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# Finite Horizon Mean Field Games on Networks 

Yves Achdou ${ }^{*}$ Manh-Khang Dao $\stackrel{\dagger}{\dagger}$ Olivier Ley ${ }_{\ddagger}^{\ddagger}$ Nicoletta Tchou ${ }^{\S}$

March 6, 2019


#### Abstract

We consider finite horizon stochastic mean field games in which the state space is a network. They are described by a system coupling a backward in time Hamilton-JacobiBellman equation and a forward in time Fokker-Planck equation. The value function $u$ is continuous and satisfies general Kirchhoff conditions at the vertices. The density $m$ of the distribution of states satisfies dual transmission conditions: in particular, $m$ is generally discontinuous across the vertices, and the values of $m$ on each side of the vertices satisfy special compatibility conditions. The stress is put on the case when the Hamiltonian is Lipschitz continuous. Existence and uniqueness are proven.


## 1 Introduction and main results

This work is the continuation of [2] which was devoted to mean field games on networks in the case of an infinite time horizon. The topic of mean field games (MFGs for short) is more and more investigated since the pioneering works $[14,15,16]$ of Lasry and Lions: it aims at studying the asymptotic behavior of stochastic differential games (Nash equilibria) as the number $N$ of agents tends to infinity. We refer to [2] for a more extended discussion on MFGs and for additional references on the analysis of the system of PDEs that stem from the model when there is no common noise.

A network (or a graph) is a set of items, referred to as vertices (or nodes or crosspoints), with connections between them referred to as edges. In the recent years, there has been an increasing interest in the investigation of dynamical systems and differential equations on networks, in particular in connection with problems of data transmission and traffic management. The literature on optimal control in which the state variable takes its values on a network is recent: deterministic control problems and related Hamilton-Jacobi equations were studied in $[1,3,11,12,20,21]$. Stochastic processes on networks and related Kirchhoff conditions at the vertices were studied in [8, 9].

The present work is devoted to finite horizon stochastic mean field games (MFGs) taking place on networks. The most important difficulty will be to deal with the transition conditions at the vertices. The latter are obtained from the theory of stochastic control in [9, 8], see Section 1.3 below. In [6], the first article on MFGs on networks, Camilli and Marchi consider a particular type of Kirchhoff condition at the vertices for the value function: this condition comes from an assumption which can be informally stated as follows: consider a vertex $\nu$ of the

[^0]network and assume that it is the intersection of $p$ edges $\Gamma_{1}, \ldots, \Gamma_{p}$, ; if, at time $\tau$, the controlled stochastic process $X_{t}$ associated to a given agent hits $\nu$, then the probability that $X_{\tau^{+}}$belongs to $\Gamma_{i}$ is proportional to the diffusion coefficient in $\Gamma_{i}$. Under this assumption, it can be seen that the density of the distribution of states is continuous at the vertices of the network. In the present work, the above mentioned assumption is not made any longer. Therefore, it will be seen below that the value function satisfies more general Kirchhoff conditions, and accordingly, that the density of the distribution of states is no longer continuous at the vertices; the continuity condition is then replaced by suitable compatibility conditions on the jumps across the vertices. A complete study of the system of differential equations arising in infinite horizon mean field games on networks with at most quadratic Hamiltonians and very general coupling costs has been supplied in a previous work, see [2].

In the present work, we focus on a more basic case, namely finite horizon MFG with globally Lipschitz Hamiltonian with rather strong assumptions on the coupling cost. This will allow us to concentrate on the difficulties induced by the Kirchhoff conditions. Therefore, this work should be seen as a first and necessary step in order to deal with more difficult situations, for example with quadratic or subquadratic Hamiltonians. We believe that treating such cases will be possible by combining the results contained in the present work with methods that can be found in [22, 23], see also [5, 13, 17] for references on Hamilton-Jacobi equations.

After obtaining the transmission conditions at the vertices for both the value function and the density, we shall prove existence and uniqueness of weak solutions of the uncoupled Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck (FP) equations (in suitable space-time Sobolev spaces), and regularity results.

The present work is organized as follows: the remainder of Section 1 is devoted to setting the problem and obtaining the system of partial differential equations and the transmission conditions at the vertices. Section 2 contains useful results on a modified heat equation in the network with general Kirchhoff conditions. Section 3 is devoted to the Fokker-Planck equation. Weak solutions are defined by using a special pair of Sobolev spaces of functions defined on the network referred to as $V$ and $W$ below. Section 4 is devoted to the HJB equation supplemented with the Kirchhoff conditions: it addresses the main difficulty of the work, consisting of obtaining regularity results for the weak solution (note that, to the best of our knowledge, such results for networks and general Kirchhoff conditions do not exist in the literature). Finally, the proofs of the main results of existence and uniqueness for the MFG system of partial differential equations are completed in Section 5.

### 1.1 Networks and function spaces

### 1.1.1 The geometry

A bounded network $\Gamma$ (or a bounded connected graph) is a connected subset of $\mathbb{R}^{n}$ made of a finite number of bounded non-intersecting straight segments, referred to as edges, which connect nodes referred to as vertices. The finite collection of vertices and the finite set of closed edges are respectively denoted by $\mathcal{V}:=\left\{\nu_{i}, i \in I\right\}$ and $\mathcal{E}:=\left\{\Gamma_{\alpha}, \alpha \in \mathcal{A}\right\}$, where $I$ and $A$ are finite sets of indices contained in $\mathbb{N}$. We assume that for $\alpha, \beta \in \mathcal{A}$, if $\alpha \neq \beta$, then $\Gamma_{\alpha} \cap \Gamma_{\beta}$ is either empty or made of a single vertex. The length of $\Gamma_{\alpha}$ is denoted by $\ell_{\alpha}$. Given $\nu_{i} \in \mathcal{V}$, the set of indices of edges that are adjacent to the vertex $\nu_{i}$ is denoted by $\mathcal{A}_{i}=\left\{\alpha \in \mathcal{A}: \nu_{i} \in \Gamma_{\alpha}\right\}$. A vertex $\nu_{i}$ is named a boundary vertex if $\sharp\left(\mathcal{A}_{i}\right)=1$, otherwise it is named a transition vertex. The set containing all the boundary vertices is named the boundary of the network and is denoted by $\partial \Gamma$ hereafter.

The edges $\Gamma_{\alpha} \in \mathcal{E}$ are oriented in an arbitrary manner. In most of what follows, we shall make
the following arbitrary choice that an edge $\Gamma_{\alpha} \in \mathcal{E}$ connecting two vertices $\nu_{i}$ and $\nu_{j}$, with $i<j$ is oriented from $\nu_{i}$ toward $\nu_{j}$ : this induces a natural parametrization $\pi_{\alpha}:\left[0, \ell_{\alpha}\right] \rightarrow \Gamma_{\alpha}=\left[\nu_{i}, \nu_{j}\right]$ :

$$
\begin{equation*}
\pi_{\alpha}(y)=\left(\ell_{\alpha}-y\right) \nu_{i}+y \nu_{j} \quad \text { for } y \in\left[0, \ell_{\alpha}\right] \tag{1.1}
\end{equation*}
$$

For a function $v: \Gamma \rightarrow \mathbb{R}$ and $\alpha \in \mathcal{A}$, we define $v_{\alpha}:\left(0, \ell_{\alpha}\right) \rightarrow \mathbb{R}$ by

$$
v_{\alpha}(y):=v \circ \pi_{\alpha}(y), \quad \text { for all } y \in\left(0, \ell_{\alpha}\right) .
$$

The function $v_{\alpha}$ is a priori defined only in $\left(0, \ell_{\alpha}\right)$. When it is possible, we extend it by continuity at the boundary by setting

$$
v_{\alpha}(0):=\lim _{y \rightarrow 0^{+}} v_{\alpha}(y) \text { and } v_{\alpha}\left(\ell_{\alpha}\right):=\lim _{y \rightarrow \ell_{\alpha}^{-}} v_{\alpha}(y) .
$$

In that latter case, we can define

$$
\left.v\right|_{\Gamma_{\alpha}}(x)= \begin{cases}v_{\alpha}\left(\pi_{\alpha}^{-1}(x)\right), & \text { if } x \in \Gamma_{\alpha} \backslash \mathcal{V}  \tag{1.2}\\ v_{\alpha}(0)=\lim _{y \rightarrow 0^{+}} v_{\alpha}(y), & \text { if } x=\nu_{i} \\ v_{\alpha}\left(\ell_{\alpha}\right)=\lim _{y \rightarrow \ell_{\alpha}^{-}} v_{\alpha}(y), & \text { if } x=\nu_{j}\end{cases}
$$

Notice that $\left.v\right|_{\Gamma_{\alpha}}$ does not coincide with the original function $v$ at the vertices in general when $v$ is not continuous.
Remark 1.1. In what precedes, the edges have been arbitrarily oriented from the vertex with the smaller index toward the vertex with the larger one. Other choices are of course possible. In particular, by possibly dividing a single edge into two, adding thereby new artificial vertices, it is always possible to assume that for all vertices $\nu_{i} \in \mathcal{V}$,

$$
\begin{equation*}
\text { either } \pi_{\alpha}(0)=\nu_{i} \text {, for all } \alpha \in \mathcal{A}_{i} \text { or } \pi_{\alpha}\left(\ell_{\alpha}\right)=\nu_{i} \text {, for all } \alpha \in \mathcal{A}_{i} . \tag{1.3}
\end{equation*}
$$

This idea was used by Von Below in [24]: some edges of $\Gamma$ are cut into two by adding artificial vertices so that the new oriented network $\bar{\Gamma}$ has the property (1.3), see Figure 1 for an example.


