



**HAL**  
open science

## Genus theory and $\epsilon$ -conjectures on p-class groups

Georges Gras

► **To cite this version:**

| Georges Gras. Genus theory and  $\epsilon$ -conjectures on p-class groups. 2019. hal-02059441v3

**HAL Id: hal-02059441**

**<https://hal.science/hal-02059441v3>**

Preprint submitted on 25 Jun 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Genus theory and $\varepsilon$ -conjectures on $p$ -class groups

Georges Gras

ABSTRACT. We suspect that the “genus part” of the class number of a number field  $K$  may be an obstruction for an “easy proof” of the classical  $p$ -rank  $\varepsilon$ -conjecture for  $p$ -class groups and, a fortiori, for a proof of the “strong  $\varepsilon$ -conjecture”:  $\#(\mathcal{C}_K \otimes \mathbb{Z}_p) \ll_{d,p,\varepsilon} (\sqrt{D_K})^\varepsilon$  for all  $K$  of degree  $d$ . We analyze the weight of genus theory in this inequality by means of an infinite family of degree  $p$  cyclic fields with many ramified primes, then we prove the  $p$ -rank  $\varepsilon$ -conjecture:  $\#(\mathcal{C}_K \otimes \mathbb{F}_p) \ll_{d,p,\varepsilon} (\sqrt{D_K})^\varepsilon$ , for  $d = p$  and the family of degree  $p$  cyclic extensions (Theorem 2.5) then sketch the case of arbitrary base fields. The possible obstruction for the strong form, in the degree  $p$  cyclic case, is the order of magnitude of the set of “exceptional”  $p$ -classes given by a well-known non-predictible algorithm, but controled thanks to recent density results due to Koymans–Pagano. Then we compare the  $\varepsilon$ -conjectures with some  $p$ -adic conjectures, of Brauer–Siegel type, about the torsion group  $\mathcal{T}_K$  of the Galois group of the maximal abelian  $p$ -ramified pro- $p$ -extension of totally real number fields  $K$ . We give numerical computations with the corresponding PARI/GP programs.

## CONTENTS

1. Introduction	2
2. A possible obstruction due to exceptional $p$ -classes	4
2.1. Definition of the family $(F_{N,p})_{N \geq 1}$	4
2.2. About the exceptional $p$ -classes	4
2.2.1. The algorithm	5
2.2.2. Examples of maximal rank	7
2.3. Estimation of $C_{p,p,\varepsilon}$ for the fields $F_{N,p}$	8
2.4. A lower bound for $C_{p,p,\varepsilon}$	10
3. PARI/GP programs computing $\mathcal{C}_{F_{N,p}}$	11
3.1. Structure of some $\mathcal{C}_{F_{N,p}}$	11
3.2. Examples of exceptional $p$ -classes	13
4. Genus theory – Abelian $p$ -ramification theory	15

---

*Date:* June 25, 2019.

*2010 Mathematics Subject Classification.* 11R29, 11R37, 08-04.

*Key words and phrases.* class groups;  $\varepsilon$ -conjecture; class field theory;  $p$ -ramification; genus theory.

4.1.	Reminders on Genus theory	15
4.1.1.	Genus field and genus number $g_{K/k}$	15
4.1.2.	Variants of the strong $\varepsilon$ -conjecture	16
4.1.3.	Computation of some successive local maxima	17
4.1.4.	Reciprocal study	19
4.2.	Reminders on $p$ -ramification theory	21
4.2.1.	Structure of the $p$ -torsion group $\mathcal{T}_K$	21
4.2.2.	The $p$ -adic Brauer–Siegel conjecture for $\mathcal{T}_K$	22
4.2.3.	Estimation of $\tilde{\mathcal{C}}_{F_{N,p},p}$	23
4.3.	$p$ -adic Brauer–Siegel conjecture versus $\varepsilon$ -conjectures	26
4.3.1.	Analysis by means of CM-fields	26
4.3.2.	Computation of $\tilde{\mathcal{C}}_{K,p}$ for imaginary quadratic fields	27
4.3.3.	Conclusion	30
4.4.	Reflection theorem and $p$ -rank $\varepsilon$ -conjectures	30
	Acknowledgments	31
	References	31

## 1. Introduction

For any number field  $K$ , we denote by  $\mathcal{C}_K$  the class group of  $K$  in the restricted sense and by  $\mathcal{C}_K \otimes \mathbb{Z}_p$  its  $p$ -class group, for any prime number  $p$ ; to avoid any ambiguity, we shall write  $\mathcal{C}_K \otimes \mathbb{F}_p$  for the “ $p$ -torsion group”, often denoted  $\mathcal{C}_K[p]$  in most papers, only giving the  $p$ -rank  $\text{rk}_p(\mathcal{C}_K)$  of  $\mathcal{C}_K$ . One knows the following classical result (weak form of theorems of Brauer, Brauer–Siegel, Tsfasman–Vladuț–Zykin [41, 44]):

*For all  $\varepsilon > 0$  there exists a constant  $C_{d,\varepsilon}$  such that  $\#\mathcal{C}_K \leq C_{d,\varepsilon} \cdot (\sqrt{D_K})^{1+\varepsilon}$ , where  $d$  is the degree of  $K$  and  $D_K$  the absolute value of its discriminant.*

But it is clear that, arithmetically, the behaviour of the  $p$ -Sylow subgroups of  $\mathcal{C}_K$  depends, in conflicting manners (regarding  $p$ ), on many parameters (signature, ramification, prime divisors of  $d$ , action of Galois groups, etc.).

After the Cohen–Lenstra–Martinet, Adam–Malle, Delaunay–Jouhet, Gerth, Koymans–Pagano,... heuristics, conjectures, or density statements, on the order and structure of  $\mathcal{C}_K \otimes \mathbb{Z}_p$  [4, 5, 1, 30, 8, 15, 28], many authors study and prove inequalities of the form  $\#(\mathcal{C}_K \otimes \mathbb{F}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^{c+\varepsilon}$ , with positive constant  $c < 1$  as small as possible (e.g., under GRH, the inequality  $\#(\mathcal{C}_K \otimes \mathbb{F}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^{1-\frac{1}{p(d-1)}+\varepsilon}$  [10, Proposition 1]; see also [14, § 1.1] for more examples and comments). The various links between this  $\varepsilon$ -conjecture and the above classical heuristics (or results) are described in [33, § 1.1, Theorem 1.2, Remark 3.3].

For a general history upon today about such inequalities, we refer to some recent papers of the bibliography (e.g., [9, 10, 14, 33, 43]) in which the reader can have a more complete list of recent contributions.

For short, we shall call “ $p$ -rank  $\varepsilon$ -conjecture” the case:

$$\#(\mathcal{C}_K \otimes \mathbb{F}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^\varepsilon,$$

and “strong  $\varepsilon$ -conjecture” the case:

$$\#(\mathcal{C}_K \otimes \mathbb{Z}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^\varepsilon.$$

These kind of results do not separate the case of totally real fields, for which we know that the class number is rather small, regarding the case of non totally real fields (see the tables in Washington’s book [42] and papers by Schoof as [36, 38], among many contributions; however, some real fields may have exceptional large class numbers regarding  $D_K$  [6, 7]). Moreover, by nature, the results that we have quoted deal with upper bounds of the  $p$ -rank  $\text{rk}_p(\mathcal{C}_K)$  and precisely we shall see that genus theory gives, when it applies, large and maximal  $p$ -ranks and possibly unbounded orders, which probably makes harder proofs by classical complex analytic way.

*Remark 1.1.* Nevertheless, many arguments and computations are in favor of the strong  $\varepsilon$ -conjecture  $\#(\mathcal{C}_K \otimes \mathbb{Z}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^\varepsilon$ , except possibly for subfamilies of density zero, even if such a conjecture is “a conjecture at infinity for fixed  $p$ ” because, as we shall see in numerical calculations, the constants  $C_{d,p,\varepsilon}$  are enormous, especially for  $d = p$ , so that, for “usual fields”, the inequalities are trivial up to some huge values of the discriminant. So it will be difficult to give convincing computations “at infinity”.

In this context, we shall only prove the case of the  $p$ -rank  $\varepsilon$ -conjecture for degree  $p$  cyclic fields  $K$  (Theorem 2.5):

**Main Theorem.** *Denote by  $\mathcal{C}_K$  the class group of any number field  $K$ . Let  $p \geq 2$  be a prime number; then the  $p$ -rank  $\varepsilon$ -conjecture saying that:  $\#(\mathcal{C}_K \otimes \mathbb{F}_p) \ll_{d,p,\varepsilon} (\sqrt{D_K})^\varepsilon$  for all  $K$  of degree  $d$ , is true for the subfamily of all cyclic extension  $F/\mathbb{Q}$  of degree  $d = p$ .*

For these fields, with Galois group  $G =: \langle \sigma \rangle$ , we shall use the exact sequence:

$$1 \rightarrow (\mathcal{C}_K \otimes \mathbb{Z}_p)^G \rightarrow \mathcal{C}_K \otimes \mathbb{Z}_p \rightarrow (\mathcal{C}_K \otimes \mathbb{Z}_p)^{1-\sigma} \rightarrow 1$$

and consider the “non-genus part”  $(\mathcal{C}_K \otimes \mathbb{Z}_p)^{1-\sigma}$  (or set of “exceptional  $p$ -classes”) as a random object for which densities results are known (see Remark 2.2).

The case  $d = p = 2$  is known from Gauss genera theory, but its generalization is not obvious since for  $p = 2$  the  $p$ -rank is canonical (given by Chevalley’s formula (1)) while for  $p > 2$  it depends on the algorithm which determines the complete structure of  $\mathcal{C}_K \otimes \mathbb{Z}_p$ ; indeed, for  $d = p = 2$  and any class  $\gamma$  such that  $\gamma^2 = 1$ , we can write  $\gamma^{\sigma-1} = \gamma^{\sigma+1-2} = N_{K/\mathbb{Q}}(\gamma) \gamma^{-2} = 1$ , which does not work for  $d = p > 2$ .

To our knowledge, for  $d = p > 2$ , only the case of cyclic cubic fields is proved [10, Corollary 1, case (3)] for the 3-rank  $\varepsilon$ -conjecture.

For CM fields, we shall try to put the “minus part” of  $\mathcal{C}_K \otimes \mathbb{Z}_p$  in “duality” with the “plus part” of the torsion group  $\mathcal{T}_K$  of the Galois group of the maximal abelian  $p$ -ramified pro- $p$ -extension of  $K$  for which, on the contrary, we know that there is no complete strong  $\varepsilon$ -conjecture because of some explicit families of density zero.

This suggests that, for any given  $p$ , the strong  $\varepsilon$ -conjecture may be true “for almost all field” in a sense to be specified.

## 2. A possible obstruction due to exceptional $p$ -classes

We shall illustrate the above comments with a family of fields with optimal  $p$ -ranks, and give numerical illustrations with PARI/GP [32] programs.

**2.1. Definition of the family  $(F_{N,p})_{N \geq 1}$ .** We consider a fixed prime  $p \geq 2$  and the sequence of all odd prime numbers  $\ell_k$ , totally split in  $\mathbb{Q}(\mu_p)$ , whence such that  $\ell_k \equiv 1 \pmod{p}$ . For  $p > 2$ , let  $F_{N,p}$  be any cyclic extension of degree  $p$ , of global conductor  $f_{N,p} = \prod_{k=1}^N \ell_k$ , for any  $N \geq 1$ . For  $p = 2$  we shall consider  $F_{N,2} := \mathbb{Q}(\sqrt{\pm \ell_1 \cdots \ell_N}) = \mathbb{Q}(\sqrt{\pm 3 \cdot 5 \cdots \ell_N})$  with  $\pm$  such that 2 be unramified, and consider the restricted sense for class groups, which is more canonical for our purpose. We may consider degree  $p$  cyclic extensions ramified at  $p$  without any modification of the forthcoming reasonings, except an useless complexity of redaction (this modifies the above conductors up to the constant factor  $p^2$  or 8).

The discriminant  $D_{N,p}$  of  $F_{N,p}$  is the product of the conductors associated to each nontrivial character of  $G_{N,p} := \text{Gal}(F_{N,p}/\mathbb{Q})$ , thus  $D_{N,p} = f_{N,p}^{p-1}$ . There are  $(p-1)^{N-1}$  such fields, all contained in  $\mathbb{Q}(\mu_{\ell_1 \cdots \ell_N})$ .

The Chevalley formula [3] gives the number of ambiguous classes (i.e., invariant by  $G_{N,p}$ ) which is equal to the genus number for cyclic fields:

$$(1) \quad \#(\mathcal{C}_{F_{N,p}} \otimes \mathbb{Z}_p)^{G_{N,p}} = p^{N-1}.$$

It is obvious that the group of ambiguous classes is  $p$ -elementary of  $p$ -rank  $N-1 \leq \text{rk}_p(\mathcal{C}_{F_{N,p}})$  and generated by the ramified primes  $\mathfrak{l}_i \mid \ell_i$ ,  $1 \leq i \leq N$ , with a (non-trivial) relation  $\prod_{i=1}^N \mathfrak{l}_i^{n_i} = (a)$ ,  $a \in F_{N,p}^\times$ , due to the classical Kummer theory over  $\mathbb{Q}(\mu_p)$  giving  $F_{N,p}(\mu_p) = \mathbb{Q}(\mu_p)(\sqrt[p]{\alpha})$ ,  $\alpha \in \mathbb{Q}(\mu_p)^\times$  depending of canonical Gauss sums.

**2.2. About the exceptional  $p$ -classes.** We know that  $\#(\mathcal{C}_{F_{N,p}} \otimes \mathbb{F}_p)$  and  $\#(\mathcal{C}_{F_{N,p}} \otimes \mathbb{Z}_p)$ , both depend of a non-predictible algorithm that we recall below; this shows that the so-called Brumer–Silverman–Duke–Zhang–Ellenberg–Venkatesh  $\varepsilon$ -conjecture on the  $p$ -ranks is not so different from the strong  $\varepsilon$ -conjecture, in a logical point of view, except that we shall see that

the  $p$ -rank is an  $O(N)$  contrary to  $\#(\mathcal{C}_{F_{N,p}} \otimes \mathbb{Z}_p)$  whose order of magnitude is unknown.

**Definition 2.1.** Let  $\delta(N) \geq 0$  be such that  $\#(\mathcal{C}_{F_{N,p}} \otimes \mathbb{F}_p) = p^{N-1+\delta(N)}$  and let  $\Delta(N) \geq \delta(N)$  be such that  $\#(\mathcal{C}_{F_{N,p}} \otimes \mathbb{Z}_p) = p^{N-1+\Delta(N)}$ .

In other words,  $p^{\Delta(N)} = \#(\mathcal{C}_{F_{N,p}} \otimes \mathbb{Z}_p)^{1-\sigma}$  measures what we shall call the set of “exceptional  $p$ -classes” (i.e., non-invariant) obtained via the classical filtration of  $\mathcal{C}_{F_{N,p}} \otimes \mathbb{Z}_p$  (a general theoretical and numerical approach was given in [19, 20] and [39] after the historical papers of Inaba [25], Redei–Reichardt [35], Fröhlich [11, 12] and others; for a wide generalization to ray class groups and a survey, see [18]). Then  $N - 1 + \delta(N)$  is the  $p$ -rank.

*Remark 2.2.* During the writing of this paper we have been informed, by Peter Koymans and Carlo Pagano, of their work [28] proving that the  $p$ -class groups of cyclic degree  $p$  fields have, under GRH, the (explicit) distribution conjectured by Frank Gerth III [15] and generalizing results of Jack Klys by means of methods developed by Etienne Fouvry–Jürgen Klüners, Alexander Smith and others.

