mathrm{res}}$ the S-class group $\mathcal{C}\ell_K^{\mathrm{res}}/\mathcal{C}\ell^{\mathrm{res}}(S)$.*

So, $\mathrm{rk}_p(\mathcal{C}\ell_K^{S\,\mathrm{res}}) = \mathrm{rk}_p(\mathcal{T}_K^{\mathrm{ord}}) - (\#S - 1)$ yields:

$$\begin{aligned}\mathrm{rk}_p(\mathcal{C}\ell_K^{\mathrm{res}}) &\leq \mathrm{rk}_p(\mathcal{C}\ell_K^{S\,\mathrm{res}}) + \mathrm{rk}_p(\mathcal{C}\ell^{\mathrm{res}}(S))\\ &\leq \mathrm{rk}_p(\mathcal{T}_K^{\mathrm{ord}}) - \big[(\#S - 1) - \mathrm{rk}_p(\mathcal{C}\ell^{\mathrm{res}}(S))\big] \leq \mathrm{rk}_p(\mathcal{T}_K^{\mathrm{ord}}) + 1,\end{aligned}$$

giving :

$$-1 \leq \mathrm{rk}_p(\mathcal{T}_K^{\mathrm{ord}}) - \mathrm{rk}_p(\mathcal{C}\ell_K^{\mathrm{res}}) \leq \#S - 1;$$

[1]The mentions "ord", "res" are related to the case $p = 2$; for $p > 2$, since $\mu_p \subset K$, the two notions coincide.

since $\#S-1$ is bounded in the family of number fields, of fixed degree d, this relation shows that any p-rank ε-conjecture, true for an invariant, is fulfilled by the other. When $\mu_p \not\subset K$, one must use the field $K(\mu_p)$ and the general reflection theorem with p-adic characters ([16, Theorem II.5.4.5], [22]).

The reflection theorem has been used in [10], in a different manner, using many split primes in $K(\mu_p)$, the notion of Arakelov class group (see e.g., [37]), and the following analytic explanation by the authors:

Roughly, the point is that small non-inert primes in a number field represent elements of the class group which tend not to satisfy any relation with small coefficients. Thus the existence of many such primes contributes significantly to the quotient of the class group by its ℓ-torsion, yielding the desired upper bounds.

It would be interesting to deepen these approaches that have connections through class field theory, complex and p-adic analytic methods.

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Villa la Gardette, 4 Chemin Château Gagnière, F–38520, Le Bourg d'Oisans.
g.mn.gras@wanadoo.fr