Figure 1: Left: the network $\Gamma$ in which the edges are oriented toward the vertex with larger index ( 4 vertices and 4 edges). Right: a new network $\tilde{\Gamma}$ obtained by adding an artificial vertex ( 5 vertices and 5 edges): the oriented edges sharing a given vertex $\nu$ either have all their starting point equal $\nu$, or have all their terminal point equal $\nu$.

### 1.1.2 Function spaces related to the space variable

The set of continuous functions on $\Gamma$ is denoted by $C(\Gamma)$ and we set

$$
=\left\{\begin{array}{l|l}
P C(\Gamma) \\
v: \Gamma \rightarrow \mathbb{R}: \text { for all } \alpha \in \mathcal{A}, & \begin{array}{l}
v_{\alpha} \in C\left(0, \ell_{\alpha}\right) \\
v_{\alpha} \text { can be extended by continuity to }\left[0, \ell_{\alpha}\right] .
\end{array}
\end{array}\right\} .
$$

By the definition of piecewise continuous functions $v \in P C(\Gamma)$, for all $\alpha \in \mathcal{A}$, it is possible to define $\left.v\right|_{\Gamma_{\alpha}}$ by (1.2) and we have $\left.v\right|_{\Gamma_{\alpha}} \in C\left(\Gamma_{\alpha}\right), v_{\alpha} \in C\left(\left[0, \ell_{\alpha}\right]\right)$.

For $m \in \mathbb{N}$, the space of $m$-times continuously differentiable functions on $\Gamma$ is defined by

$$
C^{m}(\Gamma):=\left\{v \in C(\Gamma): v_{\alpha} \in C^{m}\left(\left[0, \ell_{\alpha}\right]\right) \text { for all } \alpha \in \mathcal{A}\right\} .
$$

Notice that $v \in C^{m}(\Gamma)$ is assumed to be continuous on $\Gamma$, and that its restriction $v_{\mid \Gamma_{\alpha}}$ to each edge $\Gamma_{\alpha}$ belongs to $C^{m}\left(\Gamma_{\alpha}\right)$. The space $C^{m}(\Gamma)$ is endowed with the norm $\|v\|_{C^{m}(\Gamma)}:=$ $\sum_{\alpha \in \mathcal{A}} \sum_{k \leqslant m}\left\|\partial^{k} v_{\alpha}\right\|_{L^{\infty}\left(0, \ell_{\alpha}\right)}$. For $\sigma \in(0,1)$, the space $C^{m, \sigma}(\Gamma)$, contains the functions $v \in C^{m}(\Gamma)$ such that $\partial^{m} v_{\alpha} \in C^{0, \sigma}\left(\left[0, \ell_{\alpha}\right]\right)$ for all $\alpha \in \mathcal{A}$; it is endowed with the norm

$$
\|v\|_{C^{m, \sigma}(\Gamma)}:=\|v\|_{C^{m}(\Gamma)}+\sup _{\alpha \in \mathcal{A}} \sup _{\substack{y \neq z \\ y, z \in\left[0, \ell_{\alpha}\right]}} \frac{\left|\partial^{m} v_{\alpha}(y)-\partial^{m} v_{\alpha}(z)\right|}{|y-z|^{\sigma}} .
$$

For a positive integer $m$ and a function $v \in C^{m}(\Gamma)$, we set for $k \leqslant m$,

$$
\begin{equation*}
\partial^{k} v(x)=\partial^{k} v_{\alpha}\left(\pi_{\alpha}^{-1}(x)\right) \text { if } x \in \Gamma_{\alpha} \backslash \mathcal{V} . \tag{1.4}
\end{equation*}
$$

For a vertex $\nu$, we define $\partial_{\alpha} v(\nu)$ as the outward directional derivative of $\left.v\right|_{\Gamma_{\alpha}}$ at $\nu$ as follows:

$$
\partial_{\alpha} v(\nu):= \begin{cases}\lim _{h \rightarrow 0^{+}} \frac{v_{\alpha}(0)-v_{\alpha}(h)}{h}, & \text { if } \nu=\pi_{\alpha}(0)  \tag{1.5}\\ \lim _{h \rightarrow 0^{+}} \frac{v_{\alpha}\left(\ell_{\alpha}\right)-v_{\alpha}\left(\ell_{\alpha}-h\right)}{h}, & \text { if } \nu=\pi_{\alpha}\left(\ell_{\alpha}\right) .\end{cases}
$$

For all $i \in I$ and $\alpha \in \mathcal{A}_{i}$, setting

$$
n_{i \alpha}=\left\{\begin{align*}
1 & \text { if } \nu_{i}=\pi_{\alpha}\left(\ell_{\alpha}\right),  \tag{1.6}\\
-1 & \text { if } \nu_{i}=\pi_{\alpha}(0)
\end{align*}\right.
$$

we have

$$
\begin{equation*}
\partial_{\alpha} v\left(\nu_{i}\right)=\left.n_{i \alpha} \partial v\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)=n_{i \alpha} \partial v_{\alpha}\left(\pi_{\alpha}^{-1}\left(\nu_{i}\right)\right) . \tag{1.7}
\end{equation*}
$$

Remark 1.2. Changing the orientation of the edge does not change the value of $\partial_{\alpha} v(\nu)$ in (1.5).
We say that $v$ is Lebesgue-integrable on $\Gamma_{\alpha}$ if $v_{\alpha}$ is Lebesgue-integrable on $\left(0, \ell_{\alpha}\right)$. In this case, for all $x_{1}, x_{2} \in \Gamma_{\alpha}$,

$$
\begin{equation*}
\int_{\left[x_{1}, x_{2}\right]} v(x) d x:=\int_{\pi_{\alpha}^{-1}\left(x_{1}\right)}^{\pi_{\alpha}^{-1}\left(x_{2}\right)} v_{\alpha}(y) d y \tag{1.8}
\end{equation*}
$$

When $v$ is Lebesgue-integrable on $\Gamma_{\alpha}$ for all $\alpha \in \mathcal{A}$, we say that $v$ is Lebesgue-integrable on $\Gamma$ and we define

$$
\int_{\Gamma} v(x) d x:=\sum_{\alpha \in \mathcal{A}} \int_{0}^{\ell_{\alpha}} v_{\alpha}(y) d y .
$$

The space $L^{p}(\Gamma)=\left\{v:\left.v\right|_{\Gamma_{\alpha}} \in L^{p}\left(\Gamma_{\alpha}\right)\right.$ for all $\left.\alpha \in \mathcal{A}\right\}, p \in[1, \infty]$, is endowed with the norm $\|v\|_{L^{p}(\Gamma)}:=\left(\sum_{\alpha \in \mathcal{A}}\left\|v_{\alpha}\right\|_{L^{p}\left(0, \ell_{\alpha}\right)}^{p}\right)^{\frac{1}{p}}$ if $1 \leqslant p<\infty$, and $\max _{\alpha \in \mathcal{A}}\left\|v_{\alpha}\right\|_{L^{\infty}\left(0, \ell_{\alpha}\right)}$ if $p=+\infty$. We shall also need to deal with functions on $\Gamma$ whose restrictions to the edges are weakly-differentiable: we shall use the same notations for the weak derivatives.

Definition 1.3. For any integer $s \geqslant 1$ and any real number $p \geqslant 1$, the Sobolev space $W_{b}^{s, p}(\Gamma)$ is defined as follows

$$
W_{b}^{s, p}(\Gamma):=\left\{v: \Gamma \rightarrow \mathbb{R} \text { s.t. } v_{\alpha} \in W^{s, p}\left(0, \ell_{\alpha}\right) \text { for all } \alpha \in \mathcal{A}\right\},
$$

and endowed with the norm

$$
\|v\|_{W_{b}^{s, p}(\Gamma)}=\left(\sum_{k=1}^{s} \sum_{\alpha \in \mathcal{A}}\left\|\partial^{k} v_{\alpha}\right\|_{L^{p}\left(0, \ell_{\alpha}\right)}^{p}+\|v\|_{L^{p}(\Gamma)}^{p}\right)^{\frac{1}{p}}
$$

For $s \in \mathbb{N} \backslash\{0\}$, we also set $H_{b}^{s}(\Gamma)=W_{b}^{s, 2}(\Gamma)$ and $H^{s}(\Gamma)=C(\Gamma) \cap H_{b}^{s}(\Gamma)$.
Finally, when dealing with probability distributions in mean field games, we will often use the set $\mathcal{M}$ of probability densities, i.e., $m \in L^{1}(\Gamma), m \geqslant 0$ and $\int_{\Gamma} m(x) d x=1$.

### 1.1.3 Some space-time function spaces

The space of continuous real valued functions on $\Gamma \times[0, T]$ is denoted by $C(\Gamma \times[0, T])$.
Let $P C(\Gamma \times[0, T])$ be the space of the functions $v: \Gamma \times[0, T] \rightarrow \mathbb{R}$ such that

1. for all $t \in[0, T], v(\cdot, t)$ belongs to $P C(\Gamma)$
2. for all $\alpha \in \mathcal{A},\left.v\right|_{\Gamma_{\alpha} \times[0, T]}$ is continuous on $\Gamma_{\alpha} \times[0, T]$;

For a function $v \in P C(\Gamma \times[0, T]), \alpha \in \mathcal{A}$, we set $v_{\alpha}(y, t)=\left.v\right|_{\Gamma_{\alpha} \times[0, t]}\left(\pi_{\alpha}(y), t\right)$ for all $(y, t) \in$ $\left[0, \ell_{\alpha}\right] \times[0, T]$.