These results prove that the strong  $\varepsilon$ -conjecture for degree  $p$  cyclic fields  $K$  is true (in the meaning that this occurs with probability 1), except possibly for very sparse families of fields of density zero. One deduces this from [28, Theorem 1.1] by proving that for any map  $h : \mathbb{R} \rightarrow \mathbb{R}$ , such that  $h(X) \rightarrow \infty$  as  $X \rightarrow \infty$ , we have the following density property about the sets of exceptional  $p$ -classes of the  $p$ -class groups:

$$\lim_{X \rightarrow \infty} \left[ \frac{\#\{K : \sqrt{D_K} < X \ \& \ \#(\mathcal{C}(K) \otimes \mathbb{Z}_p)^{1-\sigma} > h(\sqrt{D_K})\}}{\#\{K : \sqrt{D_K} < X\}} \right] = 0.$$

The possible infinite “bad families” of degree  $p$  cyclic fields  $K$  are such that for some fixed  $\varepsilon$  and for any  $C > 0$ :

$$\#(\mathcal{C}(K) \otimes \mathbb{Z}_p)^{1-\sigma} > C \frac{(\sqrt{D_K})^\varepsilon}{p^{N-1}} =: C h_\varepsilon(\sqrt{D_K}),$$

where  $N$  is the number of ramified primes of  $K$ ; we shall verify Section 2.3 that these functions  $h_\varepsilon$  fulfill the condition at infinity.

But the Theorem 2.5 shall prove (unconditionally) the  $p$ -rank  $\varepsilon$ -conjecture.

**2.2.1. The algorithm.** Recall briefly the computation of  $\delta(N)$  and  $\Delta(N)$  in the purpose of some probabilistic considerations.

Let  $F$  be any degree  $p$  cyclic extension of  $\mathbb{Q}$  with  $N \geq 1$  ramified primes, put  $M := \mathcal{C}_F \otimes \mathbb{Z}_p$ , and let  $\sigma$  be a generator of  $G := \text{Gal}(F/\mathbb{Q}) \simeq \mathbb{Z}/p\mathbb{Z}$ .

Let  $I_F$  be the group of ideals of  $F$ , prime to  $p$ , and let  $M_i =: \mathcal{C}_F(\mathcal{I}_i)$ ,  $i \geq 0$ ,  $\mathcal{I}_i \subset I_F$ , defined inductively by  $M_0 := 1$ , and:

$$M_{i+1}/M_i := (M/M_i)^G, \text{ for } 0 \leq i \leq m-1,$$

where  $m \geq 1$  is the least integer  $i$  such that  $M_i = M$  (i.e., such that  $M_{i+1} = M_i$ ). Then  $M_i = \{c \in M, c^{(1-\sigma)^i} = 1\}$  for all  $i \geq 0$  and  $M_1 = M^G$ .

The sequence  $\#(M_{i+1}/M_i)$ ,  $0 \leq i \leq m$ , decreases to 1 and is bounded by  $\#M_1$  due to the injective maps  $M_{i+1}/M_i \hookrightarrow M_i/M_{i-1} \hookrightarrow \cdots \hookrightarrow M_2/M_1 \hookrightarrow M_1$  defined by the operation of  $1 - \sigma$ .

We have, for the fields  $F$ , the formulas [18, Corollary 3.7], for all  $i \geq 0$ :

$$\#(M_{i+1}/M_i) = \frac{p^{N-1}}{(\Lambda_i : \Lambda_i \cap N_{F/\mathbb{Q}}(F^\times))} \quad \& \quad \Lambda_i := \{x \in \mathbb{Q}^\times, (x) \in N_{F/\mathbb{Q}}(\mathcal{I}_i)\},$$

with  $\mathcal{I}_0 = 1$  and  $\Lambda_0 = 1$ .

The progression of the algorithm depends on the  $x \in \Lambda_i$ ,  $(x) = N_{F/\mathbb{Q}}(\mathfrak{A})$ ,  $\mathfrak{A} \in \mathcal{I}_i$ , such that  $x = N_{F/\mathbb{Q}}(y)$ ,  $y \in F^\times$ , giving the equation:

$$(y) = \mathfrak{A} \cdot \mathfrak{B}^{1-\sigma},$$

in which the solutions  $\mathfrak{B}$  (non-predictible) become new elements to be added to  $\mathcal{I}_i$  to built  $\mathcal{I}_{i+1} \supseteq \mathcal{I}_i$ , then  $\Lambda_{i+1} \supseteq \Lambda_i$ , and so on.

Put, for all  $i \geq 0$ ,  $\#(M_{i+1}/M_i) := p^{N-1-t_i^N}$ ,  $t_i^N \geq 0$ , where  $p^{t_i^N} \mid p^{N-1}$  is the index  $(\Lambda_i : \Lambda_i \cap N_{F/\mathbb{Q}}(F^\times))$  (since  $x \in \Lambda_i$  is norm of an ideal, it is locally a norm everywhere, except perhaps at the  $N$  ramified primes, but the product formula for Hasse's normic symbols gives the above divisibility). We have:

$$t_0^N = 0 \leq t_1^N \leq \cdots \leq t_m^N = N - 1.$$

Then  $\#M = \#(\mathcal{C}_F \otimes \mathbb{Z}_p) = \prod_{i \geq 0} \#(M_{i+1}/M_i) = p^{m(N-1) - \sum_{i=1}^m t_i^N}$ .

We have the following result that we shall use to test the  $\varepsilon$ -conjecture.

**Lemma 2.3** ([18, Lemma 4.2]). *Let  $F/\mathbb{Q}$  be a degree  $p$  cyclic extension with  $N \geq 1$  ramified primes. Then  $\text{rk}_p(\mathcal{C}_F) = (p-1) \cdot (N-1) - \sum_{i=1}^{p-2} t_i^N$ . Whence this yields  $\delta(N) = (p-2) \cdot (N-1) - \sum_{i=1}^{p-2} t_i^N$ .*

We see that the  $p$ -rank may vary in the interval  $[N-1, (p-1)(N-1)]$  and is always  $O(N)$  as  $N \rightarrow \infty$  which shall be of a great importance; the  $p$ -rank is equal to  $N-1$ , for all  $F/\mathbb{Q}$  of degree  $p$ , if and only if  $p=2$  as we have seen. In the same manner, the  $p^r$ -ranks are given by the expressions:

$$\sum_{i=(r-1)(p-1)}^{r(p-1)-1} (N-1-t_i^N) = (p-1) \cdot (N-1) - \sum_{i=(r-1)(p-1)}^{r(p-1)-1} t_i^N.$$

It is clear that the normic indices depend on the  $\mathbb{F}_p$ -ranks of  $N \times N$ -matrices of suitable Hilbert symbols [20, Ch. VI, §2] generalizing, for instance, Rédei's matrices for the computation of the 4-ranks of a quadratic field.

But we emphasize the fact that if some heuristics on the  $\mathbb{F}_p$ -rank of these matrices are natural, *the number of steps of the algorithm* (i.e.,  $m$ ) only depends on distribution results (see Remark 2.2).

**2.2.2. Examples of maximal rank.** In fact, for  $p > 2$ , the maximal rank, equal to  $(p-1) \cdot (N-1)$ , is very rare. We remembered that in our thesis [20, p. 39], we gave the first example of cyclic cubic field, for which  $\text{rk}_3(\mathcal{C}) = 2(N-1)$ ; this example has conductor  $f = 9 \cdot 577 \cdot 757 \cdot 991$  (the program below shows that the property is fulfilled for the eight fields of conductor  $f$ ):

```
f=3895721091 N=4 P=x^3-1298573697*x+18010351461592 C1=[9,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x-16034355152657 C1=[3,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x+15707980296811 C1=[3,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x-10132337699792 C1=[3,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x-8835062576489 C1=[3,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x-7829966535011 C1=[9,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x+912031593193 C1=[3,3,3,3,3,3]
f=3895721091 N=4 P=x^3-1298573697*x-221623244288 C1=[9,3,3,3,3,3]
```

We publish, for any further examples, a program computing all the cyclic cubic fields such that there are exceptional 3-classes; so in most cases the 3-rank is  $N$  (instead of  $N-1$ ) and many examples have a non-trivial 9-rank.

```
{bf=7;Bf=10^6;for(f=bf,Bf,e=valuation(f,3);if(e!=0 & e!=2,next);F=f/3^e;
if(Mod(F,3)!=1||core(F)!=F,next);F=factor(F);Div=component(F,1);
d=component(matsize(F),1);for(j=1,d-1,D=component(Div,j);
if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);
A=4*f-27*b^2;if(issquare(A,&a)==1,if(e==0,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);if(e==2,if(Mod(a,9)==3,a=-a);
P=x^3-f/3*x-f*a/27);N=omega(f);L=List;K=bnfinit(P,1);C8=component(K,8);
h=component(component(C8,1),1);C1=component(component(C8,1),2);
dim=component(matsize(C1),2);rho=0;for(i=1,dim,z=component(C1,i);
v=valuation(z,3);if(v>=1,rho=rho+1;listput(L,3^v));if(rho>=N,
print(" f=",f," N=",N," P=",P," C1=",L))))}
```

```
f=657 N=2 P=x^3-219*x-1241 C1=[3,3]
f=657 N=2 P=x^3-219*x+730 C1=[3,3]
(...)
f=3913 N=3 P=x^3+x^2-1304*x+17681 C1=[3,3,3]
f=3913 N=3 P=x^3+x^2-1304*x-17536 C1=[3,3,3]
(...)
f=4711 N=2 P=x^3+x^2-1570*x+19193 C1=[9,3]
f=4711 N=2 P=x^3+x^2-1570*x-13784 C1=[9,3]
f=4921 N=3 P=x^3+x^2-1640*x+23876 C1=[3,3,3]
f=4921 N=3 P=x^3+x^2-1640*x-729 C1=[3,3,3]
(...)
f=5383 N=2 P=x^3+x^2-1794*x+17744 C1=[9,3]
f=5383 N=2 P=x^3+x^2-1794*x-9171 C1=[9,3]
(..)
f=15561 N=4 P=x^3-5187*x+141778 C1=[9,3,3,3]
```

*Remark 2.4.* The Galois action of  $G = \text{Gal}(F/\mathbb{Q}) \simeq \mathbb{Z}/p\mathbb{Z}$  on the non- $p$ -part of the class group gives rise, for all prime  $q \mid \#\mathcal{C}_F$ , to:

$$\mathcal{C}_F \otimes \mathbb{Z}_q \simeq \mathbb{Z}[\mu_p] / \prod_{q \mid p} q^{n_q}, \quad n_q \geq 0,$$



whose  $\mathbb{F}_q$ -dimension is a multiple of the residue degree of  $q$  in  $\mathbb{Q}(\mu_p)/\mathbb{Q}$ , which explains the rarity of such divisibilities for large residue degrees.

**2.3. Estimation of  $C_{p,p,\varepsilon}$  for the fields  $F_{N,p}$ .** As we have explained, we do not consider degree  $p$  cyclic extensions  $F/\mathbb{Q}$  ramified at  $p$ . This shall modify the forthcoming computations by some  $O(1)$  without any consequence on the statements since  $p$  is fixed in all the sequel.

We need a lower bound of the  $k$ th prime number  $\ell_k \equiv 1 \pmod{p}$  to get an estimation of  $f_{N,p}$ .

We thank Gérald Tenenbaum for valuable indications for the good formula, from [40, Notes on Chapitre I, § 4.6], due to a result of Montgomery–Vaughan giving, for the  $k$ th prime number  $\ell_k \equiv 1 \pmod{p}$ :

$$(2) \quad \ell_k > \frac{p-1}{2} \cdot k \cdot \log\left(\frac{\ell_k}{p}\right), \text{ for all } k \geq 1.$$

Indeed, if  $\pi(x; 1, p) := \#\{\ell \leq x; \ell \equiv 1 \pmod{p}\}$  then:

$$\pi(x; 1, p) \leq \frac{2x}{(p-1) \log\left(\frac{x}{p}\right)};$$

whence the result taking  $x = \ell_k$  since  $\pi(\ell_k; 1, p) = k$ .

We intend to test, for the family  $(F_{N,p})_{N \geq 1}$  of discriminants  $D_{N,p} := D_{F_{N,p}}$ , the strong  $\varepsilon$ -conjecture, that is to say:

$$(3) \quad \#(\mathcal{A}_{F_{N,p}} \otimes \mathbb{Z}_p) =: p^{N-1+\Delta(N)} \leq C_{p,p,\varepsilon} \cdot (\sqrt{D_{N,p}})^\varepsilon,$$

where  $\Delta(N) = (m-1) \cdot (N-1) - \sum_{i=1}^{m-1} t_i^N \geq 0$ , related to the set of exceptional  $p$ -classes, has only a probabilistic value depending on  $m$  and the  $t_i^N$ .

It will be easy, from the forthcoming calculations, to test the  $p$ -rank  $\varepsilon$ -conjecture,  $\#(\mathcal{A}_{F_{N,p}} \otimes \mathbb{F}_p) \leq C_{p,p,\varepsilon} \cdot (\sqrt{D_{N,p}})^\varepsilon$ , but considering instead the weaker inequality:

$$p^{N-1+\delta(N)} = p^{(p-1)(N-1) - \sum_{i=1}^{p-2} t_i^N} \leq C_{p,p,\varepsilon} \cdot (\sqrt{D_{N,p}})^\varepsilon,$$

even in the less favorable case  $t_i^N = 0$ , for  $1 \leq i \leq p-2$  and replacing for all  $N$ ,  $D_{N,p}$  by a lower bound  $D'_{N,p}$  (in other words the existence of an inequality  $p^{(p-1)(N-1)} \leq C'_{p,p,\varepsilon} \cdot (\sqrt{D'_{N,p}})^\varepsilon$  proves the  $p$ -rank  $\varepsilon$ -conjecture for all degree  $p$  cyclic fields). We fix  $p$  and put  $D_{N,p} =: D_N =: f_N^{p-1}$ . The strong form is equivalent to prove that  $\frac{p^{N-1+\Delta(N)}}{(\sqrt{D_N})^\varepsilon}$  is bounded as  $N \rightarrow \infty$ , whence  $(N-1+\Delta(N)) \log(p) - \varepsilon \cdot \frac{p-1}{2} \sum_{k=1}^N \log(\ell_k) < \infty$ , as  $N \rightarrow \infty$ .

We then have, replacing  $\ell_k$  by a lower bound  $\ell'_k$ , to compute, using (1), (2), (3), the following quantity:

$$\begin{aligned} X(N) &:= (N - 1 + \Delta(N)) \log(p) - \varepsilon \cdot \frac{p-1}{2} \sum_{k=1}^N \log(\ell'_k) \\ &= (N - 1 + \Delta(N)) \log(p) - \varepsilon \cdot \frac{p-1}{2} \sum_{k=1}^N \log\left(\frac{p-1}{2}\right) + \log(k) + \log_2\left(\frac{\ell_k}{p}\right). \end{aligned}$$

We verify that  $\sum_{k=1}^N \log_2\left(\frac{\ell_k}{p}\right)$  can be neglected, subject to adding  $-1$  to the sum, and consider, instead:

$$\begin{aligned} X(N) &= (N - 1 + \Delta(N)) \log(p) - \varepsilon \cdot \frac{p-1}{2} \left[ -1 + \sum_{k=1}^N \left[ \log\left(\frac{p-1}{2}\right) + \log(k) \right] \right] \\ &= (N - 1 + \Delta(N)) \log(p) - \varepsilon \cdot \frac{p-1}{2} \left[ -1 + N \cdot \log\left(\frac{p-1}{2}\right) + \log(N!) \right]. \end{aligned}$$

The expression of  $N!$  leads to  $\log(N!) = N \log(N) - N + \frac{1}{2} \log(N) + O(1)$ , whence, with  $\gamma_p := \log\left(\frac{p-1}{2}\right) - 1$ :

$$\begin{aligned} X(N) &= (N - 1 + \Delta(N)) \log(p) \\ &\quad - \varepsilon \cdot \frac{p-1}{2} \left[ N \log(N) + N \cdot \gamma_p + \frac{1}{2} \log(N) + O(1) \right]. \end{aligned}$$

Now we write  $X(N)$  under the form:

$$\begin{aligned} X(N) &= N \log(p) + \Delta(N) \log(p) - \varepsilon \cdot \frac{p-1}{2} N \log(N) \\ &\quad - \log(p) - \varepsilon \cdot \frac{p-1}{2} N \cdot \gamma_p - \varepsilon \cdot \frac{p-1}{4} \log(N) - \varepsilon \cdot O(1) \\ &= N \cdot \left[ -\varepsilon \cdot \frac{p-1}{2} \log(N) + \frac{\Delta(N)}{N} \log(p) \right. \\ (4) \quad &\quad \left. + \log(p) - \frac{\log(p)}{N} - \varepsilon \cdot O(1) \right] \\ &= N \cdot \left[ -\varepsilon \cdot \frac{p-1}{2} \log(N) + \frac{\Delta(N)}{N} \log(p) \right. \\ &\quad \left. + (1 - N^{-1}) \log(p) - \varepsilon \cdot O(1) \right]. \end{aligned}$$

Replacing  $\Delta(N)$  by the maximal value  $(p-2)(N-1)$  of  $\delta(N)$ , the dominant term  $-\varepsilon \cdot \frac{p-1}{2} \log(N)$  ensures the existence of a positive constant  $C_\varepsilon$  since  $\frac{(p-1)(N-1)}{N} \log(p) = O(1)$  giving:

$$(5) \quad X_0(N) = -\varepsilon \frac{p-1}{2} N \log(N) + N \left[ (p-1) \log(p) - \varepsilon O(1) - o(1) \right].$$

It is easy to verify that  $X_0(N)$ , as function of  $N$ , admits, for an  $N_0 \gg 0$ , a computable maximum, only depending on  $p$  and  $\varepsilon$  (e.g., for  $p = 7$ ,  $\varepsilon = 0.1$ ,  $N_0 \approx 2935394 \cdot 10^{10}$ ,  $X_0(N_0) \approx 88 \cdot 10^{14}$ ).