For two nonnegative integers $m$ and $n$, let $C^{m, n}(\Gamma \times[0, T])$ be the space of continuous real valued functions $v$ on $\Gamma \times[0, T]$ such that for all $\alpha \in \mathcal{A},\left.v\right|_{\Gamma_{\alpha} \times[0, T]} \in C^{m, n}\left(\Gamma_{\alpha} \times[0, T]\right)$. For $\sigma \in(0,1), \tau \in(0,1)$, we define in the same manner $C^{m+\sigma, n+\tau}(\Gamma \times[0, T])$

Useful results on continuous and compact embeddings of space-time function spaces are given in Appendix A.

### 1.2 A class of stochastic processes on $\Gamma$

After rescaling the edges, it may be assumed that $\ell_{\alpha}=1$ for all $\alpha \in \mathcal{A}$. Let $\mu_{\alpha}, \alpha \in \mathcal{A}$ and $p_{i \alpha}, i \in I, \alpha \in \mathcal{A}_{i}$ be positive constants such that $\sum_{\alpha \in \mathcal{A}_{i}} p_{i \alpha}=1$. Consider also a real valued function $a \in P C(\Gamma \times[0, T])$, such that, for all $\alpha \in \mathcal{A}, t \in[0, T],\left.a\right|_{\Gamma_{\alpha}}(\cdot, t)$ belongs to $C^{1}\left(\Gamma_{\alpha}\right)$.

As in Remark 1.1, we make the assumption (1.3) by possibly adding artificial nodes: if $\nu_{i}$ is such an artificial node, then $\sharp\left(\mathcal{A}_{i}\right)=2$, and we assume that $p_{i \alpha}=1 / 2$ for $\alpha \in \mathcal{A}_{i}$. The diffusion parameter $\mu$ has the same value on the two sides of an artificial vertex. Similarly, the function $a$ does not have jumps across an artificial vertex.

Consider a Brownian motion ( $W_{t}$ ) defined on the real line. Following Freidlin and Sheu ([8]), we know that there exists a unique Markov process on $\Gamma$ with continuous sample paths that can be written $\left(X_{t}, \alpha_{t}\right)$ where $X_{t} \in \Gamma_{\alpha_{t}}$ (if $X_{t}=\nu_{i}, i \in I, \alpha_{t}$ is arbitrarily chosen as the smallest index in $\mathcal{A}_{i}$ ) such that, defining the process $x_{t}=\pi_{\alpha_{t}}\left(X_{t}\right)$ with values in $[0,1]$,

- we have

$$
\begin{equation*}
d x_{t}=\sqrt{2 \mu_{\alpha_{t}}} d W_{t}+a_{\alpha_{t}}\left(x_{t}, t\right) d t+d \ell_{i, t}+d h_{i, t}, \tag{1.9}
\end{equation*}
$$

- $\ell_{i, t}$ is continuous non-decreasing process (measurable with respect to the $\sigma$-field generated by $\left.\left(X_{t}, \alpha_{t}\right)\right)$ which increases only when $X_{t}=\nu_{i}$ and $x_{t}=0$,
- $h_{i, t}$ is continuous non-increasing process (measurable with respect to the $\sigma$-field generated by $\left.\left(X_{t}, \alpha_{t}\right)\right)$ which decreases only when $X_{t}=\nu_{i}$ and $x_{t}=1$,
and for all function $v \in C^{2,1}(\Gamma \times[0, T])$ such that

$$
\begin{equation*}
\sum_{\alpha \in \mathcal{A}_{i}} p_{i \alpha} \partial_{\alpha} v\left(\nu_{i}, t\right)=0, \text { for all } i \in I, t \in[0, T], \tag{1.10}
\end{equation*}
$$

the process

$$
\begin{equation*}
M_{t}=v\left(X_{t}, t\right)-\int_{0}^{t}\left(\partial_{t} v\left(X_{s}, s\right)+\mu_{\alpha_{s}} \partial^{2} v\left(X_{s}, s\right)+\left.a\right|_{\Gamma_{\alpha_{s}}}\left(X_{s}, s\right) \partial v\left(X_{s}, s\right)\right) d s \tag{1.11}
\end{equation*}
$$

is a martingale, i.e.,

$$
\begin{equation*}
\mathbb{E}\left(M_{t} \mid X_{s}\right)=M_{s}, \quad \text { for all } 0 \leqslant s<t \leqslant T . \tag{1.12}
\end{equation*}
$$

For what follows, it will be convenient to set

$$
\begin{equation*}
D:=\left\{u \in C^{2}(\Gamma): \sum_{\alpha \in \mathcal{A}_{i}} p_{i \alpha} \partial_{\alpha} u\left(\nu_{i}\right)=0, \text { for all } i \in I\right\} . \tag{1.13}
\end{equation*}
$$

Remark 1.4. Note that in (1.10), the condition at boundary vertices boils down to a Neumann condition.
Remark 1.5. The assumption that all the edges have unit length is not restrictive, because we can always rescale the constants $\mu_{\alpha}$ and the piecewise continuous function $a$.

The goal is to derive the boundary value problem satisfied by the law of the stochastic process $X_{t}$. Since the derivation here is formal, we assume that the law of the stochastic process $X_{t}$ is a measure which is absolutely continuous with respect to the Lebesgue measure on $\Gamma$ and regular enough so that the following computations make sense. Let $m(x, t)$ be its density. We have

$$
\begin{equation*}
\mathbb{E}\left[v\left(X_{t}, t\right)\right]=\int_{\Gamma} v(x, t) m(x, t) d x, \quad \text { for all } v \in P C(\Gamma \times[0, T]) . \tag{1.14}
\end{equation*}
$$

Consider $u \in C^{2,1}(\Gamma \times[0, T])$ such that for all $t \in[0, T], u(\cdot, t) \in D$. Then, from (1.11)-(1.12), we see that

$$
\begin{equation*}
\mathbb{E}\left[u\left(X_{t}, t\right)\right]=\mathbb{E}\left[u\left(X_{0}, 0\right)\right]+\mathbb{E}\left[\int_{0}^{t}\left(\partial_{t} u\left(X_{s}, s\right)+\mu_{\alpha_{s}} \partial^{2} u\left(X_{s}, s\right)+\left.a\right|_{\Gamma_{\alpha_{s}}}\left(X_{s}, s\right) \partial u\left(X_{s}, s\right)\right) d s\right] . \tag{1.15}
\end{equation*}
$$

Using (1.14) and taking the time-derivative of each member of (1.15), we obtain

$$
\int_{\Gamma} \partial_{t}(u m)(x, t) d x=\mathbb{E}\left(\partial_{t} u\left(X_{t}, t\right)+\mu_{\alpha_{s}} \partial^{2} u\left(X_{t}, t\right)+\left.a\right|_{\Gamma_{\alpha_{s}}}\left(X_{t}, t\right) \partial u\left(X_{t}, t\right)\right) .
$$

Using again (1.14), we get

$$
\begin{equation*}
\int_{\Gamma}\left(\mu \partial^{2} u(x, t)+a(x, t) \partial u(x, t)\right) m(x, t) d x=\int_{\Gamma} u(x, t) \partial_{t} m(x, t) d x \tag{1.16}
\end{equation*}
$$

By integration by parts, recalling (1.3), we get

$$
\begin{align*}
0= & \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_{\alpha}}\left(\partial_{t} m(x, t)-\mu_{\alpha} \partial^{2} m(x, t)+\partial(a m)(x, t)\right) u(x, t) d x \\
& -\left.\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_{i}}\left[\left.\left.n_{i \alpha} a\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) m\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)-\mu_{\alpha} \partial_{\alpha} m\left(\nu_{i}, t\right)\right] u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) \\
& -\left.\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_{i}} \mu_{\alpha} m\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) \partial_{\alpha} u\left(\nu_{i}, t\right) \tag{1.17}
\end{align*}
$$

where $n_{i \alpha}$ is defined in (1.6).
We choose first, for every $\alpha \in \mathcal{A}$, a smooth function $u$ which is compactly supported in $\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times[0, T]$. Hence $\left.u\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)=0$ and $\partial_{\beta} u\left(\nu_{i}, t\right)=0$ for all $i \in I, \beta \in \mathcal{A}_{i}$. Notice that $u(\cdot, t) \in D$. It follows that $m$ satisfies

$$
\begin{equation*}
\left(\partial_{t} m-\mu_{\alpha} \partial^{2} m+\partial(m a)\right)(x, t)=0, \quad \text { for } x \in \Gamma_{\alpha} \backslash \mathcal{V}, t \in(0, T), \alpha \in \mathcal{A} \tag{1.18}
\end{equation*}
$$

For a smooth function $\chi:[0, T] \rightarrow \mathbb{R}$ compactly supported in $(0, T)$, we may choose for every $i \in I$, a smooth function $u$ such that $u\left(\nu_{j}, t\right)=\chi(t) \delta_{i, j}$ for all $t \in[0, T], j \in I$ and $\partial_{\alpha} u\left(\nu_{j}, t\right)=0$ for all $t \in[0, T], j \in I$ and $\alpha \in \mathcal{A}_{j}$, we infer a condition for $m$ at the vertices,

$$
\begin{equation*}
\left.\left.\sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha} a\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) m\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)-\mu_{\alpha} \partial_{\alpha} m\left(\nu_{i}, t\right)=0 \quad \text { for all } i \in I, t \in(0, T) \tag{1.19}
\end{equation*}
$$

This condition is called a transmission condition if $\nu_{i}$ is a transition vertex and reduces to a Robin boundary condition when $\nu_{i}$ is a boundary vertex.

Finally, for a smooth function $\chi:[0, T] \rightarrow \mathbb{R}$ compactly supported in $(0, T)$, for every transition vertex $\nu_{i} \in \mathcal{V} \backslash \partial \Gamma$ and $\alpha, \beta \in \mathcal{A}_{i}$, we choose $u$ such that

- $u(\cdot, t) \in D$
- $\partial_{\alpha} u\left(\nu_{i}, t\right)=\chi(t) / p_{i \alpha}, \partial_{\beta} u\left(\nu_{i}\right)=-\chi(t) / p_{i \beta}, \partial_{\gamma} u\left(\nu_{i}\right)=0$ if $\gamma \in \mathcal{A}_{i} \backslash\{\alpha, \beta\}$
- The directional derivatives of $u$ at the vertices $\nu \neq \nu_{i}$ are 0 .

Using such a test-function in (1.17) yields a jump condition for $m$,

$$
\frac{\left.m\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)}{\gamma_{i \alpha}}=\frac{\left.m\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)}{\gamma_{i \beta}}, \quad \text { for all } \alpha, \beta \in \mathcal{A}_{i}, \nu_{i} \in \mathcal{V}, t \in(0, T)
$$

in which

$$
\begin{equation*}
\gamma_{i \alpha}=\frac{p_{i \alpha}}{\mu_{\alpha}}, \quad \text { for all } i \in I, \alpha \in \mathcal{A}_{i} \tag{1.20}
\end{equation*}
$$

Summarizing, we get the following boundary value problem for $m$ (recall that the coefficients $n_{i \alpha}$ are defined in (1.6)):

$$
\left\{\begin{align*}
\partial_{t} m-\mu_{\alpha} \partial^{2} m+\partial(m a)=0, & (x, t) \in\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T), \alpha \in \mathcal{A}  \tag{1.21}\\
\sum_{\alpha \in \mathcal{A}_{i}} \mu_{\alpha} \partial_{\alpha} m\left(\nu_{i}, t\right)-\left.\left.n_{i \alpha} a\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right) m\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=0, & t \in(0, T), \nu_{i} \in \mathcal{V} \\
\frac{\left.m\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)}{\gamma_{i \alpha}}=\frac{\left.m\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)}{\gamma_{i \beta}}, & t \in(0, T), \alpha, \beta \in \mathcal{A}_{i}, \nu_{i} \in \mathcal{V} \\
m(x, 0)=m_{0}(x), & x \in \Gamma .
\end{align*}\right.
$$

### 1.3 Formal derivation of the MFG system on $\Gamma$

Here we aim at obtaining the MFG system of forward-backward partial differential equations on the network, at least formally. The assumptions that we are going to make below on the optimal control problem are a little restrictive, for two reasons: first, we wish to avoid some technicalities linked to the measurability of the control process; second, the assumptions on the costs must be consistent with the assumptions that we shall make on the Hamiltonian, see Section 1.4.2 below. In particular, we shall impose that the Hamiltonian is globally Lipschitz continuous. More general and difficult cases, e.g., quadratic Hamiltonians, will be the subject of a future work.

Consider a continuum of indistinguishable agents moving on the network $\Gamma$. The state of a representative agent at time $t$ is a time-continuous controlled stochastic process $X_{t}$ as defined in Section 1.2, where the control is the drift $a_{t}$, supposed to be in the form $a_{t}=a\left(X_{t}, t\right)$. Let $m(\cdot, t)$ be the probability measure on $\Gamma$ that describes the distribution of states at time $t$.

For a representative agent, the optimal control problem is of the form

$$
\begin{equation*}
v(x, t)=\inf _{a_{s}} \mathbb{E}_{x t}\left[\int_{t}^{T}\left(L\left(X_{s}, a_{s}\right)+\mathscr{V}[m(\cdot, t)]\left(X_{s}\right)\right) d s+v_{T}\left(X_{T}\right)\right], \tag{1.22}
\end{equation*}
$$

where $\mathbb{E}_{x t}$ stands for the expectation conditioned by the event $X_{t}=x$.
We discuss the ingredients appearing in (1.22):

- We assume that the control is in a feeback form $a_{t}=a\left(X_{t}, t\right)$ where the function $a$, defined on $\Gamma \times[0, T]$, is sufficiently regular in the edges of the network. Then, almost surely if $X_{t} \in \Gamma_{\alpha} \backslash \mathcal{V}$,

$$
d \pi_{\alpha}^{-1}\left(X_{t}\right)=a_{\alpha}\left(\pi_{\alpha}^{-1}\left(X_{t}\right), t\right) d t+\sqrt{2 \mu_{\alpha}} d W_{t} .
$$

An informal way to describe the behavior of the process at the vertices is as follows: if $X_{t}$ hits $\nu_{i} \in \mathcal{V}$, then it enters $\Gamma_{\alpha}, \alpha \in \mathcal{A}_{i}$ with probability $p_{i \alpha}>0$ ( $p_{i \alpha}$ was introduced in Section 1.2). We assume that there is an optimal feedback law $a^{\star}$.

- We assume that for all $\alpha \in \mathcal{A}, a_{\alpha}$ maps $\left[0, \ell_{\alpha}\right] \times[0, T]$ to a compact interval $A_{\alpha}=\left[\underline{a_{\alpha}}, \overline{,_{\alpha}}\right]$.
- The contribution of the control to the running cost involves the Lagrangian $L$, i.e., a real valued function defined on $\cup_{\alpha \in \mathcal{A}}\left(\Gamma_{\alpha} \backslash \mathcal{V} \times A_{\alpha}\right)$. If $x \in \Gamma_{\alpha} \backslash \mathcal{V}$ and $a \in A_{\alpha}$, then $L(x, a)=$ $L_{\alpha}\left(\pi_{\alpha}^{-1}(x), a\right)$, where $L_{\alpha}$ is a continuous real valued function defined on $\left[0, \ell_{\alpha}\right] \times A_{\alpha}$. We assume that $L_{\alpha}(x, \cdot)$ is strictly convex on $A_{\alpha}$.
- The contribution of the distribution of states to the running ccost involves the coupling cost operator, which can either be nonlocal, i.e., $\mathscr{V}: \mathcal{P}(\Gamma) \rightarrow \mathcal{C}^{2}(\Gamma)$ (where $\mathcal{P}(\Gamma)$ is the set of Borel probability measures on $\Gamma$ ), or local, i.e., $\mathscr{V}[m](x)=F(m(x))$ for a continuous function $F: \mathbb{R}^{+} \rightarrow \mathbb{R}$, assuming that $m$ is absolutely continuous with respect to the Lebesgue measure and identifying with its density.
- The last term is the terminal $\operatorname{cost} v_{T}$, which depends only on the state variable for simplicity.

Under suitable additional assumptions, Ito calculus as in [8, 9] and the dynamic programming principle lead to the following HJB equation on $\Gamma$, more precisely the following boundary value
problem

$$
\begin{cases}-\partial_{t} v-\mu_{\alpha} \partial^{2} v+H(x, \partial v)=\mathscr{V}[m(\cdot, t)](x), & \text { in }\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T), \alpha \in \mathcal{A},  \tag{1.23}\\ \sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha} \partial_{\alpha} v\left(\nu_{i}, t\right)=0, & \text { if }\left(\nu_{i}, t\right) \in \mathcal{V} \times(0, T), \\ \left.v\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=\left.v\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right) & \text { for all } \nu_{i} \in \mathcal{V}, t \in(0, T) \alpha, \beta \in \mathcal{A}_{i}, \\ v(x, T)=v_{T}(x) & \text { in } \Gamma .\end{cases}
$$

We refer to $[14,15,16]$ for the interpretation of the value function $v$. Let us comment the different equations in (1.23):

1. The first equation is a HJB equation the Hamiltonian $H$ of which is a real valued function defined on $\left(\cup_{\alpha \in \mathcal{A}} \Gamma_{\alpha} \backslash \mathcal{V}\right) \times \mathbb{R}$ given by

$$
\begin{equation*}
H(x, p)=\sup _{a \in A_{\alpha}}\left\{-a p-L_{\alpha}\left(\pi_{\alpha}^{-1}(x), a\right)\right\} \quad \text { for } x \in \Gamma_{\alpha} \backslash \mathcal{V} \text { and } p \in \mathbb{R} . \tag{1.24}
\end{equation*}
$$

We assume that $L$ is such that the Hamiltonians $\left.H\right|_{\Gamma_{\alpha} \times \mathbb{R}}$ are Lipschitz continuous with respect to $p$ and $C^{1}$.
2. The second equation in (1.23) is a Kirchhoff transmission condition (or Neumann boundary condition if $\nu_{i} \in \partial \Gamma$ ); it is the consequence of the assumption on the behavior of $X_{s}$ at vertices. It involves the positive constants $\gamma_{i \alpha}$ defined in (1.20).
3. The third condition means in particular that $v$ is continuous at the vertices.
4. The fourth condition is a terminal condition for the backward in time HJB equation.