This proves the  $p$ -rank  $\varepsilon$ -conjecture for the family  $(F_{N,p})_{N \geq 1}$ , even assuming always a maximal  $p$ -rank  $(p-1)(N-1)$ . Whence we can state:

**Theorem 2.5.** *Let  $p \geq 2$  be a given prime number.*

(i) *The  $p$ -rank  $\varepsilon$ -conjecture on the existence, for all  $\varepsilon > 0$ , of a constant  $C_{d,p,\varepsilon}$  such that  $\#(\mathcal{C}_K \otimes \mathbb{F}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^\varepsilon$  for all  $K$  of degree  $d$ , is true for the subfamily of all cyclic extension  $F/\mathbb{Q}$  of degree  $d = p$ .*

(ii) *For any cyclic extension  $F/\mathbb{Q}$  of degree  $p$ , let  $\Delta(N) \geq 0$  be defined by  $\#(\mathcal{C}_F \otimes \mathbb{Z}_p) = p^{N-1+\Delta(N)}$ , where  $N$  is the number of ramified primes. Then the strong  $\varepsilon$ -conjecture,  $\#(\mathcal{C}_K \otimes \mathbb{Z}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^\varepsilon$ , is true for the family of degree  $p$  cyclic extensions under the condition  $\Delta(N) \ll N \log(N)$ .*

**Proof.** Consider arbitrary prime numbers  $\ell_{n_i} \equiv 1 \pmod{p}$ ,  $1 \leq i \leq N$ , and a field  $K$ , cyclic of degree  $p$ , of conductor  $\prod_{i=1}^N \ell_{n_i}$ . Thus  $D_K \geq D_{F_N}$  and the maximal  $p$ -rank of  $\mathcal{C}_K$  is still  $(p-1)(N-1)$ . We have seen that if we put  $\delta(N) = (p-2)(N-1)$  in (4), giving (5), the main term ensures the existence of the constant  $C_{p,p,\varepsilon}$ . We omit the details when  $p^2$  (or 8) divides the conductor.  $\square$

Very probably, considering the Remark 2.2,  $\Delta(N)$  is almost all the time of order much less than  $N \log(N)$  since  $p^r$ -ranks are in practice very rare for  $r \geq 2$  as well as a maximal  $p$ -rank equal to  $(p-1)(N-1)$  (see § 2.2.2); but the dominant terms in (4) being  $\Delta(N) \log(p) - \varepsilon \cdot \frac{p-1}{2} N \log(N)$ , this may give some trouble for the proof of an universal strong  $\varepsilon$ -conjecture assuming for instance  $\Delta(N) = O(1) \log(N)$ , or more, for infinite families of degree  $p$  cyclic fields, even of density zero.

**2.4. A lower bound for  $C_{p,p,\varepsilon}$ .** This part is not essential for our purpose, but it will show that a lower bound of  $C_{p,p,\varepsilon}$  is of the same order of magnitude as for the upper bound deduced from (5). We shall use the following property which may be justified from results given in [31]. For all  $k \geq 1$ , the  $k$ th prime  $\ell_k \equiv 1 \pmod{p}$  fulfills, for some constants  $c_p$  only depending on  $p$ , the inequality  $\ell_k < c_p ((p-1)k) \cdot \log((p-1)k) < c_p ((p-1)k)^2$ . Then:

$$\begin{aligned} (N-1 + \delta(N)) \log(p) &\leq \log(C_\varepsilon) + \varepsilon \frac{p-1}{2} \sum_{k=1}^N \log(\ell_k) \\ &\leq \log(C_\varepsilon) + \varepsilon \frac{p-1}{2} \sum_{k=1}^N [\log(c_p) + 2 \log(p-1) + 2 \log(k)], \\ &\leq \log(C_\varepsilon) + \varepsilon \frac{p-1}{2} \cdot [N \gamma'_p + 2 \log(N!)] \end{aligned}$$

where  $\gamma'_p := \log(c_p) + 2 \log(p-1) > 0$ . Whence, from the value of  $\log(N!)$ :

$$(N-1 + \delta(N)) \log(p) \leq \log(C_\varepsilon) + \varepsilon \frac{p-1}{2} [2N \log(N) + N \gamma''_p + \log(N) + O(1)],$$

with  $\gamma''_p = \gamma'_p - 2 > 0$ . Thus:

$$\begin{aligned} \log(C_\varepsilon) &\geq (N-1 + \delta(N)) \log(p) - \varepsilon (p-1) N \log(N) \\ (6) \quad &\quad - \varepsilon \frac{p-1}{2} N \gamma''_p - \varepsilon \frac{p-1}{2} \log(N) - \varepsilon O(1) \\ &\geq N \cdot \left[ -\varepsilon (p-1) \log(N) + \frac{\delta(N)}{N} \log(p) + O(1) \right]. \end{aligned}$$

*Remarks 2.6.* (i) As soon as we replace  $\varepsilon$  by  $c + \varepsilon$ ,  $0 < c < 1$ , as it is done in much papers giving general proofs for  $\#(\mathcal{C}_K \otimes \mathbb{F}_p) \leq C_{d,p,c,\varepsilon} \cdot (\sqrt{D_K})^{c+\varepsilon}$ , the above computations becomes, replacing  $\Delta(N)$  by  $\delta(N)$ :

$$\begin{aligned} \log(C_{p,p,c,\varepsilon}) &\leq N \cdot \left[ - (c + \varepsilon) \cdot \frac{p-1}{2} \log(N) + \frac{\delta(N)}{N} \log(p) + O(1) \right] \\ \log(C_{p,p,c,\varepsilon}) &\geq N \cdot \left[ - (c + \varepsilon) \cdot (p-1) \log(N) + \frac{\delta(N)}{N} \log(p) + O(1) \right]. \end{aligned}$$

For  $\varepsilon \rightarrow 0$ , the formulas (5) and (6) give a limit value  $C_{p,c}$  of  $C_{p,p,c,\varepsilon}$  such that  $\log(C_{p,c}) \approx (N-1 + \delta(N)) \log(p) - c \frac{p-1}{2} N \log(N) - N O(1)$ , rapidly negative as  $N$  increases, giving for degree  $p$  cyclic fields, an obvious proof of the “ $p$ -rank  $(c + \varepsilon)$ -property”.

(ii) If we replace  $\mathbb{Q}$  by a number field  $k$  and the fields  $F_{N,p}$  by the degree  $p$  cyclic extensions  $F_{k,N,p}$  of  $k$  with  $N$  ramified prime ideals of  $k$ , the details of computations are more complicate, but the results and comments are similar since the  $p$ -rank of  $\mathcal{C}_{F_{k,N,p}}$  is still equivalent to  $O(N)$  because of the exact sequence of genus theory and the inequalities [16, Theorem IV.4.5.1]:

$$N - c_{k,p} \leq \text{rk}_p(\mathcal{C}_{F_{k,N,p}}^G) \leq N + c'_{k,p},$$

where the constants  $c_{k,p}$ ,  $c'_{k,p}$  depend on the  $p$ -classes and units of  $k$ . Then, the general algorithm computing the  $p$ -rank via the filtration  $(M_i)_{i \geq 0}$  is identical up to similar modifications (see [18, §4.4]) and shall give the  $p$ -rank  $\varepsilon$ -conjecture for the relative degree  $p$  cyclic case without too much difficulties. The case of abelian  $p$ -extensions may be accessible from the  $p^n$ -cyclic cases, with some effort...

The proof of the strong  $\varepsilon$ -conjecture, for the degree  $p$  cyclic extensions of  $k$ , remains, theoretically, open, but is clearly not a folk conjecture for such very particular real fields because of the possible generalization of the density results of [28] and the conclusion, about the possible existence of pathological families of density zero, is still relevant. Meanwhile a particular study of the algorithm giving  $\Delta(N)$ , independently of any density results, should be a crucial step for many questions in number theory.

### 3. PARI/GP programs computing $\mathcal{C}_{F_{N,p}}$

**3.1. Structure of some  $\mathcal{C}_{F_{N,p}}$ .** The following numerical results show that exceptional  $p$ -classes (indicated by \*) are not excessively frequent for these fields. We examine the cases  $p = 3, 5, 7$ , then  $p = 2$  (one must precise  $p$  and the conductor  $f = \ell_1 \cdots \ell_N$ ,  $\ell_i \equiv 1 \pmod{p}$ ), in the following program giving all the  $(p-1)^{N-1}$  fields of conductor  $f$ :

```
{p=3;f=7*13*19*31*37;V=polsubcyclo(f,p);d=matsize(V);d=component(d,2);
for(k=1,d,P=component(V,k);if(nfdisc(P)!=f^(p-1),next);K=bnfinit(P,1);
C8=component(K,8);C81=component(C8,1);h=component(C81,1);
C1=component(C81,2);print("p=",p," f=",f," P=",P," C1=",C1)}}
```

```
p=3 f=1983163=7*13*19*31*37
```

$P=x^3+x^2-661054*x+49725976$	$C1=[6,6,3,3]$
$P=x^3+x^2-661054*x+198463201$	$C1=[3,3,3,3]$
$P=x^3+x^2-661054*x+97321888$	$C1=[39,3,3,3,3]*$
$P=x^3+x^2-661054*x-186270421$	$C1=[3,3,3,3]$
$P=x^3+x^2-661054*x-79179619$	$C1=[6,6,3,3]$
$P=x^3+x^2-661054*x-188253584$	$C1=[3,3,3,3,3]*$
$P=x^3+x^2-661054*x+138968311$	$C1=[3,3,3,3]$
$P=x^3+x^2-661054*x-146607161$	$C1=[3,3,3,3,3]*$
$P=x^3+x^2-661054*x-158506139$	$C1=[3,3,3,3]$
$P=x^3+x^2-661054*x-140657672$	$C1=[6,6,3,3]$
$P=x^3+x^2-661054*x+186564223$	$C1=[3,3,3,3]$
$P=x^3+x^2-661054*x+81456584$	$C1=[3,3,3,3]$
$P=x^3+x^2-661054*x+206395853$	$C1=[3,3,3,3]$
$P=x^3+x^2-661054*x-206102051$	$C1=[12,12,3,3]$
$P=x^3+x^2-661054*x+2130064$	$C1=[3,3,3,3,3]*$
$P=x^3+x^2-661054*x+27911183$	$C1=[3,3,3,3,3]*$

$p=3 \quad f=85276009=7*13*19*31*37*43$

$P=x^3-x^2-28425336*x+58104545836$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+56995957719$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-7472705085$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+10264704787$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+51623569152$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+30901498965$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-50622365639$	$C1=[6,6,3,3,3]$
$P=x^3-x^2-28425336*x-29132811371$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-46614393216$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-33226059803$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-56506410260$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-12248161589$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+23994142236$	$C1=[6,6,3,3,3]$
$P=x^3-x^2-28425336*x+52220501215$	$C1=[3,3,3,3,3,3]*$
$P=x^3-x^2-28425336*x+11799672949$	$C1=[6,6,3,3,3]$
$P=x^3-x^2-28425336*x-57529722368$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-21287418543$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-47040773261$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+36273887532$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-53436473936$	$C1=[6,6,3,3,3,3]*$
$P=x^3-x^2-28425336*x+40964068027$	$C1=[21,3,3,3,3]$
$P=x^3-x^2-28425336*x+45824800540$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+56313749647$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+13078813084$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+39940755919$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-37830964289$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+57848717809$	$C1=[6,6,3,3,3]$
$P=x^3-x^2-28425336*x-9775157328$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-2526696563$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-58126654431$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x+28428494704$	$C1=[3,3,3,3,3]$
$P=x^3-x^2-28425336*x-51986781783$	$C1=[3,3,3,3,3]$

$p=5 \quad f=13981=11*31*41$

$P=x^5+x^4-5592x^3+46417x^2+4301003x-26664769$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3+46417x^2+2623283x-26664769$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3-261165x^2-4479065x-26832541$	$Cl=[55, 5, 5]^*$
$P=x^5+x^4-5592x^3+32436x^2+1518784x+1814528$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3-205241x^2-2074333x-6028813$	$Cl=[5, 5, 5]^*$
$P=x^5+x^4-5592x^3-135336x^2+847696x+1143440$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3-205241x^2-1878599x+234675$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3+74379x^2+3993421x-7035445$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3+158265x^2-424575x-12851541$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3+130303x^2+442247x+346523$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3+46417x^2+1644613x-16878069$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3+214189x^2-2493763x+5715227$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3+74379x^2+3238447x-28174717$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3+32436x^2+5992704x-2659392$	$Cl=[5, 5, 5]^*$
$P=x^5+x^4-5592x^3-37469x^2+4384889x-11397517$	$Cl=[5, 5]$
$P=x^5+x^4-5592x^3-9507x^2+7432747x+1185383$	$Cl=[5, 5]$

**3.2. Examples of exceptional  $p$ -classes.** If one drops the invariant classes from the class group, this gives a very small component (often equal to 1); for instance we list some fields of the above tables with exceptional  $p$ -classes and/or nontrivial non- $p$ -parts of the class group (for  $p = 3, 5, 7, 11$ ):

$p=3$	
$f=1983163$	$P=x^3+x^2-661054x+97321888 \quad Cl=[39, 3, 3, 3, 3]=[13]x[3, 3, 3, 3, 3]^*$
$f=1983163$	$P=x^3+x^2-661054x-206102051 \quad Cl=[12, 12, 3, 3]=[4, 4]x[3, 3, 3, 3]$
$f=85276009$	$P=x^3-x^2-28425336x-53436473936 \quad Cl=[6, 6, 3, 3, 3, 3]=[2, 2]x[3, 3, 3, 3, 3]^*$
$p=5$	
$f=13981$	$P=x^5+x^4-5592x^3-261165x^2-4479065x-26832541 \quad Cl=[55, 5, 5]=[11]x[5, 5, 5]^*$
$f=13981$	$P=x^5+x^4-5592x^3-205241x^2-2074333x-6028813 \quad Cl=[5, 5, 5]^*$
$f=13981$	$P=x^5+x^4-5592x^3+32436x^2+5992704x-2659392 \quad Cl=[5, 5, 5]^*$
$p=7$	
$f=88537$	$P=x^7+x^6-37944x^5-1134719x^4+324123632x^3+13095064100x^2-393352790753x-8536744545007 \quad Cl=[301, 7]=[43]x[7, 7]$
$f=10004681$	$P=x^7-x^6-4287720x^5-1266715121x^4+5127549957760x^3+2650344024068794x^2-951078919604894529x-488606218944681147667 \quad Cl=[791, 7, 7]=[113]x[7, 7, 7]$
$f=10004681$	$P=x^7-x^6-4287720x^5+3455494311x^4+176273349584x^3-471685834336278x^2-36087798097778993x+6487901368894795147 \quad Cl=[7, 7, 7]^*$
$f=10004681$	$P=x^7-x^6-4287720x^5+2985274304x^4+1830007100160x^3-1788009977372784x^2+233770322355404864x-5599630780142239232 \quad Cl=[14, 14, 14]=[2, 2, 2]x[7, 7, 7]$
$f=10004681$	$P=x^7-x^6-4287720x^5+6276814353x^4-4047542893720x^3+1360785664233294x^2-232635292693132049x+15953699891990750023 \quad Cl=[7, 7, 7]^*$
$f=10004681$	$P=x^7-x^6-4287720x^5-6169008811x^4-3865457699520x^3-1247559831026016x^2-202130636944756129x-12969698603184144677$

$C1 = [7, 7, 7, 7] *$   
 $f = 10004681 \quad P = x^7 - x^6 - 4287720x^5 + 1224450448x^4 + 5079527488960x^3 - 2161274540764336x^2 - 1677161713287529664x + 812143879436422435328$   
 $C1 = [7, 7, 7, 7] *$   
 $f = 10004681 \quad P = x^7 - x^6 - 4287720x^5 - 1306733845x^4 + 4849019638720x^3 + 2420349994235592x^2 - 659732951886568641x - 207964718993797238079$   
 $C1 = [7, 7, 7, 7] *$   
 $f = 10004681 \quad P = x^7 - x^6 - 4287720x^5 + 5756570941x^4 - 3107102879720x^3 + 760318637129246x^2 - 74394888056758073x + 1594979915105904419$   
 $C1 = [7, 7, 7, 7]$   
 $f = 10004681 \quad P = x^7 - x^6 - 4287720x^5 - 1746939809x^4 + 2392910471944x^3 + 1648023037138232x^2 + 241022177190387487x - 9669121620934915453$   
 $C1 = [7, 7, 7, 7] *$   
 $f = 10004681 \quad P = x^7 - x^6 - 4287720x^5 - 4188081973x^4 - 284662313448x^3 + 706702291060440x^2 + 90226184532822239x - 3998693323498243787$   
 $C1 = [7, 7, 7, 7] *$   
 $f = 10004681 \quad P = x^7 - x^6 - 4287720x^5 - 2227164497x^4 + 1569805356712x^3 + 369664737597974x^2 - 166416491455189217x + 3813962316737479895$   
 $C1 = [7, 7, 7, 7] *$   
 $f = 10004681 \quad P = x^7 - x^6 - 4287720x^5 - 3307670045x^4 + 1296557509240x^3 + 1587033521303494x^2 + 232482552302284071x - 10617355312468435915$   
 $C1 = [203, 7, 7] = [203]x[7, 7, 7]$   
 $f = 10004681 \quad P = x^7 - x^6 - 4287720x^5 + 1654651731x^4 + 4187910318240x^3 - 1965195178959414x^2 - 1012216801097102473x + 451605062388713519719$   
 $C1 = [14, 14, 14] = [2, 2, 2]x[7, 7, 7]$   
 $f = 10004681 \quad P = x^7 - x^6 - 4287720x^5 + 1544600240x^4 + 4818925558272x^3 - 2343209264562096x^2 - 1609606088655340480x + 886992887186768275456$   
 $C1 = [203, 7, 7] = [203]x[7, 7, 7]$   
 $f = 10004681 \quad P = x^7 - x^6 - 4287720x^5 - 1306733845x^4 + 4427622475000x^3 + 1798213808544282x^2 - 1313378138040048441x - 613859706212867719587$   
 $C1 = [301, 7, 7] = [301]x[7, 7, 7]$   
 $p = 11$   
 $f = 137149 \quad P = x^{11} + x^{10} - 62340x^9 - 2099173x^8 + 998038116x^7 + 30321726924x^6 - 5078707527329x^5 + 3275334221180x^4 + 8066546096404148x^3 - 214723422858644515x^2 + 1902599837479513519x - 4121588229203611219$   
 $C1 = [253, 11] = [23]x[11, 11]$   
 $f = 137149 \quad P = x^{11} + x^{10} - 62340x^9 - 727683x^8 + 1217887963x^7 + 33409088063x^6 - 7760886906947x^5 - 350751766601032x^4 + 3398347545513222x^3 + 236507399684756272x^2 + 2593585988882665302x + 8529384350363670191$   
 $C1 = [979, 11] = [89]x[11, 11]$   
 $f = 137149 \quad P = x^{11} + x^{10} - 62340x^9 - 1550577x^8 + 1265615815x^7 + 66889353347x^6 - 7866931610939x^5 - 712972865698216x^4 - 12900860936489076x^3 + 143276981594922336x^2 + 1596818122984871186x + 3178173588229813309$   
 $C1 = [737, 11] = [67]x[11, 11]$   
 $f = 137149 \quad P = x^{11} + x^{10} - 62340x^9 - 7310835x^8 + 70636578x^7 + 43296296622x^6 + 1378934348258x^5 - 28471672310749x^4 - 944763467217249x^3 + 12433374265353417x^2 + 31266667418235948x - 324164722946199831$   
 $C1 = [253, 11] = [23]x[11, 11]$   
 $f = 137149 \quad P = x^{11} + x^{10} - 62340x^9 - 727683x^8 + 1153290784x^7 + 18055120364x^6 - 7130815486262x^5 - 134781314432095x^4 + 13777568483843493x^3 + 310799275320778321x^2 + 883774494827373474x - 5728549445587601897$   
 $C1 = [253, 11] = [23]x[11, 11]$

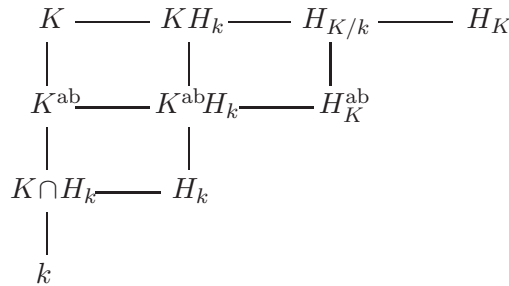
For  $p = 2$ ,  $F_{N,2}$  may be real or complex; as we know, the 2-rank is always  $N - 1$  and any exceptional classes give non-trivial 4-ranks. The PARI/GP instruction `bnfnarrow` allows the 2-structure in the restricted sense:

```
{m=1;for(N=2,100,e1=prime(N);m=(-1)^((e1-1)/2)*e1*m;P=x^2-m;
K=bnfinit(P,1);L=bnfnarrow(K);C1=component(L,2);print("m=",m," C1=",C1))}
m=-15 C1=[2]
m=+105 C1=[2,2]
m=-1155 C1=[2,2,2]
m=-15015 C1=[12,2,2,2]
m=-255255 C1=[16,2,2,2,2]*
m=+4849845 C1=[4,2,2,2,2,2]*
m=-111546435 C1=[42,2,2,2,2,2,2]
m=-3234846615 C1=[308,2,2,2,2,2,2,2]*
m=+100280245065 C1=[2,2,2,2,2,2,2,2,2]
m=+3710369067405 C1=[34,2,2,2,2,2,2,2,2,2]
m=+152125131763605 C1=[2,2,2,2,2,2,2,2,2,2,2]
m=-6541380665835015 C1=[28284,2,2,2,2,2,2,2,2,2,2,2]*
m=+307444891294245705 C1=[14,2,2,2,2,2,2,2,2,2,2,2,2]
m=+16294579238595022365 C1=[2,2,2,2,2,2,2,2,2,2,2,2,2,2]
m=-961380175077106319535 C1=[1210734,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
m=-58644190679703485491635 C1=[1622526,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
m=+3929160775540133527939545 C1=[2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
m=-278970415063349480483707695
C1=[83911452,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]*
m=-20364840299624512075310661735
C1=[362626834,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
m=+1608822383670336453949542277065
C1=[4,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]*
m=-133532257844637925677812008996395
C1=[2322692420,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]*
```

#### 4. Genus theory – Abelian $p$ -ramification theory

**4.1. Reminders on Genus theory.** For wide information on genus theory, see for example [12, 13, 16, 27, 29, 34].

**4.1.1. Genus field and genus number  $g_{K/k}$ .** Introduce the genus field according to an extension  $K/k$  (we shall take  $k = \mathbb{Q}$  in the sequel) which is the maximal subextension  $H_{K/k}$  of  $H_K$  (in the restricted sense) equal to the compositum of  $K$  with an abelian extension of  $k$ :





Thus  $H_{K/k}$  is for instance equal to the compositum of  $K$  with  $H_K^{\text{ab}}$  (the maximal abelian subextension of  $H_K/k$ ), according to the diagram above, where  $K^{\text{ab}}$  is the maximal abelian subextension of  $K/k$ . The genus number is  $g_{K/k} := [H_{K/k} : KH_k]$ . When  $K/k$  is cyclic, the genus number  $g_{K/k}$  is equal to the number of invariant classes by  $\text{Gal}(K/k)$  given by Chevalley's formula (1). In the general Galois case, we have the similar expression

$$g_{K/k} = \frac{\#\mathcal{C}_k \cdot \prod_{\mathfrak{l}} e_{\mathfrak{l}}^{\text{ab}}}{[K^{\text{ab}} : k] \cdot (E_k^{\text{pos}} : E_k^{\text{pos}} \cap \mathcal{N}_{K/k})},$$

where  $E_k^{\text{pos}}$  is the group of *totally positive* units of  $k$ ,  $e_{\mathfrak{l}}^{\text{ab}}$  the index of ramification of  $\mathfrak{l}$  in  $K^{\text{ab}}/k$  and  $\mathcal{N}_{K/k}$  the group of local norms in  $K/k$ . A general formula does exist for non-Galois fields (e.g., [16, Theorem IV.4.2 & Corollaries]).

**4.1.2. Variants of the strong  $\varepsilon$ -conjecture.** Since the  $p$ -genus group of  $K$  may be an obstruction to the strong  $\varepsilon$ -conjecture, we may consider, in the exact sequence  $1 \rightarrow \mathcal{C}'_K \otimes \mathbb{Z}_p \rightarrow \mathcal{C}_K \otimes \mathbb{Z}_p \rightarrow \text{Gal}(H_{K/\mathbb{Q}}/K) \otimes \mathbb{Z}_p \rightarrow 1$ , the number  $\#(\mathcal{C}'_K \otimes \mathbb{Z}_p)$  (giving the number of exceptional  $p$ -classes instead of the whole  $p$ -class group) and propose the following form of the  $\varepsilon$ -conjecture:

**Conjecture 4.1.** *For a number field  $K$ , let  $H_K$  be its Hilbert's class field,  $H_{K/\mathbb{Q}}$  its genus field and  $\mathcal{C}'_K := \text{Gal}(H_K/H_{K/\mathbb{Q}})$ . Let  $p$  be a prime number. For all  $\varepsilon > 0$  there exists  $C'_{d,p,\varepsilon}$  such that  $\#(\mathcal{C}'_K \otimes \mathbb{Z}_p) \leq C'_{d,p,\varepsilon} \cdot (\sqrt{D_K})^\varepsilon$  holds for all  $K$  of degree  $d$ , except possibly for sparse families of density zero.*

One may ask what happens for a “global” strong  $\varepsilon$ -conjecture on the form:

$$\#\mathcal{C}_K \leq \widehat{C}_{d,\varepsilon} \cdot (\sqrt{D_K})^\varepsilon.$$

The paper [6] of Ryan Daileda recalls some results by Littlewood showing that (under GRH) there exist imaginary quadratic fields with arbitrary large discriminant for which  $\#\mathcal{C}_K \geq c \cdot \sqrt{D_K} \log_2(D_K)$ , where  $c$  is an absolute constant. For real quadratic fields a result of Montgomery and Weinberger is that there exist real quadratic fields with arbitrary large discriminant whose class numbers satisfy  $\#\mathcal{C}_K \geq c \cdot \sqrt{D_K} \frac{\log_2(D_K)}{\log(D_K)}$ . Analogous results are known for cyclic cubic fields and Daileda proves that there exists an absolute constant  $c > 0$  so that there are totally real non-abelian cubic fields, with arbitrary large discriminant satisfying  $\#\mathcal{C}_K \geq c \cdot \sqrt{D_K} \left( \frac{\log_2(D_K)}{\log(D_K)} \right)^2$ . All this has been generalized to CM number fields in [7].

So, if we consider an inequality of “strong global  $\varepsilon$ -conjecture” type:

$$\#\mathcal{C}_K \leq \widehat{C}_{d,\varepsilon} \cdot (\sqrt{D_K})^\varepsilon, \quad \text{for all } K \text{ of degree } d,$$

any *infinite family*  $\mathcal{K}$  of fields  $K$  such that  $\#\mathcal{C}_K \geq c \cdot \sqrt{D_K}$ , for a constant  $c > 0$ , independent of  $K \in \mathcal{K}$ , yields:

$$\log(\widehat{C}_{d,\varepsilon}) \geq \log(c) + (1 - \varepsilon) \cdot \log(\sqrt{D_K}),$$

which is absurd. In other words, there is in general no strong global  $\varepsilon$ -conjecture; nevertheless the question of the strong form we have considered

(for  $p$  fixed and for  $\mathcal{C}_K$  or  $\mathcal{C}'_K$ ):

$$\#(\mathcal{C}_K \otimes \mathbb{Z}_p) \leq C_{d,p,\varepsilon} \cdot (\sqrt{D_K})^\varepsilon,$$

depends on the finiteness of fields  $K$  such that  $\#(\mathcal{C}_K \otimes \mathbb{Z}_p) \geq c_p \cdot \sqrt{D_K}$ . For instance, in the quadratic case and  $p > 2$ , this should give some  $p$ -class groups for which the  $p$ -rank and/or the exponent tend to infinity with  $D_K$ ; even in the imaginary case, this may occur only for very sparse families.

**4.1.3. Computation of some successive local maxima.** The following program, for imaginary quadratic fields, allows a study of this question by computing the successive local maxima of  $C'_{2,\varepsilon}$ , in  $C$ , the discriminant and the class number obtained for each local maximum, in  $D$  and  $h$ , respectively:

```
{eps=0.05;Cm=0;BD=2;BD=10^9;for(D=bD,BD,e=valuation(D,2);M=D/2^e;
if(core(M)!=M,next);if((e==1||e>3)||e%4==0&&Mod(M,4)!=-1)||e%4==2&&Mod(M,4)!=1),
next);h=qfbclassno(-D);N=omega(D);C=h/(2^(N-1)*(sqrt(D)^eps));
if(C>Cm,Cm=C;print("D=",D," h=",h," C=",C))}
```

As far as the program has run, it slows down between  $C = 6 \cdot 10^4$  and  $C = 7 \cdot 10^4$ , but no conclusion is possible as expected from the above. But the most spectacular fact, that we have discover, is that, for each local maximum  $C$ , the corresponding discriminant is prime (whence  $h$  odd) whatever  $\varepsilon$ , as shown by this short excerpt:

D	h	C
-3	1	0.972908434869468710702241668941166407
-23	3	2.773818617890694606606085132125197163
-47	5	4.541167885124564220325740509229014479
-71	7	6.292403751297605635733619062872115785
-167	11	9.678872599268429560299054329160821597
-191	13	11.400332501352005304200415816510168367
-239	15	13.080709822134822456679612679136456819
-311	19	16.460180420909375330798097085967676763
-431	21	18.045019802162182082161592477498679286
-479	25	21.425532320359474690178248184779886979
(...)		
-1118314391	77395	45972.572539103313552220923397893022055
-1130984399	77697	46138.963607764612827282941613333974355
-1139075159	78141	46394.356488699687700451681236043961611
-1184068679	81267	48203.636807716833174323833829210758624
-1229647319	81419	48248.214898940545998582986401841694028
-1237871879	82171	48685.729279564354497202266261569081321
-1250370239	83503	49462.50565126222727040029721940389296

We have no counterexample in the selected interval  $D \leq 2 \cdot 10^9$  and no serious explanation, but if we test the local maxima of  $\frac{\#\mathcal{C}_K}{(\sqrt{D_K})^\varepsilon}$  instead of  $\frac{2^{-(N-1)} \#\mathcal{C}_K}{(\sqrt{D_K})^\varepsilon}$ , we have for instance the following normal behaviour:

D=[3, 1; 5, 1]	h=2	C=1.8690792417830016333288729232969402612
D=Mat([23, 1])	h=3	C=2.7738186178906946066060851321251971632
D=[3, 1; 13, 1]	h=4	C=3.6499202570298105632113852382187054319
D=Mat([47, 1])	h=5	C=4.5411678851245642203257405092290144795

```

D=Mat([71, 1])          h=7      C=6.2924037512976056357336190628721157852
D=[5, 1; 19, 1]        h=8      C=7.1391564091327201771370767959908455517
D=[7, 1; 17, 1]        h=10     C=8.8738345254872857598506328581670649858
(...)
D=Mat([63839, 1])      h=423    C=320.78438186581184951419633278279394585
D=[23, 1; 2833, 1]     h=424    C=321.37826092893978747476207092089453868
D=[113, 1; 607, 1]    h=434    C=328.53606501556853121813362584226004750
D=[19, 1; 23, 1; 163, 1] h=440    C=332.76370450428389891215784283045533831
D=[41, 1; 1831, 1]    h=454    C=342.90123385517606193005461615124054801
D=[19, 1; 37, 1; 113, 1] h=468    C=352.97586311634473415004028909597003881
D=[43, 1; 1973, 1]    h=480    C=361.43179021472306392157190052394995551
D=Mat([88079, 1])     h=487    C=366.35924203761999218970143789845358049

```

In the same way, if we compute the successive maxima of the 3-class groups, we obtain a similar result:

```

{p=3; bD=1; BD=10^9; Cm=0; for(D=bD, BD, e=valuation(D, 2); M=D/2^e;
if(core(M)!=M, next); if((e==1 || e>3) || (e==0&Mod(M, 4)!=-1) || (e==2&Mod(M, 4)!=1),
next); h=qfbclassno(-D); hp=p^valuation(h, p); Cp=hp; if(Cp>Cm, Cm=Cp;
C=log(Cp)/log(sqrt(D)); print("D=", -D, " h=", h, " hp=", hp, " C=", C))}
D=-23      h=3      hp=3      C=0.70075861284442195481324
D=-199     h=9      hp=9      C=0.83019007976763598642971
D=-983     h=27     hp=27     C=0.95661698654993161545339
D=-3671    h=81     hp=81     C=1.07074359233325762042197
D=-29399   h=243    hp=243    C=1.06778367209896382287404
D=-178559  h=729    hp=729    C=1.09019287826209803280171
D=-2102999 h=2187   hp=2187   C=1.05643959875714455523718
D=-14868719 h=6561   hp=6561   C=1.06436822551851827813563
D=-98311919 h=19683  hh=19683  C=1.07451592116950263349372

```

Then, for  $p = 2$ :

```

{p=2; bD=1; BD=10^9; Cm=0; for(D=bD, BD, e=valuation(D, 2); M=D/2^e;
if(core(M)!=M, next); if((e==1 || e>3) || (e==0&Mod(M, 4)!=-1) || (e==2&Mod(M, 4)!=1),
next); h=qfbclassno(-D); hp=p^valuation(h, p); Cp=hp; if(Cp>Cm, Cm=Cp;
C=log(Cp)/log(sqrt(D)); print("D=", -D, " h=", h, " hp=", hp, " C=", C))}
D=-15      h=2      hp=2      C=0.511916049619630978775355357
D=-39      h=4      hp=4      C=0.756801438067480149325544162
D=-95      h=8      hp=8      C=0.913262080279460212705801846
D=-399     h=16     hp=16     C=0.925899677503555682939700450
D=-791     h=32     hp=32     C=1.038687593312750474942887870
D=-2519    h=64     hp=64     C=1.062075159346033035976072133
D=-10295   h=128    hp=128    C=1.050289653382181398975491576
D=-39431   h=256    hp=256    C=1.048009122470377471769618833
D=-132599  h=512    hp=512    C=1.057783767181715434360601717
D=-328319  h=1024   hp=1024   C=1.091420745999194423260975917
D=-1333631 h=2048   hp=2048   C=1.081244297733198664388474474
D=-4599839 h=4096   hp=4096   C=1.084346236368631648159879902
D=-18855359 h=8192   hp=8192   C=1.075781736259555689965062133
D=-63836951 h=16384  hp=16384  C=1.079918254667737276882538104
D=-266675639 h=32768  hp=32768  C=1.071791801714607295960939150
D=-966467519 h=65536  hp=65536  C=1.072093388756179498237226639

```

We shall examine elsewhere all these strange phenomena which seems valid for all  $p$  and suggest the existence of families for which the  $p$ -part of the class number has maximal values, so that  $\frac{\log(\#(\mathcal{C}_K \otimes \mathbb{Z}_p))}{\log(\sqrt{D_K})} \rightarrow 1$  as  $D_K \rightarrow \infty$ .

**4.1.4. Reciprocal study.** To try to suggest the existence of analogous families giving huge  $p$ -class groups, a trick is to consider normic equations, in integers  $a, b$ , of the form:

$$a^2 + m b^2 = 4 \cdot q^{p\rho}, \quad \gcd(a, b) \in \{1, 2\},$$

where  $q > 1$  is any fixed integer and  $\rho$  an exponent as large as possible; then when  $a \geq 1$  increases, we deduce  $b$  and the square free integer  $m$ . This kind of experiment has shown, in [17, § 5.3], that there exist huge discriminants giving interesting  $p$ -adic invariants. Moreover the function:

$$C_{K,p} := \frac{\log(\#(\mathcal{C}_K \otimes \mathbb{Z}_p))}{\log(\sqrt{D_K})},$$

giving (for any  $\varepsilon > 0$ ):

$$\log(C_\varepsilon) \geq (C_{K,p} - \varepsilon) \cdot \log(\sqrt{D_K}),$$

may constitute an obstruction for the strong  $\varepsilon$ -conjecture as soon as:

$$\liminf_{K \in \mathcal{K}} C_{K,p} > 0$$

for an infinite subfamily  $\mathcal{K}$  of the set of imaginary quadratic fields. But a priori, this does not affect the  $p$ -rank  $\varepsilon$ -conjecture.

Of course, the right member of the normic equation being rapidly too large when  $\rho$  increases, the experimentation is very limited regarding PARI/GP possibilities. However, even for small values of  $\rho$  predicting, a priori,  $p$ -classes of order around  $p^\rho$  we obtain much large orders, and the following numerical results may be convincing enough about the infiniteness of such utmost examples.

```
{p=3;rho=4;q=2;Y=4*q^(p^rho);ba=1;Ba=sqrt(Y);H=1;for(a=ba,Ba,B=Y-a^2;
m=core(B);D=m;if(Mod(m,4)!=-1,D=4*m);b=component(core(B,1),2);
if(gcd(a,b)>2,next);h=qfbclassno(-D);vh=valuation(h,p);hp=p^vh;
if(hp>H,H=hp;Cp=log(hp)/log(sqrt(D));Hp=component(quadclassunit(-D),2);
d=component(matsize(Hp),2);L=List;for(k=1,d,c=component(Hp,k);
w=valuation(c,p);if(valuation(c,p)!=0,listput(L,p^w)));
print("D=",D," a=",a," b=",b," Cp=",Cp," hp=",hp," Hp=",L))}
rho=4
D=9671406556917033397649407 a=1 b=1 Cp=0.152767 hp=81 Hp=[81]
D=197375644018714967298967 a=5 b=7 Cp=0.204814 hp=243 Hp=[243]
D=9671406556917033397648447 a=31 b=1 Cp=0.229151 hp=729 Hp=[729]
D=9671406556917033397648319 a=33 b=1 Cp=0.267343 hp=2187 Hp=[729,3]
D=9671406556917033397644647 a=69 b=1 Cp=0.305534 hp=6561 Hp=[243,27]
D=9671406556917033397435039 a=463 b=1 Cp=0.343726 hp=19683 Hp=[19683]
D=9671406556917033397373783 a=525 b=1 Cp=0.381918 hp=59049 Hp=[59049]
D=9671406556917033395993039 a=1287 b=1 Cp=0.420110 hp=177147 Hp=[177147]
D=9671406556917033372819119 a=4983 b=1 Cp=0.496494 hp=1594323 Hp=[531441,3]
```

```

D=9671406556917022018093783 a=106675 b=1 Cp=0.534686 hp=4782969
Hp=[1594323,3]
{p=2;rho=6;q=2;Y=4*q^(p^rho);ba=1;Ba=sqrt(Y);H=1;for(a=ba,Ba,B=Y-a^2;
m=core(B);D=m;if(Mod(m,4)!=-1,D=4*m);b=component(core(B,1),2);
if(gcd(a,b)>2,next);h=qfbcassno(-D);vh=valuation(h,p);hp=p^vh;
if(hp>H,H=hp;Cp=log(hp)/log(sqrt(D));Hp=component(quadclassunit(-D),2);
d=component(matsize(Hp),2);L=List;for(k=1,d,c=component(Hp,k);
w=valuation(c,p);if(valuation(c,p)!=0,listput(L,p^w)));
print("D=",D," a=",a," b=",b," Cp=",Cp," hp=",hp," Hp=",L))}
rho=6
D=8198552921648689607 a=1 b=3 Cp=0.445646 hp=16384 Hp=[512,2,2,2,2,2]
D=18446744073709551615 a=2 b=2 Cp=0.468750 hp=32768 Hp=[512,4,2,2,2,2]
D=73786976294838206415 a=7 b=1 Cp=0.575757 hp=524288 Hp=[32768,4,2,2,2]
D=73786976294838175135 a=177 b=1 Cp=0.636363 hp=2097152 Hp=[8192,16,2,2,2,2]
D=73786976294831146815 a=2657 b=1 Cp=0.666666 hp=4194304
Hp=[16384,8,2,2,2,2,2]
D=8198552921599834167 a=20969 b=3 Cp=0.732133 hp=8388608
Hp=[65536,4,2,2,2,2,2]
D=73786976293564644495 a=35687 b=1 Cp=0.787878 hp=67108864
Hp=[2048,512,2,2,2,2,2,2]
D=73786976290585731943 a=65211 b=1 Cp=0.969696 hp=4294967296
Hp=[33554432,4,2,2,2,2,2]

{p=2;rho=6;q=3;Y=4*q^(p^rho);ba=1;Ba=sqrt(Y);H=1;for(a=ba,Ba,B=Y-a^2;
m=core(B);D=m;if(Mod(m,4)!=-1,D=4*m);b=component(core(B,1),2);
if(gcd(a,b)>2,next);h=qfbcassno(-D);vh=valuation(h,p);hp=p^vh;
if(hp>H,H=hp;Cp=log(hp)/log(sqrt(D));Hp=component(quadclassunit(-D),2);
d=component(matsize(Hp),2);L=List;for(k=1,d,c=component(Hp,k);
w=valuation(c,p);if(valuation(c,p)!=0,listput(L,p^w)));
print("D=",D," a=",a," b=",b," Cp=",Cp," hp=",hp," Hp=",L))}
rho=6
D=13734735281170049938631396357123 a=1 b=1 Cp=0.154682 hp=256 Hp=[64,2,2]
D=53651309692070507572778892020 a=2 b=32 Cp=0.272429 hp=8192
Hp=[64,4,4,2,2,2,2]
D=13734735281170049938631396357108 a=4 b=2 Cp=0.270694 hp=16384
Hp=[64,8,2,2,2,2,2,2]
D=549389411246801997545255854283 a=7 b=5 Cp=0.303662 hp=32768
Hp=[64,4,2,2,2,2,2,2,2]
D=38046358119584625868785031460 a=8 b=38 Cp=0.379179 hp=262144
Hp=[512,8,2,2,2,2,2,2,2]
D=13734735281170049938631396327195 a=173 b=1 Cp=0.464048 hp=16777216
Hp=[32768,8,4,2,2,2,2,2]

```

We note the exceptional case  $D_K = 73786976290585731943$  with

$$\mathcal{C}_K \otimes \mathbb{Z}_2 \simeq \mathbb{Z}/2^{25}\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^5,$$

giving the large value  $C_{K,2} = 0.969696$ .

One computes that the group  $\mathcal{T}_K$  that we shall study in the next section is isomorphic to  $\mathbb{Z}/2^2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^6$  which yields  $[\tilde{K} \cap H_K : K] = 2^{25}$ , where  $\tilde{K}$  is the compositum of the  $\mathbb{Z}_2$ -extensions of  $K$ .

Many families are described by means of parametrized radicals as the family of fields  $K = \mathbb{Q}(\sqrt{k^2 - q^n})$  with any prime  $q \neq 2$ ,  $k^2 - q^n < 0$ , in which  $\mathcal{C}_K$  has, under some conditions on the parameters, an element of order  $n$  (see [2, Theorem 3.1] and its bibliography); applied to  $n = p^r$ , we get, for  $C_{K,p}$ , the upper bound  $\frac{O(1)r}{p^r} \rightarrow 0$  as  $r \rightarrow \infty$ , not sufficient to give “bad families”.

It is difficult to say if some of the above huge discriminants may be obtained with explicit parametrized expressions.

**4.2. Reminders on  $p$ -ramification theory.** We intend to give now some analogies with the torsion group  $\mathcal{T}_K$  of the Galois group of the maximal abelian  $p$ -ramified (i.e., unramified outside  $p$  and  $\infty$ ) pro- $p$ -extension of  $K$ ; this extension contains the  $p$ -Hilbert class field of  $K$  (in the ordinary sense) and the compositum of the  $\mathbb{Z}_p$ -extensions of  $K$ . This Galois group introduces the “normalized”  $p$ -adic regulator of  $K$  defined in [21, §5].

Since in this section the non- $p$ -part of the class group does not intervene, unless otherwise stated, we shall put, by abuse of notation,  $\mathcal{C}_K := \mathcal{C}_K \otimes \mathbb{Z}_p$ .

**4.2.1. Structure of the  $p$ -torsion group  $\mathcal{T}_K$ .** Let  $K$  be any number field and let  $p \geq 2$  be a prime number; we denote by  $\mathfrak{p} \mid p$  the prime ideals of  $K$  dividing  $p$ . Consider the group  $E_K$  of  $p$ -principal global units of  $K$  (i.e., units  $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p} \mid p} \mathfrak{p}}$ ). For each  $\mathfrak{p} \mid p$ , let  $K_{\mathfrak{p}}$  be the  $\mathfrak{p}$ -completion of  $K$  and  $\bar{\mathfrak{p}}$  the corresponding prime ideal of the ring of integers of  $K_{\mathfrak{p}}$ ; then let:

$$U_K := \left\{ u \in \bigoplus_{\mathfrak{p} \mid p} K_{\mathfrak{p}}^{\times}, u = 1 + x, x \in \bigoplus_{\mathfrak{p} \mid p} \bar{\mathfrak{p}} \right\} \quad \& \quad W_K := \text{tor}_{\mathbb{Z}_p}(U_K),$$

the  $\mathbb{Z}_p$ -module of principal local units at  $p$  and its torsion subgroup.

We consider the diagonal embedding  $E_K \otimes \mathbb{Z}_p \rightarrow U_K$  whose image is  $\bar{E}_K$ , the topological closure of  $E_K$  in  $U_K$ .

We assume in this paper that  $K$  satisfies the Leopoldt conjecture at  $p$ . Whence the following  $p$ -adic result ([21, Lemma 3.1, Corollary 3.2], [16, Lemma III.4.2.4], [26, Définition 2.11, Proposition 2.12]):

**Lemma 4.2.** *Let  $\mu_K$  be the group of global roots of unity of  $p$ -power order of  $K$ . Under the Leopoldt conjecture for  $p$  in  $K$ , we have  $\text{tor}_{\mathbb{Z}_p}(\bar{E}_K) = \mu_K$  and the exact sequence (where  $\log$  is the  $p$ -adic logarithm):*

$$1 \rightarrow W_K / \mu_K \longrightarrow \text{tor}_{\mathbb{Z}_p}(U_K / \bar{E}_K) \xrightarrow{\log} \text{tor}_{\mathbb{Z}_p}(\log(U_K) / \log(\bar{E}_K)) \rightarrow 0.$$

Put  $\mathcal{W}_K := W_K / \mu_K$  &  $\mathcal{R}_K := \text{tor}_{\mathbb{Z}_p}(\log(U_K) / \log(\bar{E}_K))$ . Then the above exact sequence becomes  $1 \rightarrow \mathcal{W}_K \longrightarrow \text{tor}_{\mathbb{Z}_p}(U_K / \bar{E}_K) \xrightarrow{\log} \mathcal{R}_K \rightarrow 0$ .