If (1.23) has a smooth solution, then it provides a feedback law for the optimal control problem, i.e.,

$$
a^{\star}(x, t)=-\partial_{p} H(x, \partial v(x, t)) .
$$

At the MFG equilibrium, $m$ is the density of the invariant measure associated with the optimal feedback law, so, according to Section 1.2, it satisfies (1.21), where $a$ is replaced by $a^{\star}=-\partial_{p} H(x, \partial v(x, t))$. We end up with the following system:

$$
\begin{cases}-\partial_{t} v-\mu_{\alpha} \partial^{2} v+H(x, \partial v)=\mathscr{V}[m(\cdot, t)](x), & (x, t) \in\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T), \alpha \in \mathcal{A},  \tag{1.25}\\ \partial_{t} m-\mu_{\alpha} \partial^{2} m-\partial\left(m \partial_{p} H(x, \partial v)\right)=0, & (x, t) \in\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T), \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha} \partial_{\alpha} v\left(\nu_{i}, t\right)=0, & \left(\nu_{i}, t\right) \in \mathcal{V} \times(0, T), \\ \sum_{\alpha \in \mathcal{A}_{i}} \mu_{\alpha} \partial_{\alpha} m\left(\nu_{i}, t\right)+\left.n_{i \alpha} \partial_{p} H^{\alpha}\left(\nu_{i},\left.\partial v\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right) m\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=0, & \left(\nu_{i}, t\right) \in \mathcal{V} \times(0, T), \\ \left.v\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=\left.v\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right), \frac{\left.m\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)}{\gamma_{i \alpha}}=\frac{\left.m\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)}{\gamma_{i \beta}}, & \alpha, \beta \in \mathcal{A}_{i},\left(\nu_{i}, t\right) \in \mathcal{V} \times(0, T), \\ v(x, T)=v_{T}(x), m(x, 0)=m_{0}(x) & x \in \Gamma,\end{cases}
$$

where $H^{\alpha}:=\left.H\right|_{\Gamma_{\alpha} \times \mathbb{R}}$. At a vertex $\nu_{i}, i \in I$, the transmission conditions for both $v$ and $m$ consist of $d_{\nu_{i}}=\sharp\left(\mathcal{A}_{i}\right)$ linear relations, which is the appropriate number of relations to have a well posed problem. If $\nu_{i} \in \partial \Gamma$, there is of course only one Neumann like condition for $v$ and for $m$.

### 1.4 Assumptions and main results

Before giving the precise definition of solutions of the MFG system (1.25) and stating our result, we need to introduce some suitable functions spaces.

### 1.4.1 Function spaces related to the Kirchhoff conditions

The following function spaces will be the key ingredients in order to build weak solutions of (1.25).
Definition 1.6. We define two Sobolev spaces: $V:=H^{1}(\Gamma)$, and

$$
\begin{equation*}
W:=\left\{w: \Gamma \rightarrow \mathbb{R}: w \in H_{b}^{1}(\Gamma) \text { and } \frac{\left.w\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)}{\gamma_{i \alpha}}=\frac{\left.w\right|_{\Gamma_{\beta}}\left(\nu_{i}\right)}{\gamma_{i \beta}} \text { for all } i \in I, \alpha, \beta \in \mathcal{A}_{i}\right\}, \tag{1.26}
\end{equation*}
$$

which is a subspace of $H_{b}^{1}(\Gamma)$.
Definition 1.7. Let the function $\varphi \in W$ be defined as follows:

$$
\left\{\begin{array}{l}
\varphi_{\alpha} \text { is affine on }\left(0, \ell_{\alpha}\right)  \tag{1.27}\\
\left.\varphi\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)=\gamma_{i \alpha}, \text { if } \alpha \in \mathcal{A}_{i}, \\
\varphi \text { is constant on the edges } \Gamma_{\alpha} \text { which touch the boundary of } \Gamma .
\end{array}\right.
$$

Note that $\varphi$ is positive and bounded. We set $\bar{\varphi}=\max _{\Gamma} \varphi, \underline{\varphi}=\min _{\Gamma} \varphi$.
Remark 1.8. One can see that $v \in V \longmapsto v \varphi$ is an isomorphism from $V$ onto $W$ and $w \in W \longmapsto$ $w \varphi^{-1}$ is the inverse isomorphism.

Definition 1.9. Let the function space $\mathcal{W} \subset W$ be defined as follows:

$$
\begin{equation*}
\mathcal{W}:=\left\{m: \Gamma \rightarrow \mathbb{R}: m_{\alpha} \in C^{1}\left(\left[0, \ell_{\alpha}\right]\right) \text { and } \frac{\left.m\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)}{\gamma_{i \alpha}}=\frac{\left.m\right|_{\Gamma_{\beta}}\left(\nu_{i}\right)}{\gamma_{i \beta}} \text { for all } i \in I, \alpha, \beta \in \mathcal{A}_{i}\right\} . \tag{1.28}
\end{equation*}
$$

### 1.4.2 Running assumptions (H)

(Diffusion constants) $\left(\mu_{\alpha}\right)_{\alpha \in \mathcal{A}}$ is a family of positive numbers.
(Jump coefficients) $\left(\gamma_{i \alpha}\right)_{\alpha \in \mathcal{A}_{i}}$ is a family of positive numbers such that $\sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha}=1$.
(Hamiltonian) The Hamiltonian $H$ is defined by the collection $H^{\alpha}:=\left.H\right|_{\Gamma_{\alpha} \times \mathbb{R}}, \alpha \in \mathcal{A}$ : we assume that

$$
\begin{array}{lr}
H^{\alpha} \in C^{1}\left(\Gamma_{\alpha} \times \mathbb{R}\right), & \text { for any } x \in \Gamma_{\alpha}, \\
H^{\alpha}(x, \cdot) \text { is convex in } p, & \text { for any }(x, p) \in \Gamma_{\alpha} \times \mathbb{R}, \\
H^{\alpha}(x, p) \leqslant C_{0}(|p|+1), & \text { for any }(x, p) \in \Gamma_{\alpha} \times \mathbb{R}, \\
\left|\partial_{p} H^{\alpha}(x, p)\right| \leqslant C_{0}, & \text { for any }(x, p) \in \Gamma_{\alpha} \times \mathbb{R},
\end{array}
$$

for a constant $C_{0}$ independent of $\alpha$.
(Coupling operator) We assume that $\mathscr{V}$ is a continuous map from $L^{2}(\Gamma)$ to $L^{2}(\Gamma)$, such that for all $m \in L^{2}(\Gamma)$,

$$
\begin{equation*}
\|\mathscr{V}[m]\|_{L^{2}(\Gamma)} \leqslant C\left(\|m\|_{L^{2}(\Gamma)}+1\right) \tag{1.34}
\end{equation*}
$$

Note that such an assumption is satisfied by local operators of the form $\mathscr{V}[m](x)=$ $F(m(x))$ where $F$ is a Lipschitz-continuous function.
(Initial and terminal data) $m_{0} \in L^{2}(\Gamma) \cap \mathcal{M}$ and $v_{T} \in H^{1}(\Gamma)$.
The above set of assumptions, referred to as $(\mathrm{H})$, will be the running assumptions hereafter. We will use the following notation: $\underline{\mu}:=\min _{\alpha \in \mathcal{A}} \mu_{\alpha}>0$ and $\bar{\mu}:=\max _{\alpha \in \mathcal{A}} \mu_{\alpha}$.

### 1.4.3 Strictly increasing coupling

We will also say that the coupling $\mathscr{V}$ is strictly increasing if, for any $m_{1}, m_{2} \in \mathcal{M} \cap L^{2}(\Gamma)$,

$$
\int_{\Gamma}\left(m_{1}-m_{2}\right)\left(\mathscr{V}\left[m_{1}\right]-\mathscr{V}\left[m_{2}\right]\right) d x \geqslant 0
$$

and equality implies $m_{1}=m_{2}$.

### 1.4.4 Stronger assumptions on the coupling operator

We will sometimes need to strengthen the assumptions on the coupling operator, namely that $\mathscr{V}$ has the following smoothing properties:
$\mathscr{V}$ maps the topological dual of $W$ to $H_{b}^{1}(\Gamma)$; more precisely, $\mathscr{V}$ defines a Lipschitz map from $W^{\prime}$ to $H_{b}^{1}(\Gamma)$.

Note that such an assumption is not satisfied by local operators.

### 1.4.5 Definition of solutions and main result

Definition 1.10. (solutions of the MFG system) A weak solution of the Mean Field Games system (1.25) is a pair $(v, m)$ such that

$$
\begin{aligned}
& v \in L^{2}\left(0, T ; H^{2}(\Gamma)\right) \cap C([0, T] ; V), \partial_{t} v \in L^{2}\left(0, T ; L^{2}(\Gamma)\right) \\
& m \in L^{2}(0, T ; W) \cap C\left((0, T] ; L^{2}(\Gamma) \cap \mathcal{M}\right), \partial_{t} m \in L^{2}\left(0, T ; V^{\prime}\right)
\end{aligned}
$$

$v$ satisfies

$$
\left\{\begin{aligned}
&-\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_{\alpha}}\left[\partial_{t} v(x, t) \mathrm{w}(x)\right.\left.+\mu_{\alpha} \partial v(x, t) \partial \mathrm{w}(x)+H(x, \partial v(x, t)) \mathrm{w}(x)\right] d x \\
&=\int_{\Gamma} \mathscr{V}[m(\cdot, t)](x) \mathrm{w}(x) d x, \quad \text { for all } \mathrm{w} \in W, \text { a.e. } t \in(0, T) \\
& v(x, T)=v_{T}(x) \text { for a.e. } x \in \Gamma
\end{aligned}\right.
$$

and $m$ satisfies

$$
\left\{\begin{array}{l}
\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_{\alpha}}\left[\partial_{t} m(x, t) v(x) d x+\mu_{\alpha} \partial m(x, t) \partial v(x)+\partial_{p} H(x, \partial v(x, t)) m(x, t) \partial v(x)\right] d x \\
m(x, 0)=m_{0}(x) \text { for a.e. } x \in \Gamma, \quad \text { for all } \mathrm{v} \in V, \text { a.e. } t \in(0, T)
\end{array}\right.
$$

where $V$ and $W$ are introduced in Definition 1.6.

We are ready to state the main result:
Theorem 1.11. Under assumptions (H),
(i) (Existence) There exists a weak solution ( $v, m$ ) of (1.25).
(ii) (Uniqueness) If $\mathscr{V}$ is strictly increasing (see 1.4.3), then the solution is unique.
(iii) (Regularity) If $\mathscr{V}$ satisfies furthermore the stronger assumptions made in Section 1.4.4 and if $v_{T} \in C^{2+\eta}(\Gamma) \cap D$ for some $\eta \in(0,1)$ ( $D$ is given in (1.13)), then $v \in C^{2,1}(\Gamma \times[0, T])$. Moreover, if for all $\alpha \in \mathcal{A}, \partial_{p} H^{\alpha}(x, p)$ is a Lipschitz function defined in $\Gamma_{\alpha} \times \mathbb{R}$, and if $m_{0} \in W$, then $m \in C([0, T] ; W) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right) \cap L^{2}\left(0, T ; H_{b}^{2}(\Gamma)\right)$.

## 2 Preliminary: a modified heat equation on the network with general Kirchhoff conditions

This section contains results on the solvability of some linear boundary value problems with terminal condition, that will be useful in what follows. Consider

$$
\begin{cases}-\partial_{t} v-\mu_{\alpha} \partial^{2} v=h, & \text { in }\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T), \alpha \in \mathcal{A},  \tag{2.1}\\ \left.v\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=\left.v\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right), & t \in(0, T) \alpha, \beta \in \mathcal{A}_{i}, \nu_{i} \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha} \partial_{\alpha} v\left(\nu_{i}, t\right)=0, & t \in(0, T), \nu_{i} \in \mathcal{V}, \\ v(x, T)=v_{T}(x), & x \in \Gamma,\end{cases}
$$

where $h \in L^{2}\left(0, T ; W^{\prime}\right)$ and $v_{T} \in L^{2}(\Gamma)$.
Definition 2.1. If $v_{T} \in L^{2}(\Gamma)$ and $h \in L^{2}\left(0, T ; W^{\prime}\right)$, a weak solution of (2.1) is a function $v \in L^{2}(0, T ; V) \cap C\left([0, T] ; L^{2}(\Gamma)\right)$ such that $\partial_{t} v \in L^{2}\left(0, T ; W^{\prime}\right)$ and

$$
\left\{\begin{array}{l}
-\left\langle\partial_{t} v(t), w\right\rangle_{W^{\prime}, W}+\mathscr{B}(v(\cdot, t), w)=\langle h(t), w\rangle_{W^{\prime}, W} \quad \text { for all } w \in W \text { and a.e. } t \in(0, T),  \tag{2.2}\\
v(x, T)=v_{T}(x),
\end{array}\right.
$$

where $\mathscr{B}: V \times W \rightarrow \mathbb{R}$ is the bilinear form defined as follows:

$$
\mathscr{B}(v, w):=\int_{\Gamma} \mu \partial v \partial w d x=\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_{\alpha}} \mu_{\alpha} \partial v \partial w d x .
$$

We use the Galerkin's method (see [7]), i.e., we construct solutions of some finite-dimensional approximations to (2.1).

Recall that $\varphi$ has been defined in Definition 1.7. We notice first that the symmetric bilinear form $\widehat{\mathscr{B}}(u, v):=\int_{\Gamma} \mu \varphi \partial u \partial v$ is such that $(u, v) \mapsto(u, v)_{L^{2}(\Gamma)}+\widehat{\mathscr{B}}(u, v)$ is an inner product in $V$ equivalent to the standard inner product in $V$, namely $(u, v)_{V}=(u, v)_{L^{2}(\Gamma)}+\int_{\Gamma} \partial u \partial v$. Therefore, by standard Fredholm's theory, there exist

- a non decreasing sequence of nonnegative real numbers $\left(\lambda_{k}\right)_{k=1}^{\infty}$, that tends to $+\infty$ as $k \rightarrow \infty$
- A Hilbert basis $\left(\mathrm{v}_{k}\right)_{k=1}^{\infty}$ of $L^{2}(\Gamma)$, which is also a a total sequence of $V$ (and orthogonal if $V$ is endowed with the scalar product $\left.(u, v)_{L^{2}(\Gamma)}+\widehat{\mathscr{B}}(u, v)\right)$,
such that

$$
\begin{equation*}
\widehat{\mathscr{B}}\left(\mathrm{v}_{k}, v\right)=\lambda_{k}\left(\mathrm{v}_{k}, v\right)_{L^{2}(\Gamma)} \quad \text { for all } v \in V . \tag{2.3}
\end{equation*}
$$

Note that

$$
\int_{\Gamma} \mu \partial v_{k} \partial v_{\ell} \varphi d x= \begin{cases}\lambda_{k} & \text { if } k=\ell \\ 0 & \text { if } k \neq \ell\end{cases}
$$

Note also that $\mathrm{v}_{k}$ is a weak solution of

$$
\begin{cases}-\mu_{\alpha} \partial\left(\varphi \partial \mathrm{v}_{k}\right)=\lambda_{k} \mathrm{v}_{k}, & \text { in } \Gamma_{\alpha} \backslash \mathcal{V}, \alpha \in \mathcal{A}  \tag{2.4}\\ \left.\mathrm{v}_{k}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)=\left.\mathrm{v}_{k}\right|_{\Gamma_{\beta}}\left(\nu_{i}\right), & \alpha, \beta \in \mathcal{A}_{i} \\ \sum_{\alpha \in \mathcal{A}} \gamma_{i \alpha} \mu_{\alpha} \partial_{\alpha} \mathrm{v}_{k}\left(\nu_{i}\right)=0, & \nu_{i} \in \mathcal{V}\end{cases}
$$

which implies that $\mathrm{v}_{k} \in C^{2}(\Gamma)$.
Finally, by Remark 1.8, the sequence $\left(\varphi \vee_{k}\right)_{k=1}^{\infty}$ is a total family in $W$ (but is not orthogonal if $W$ is endowed with the standard inner product).

Lemma 2.2. For any positive integer $n$, there exist $n$ absolutely continuous functions $y_{k}^{n}$ : $[0, T] \rightarrow \mathbb{R}, k=1, \ldots, n$, and a function $v_{n}:[0, T] \rightarrow L^{2}(\Gamma)$ of the form

$$
\begin{equation*}
v_{n}(x, t)=\sum_{k=1}^{n} y_{k}^{n}(t) \mathrm{v}_{k}(x), \tag{2.5}
\end{equation*}
$$

such that

$$
\begin{equation*}
y_{k}^{n}(T)=\int_{\Gamma} v_{T} v_{k} d x, \quad \text { for } k=1, \ldots, n \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
-\frac{d}{d t}\left(v_{n}, \mathrm{v}_{k} \varphi\right)_{L^{2}(\Gamma)}+\mathscr{B}\left(v_{n}, \mathrm{v}_{k} \varphi\right)=\left\langle h(t), \mathrm{v}_{k} \varphi\right\rangle, \quad \text { for a.a. } t \in(0, T), \text { for all } k=1, \ldots, n . \tag{2.7}
\end{equation*}
$$

Proof of Lemma 2.2. For $n \geqslant 1$, we consider the symmetric $n$ by $n$ matrix $M_{n}$ defined by

$$
\left(M_{n}\right)_{k \ell}=\int_{\Gamma} \mathrm{v}_{k} \mathrm{v}_{\ell} \varphi d x .
$$

Since $\varphi$ is positive and $\left(\mathrm{v}_{k}\right)_{k=1}^{\infty}$ is a Hilbert basis of $L^{2}(\Gamma)$, we can check that $M_{n}$ is a positive definite matrix and there exist two constants $c, C$ independent of $n$ such that

$$
\begin{equation*}
c|\xi|^{2} \leqslant \sum_{k, \ell=1}^{n}\left(M_{n}\right)_{k \ell} \xi_{k} \xi_{\ell} \leqslant C|\xi|^{2}, \quad \text { for all } \xi \in \mathbb{R}^{n} \tag{2.8}
\end{equation*}
$$

Looking for $v_{n}$ of the form (2.5), and setting $Y=\left(y_{1}^{n}, \ldots, y_{n}^{n}\right)^{T}, \dot{Y}=\left(\frac{d}{d t} y_{1}^{n}, \ldots, \frac{d}{d t} y_{n}^{n}\right)^{T}$, (2.7) implies that we have to solve the following system of differential equations

$$
-M_{n} \dot{Y}+B Y=F_{n}, \quad Y(T)=\left(\int_{\Gamma} v_{T} \mathrm{v}_{1}, \ldots, \int_{\Gamma} v_{T} \mathrm{v}_{n}\right)^{T}
$$

where $B_{k \ell}=\mathscr{B}\left(\mathrm{v}_{\ell}, \mathrm{v}_{k} \varphi\right)$ and $F_{n}(t)=\left(\left\langle h(t), \mathrm{v}_{1} \varphi\right\rangle, \ldots,\left\langle h(t), \mathrm{v}_{n} \varphi\right\rangle\right)^{T}$. Since the matrix $M_{n}$ is invertible, the ODE system has a unique absolutely continuous solution. The lemma is proved.