Let  $\tilde{K}$  be the compositum of the  $\mathbb{Z}_p$ -extensions,  $H_K$  the  $p$ -Hilbert class field and  $H_K^{\text{pr}}$  the maximal Abelian  $p$ -ramified pro- $p$ -extension, of  $K$ . Then let  $H_K^{\text{bp}}$  be the Bertrandias–Payan field (compositum of the  $p$ -cyclic extensions of  $K$  embeddable in  $p$ -cyclic extensions of arbitrary large degree).

In the following diagram, class field theory yields:

$$\mathrm{Gal}(H_K^{\mathrm{pr}}/H_K) \simeq U_K/\overline{E}_K \text{ and } \mathrm{Gal}(H_K^{\mathrm{pr}}/H_K^{\mathrm{bp}}) \simeq \mathcal{W}_K.$$

We denote by  $\tilde{\mathcal{C}}_K$  the subgroup of the  $p$ -class group  $\mathcal{C}_K$  corresponding to  $\mathrm{Gal}(H_K/\tilde{K} \cap H_K)$  by class field theory.

Then  $\mathcal{R}_K$  is isomorphic to  $\mathrm{Gal}(H_K^{\mathrm{bp}}/\tilde{K}H_K)$ :

$$\begin{array}{ccccc}
 & & \mathcal{T}_K & & \\
 & \tilde{K} & \xrightarrow{\simeq \tilde{\mathcal{C}}_K} & \tilde{K}H_K & \xrightarrow{\simeq \mathcal{R}_K} & H_K^{\mathrm{bp}} & \xrightarrow{\simeq \mathcal{W}_K} & H_K^{\mathrm{pr}} \\
 & \downarrow & & \downarrow & & & & \uparrow \\
 & \tilde{K} \cap H_K & \xrightarrow{\simeq \mathcal{C}_K} & H_K & \xrightarrow{\simeq U_K/\overline{E}_K} & & & \\
 & \downarrow & & \downarrow & & & & \\
 & K & & & & & & 
 \end{array}$$

We have  $\#\mathcal{T}_K = \#\tilde{\mathcal{C}}_K \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K$  and the following inequalities:

$$\begin{aligned}
 (7) \quad \mathrm{rk}_p(\mathcal{T}_K) &\leq \mathrm{rk}_p(\tilde{\mathcal{C}}_K) + \mathrm{rk}_p(\mathcal{R}_K) + \mathrm{rk}_p(\mathcal{W}_K) \\
 &\leq \mathrm{rk}_p(\mathcal{C}_K) + r_1 + r_2 - 1 + \#S_K,
 \end{aligned}$$

where  $(r_1, r_2)$  is the signature of  $K$  and  $S_K$  the set of  $p$ -places of  $K$ . So, for a constant degree  $d$ , the  $p$ -rank  $\varepsilon$ -conjecture for the  $p$ -class groups implies the  $p$ -rank  $\varepsilon$ -conjecture for the torsion groups  $\mathcal{T}_K$  and conversely since we have the other inequality:

$$\begin{aligned}
 (8) \quad \mathrm{rk}_p(\mathcal{C}_K) &\leq \mathrm{rk}_p(\tilde{\mathcal{C}}_K) + \mathrm{rk}_p(\mathrm{Gal}(\tilde{K} \cap H_K/K)) \\
 &\leq \mathrm{rk}_p(\tilde{\mathcal{C}}_K) + r_2 + 1 \leq \mathrm{rk}_p(\mathcal{T}_K) + r_2 + 1.
 \end{aligned}$$

For more precise rank formulas for  $\mathcal{T}_K$ , see [16, Corollary III.4.2.3] and the reflection theorem that we shall recall in §4.4.

**4.2.2. The  $p$ -adic Brauer–Siegel conjecture for  $\mathcal{T}_K$ .** We have proposed in [17], for the totally real case, after extensive numerical computations, the following conjecture:

**Conjecture 4.3.** *Let  $p \geq 2$  be prime and let  $d$  be a given degree. For any number field  $K$  (under Leopoldt’s conjecture), let  $\mathcal{T}_K$  be the torsion group of the Galois group of the maximal abelian  $p$ -ramified pro- $p$ -extension of  $K$ . There exists a constant  $\tilde{\mathcal{C}}_{d,p}$  such that:*

$$\#\mathcal{T}_K \leq (\sqrt{D_K})^{\tilde{\mathcal{C}}_{d,p}}, \text{ for all } K \text{ totally real of degree } d.$$

We put, for  $p$  fixed and for any totally real number field  $K$ :

$$(9) \quad \tilde{\mathcal{C}}_{K,p} := \frac{\log(\#\mathcal{T}_K)}{\log(\sqrt{D_K})} \leq \tilde{\mathcal{C}}_{d,p}.$$

In practice,  $\tilde{\mathcal{C}}_{K,p}$  may be much smaller than 1 (and it is often 0), except very sparse cases as that of  $K = \mathbb{Q}(\sqrt{19})$  and  $p = 13599893$ , for which  $\mathcal{T}_K = \mathcal{R}_K \simeq \mathbb{Z}/p\mathbb{Z}$ , whence  $\tilde{\mathcal{C}}_{K,p} = \frac{\log(\#\mathcal{T}_K)}{\log(\sqrt{4 \times 19})} = \frac{\log(13599893)}{\log(\sqrt{4 \times 19})} = 7.5855$ .

But  $\mathcal{C}_K = 1$ .

**4.2.3. Estimation of  $\tilde{\mathcal{C}}_{F_N,p}$ .** Put  $F := F_{N,p}$  and  $\tilde{\mathcal{C}}_{F_N,p} =: \tilde{\mathcal{C}}_F$  for  $p$  fixed. Then, from the computations in Subsection 2.3, using for  $\mathcal{T}_F$  the analog of Chevalley's formula given in [16, Theorem IV.3.3], we can put similarly  $\#\mathcal{T}_F =: p^{N-r+\tilde{\Delta}(N)}$ ,  $\tilde{\Delta}(N) \geq 0$ , where  $r \geq 0$  depends on  $p$ -adic properties of the ramified primes  $\ell \neq p$ , and we may estimate that, as  $N \rightarrow \infty$ :

$$\tilde{\mathcal{C}}_F \approx \frac{(N-r+\tilde{\Delta}(N))\log(p)}{\frac{p-1}{2} [N\log(N) + N\gamma_p + \frac{1}{2}\log(N) + O(1)]} \sim c_p \cdot \frac{1+o(1)}{\log(N)},$$

where  $c_p = \frac{2\log(p)}{p-1}$  and assuming a small order of magnitude of  $\tilde{\Delta}(N)$ .

Give a program computing (in Cp)  $\tilde{\mathcal{C}}_F$ ; in the imaginary quadratic case for  $p = 2$ , the conjectural inequality implies  $\#(\tilde{\mathcal{C}}_F) \leq (\sqrt{D_F})^{\tilde{\mathcal{C}}_F}$  (indeed,  $\#\mathcal{R}_F = 1$  and  $\#\mathcal{T}_F = \#\tilde{\mathcal{C}}_F \cdot \mathcal{W}_F$ , where  $\mathcal{W}_F = 2$  (resp. 1) if  $m \equiv \pm 1 \pmod{8}$  (resp. if not)). So  $\mathcal{C}_F$  (in Clres) may be larger than  $\#\mathcal{T}_F$  (in Tor):

```
{p=2;n=12;m=1;for(N=2,100,el=prime(N);m=(-1)^((el-1)/2)*el*m;P=x^2-m;
K=bnfinit(P,1);D=abs(m);Kpn=bnrinit(K,p^n);r=1;if(m<0,r=2);L=List;
Hpn=component(component(Kpn,5),2);e=component(matsize(Hpn),2);T=1;
for(k=1,e-r,c=component(Hpn,e-k+1);if(Mod(c,p)==0,q=p^valuation(c,p);
T=T*q;listinsert(L,q,1));C8=component(K,8);C81=component(C8,1);
h=component(C81,1);Clord=component(C81,2);K=bnfnarrow(K);
Cl=component(K,2);print("m=",m," Clres=",Clres," Clord =",Clord);
print("Structure of T=",L);print("#Tor=",T," Cp=",log(T)/log(sqrt(D)))}
```

```
m=-15 Clres=[2]
Structure of Tor=[2]
#Tor=2 Cp=0.51191604961963097877535535772960454081
m=105 Clres=[2,2]=[2]xClord
Structure of Tor=[2,2]
#Tor=4 Cp=0.59574824743531323067786608868687642325
m=-1155 Clres=[2,2,2]
Structure of Tor=[2,2,2]
#Tor=8 Cp=0.58975726471501581115878339498474155345
m=-15015 Clres=[12,2,2,2]
Structure of Tor=[2,2,2,2]
#Tor=16 Cp=0.57661327808675875001115538902772596330
m=-255255 Clres=[16,2,2,2,2]
Structure of Tor=[2,2,2,2,2]
#Tor=32 Cp=0.55674390390043840097934284424073618196
```



```

m=4849845   Clres=[4,2,2,2,2,2]=[2]xClord
Structure of Tor=[4,2,2,2,2,2]
#Tor=128     Cp=0.63036067699149527703810495075838580918
m=-111546435   Clres=[42,2,2,2,2,2,2]
Structure of Tor=[2,2,2,2,2,2,2]
#Tor=128     Cp=0.52369594802182316940823598390366865409
m=-3234846615   Clres=[308,2,2,2,2,2,2,2]
Structure of Tor=[4,2,2,2,2,2,2,2]
#Tor=512     Cp=0.56978162829823280646049502887398109011
m=100280245065   Clres=[2,2,2,2,2,2,2,2]=[2]xClord
Structure of Tor=[2,2,2,2,2,2,2,2]
#Tor=512     Cp=0.49254011775311327297187105815891979776
m=3710369067405   Clres=[34,2,2,2,2,2,2,2,2]=[2]xClord
Structure of Tor=[2,2,2,2,2,2,2,2,2]
#Tor=1024    Cp=0.47898799604336105240349464751069989167
m=152125131763605   Clres=[2,2,2,2,2,2,2,2,2,2]=[2]xClord
Structure of Tor=[4,2,2,2,2,2,2,2,2,2]
#Tor=4096    Cp=0.50942162732997733185991268818214459006
m=-6541380665835015   Clres=[28284,2,2,2,2,2,2,2,2,2,2]
Structure of Tor=[2,2,2,2,2,2,2,2,2,2,2]
#Tor=4096    Cp=0.45680772041407406738242757697825579415
m=307444891294245705   Clres=[14,2,2,2,2,2,2,2,2,2,2,2]=[2]xClord
Structure of Tor=[2,2,2,2,2,2,2,2,2,2,2,2]
#Tor=8192    Cp=0.44755741322126195443312989366082458089
m=16294579238595022365   Clres=[2,2,2,2,2,2,2,2,2,2,2,2,2]=[2]xClord
Structure of Tor=[32,2,2,2,2,2,2,2,2,2,2,2,2]
#Tor=262144  Cp=0.56407742575833976013164891102271736655
m=-961380175077106319535   Clres=[1210734,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
Structure of Tor=[2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
#Tor=32768   Cp=0.43039341330765014032907101774855577316
m=-58644190679703485491635   Clres=[1622526,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
Structure of Tor=[2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
#Tor=65536   Cp=0.42308787191112227068703992014107942867

```

We do not know an algorithm computing the “exceptional elements” of  $\mathcal{T}_F$  as for  $p$ -class groups.

The case  $p = 3$  is similar and gives for instance for the 16 cyclic cubic fields of conductor  $f = 7 \cdot 13 \cdot 19 \cdot 31 \cdot 37$ :

```

P=x^3+x^2-661054*x+49725976   Cl=[3,3,3,3,3]
Structure of Tor=[9,3,3,3,3]
#Tor=729   Cp=0.45459180523024141673723101157712338880
P=x^3+x^2-661054*x+198463201   Cl=[6,6,3,3]
Structure of Tor=[9,3,3,3,3]
#Tor=729   Cp=0.45459180523024141673723101157712338880
P=x^3+x^2-661054*x+97321888   Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243   Cp=0.37882650435853451394769250964760282400
P=x^3+x^2-661054*x-186270421   Cl=[39,3,3,3,3]
Structure of Tor=[9,3,3,3,3]
#Tor=729   Cp=0.45459180523024141673723101157712338880
P=x^3+x^2-661054*x-79179619   Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3,3]

```

```

#Tor=81    Cp=0.30306120348682761115815400771808225920
P=x^3+x^2-661054*x-188253584    Cl=[6,6,3,3]
Structure of Tor=[9,3,3,3,3]
#Tor=729    Cp=0.45459180523024141673723101157712338880
P=x^3+x^2-661054*x+138968311    Cl=[3,3,3,3,3]
Structure of Tor=[3,3,3,3]
#Tor=81    Cp=0.30306120348682761115815400771808225920
P=x^3+x^2-661054*x-146607161    Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243    Cp=0.37882650435853451394769250964760282400
P=x^3+x^2-661054*x-158506139    Cl=[3,3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243    Cp=0.37882650435853451394769250964760282400
P=x^3+x^2-661054*x-140657672    Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243    Cp=0.37882650435853451394769250964760282400
P=x^3+x^2-661054*x+186564223    Cl=[6,6,3,3]
Structure of Tor=[3,3,3,3]
#Tor=81    Cp=0.30306120348682761115815400771808225920
P=x^3+x^2-661054*x+81456584    Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243    Cp=0.37882650435853451394769250964760282400
P=x^3+x^2-661054*x+206395853    Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3]
#Tor=81    Cp=0.30306120348682761115815400771808225920
P=x^3+x^2-661054*x-206102051    Cl=[3,3,3,3]
Structure of Tor=[3,3,3,3]
#Tor=81    Cp=0.30306120348682761115815400771808225920
P=x^3+x^2-661054*x+2130064    Cl=[12,12,3,3]
Structure of Tor=[3,3,3,3,3,3]
#Tor=729    Cp=0.45459180523024141673723101157712338880
P=x^3+x^2-661054*x+27911183    Cl=[3,3,3,3,3]
Structure of Tor=[3,3,3,3,3]
#Tor=243    Cp=0.37882650435853451394769250964760282400

```

Now give some examples for  $p = 2$  and  $p = 3$  where the constant  $\tilde{C}_F$  is rather large; the degree  $p$  cyclic extensions  $F$  are not necessary of the form  $F_N$ .

(i)  $p = 2$ ,  $m$  divides  $5 \cdot 7 \cdots 41 \cdot 43$ .

```

m=5005 Clres=[2,2,2] Clord =[2,2]
Structure of Tor=[4,2,2]
#Tor=16    Cp=0.65098051255057821664489247470558390764
m=1078282205 Clres=[4,2,2,2,2,2,2] Clord =[4,2,2,2,2,2,2]
Structure of Tor=[32,4,2,2,2,2,2]
#Tor=4096    Cp=0.79983769534579979852611700457236749058
m=215656441 Clres=[2,2,2,2,2,2,2] Clord =[2,2,2,2,2,2]
Structure of Tor=[8,2,2,2,2,2]
#Tor=256    Cp=0.57794783195778051534152523962133180636
m=436092044389001 Clres=[2,2,2,2,2,2,2,2,2,2,2] Clord =[2,2,2,2,2,2,2,2,2,2,2]
Structure of Tor=[32,2,2,2,2,2,2,2,2,2,2]
#Tor=16384    Cp=0.57575701869336876560350400103707496594
m=46189 Clres=[2,2,2] Clord =[2,2]
Structure of Tor=[16,2,2]

```

```

#Tor=64      Cp=0.77443028965455279095387556052161745644
m=221 Clres=[4] Clord =[2]
Structure of Tor=[16]
#Tor=16      Cp=1.0272342185833848333397010211662592994
m=435656388001 Clres=[2,2,2,2,2,2] Clord =[2,2,2,2,2,2]
Structure of Tor=[8,2,2,2,2,2]
#Tor=512     Cp=0.46554453503678235229309396294036417544

```

We note the case of  $m = 221$  for which  $\mathcal{C}_F^{\text{res}} \simeq \mathbb{Z}/4\mathbb{Z}$ ,  $\mathcal{C}_F^{\text{ord}} \simeq \mathbb{Z}/2\mathbb{Z}$ , but  $\mathcal{T}_F \simeq \mathbb{Z}/16\mathbb{Z}$  due to the 2-adic regulator. This gives the exceptional value  $\tilde{\mathcal{C}}_{F,2} \approx 1.02723422$ .

(ii)  $p = 3$ ,  $f$  divides  $13 \cdot 19 \cdot 31 \cdot 37 \cdot 43$ .

```

f=10621 P=x^3+x^2-3540*x-60579 Cl=[3,3]
Structure of Tor=[9,9,3]
#Tor=243 Cp=0.33667761382504192691963484073748684274
P=x^3+x^2-4060762*x-3150249179 Cl=[3,3,3,3]
Structure of Tor=[27,27,3,3,3]
#Tor=19683 Cp=0.60601970488507546845534271332747631693
f=12182287 P=x^3+x^2-4060762*x+187697459 Cl=[9,9,3,3,3]
Structure of Tor=[9,9,3,3,3]
#Tor=2187 Cp=0.47134865935505869768748877703248157983
f=12182287 P=x^3+x^2-4060762*x+2380509119 Cl=[3,3,3,3,3]
Structure of Tor=[9,3,3,3,3]
#Tor=729 Cp=0.40401313659005031230356180888498421129
f=12182287 P=x^3+x^2-4060762*x-2309671376 Cl=[3,3,3,3,3]
Structure of Tor=[9,9,3,3,3]
#Tor=2187 Cp=0.47134865935505869768748877703248157983
f=641173 P=x^3-x^2-213724*x-29968901 Cl=[6,6,3]
Structure of Tor=[9,3,3,3]
#Tor=243 Cp=0.33667761382504192691963484073748684274
f=392977 P=x^3-x^2-130992*x+6826156 Cl=[3,3,3]
Structure of Tor=[9,3,3,3]
#Tor=243 Cp=0.33667761382504192691963484073748684274
f=7657 P=x^3+x^2-2552*x+47360 Cl=[9,3,3]
Structure of Tor=[9,9,3]
#Tor=243 Cp=0.33667761382504192691963484073748684274

```

**4.3.  $p$ -adic Brauer–Siegel conjecture versus  $\varepsilon$ -conjectures.** The  $p$ -adic Brauer–Siegel Conjecture 4.3 concerns more essentially the totally real case for the following reasons which are yet given by the rank inequalities (7) and (8).

**4.3.1. Analysis by means of CM-fields.** We have, with obvious notations and  $p \neq 2$ :

$$\begin{aligned} \mathcal{C}_K &= \mathcal{C}_K^- \oplus \mathcal{C}_K^+, & \mathcal{T}_K &= \mathcal{T}_K^- \oplus \mathcal{T}_K^+, \\ \mathcal{R}_K &= \mathcal{R}_K^- \oplus \mathcal{R}_K^+, & \mathcal{W}_K &= \mathcal{W}_K^- \oplus \mathcal{W}_K^+; \end{aligned}$$

but we have the following properties which explain the differences between real fields and non real ones (under the Leopoldt conjecture):

- (i)  $\#\mathcal{R}_K^- = 1$  since all the units of infinite order of  $K$  are real;
- (ii)  $\#\mathcal{C}_K^- = \#\tilde{\mathcal{C}}_K^- \cdot \#\text{Gal}(\tilde{K} \cap H_K/K)^-$  where  $\text{Gal}(\tilde{K} \cap H_K/K)^- \simeq \mathcal{C}_K^-/\tilde{\mathcal{C}}_K^-$  may be large (but with bounded rank) since  $\text{Gal}(\tilde{K}/K)^- \simeq \mathbb{Z}_p^{\frac{d}{2}}$  contrary to  $\text{Gal}(\tilde{K}/K)^+ \simeq \mathbb{Z}_p$ ;
- (iii)  $\#\mathcal{T}_K^- = \#\tilde{\mathcal{C}}_K^- \cdot \#\mathcal{W}_K^-$  is essentially equal to  $\#\tilde{\mathcal{C}}_K^-$  since  $\mathcal{W}_K^-$  is most often trivial and does not intervene in estimations of class groups for  $d$  fixed;
- (iv)  $\#\mathcal{R}_K^+$  is the main  $p$ -adic invariant which may be nontrivial for much primes  $p$ , even if we have conjectured in [24] that it is trivial for all  $p$  large enough;
- (v)  $\#\mathcal{C}_K^+$  is essentially equal to  $\#\tilde{\mathcal{C}}_K^+$  since the part of the Hilbert class field, contained in the cyclotomic  $\mathbb{Z}_p$ -extension, is very limited;
- (vi)  $\#\mathcal{T}_K^+ = \#\tilde{\mathcal{C}}_K^+ \cdot \#\mathcal{R}_K^+ \cdot \#\mathcal{W}_K^+$  is thus essentially the product  $\#\mathcal{C}_K^+ \cdot \#\mathcal{R}_K^+$ .

So if we assume that  $\#\mathcal{T}_K^+$  is controled, we may consider that  $\#\mathcal{C}_K^+$  is much less than  $\#\mathcal{T}_K^+$  because of the regulator; then, since  $\#\mathcal{T}_K^-$  is independent of any regulator, it is mesured by a divisor of  $\#\mathcal{C}_K^-$  (equal to  $\#\tilde{\mathcal{C}}_K^-$ ) and by  $\#\mathcal{W}_K^-$  controled, which explains (from the factor  $[\tilde{K} \cap H_K : K]$ ) a bigger order of magnitude of  $\#\mathcal{C}_K^-$  regarding  $\#\mathcal{T}_K^-$  than  $\#\mathcal{C}_K^+$  regarding  $\#\mathcal{T}_K^+$ .

**4.3.2. Computation of  $\tilde{\mathcal{C}}_{K,p}$  for imaginary quadratic fields.** We shall illustrate the cases  $p = 2$ , then  $p = 3$ , in various intervals of negative discriminants to observe the local decreasing of the variable  $\tilde{\mathcal{C}}_{K,p}$  (9).

Note that for  $p > 3$ , the group  $\mathcal{W}_K$  is trivial contrary to the cases  $p = 2$  and 3 where  $\mathcal{W}_K$  may be  $\mathbb{Z}/p\mathbb{Z}$ , which must probably be discarded in our considerations.

**(a) Case  $p = 2$ .** The case  $p = 2$  is interesting because of the influence of exceptional classes and gives  $(v_p(\#\mathcal{T}_K)$  in  $\text{vptor}$ ,  $\tilde{\mathcal{C}}_{K,p}$  in  $\text{Cp}$ ):

```
{p=2; bD=10^8; BD=2*10^8; Lp=log(p); vp=0; n=20;
for(D=bD, BD, e=valuation(D, 2); M=D/2^e; if(core(M) != M, next);
if((e==1 || e>3) || (e==0 & Mod(M, 4) != -1) || (e==2 & Mod(M, 4) == -1), next);
m=D; if(e!=0, m=D/4); P=x^2+m; K=bnfinit(P, 1); Kpn=bnrinit(K, p^n);
C5=component(Kpn, 5); Hpn0=component(C5, 1); Hpn=component(C5, 2);
Hpn1=component(Hpn, 1); vptor=valuation(Hpn0/Hpn1, p)-(n-1);
if(vptor > vp, vp=vptor); if(vptor >= vp, Cp=vptor*Lp/log(sqrt(D)));
print("D=", -D, " m=", -m, " vptor=", vptor, " Cp=", Cp))}
```

```
p=2, Interval [10^6, 2*10^6]
D=-1000011 m=-1000011 vptor=3 Cp=0.301029755983435929933445793
D=-1000020 m=-250005 vptor=3 Cp=0.3010295598834164958938994188
D=-1000036 m=-250009 vptor=4 Cp=0.4013722816881976053812061427
D=-1000132 m=-250033 vptor=5 Cp=0.5017118661610285687682449315
(...)
D=-1003620 m=-250905 vptor=5 Cp=0.5015854691511746133432777519
D=-1003940 m=-250985 vptor=6 Cp=0.6018886779424325532238667109
```

```

D=-1005843 m=-1005843 vptor=7 Cp=0.702107244840955486811297669
D=-1007492 m=-251873 vptor=8 Cp=0.8023131911871206028276870051
(...)
D=-1327972 m=-331993 vptor=8 Cp=0.7865966565831969109249007496
D=-1345476 m=-336369 vptor=9 Cp=0.8841001125437591214944446738
D=-1347524 m=-336881 vptor=10 Cp=0.982227596578129040877631145

```

p=2, Interval  $[10^7, 2*10^7]$

```

D=-10000004 m=-2500001 vptor=3 Cp=0.25802570416574861895099915
(...)
D=-10000136 m=-2500034 vptor=3 Cp=0.25802549285588586651933610
D=-10000212 m=-2500053 vptor=4 Cp=0.34403382825873844184039068
D=-10000228 m=-2500057 vptor=5 Cp=0.43004224263528166242354902
(...)
D=-10001220 m=-2500305 vptor=5 Cp=0.43003959612044524303603555
D=-10001355 m=-10001355 vptor=7 Cp=0.6020549303754533561270683
(...)
D=-10028164 m=-2507041 vptor=10 Cp=0.8599356519990630566433445
D=-11423624 m=-2855906 vptor=10 Cp=0.8530415407428446759627785
D=-11434244 m=-2858561 vptor=11 Cp=0.9382920445879771130663980
D=-19227908 m=-4806977 vptor=11 Cp=0.9092149504336010969244116

```

p=2, Interval  $[10^8, 2*10^8]$

```

D=-100000011 m=-100000011 vptor=2 Cp=0.15051499693318294862950
D=-100000020 m=-25000005 vptor=3 Cp=0.225772494296692430574969
(...)
D=-100000072 m=-25000018 vptor=3 Cp=0.225772487923331962924891
D=-100000120 m=-25000030 vptor=4 Cp=0.301029976053644338278634
(...)
D=-100000228 m=-25000057 vptor=4 Cp=0.301029958404363471726365
D=-100000324 m=-25000081 vptor=6 Cp=0.451544914074197327895496
(...)
D=-100009811 m=-100009811 vptor=6 Cp=0.45154258866246136591601
D=-100009988 m=-25002497 vptor=7 Cp=0.526799636159210448373252
(...)
D=-100042692 m=-25010673 vptor=9 Cp=0.677301796362621931199437
D=-100120215 m=-100120215 vptor=11 Cp=0.8277784989602807429303
D=-100703939 m=-100703939 vptor=11 Cp=0.8275173634473368234393
D=-101091716 m=-25272929 vptor=13 Cp=0.97777114254342282551717
D=-196241540 m=-49060385 vptor=13 Cp=0.94380528108729550144090

```

One sees some influence of genus theory since, for  $D = -101091716$ , we have  $\tilde{C}_{K,2} \approx 0.977771$  because of  $\#\mathcal{T}_K = 2^{13}$ , but to be put in relation with  $\tilde{C}_{K,2} \approx 0.982227$  for  $D = -1347524$ , of the first interval, with  $\#\mathcal{T}_K = 2^{10}$ .

The structure of the class group given by PARI/GP is  $[1024, 2, 2, 2]$ .

Then consider the program computing the structure of  $\mathcal{T}_K$  [23, Programme I, § 3.2] that we recall for the convenience of the reader (choose p, nt such that  $p^{nt}$  be a multiple of the exponent of  $\mathcal{T}_K$ , then the polynomial P):

```

{p=2;nt=32;P=x^2+101091716;K=bnfinit(P,1);Kpn=bnrinit(K,p^nt);
S=component(component(Kpn,1),7);r=component(component(S,2),2)+1;

```

```
Hpn=component(component(Kpn,5),2);L=List;e=component(matsize(Hpn),2);
R=0;for(k=1,e-r,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;
listinsert(L,p^valuation(c,p),1));print("Structure of T: ",L)}
```

Then we obtain  $\mathcal{T}_K \simeq \mathbb{Z}/2^{10}\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . The difference comes from  $\mathcal{W}_K \simeq \mathbb{Z}/2\mathbb{Z}$  (since  $-101091716 \equiv -1 \pmod{16}$ ) and  $[\tilde{K} \cap H_K : K] = 2$ .

**(b) Case  $p = 3$ .**

```
{p=3;bd=10^6;BD=2*10^6;Lp=log(p);vp=0;n=8;
for(D=bd,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3)|| (e==0 & Mod(M,4)!=-1)|| (e==2 & Mod(M,4)==-1),next);
m=D;if(e!=0,m=D/4);P=x^2+m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p)-(n-1);
if(vptor>vp,vp=vptor);if(vptor>=vp,Cp=vptor*Lp/log(sqrt(D));
print("D=", -D, " m=", -m, " vptor=", vptor, " Cp=", Cp))}
p=3, Interval [10^6, 2*10^6]
D=-1000011 m=-1000011 vptor=1 Cp=0.1590402916116620131420348520
D=-1000020 m=-250005 vptor=1 Cp=0.1590401880079362819278196125
D=-1000043 m=-1000043 vptor=1 Cp=0.1590399232477087416304663599
(...)
D=-1020548 m=-255137 vptor=3 Cp=0.4764198506806243371783107010
D=-1021332 m=-255333 vptor=4 Cp=0.6351912130796763789096457580
D=-1022687 m=-1022687 vptor=5 Cp=0.7939129439088033794338891302
(...)
D=-1898859 m=-1898859 vptor=5 Cp=0.7599296140003213455753306574
p=3, Interval [10^7, 10^7+10^6]
D=-10002927 m=-10002927 vptor=3 Cp=0.40895365007950900178024254
D=-10003224 m=-2500806 vptor=4 Cp=0.54527052902405852884991426
(...)
D=-10065279 m=-10065279 vptor=4 Cp=0.54506139909286008239956811
D=-10066440 m=-2516610 vptor=5 Cp=0.68132187532448910070906763
(...)
D=-14316744 m=-3579186 vptor=5 Cp=0.66675746065456780942779488
D=-14547531 m=-14547531 vptor=6 Cp=0.79933316809564910995167969
(...)
D=-19767512 m=-4941878 vptor=6 Cp=0.78474406738375920976115602
p=3, Interval [10^8, 10^8+10^7]
D=-100075971 m=-100075971 vptor=4 Cp=0.477101585455621088296257
D=-100080003 m=-100080003 vptor=5 Cp=0.596375677517090310811391
(...)
D=-100787315 m=-100787315 vptor=5 Cp=0.596147767754605081482216
D=-100867844 m=-25216961 vptor=6 Cp=0.715346318656502973478053
(...)
D=-117344127 m=-117344127 vptor=6 Cp=0.709521342553855083402930
D=-119846559 m=-119846559 vptor=7 Cp=0.826835890443221508331985
(...)
D=-135140024 m=-33785006 vptor=7 Cp=0.821531794828164970116186
D=-136159455 m=-136159455 vptor=8 Cp=0.938516745792290367614873
```

For  $D_K = -136159455$ , we get  $\mathcal{W}_K \simeq \mathbb{Z}/3\mathbb{Z}$ ,  $\mathcal{T}_K \simeq \mathbb{Z}/3^7\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  and  $\mathcal{C}_K \simeq \mathbb{Z}/3^6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  (using the instruction `quadclassunit(-136159455)`).

**4.3.3. Conclusion.** Consider a prime  $p$  and a fixed degree  $d$ . Assume, tentatively, a strong  $\varepsilon$ -conjecture for the groups  $\mathcal{T}_K$  in the case of totally real number fields  $K$  of degree  $d$ , which implies a strong  $\varepsilon$ -conjecture for the  $p$ -class groups. We then have the existence, for all  $\varepsilon > 0$ , of a constant  $\tilde{C}_{d,p,\varepsilon}$  such that:

$$\log(\#\mathcal{T}_K) \leq \log(\tilde{C}_{d,p,\varepsilon}) + \varepsilon \cdot \log(\sqrt{D_K});$$

then introduce the function  $\tilde{C}_{K,p}$ :

$$(10) \quad \tilde{C}_{K,p} := \frac{\log(\#\mathcal{T}_K)}{\log(\sqrt{D_K})} \leq \frac{\log(\tilde{C}_{d,p,\varepsilon})}{\log(\sqrt{D_K})} + \varepsilon.$$

So, when  $D_K \rightarrow \infty$ , we get  $\tilde{C}_{K,p} = \varepsilon + o(1)$ . But in [17, § 5.3], we have proved that there exist explicit infinite families of real quadratic fields for which  $\tilde{C}_{K,p} \approx 1$  (contradiction).