We propose to send $n$ to $+\infty$ and show that a subsequence of $\left\{v_{n}\right\}$ converges to a solution of (2.1). Hence, we need some uniform estimates for $\left\{v_{n}\right\}$.

Lemma 2.3. There exists a constant $C$ depending only on $\Gamma,\left(\mu_{\alpha}\right)_{\alpha \in \mathcal{A}}, T$ and $\varphi$ such that

$$
\left\|v_{n}\right\|_{L^{\infty}\left(0, T ; L^{2}(\Gamma)\right)}+\left\|v_{n}\right\|_{L^{2}(0, T ; V)}+\left\|\partial_{t} v_{n}\right\|_{L^{2}\left(0, T ; W^{\prime}\right)} \leqslant C\left(\|h\|_{L^{2}\left(0, T ; W^{\prime}\right)}+\left\|v_{T}\right\|_{L^{2}(\Gamma)}\right) .
$$

Proof of Lemma 2.3. Multiplying (2.7) by $y_{k}^{n}(t) e^{\lambda t}$ for a positive constant $\lambda$ to be chosen later, summing for $k=1, \ldots, n$ and using the formula (2.5) for $v_{n}$, we get

$$
-\int_{\Gamma} \partial_{t} v_{n} v_{n} e^{\lambda t} \varphi d x+\int_{\Gamma} \mu \partial v_{n} \partial\left(v_{n} e^{\lambda t} \varphi\right) d x=e^{\lambda t}\left\langle h(t), v_{n} \varphi\right\rangle_{W^{\prime}, W}
$$

and

$$
-\int_{\Gamma}\left[\partial_{t}\left(\frac{v_{n}^{2}}{2} e^{\lambda t}\right)-\frac{\lambda}{2} v_{n}^{2} e^{\lambda t}\right] \varphi d x+\int_{\Gamma} \mu\left(\partial v_{n}\right)^{2} e^{\lambda t} \varphi d x+\int_{\Gamma} \mu \partial v_{n} v_{n} e^{\lambda t} \partial \varphi d x=e^{\lambda t}\left\langle h(t), v_{n} \varphi\right\rangle .
$$

Integrating both sides from $s$ to $T$, we obtain

$$
\begin{aligned}
& \int_{\Gamma}\left(\frac{v_{n}^{2}(x, s)}{2} e^{\lambda s}-\frac{v_{n}^{2}(x, T)}{2} e^{\lambda T}\right) \varphi d x+\frac{\lambda}{2} \int_{s}^{T} \int_{\Gamma} v_{n}^{2} e^{\lambda t} \varphi d x d t \\
& +\int_{s}^{T} \int_{\Gamma} \mu\left(\partial v_{n}\right)^{2} e^{\lambda t} \varphi d x d t+\int_{s}^{T} \int_{\Gamma} \mu \partial v_{n} v_{n} e^{\lambda t} \partial \varphi d x d t \\
= & \int_{s}^{T} e^{\lambda t}\left\langle h(t), v_{n}(t) \varphi\right\rangle d t \\
\leqslant & C \int_{s}^{T} e^{\lambda t}\|h(t)\|_{W^{\prime}}\left\|v_{n}(t)\right\|_{V} d t \\
\leqslant & \frac{1}{2} \int_{s}^{T} \int_{\Gamma} \mu\left(\left(\partial v_{n}\right)^{2}+v_{n}^{2}\right) e^{\lambda t} \varphi d x d t+\frac{C^{2}}{2 \underline{\mu}} \int_{s}^{T} e^{\lambda t}\|h(t)\|_{W^{\prime}}^{2} d t,
\end{aligned}
$$

where $C$ is positive constant depending on $\varphi$, because of Remark 1.8. Therefore,

$$
\begin{aligned}
& e^{\lambda s} \int_{\Gamma} \frac{v_{n}^{2}(x, s)}{2} \varphi d x+\frac{1}{4} \int_{s}^{T} \int_{\Gamma} \mu\left(\partial v_{n}\right)^{2} \varphi e^{\lambda t} d x d t+\left(\frac{\lambda}{2}-\frac{\bar{\mu}}{2}-\bar{\mu} \frac{\|\partial \varphi\|_{L^{\infty}(\Gamma)}^{2}}{\underline{\varphi}^{2}}\right) \int_{s}^{T} \int_{\Gamma} v_{n}^{2} e^{\lambda t} \varphi d x d t \\
\leqslant & e^{\lambda T} \int_{\Gamma} \frac{v_{n}^{2}(x, T)}{2} \varphi d x+\frac{C^{2}}{2 \underline{\mu}} e^{\lambda T} \int_{s}^{T}\|h(t)\|_{W^{\prime}}^{2} d t .
\end{aligned}
$$

Choosing $\lambda \geqslant 1 / 2+\bar{\mu}+2 \bar{\mu}\|\partial \varphi\|_{L^{\infty}(\Gamma)}^{2} / \underline{\varphi}^{2}$ and noticing that $\int_{\Gamma} v_{n}^{2}(x, T) \varphi d x$ is bounded by $\bar{\varphi} \int_{\Gamma} v_{T}^{2} d x$ from (2.6), it follows that

$$
\begin{align*}
& \int_{\Gamma} v_{n}^{2}(x, s) \varphi d x+\int_{s}^{T} \int_{\Gamma} v_{n}^{2} \varphi d x d t+\int_{s}^{T} \int_{\Gamma} \mu\left(\partial v_{n}\right)^{2} \varphi d x d t \\
\leqslant & 2 e^{\lambda T}\left(\frac{C^{2}}{\underline{\mu}}\|h\|_{L^{2}\left(0, T ; W^{\prime}\right)}^{2}+\bar{\varphi} \int_{\Gamma} v_{T}^{2} d x\right) . \tag{2.9}
\end{align*}
$$

Estimate of $v_{n}$ in $L^{\infty}\left(0, T ; L^{2}(\Gamma)\right)$ and $L^{2}(0, T ; V)$. From (2.9), it is straightforward to see that

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{\infty}\left(0, T ; L^{2}(\Gamma)\right)}+\left\|v_{n}\right\|_{L^{2}(0, T ; V)} \leqslant C\left(\|h\|_{L^{2}\left(0, T ; W^{\prime}\right)}+\left\|v_{T}\right\|_{L^{2}(\Gamma)}\right) \tag{2.10}
\end{equation*}
$$

for some constant $C$ depending only on $\left(\mu_{\alpha}\right)_{\alpha \in \mathcal{A}}, \varphi$ and $T$.
Estimate $\partial_{t} v_{n}$ in $L^{2}\left(0, T ; W^{\prime}\right)$. Consider the closed subspace $G_{1}$ of $W$ defined by $G_{1}=$ $\left\{w \in W: \int_{\Gamma} \mathrm{v}_{k} w d x=0\right.$ for all $\left.k \leqslant n\right\}$. It has a finite co-dimension equal to $n$. Consider also the $n$-dimensional subspace $G_{2}=\operatorname{span}\left\{\mathrm{v}_{1} \varphi, \ldots, \mathrm{v}_{n} \varphi\right\}$ of $W$. The invertibility of the matrix $M_{n}$ introduced in the proof of Lemma 2.2 implies that $G_{1} \cap G_{2}=\{0\}$. This implies that $W=G_{1} \oplus G_{2}$. For $w \in W$, we can write $w$ of the form $w=w_{n}+\hat{w}_{n}$, where $w_{n} \in G_{2}$ and $\hat{w}_{n} \in G_{1}$. Hence, for a.e. $t \in[0, T]$, from (2.5) and (2.7), one gets

$$
\begin{equation*}
\left\langle\partial_{t} v_{n}(t), w\right\rangle_{W^{\prime}, W}=\frac{d}{d t}\left(\int_{\Gamma} v_{n} w d x\right)=\frac{d}{d t}\left(\int_{\Gamma} v_{n} w_{n} d x\right)=-\left\langle h(t), w_{n}\right\rangle_{W^{\prime}, W}+\int_{\Gamma} \mu \partial v_{n} \partial w_{n} d x . \tag{2.11}
\end{equation*}
$$

Since there exists a constant $C$ independent of $n$ such that $\left\|w_{n}\right\|_{W} \leqslant C\|w\|_{W}$, it follows that

$$
\left\|\partial_{t} v_{n}(t)\right\|_{W^{\prime}} \leqslant C\left(\|h(t)\|_{W^{\prime}}+\bar{\mu}\left\|v_{n}(t)\right\|_{V}\right),
$$

for almost every $t$, and therefore, from (2.10), we obtain

$$
\left\|\partial_{t} v_{n}(t)\right\|_{L^{2}\left(0, T ; W^{\prime}\right)}^{2} \leqslant C\left(\|h\|_{L^{2}\left(0, T ; W^{\prime}\right)}^{2}+\left\|v_{T}\right\|_{L^{2}(\Gamma)}^{2}\right)
$$

for a constant $C$ independent of $n$.
Theorem 2.4. There exists a unique solution $v$ of (2.1), which satisfies

$$
\begin{equation*}
\|v\|_{L^{\infty}\left(0, T ; L^{2}(\Gamma)\right)}+\|v\|_{L^{2}(0, T ; V)}+\left\|\partial_{t} v\right\|_{L^{2}\left(0, T ; W^{\prime}\right)} \leqslant C\left(\|h\|_{L^{2}\left(0, T ; W^{\prime}\right)}+\left\|v_{T}\right\|_{L^{2}(\Gamma)}\right), \tag{2.12}
\end{equation*}
$$

where $C$ is a constant that depends only on $\Gamma,\left(\mu_{\alpha}\right)_{\alpha \in \mathcal{A}}, T$ and $\varphi$.
Proof of Theorem 2.4. From Lemma 2.3, the sequence $\left(v_{n}\right)_{n \in \mathbb{N}}$ is bounded in $L^{2}(0, T ; V)$ and the sequence $\left(\partial_{t} v_{n}\right)_{n \in \mathbb{N}}$ is bounded in $L^{2}\left(0, T ; W^{\prime}\right)$. Hence, up to the extraction of a subsequence, there exists a function $v$ such that $v \in L^{2}(0, T ; V), \partial_{t} v \in L^{2}\left(0, T ; W^{\prime}\right)$ and

$$
\begin{cases}v_{n} \rightharpoonup v & \text { weakly in } L^{2}(0, T ; V)  \tag{2.13}\\ \partial_{t} v_{n} \rightharpoonup \partial_{t} v & \text { weakly in } L^{2}\left(0, T ; W^{\prime}\right) .\end{cases}
$$