We obtain, generalizing to arbitrary degrees, the following heuristic:

*There is no absolute strong  $\varepsilon$ -conjecture for the  $\mathcal{T}_K$  groups of totally real number fields  $K$  and  $\mathcal{T}_K$  is essentially governed by the normalized  $p$ -adic regulator  $\mathcal{R}_K$ . Nevertheless, as for  $p$ -class groups, one may conjecture that the exceptions to the strong  $\varepsilon$ -conjecture are due to sparse subfamilies of density zero.*

Recall that, from (7) and (8), the  $p$ -rank  $\varepsilon$ -conjecture for the  $\mathcal{T}_K$  does exist if and only if the  $p$ -rank  $\varepsilon$ -conjecture does exist for the  $p$ -class groups  $\mathcal{C}_K$ .

**4.4. Reflection theorem and  $p$ -rank  $\varepsilon$ -conjectures.** Another justification of the above comments is to recall the reflection theorem [22] which exchanges, roughly speaking, “imaginary components” of  $p$ -class groups  $\mathcal{C}$  with “real components” of  $p$ -torsion groups  $\mathcal{T}$ , and conversely, subject to consider fields  $K$  containing the group  $\mu_p$  of  $p$ th roots of unity.

In full generality, the following result precises (7) and (8) when  $\mu_p \subset K$ :

**Proposition 4.4** ([16, Theorem III.4.2.2]). *Let  $K$  be a number field containing  $\mu_p$  and fulfilling Leopoldt’s conjecture at  $p$ . Then we have the rank formula (reflection theorem)<sup>1</sup>  $\text{rk}_p(\mathcal{T}_K^{\text{ord}}) = \text{rk}_p(\mathcal{C}_K^{S,\text{res}}) + \#S - 1$ , where  $S$  is the set of  $p$ -places of  $K$  and  $\mathcal{C}_K^{S,\text{res}}$  the  $S$ -class group  $\mathcal{C}_K^{\text{res}}/\mathcal{C}_K^{\text{res}}(S)$ .*

So,  $\text{rk}_p(\mathcal{C}_K^{S,\text{res}}) = \text{rk}_p(\mathcal{T}_K^{\text{ord}}) - (\#S - 1)$  yields:

$$\begin{aligned} \text{rk}_p(\mathcal{C}_K^{\text{res}}) &\leq \text{rk}_p(\mathcal{C}_K^{S,\text{res}}) + \text{rk}_p(\mathcal{C}_K^{\text{res}}(S)) \\ &\leq \text{rk}_p(\mathcal{T}_K^{\text{ord}}) - [(\#S - 1) - \text{rk}_p(\mathcal{C}_K^{\text{res}}(S))] \leq \text{rk}_p(\mathcal{T}_K^{\text{ord}}) + 1, \end{aligned}$$

giving :

$$-1 \leq \text{rk}_p(\mathcal{T}_K^{\text{ord}}) - \text{rk}_p(\mathcal{C}_K^{\text{res}}) \leq \#S - 1;$$

---

<sup>1</sup>The mentions “ord”, “res” are related to the case  $p = 2$ ; for  $p > 2$ , since  $\mu_p \subset K$ , the two notions coincide.

since  $\#S - 1$  is bounded in the family of number fields, of fixed degree  $d$ , this relation shows that any  $p$ -rank  $\varepsilon$ -conjecture, true for an invariant, is fulfilled by the other. When  $\mu_p \not\subset K$ , one must use the field  $K(\mu_p)$  and the general reflection theorem with  $p$ -adic characters ([16, Theorem II.5.4.5], [22]).

The reflection theorem has been used in [10], in a different manner, using many split primes in  $K(\mu_p)$ , the notion of Arakelov class group (see e.g., [37]), and the following analytic explanation by the authors:

*Roughly, the point is that small non-inert primes in a number field represent elements of the class group which tend not to satisfy any relation with small coefficients. Thus the existence of many such primes contributes significantly to the quotient of the class group by its  $\ell$ -torsion, yielding the desired upper bounds.*

It would be interesting to deepen these approaches that have connections through class field theory, complex and  $p$ -adic analytic methods.

## Acknowledgments

We thank Peter Koymans and Carlo Pagano for fruitful information about various approaches (density results v.s.  $\varepsilon$ -conjectures) concerning  $p$ -class groups in degree  $p$  cyclic fields.

## References

- [1] ADAM, M.; MALLE, G. A class group heuristic based on the distribution of 1-eigenspaces in matrix groups, *J. Number Theory* **149** (2015), 225–235. [MR3296009](#) [Zbl 1371.11145](#)
- [2] BANERJEE, K.; HOQUE, A. From Picard groups of hyperelliptic surfaces to class groups of families of quadratic fields (preprint 2019). <https://arxiv.org/pdf/1903.04210>
- [3] CHEVALLEY, C. Sur la théorie du corps de classes dans les corps finis et les corps locaux, *Jour. of the Fac. of Sc., Tokyo, Sec. I*, **2** (1933), 365–476. [Zbl 0008.05301](#)
- [4] COHEN, H.; LENSTRA JR., H.W. Heuristics on class groups of number fields, *In Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983)*, 33–62, Lecture Notes in Math. **1068**, Springer, Berlin, 1984. [MR0756082](#) [Zbl 0558.12002](#)
- [5] COHEN, H.; MARTINET, J. Étude heuristique des groupes de classes des corps de nombres, *Journal für die Reine und Angewandte Mathematik* **404** (1990), 39–76. [MR1037430](#) [Zbl 0699.12016](#) <https://eudml.org/doc/153196>
- [6] DAILEDÀ, R.C. Non-abelian number fields with very large class numbers, *Acta Arithmetica*, **125** (2006), no. 3, 215–255. [MR2276192](#) [Zbl 1158.11044](#)
- [7] DAILEDÀ, R.C.; KRISHNAMOORTHY, R.; MALYSHEV, A. Maximal class numbers of CM number fields, *J. Number Theory* **130** (2010), no. 4, 936–943. [MR2600412](#) [Zbl 1185.11065](#) <https://doi.org/10.1016/j.jnt.2009.09.013>
- [8] DELAUNAY, C.; JOUHET, F. The Cohen–Lenstra heuristics, moments and  $p^j$ -ranks of some groups, *Acta Arithmetica* (to appear). <https://arxiv.org/pdf/1303.7337>
- [9] ELLENBERG, J.S.; PIERCE, L.B.; WOOD, M.M. On  $\ell$ -torsion in class groups of number fields (preprint 2017). <https://arxiv.org/pdf/1606.06103>
- [10] ELLENBERG, J.S.; VENKATESH, A. Reflection principles and bounds for class group torsion, *Int. Math. Res. Not.* **2007** (2007), no. 1. [MR2331900](#) [Zbl 1130.11060](#) <https://doi.org/10.1093/imrn/rnm002>



- [11] FRÖHLICH, A. The generalization of a theorem of L. Rédei's, *Qart. Jour. of math. Oxford* **2** (1954), no. 5, 13–140. MR0066425 Zbl 0058.26903  
<https://doi.org/10.1093/qmath/5.1.130>
- [12] FRÖHLICH, A. The genus field and genus group in finite number fields I, *Mathematika* **6** (1959), 40–46. MR0113868 Zbl 0202.33005  
<https://doi.org/10.1112/S0025579300001911>
- [13] FURUTA, Y. The genus field and genus number in algebraic number fields, *Nagoya Math. J.* **29** (1967), 281–285. MR0209260 Zbl 0166.05901  
<https://doi.org/10.1017/S0027763000024387>.
- [14] FREI, C.; WIDMER, M. Average bounds for the  $p$ -torsion in class groups of cyclic extensions (preprint 2018). <https://arxiv.org/pdf/1709.09934>.
- [15] GERTH, III, F. Densities for certain  $\ell$ -ranks in cyclic fields of degree  $\ell^n$ , *Compos. Math.* **60** (1986), no. 3, 295–322. MR869105 Zbl 0608.12008  
[http://www.numdam.org/article/CM\\_1986\\_\\_60\\_3\\_295\\_0.pdf](http://www.numdam.org/article/CM_1986__60_3_295_0.pdf)
- [16] GRAS, G. Class Field Theory: from theory to practice, corr. 2nd ed., *Springer Monographs in Mathematics*, Springer (2005), xiii+507 pages. MR1941965 Zbl 1019.11032 <https://doi.org/10.1007/978-3-662-11323-3>
- [17] GRAS, G. Heuristics and conjectures in direction of a  $p$ -adic Brauer–Siegel theorem, *Math. Comp.* (2018). <https://doi.org/10.1090/mcom/3395>
- [18] GRAS, G. Invariant generalized ideal classes – structure theorems for  $p$ -class groups in  $p$ -extensions, *Proc. Indian Acad. Sci. Math. Sci.* **127** (2017), no. 1, 1–34. MR3605238 Zbl 06731203 <https://doi.org/10.1007/s12044-016-0324-1>
- [19] GRAS, G. Sur les  $p$ -classes d'idéaux dans les extensions cycliques relatives de degré premier  $p$ , I, *Annales de l'Institut Fourier* **23** (1973), no. 3, 1–48. MR360519 Zbl 0276.12013 <https://doi.org/10.5802/aif.471>
- [20] GRAS, G. Sur les  $p$ -classes d'idéaux dans les extensions cycliques relatives de degré premier  $p$ , II, *Annales de l'Institut Fourier* **23** (1973), no. 4, 1–44. MR360519 Zbl 0276.12013 <https://doi.org/10.5802/aif.480>
- [21] GRAS, G. The  $p$ -adic Kummer–Leopoldt Constant: Normalized  $p$ -adic Regulator, *Int. J. Number Theory* **14** (2018), no. 2, 329–337. MR3761496 Zbl 1393.11071  
<https://doi.org/10.1142/S1793042118500203>
- [22] GRAS, G. Théorèmes de réflexion, *J. Théor. Nombres Bordeaux* **10** (1998), no. 2, 399–499. MR1828251 Zbl 0949.11058  
[http://www.numdam.org/item/JTNB\\_1998\\_\\_10\\_2\\_399\\_0/](http://www.numdam.org/item/JTNB_1998__10_2_399_0/)
- [23] GRAS, G. On  $p$ -rationality of number fields. Applications – PARI/GP programs, *Publ. Math. Fac. Sci. Besançon (Théorie des Nombres)*, Années 2017/2018. <https://arxiv.org/pdf/1709.06388> <https://arxiv.org/pdf/1904.10707>
- [24] GRAS, G. Les  $\theta$ -régulateurs locaux d'un nombre algébrique : Conjectures  $p$ -adiques, *Canadian Journal of Mathematics* **68** (2016), no. 3, 571–624. MR3492629 Zbl 1351.11033 <http://dx.doi.org/10.4153/CJM-2015-026-3>
- [25] INABA, E. Über die Struktur der  $\ell$ -klassengruppe zyklischer Zahlkörper von Primzahlgrad  $\ell$ , *J. Fac. Sci. Univ. Tokyo, Sect. I* **4** (1940), 61–115. Zbl 0024.01002
- [26] JAULENT, J.-F. Théorie  $\ell$ -adique globale du corps de classes, *J. Théorie des Nombres de Bordeaux* **10** (1998), no. 2, 355–397. [http://www.numdam.org/article/JTNB\\_1998\\_\\_10\\_2\\_355\\_0.pdf](http://www.numdam.org/article/JTNB_1998__10_2_355_0.pdf) MR1828250 Zbl 0938.11052
- [27] JAULENT, J.-F. Unités et classes dans les extensions métabéliennes de degré  $np^s$  sur un corps de nombres algébriques, *Ann. Inst. Fourier (Grenoble)* **31** (1981), no. 1, ix–x, 39–62. MR0613028 Zbl 0436.12007  
[http://www.numdam.org/article/AIF\\_1981\\_\\_31\\_1\\_39\\_0.pdf/](http://www.numdam.org/article/AIF_1981__31_1_39_0.pdf/)
- [28] KOYMANS, P.; PAGANO, C. On the distribution of  $\mathcal{A}(K)[\ell^\infty]$  for degree  $\ell$  cyclic fields (preprint 2018) <https://arxiv.org/pdf/1812.06884>

- [29] MAIRE, C. Genus theory and governing fields, *New York J. Math.* **24** (2018), 1056–1067. MR3890964 Zbl 06982028 <http://nyjm.albany.edu/j/2018/24-50.html>
- [30] MALLE, G. On the distribution of class groups of number fields, *Experiment. Math.* **19** (2010), 465–474. <https://projecteuclid.org/euclid.em/1317758105> MR2011m:11224 Zbl 1297.11139
- [31] MAYNAR, J. On the Brun–Titchmarsh theorem, *Acta Arithmetica* **157** (2013), No. 3, 249–296. MR3019418 Zbl 1321.11099 <https://eudml.org/doc/279293>
- [32] The PARI Group, PARI/GP, version 2.9.0, *Université de Bordeaux* (2016). <http://pari.math.u-bordeaux.fr/>
- [33] PIERCE, L.B.; TURNAGE-BUTTERBAUGH, C.L.; WOOD, M.M. On a conjecture for  $p$ -torsion in class groups of number fields: from the perspective of moments (preprint 2019). <https://arxiv.org/pdf/1902.02008>
- [34] RAZAR, M.J. Central and genus class fields and the Hasse norm theorem, *Compositio Math.* **35** (1977), no. 3, 281–298. MR0466073 Zbl 0376.12006 [http://www.numdam.org/item?id=CM\\_1977\\_\\_35\\_3\\_281\\_0\\_1057](http://www.numdam.org/item?id=CM_1977__35_3_281_0_1057)
- [35] REDEI, L.; REICHARDT, H. Die Anzahl der durch 4 teilbaren Invarianten der Klassengruppe eines beliebigen quadratischen Zahlkörpers, *J. f.d.r.u.a. Math.* **170** (1933). Zbl 0007.39602
- [36] SCHOOF, R. Class numbers of real cyclotomic fields of prime conductor, *Math. Comp.* **72** (2003), no. 242, 913–937. MR1954975 Zbl 1052.11071 <https://doi.org/10.1090/S0025-5718-02-01432-1>
- [37] SCHOOF, R. Minus class groups of the fields of the  $\ell$ -th roots of unity, *Math. Comp.* **67** (1998), no. 67, 1225–1245. <http://www.mat.uniroma2.it/~schoof/minus.pdf> MR1458225 Zbl 0902.11043
- [38] SCHOOF, R. Computing Arakelov class groups, Buhler, J.P. (ed.) et al., Algorithmic Number Theory, *Cambridge University Press, MSRI Publications* **44** (2008), 447–495. MR2467554 Zbl 1188.11076 <https://www.mat.uniroma2.it/~schoof/14schoof.pdf>
- [39] STEVENHAGEN, P. Rédei-matrices and applications, *Number Theory Paris 1992–93, Cambridge University Press, London Math. Soc. Lecture Notes, Cambridge* (1995), 245–260. <https://doi.org/10.1017/CB09780511661990.015>
- [40] TENENBAUM, G. Introduction à la théorie analytique et probabiliste des nombres, 3<sup>e</sup> édition revue et augmentée, *Coll. Échelles*, Belin 2008.
- [41] TSFASMAN, M.; VLADUȚ, S. Infinite global fields and the generalized Brauer–Siegel theorem, Dedicated to Yuri I. Manin on the occasion of his 65th birthday, *Moscow Math. J.* **2** (2002), no. 2, 329–402. MR1944510 Zbl 1004.11037 <http://www.ams.org/distribution/mmj/vol2-2-2002/tsfasman-vladuts.pdf>
- [42] WASHINGTON, L.C. Introduction to cyclotomic fields, *GTM* **83**, Springer-Verlag, New York 1997, xiv+487 pp. MR1421575 Zbl 0484.12001
- [43] WIDMER, M. Bounds for the  $p$ -torsion in class groups, *Bull. Lond. Math. Soc.* **50** (2018), 124–13. <https://doi.org/10.1112/blms.12113>
- [44] ZYKIN, A.I. Brauer–Siegel and Tsfasman–VladuȚ theorems for almost normal extensions of global fields, *Moscow Math. J.* **5** (2005), no. 4, 961–968. MR2267316 Zbl 1125.11062 <http://www.ams.org/distribution/mmj/vol5-4-2005/zykin.pdf>

VILLA LA GARDETTE, 4 CHEMIN CHÂTEAU GAGNIÈRE, F–38520, LE BOURG D’OISANS.  
g.mn.gras@wanadoo.fr