Fix an integer $N$ and choose a function $\bar{v} \in C^{1}([0, T] ; V)$ having the form

$$
\begin{equation*}
\bar{v}(t)=\sum_{k=1}^{N} d_{k}(t) \mathrm{v}_{k}, \tag{2.14}
\end{equation*}
$$

where $d_{1}, \ldots, d_{N}$ are given real valued $C^{1}$ functions defined in $[0, T]$. For all $n \geqslant N$, multiplying (2.7) by $d_{k}(t)$, summing for $k=1, \ldots, n$ and integrating over $(0, T)$ leads to

$$
\begin{equation*}
-\int_{0}^{T} \int_{\Gamma} \partial_{t} v_{n} \bar{v} \varphi d x d t+\int_{0}^{T} \int_{\Gamma} \mu \partial v_{n} \partial(\bar{v} \varphi) d x d t=\int_{0}^{T}\langle h, \bar{v} \varphi\rangle d t . \tag{2.15}
\end{equation*}
$$

Letting $n \rightarrow+\infty$, we obtain from (2.13) that

$$
\begin{equation*}
-\int_{0}^{T}\left\langle\partial_{t} v, \bar{v} \varphi\right\rangle d t+\int_{0}^{T} \int_{\Gamma} \mu \partial v \partial(\bar{v} \varphi) d x d t=\int_{0}^{T}\langle h, \bar{v} \varphi\rangle d t . \tag{2.16}
\end{equation*}
$$

Since the functions of the form (2.14) are dense in $L^{2}(0, T ; V)$, (2.16) holds for all test function $\bar{v} \in L^{2}(0, T ; V)$. Recalling the isomorphism $\bar{v} \in V \mapsto \bar{v} \varphi \in W$ (see Remark 1.8), we obtain that, for all $w \in W$ and $\psi \in C_{c}^{1}(0, T)$,

$$
-\int_{0}^{T}\left\langle\partial_{t} v, w\right\rangle \psi d t+\int_{0}^{T} \int_{\Gamma} \mu \partial v \partial w \psi d x d t=\int_{0}^{T}\langle h, w\rangle \psi d t .
$$

This implies that, for a.e. $t \in(0, T)$,

$$
-\left\langle\partial_{t} v, w\right\rangle+\mathscr{B}(v, w)=\langle h, w\rangle \quad \text { for all } w \in W .
$$

Using [19, Theorem 3.1] (or the same argument as in [7, pages 287-288]), we see that $v \in$ $C\left([0, T] ; L_{\varphi}^{2}(\Gamma)\right)$, where $L_{\varphi}^{2}(\Gamma)=\left\{w: \Gamma \rightarrow \mathbb{R}: \int_{\Gamma} w^{2} \varphi d x<+\infty\right\}$, and since $\varphi$ is bounded from below and from above by positive numbers, $L_{\varphi}^{2}(\Gamma)=L^{2}(\Gamma)$ with equivalent norms. Moreover,

$$
\max _{0 \leqslant t \leqslant T}\|v(\cdot, t)\|_{L^{2}(\Gamma)} \leqslant C\left(\left\|\partial_{t} v\right\|_{L^{2}\left(0, T ; W^{\prime}\right)}+\|v\|_{L^{2}(0, T ; V)}\right) .
$$

We are now going to prove $v(T)=v_{T}$. For all $\bar{v} \in C^{1}([0, T] ; V)$ of the form (2.14) and such that $\bar{v}(0)=0$, we deduce from (2.15) and (2.16) that

$$
\begin{aligned}
& -\int_{0}^{T} \int_{\Gamma} \partial_{t} \bar{v} v_{n} \varphi d x d t-\int_{\Gamma} \bar{v}(T) v_{n}(T) \varphi d x+\int_{0}^{T} \int_{\Gamma} \mu \partial v_{n} \partial(\bar{v} \varphi) d x d t \\
= & -\int_{0}^{T} \int_{\Gamma} \partial_{t} \bar{v} v \varphi d x d t-\int_{\Gamma} \bar{v}(T) v(T) \varphi d x+\int_{0}^{T} \int_{\Gamma} \mu \partial v \partial(\bar{v} \varphi) d x d t .
\end{aligned}
$$

We know that $v_{n}(T) \rightarrow v_{T}$ in $L^{2}(\Gamma)$. Then, using (2.13), we obtain

$$
\int_{\Gamma} \bar{v}(T) v_{T} \varphi d x=\int_{\Gamma} \bar{v}(T) v(T) \varphi d x .
$$

Since the functions of the form $\sum_{k=1}^{N} d_{k}(T) \mathrm{v}_{k}$ are dense in $L^{2}(\Gamma)$, we conclude that $v(T)=v_{T}$.
In order to prove the energy estimate (2.12), we use $v e^{\lambda t} \varphi$ as a test function in (2.2) and apply similar arguments as in the proof of Lemma 2.3 for $\lambda$ large enough, we get (2.12).

Finally, if $h=0$ and $v_{T}=0$, by the energy estimate for $v$ in (2.12), we deduce that $v=0$. Uniqueness is proved.

Theorem 2.5. If $v_{T} \in V$ and $h \in L^{2}(\Gamma \times(0, T))$, then the unique solution $v$ of (2.1) satisfies $v \in L^{2}\left(0, T ; H^{2}(\Gamma)\right) \cap C([0, T] ; V)$ and $\partial_{t} v \in L^{2}\left(0, T ; L^{2}(\Gamma)\right)$. Moreover,

$$
\begin{equation*}
\|v\|_{L^{\infty}(0, T ; V)}+\|v\|_{L^{2}\left(0, T ; H^{2}(\Gamma)\right)}+\left\|\partial_{t} v\right\|_{L^{2}\left(0, T ; L^{2}(\Gamma)\right)} \leqslant C\left(\|h\|_{L^{2}\left(0, T ; L^{2}(\Gamma)\right)}+\left\|v_{T}\right\|_{V}\right) \tag{2.17}
\end{equation*}
$$

for a positive constant $C$ that depends only on $\Gamma,\left(\mu_{\alpha}\right)_{\alpha \in \mathcal{A}}, T$ and $\varphi$.
Proof of Theorem 2.5. It is enough to prove estimate (2.17) for $v_{n}$.
Multiplying (2.7) by $-\frac{d}{d t} y_{k}^{n}$, summing for $k=1, \ldots, n$ and using (2.5) leads to

$$
\int_{\Gamma}\left(\partial_{t} v_{n}\right)^{2} \varphi d x-\int_{\Gamma} \mu \partial v_{n} \partial\left(\partial_{t} v_{n} \varphi\right) d x=-\int_{\Gamma} h \partial_{t} v_{n} \varphi d x
$$

hence

$$
\int_{\Gamma}\left(\partial_{t} v_{n}\right)^{2} \varphi d x-\int_{\Gamma} \mu \partial_{t} \frac{\left(\partial v_{n}\right)^{2}}{2} \varphi d x-\int_{\Gamma} \mu \partial v_{n} \partial_{t} v_{n} \partial \varphi d x=-\int_{\Gamma} h \partial_{t} v_{n} \varphi d x
$$

Multiplying by $e^{\lambda t}$ where $\lambda$ will chosen later, and taking the integral from $s$ to $T$, we obtain

$$
\begin{align*}
& \int_{s}^{T} \int_{\Gamma}\left(\partial_{t} v_{n}\right)^{2} e^{\lambda t} \varphi d x d t-\int_{\Gamma} \frac{\mu}{2}\left[\left(\partial v_{n}(T)\right)^{2} e^{\lambda T}-\left(\partial v_{n}(s)\right)^{2} e^{\lambda s}\right] \varphi d x \\
& +\lambda \int_{s}^{T} \int_{\Gamma} \frac{\mu}{2}\left(\partial v_{n}\right)^{2} e^{\lambda t} \varphi d x d t-\int_{s}^{T} \int_{\Gamma} \mu \partial v_{n} \partial_{t} v_{n} e^{\lambda t} \partial \varphi d x d t \\
= & -\int_{S}^{T} \int_{\Gamma} h \partial_{t} v_{n} e^{\lambda t} \varphi d x d t \\
\leqslant & \frac{1}{2} \int_{s}^{T} \int_{\Gamma} h^{2} e^{\lambda t} \varphi d x d t+\frac{1}{2} \int_{s}^{T} \int_{\Gamma}\left(\partial_{t} v_{n}\right)^{2} e^{\lambda t} \varphi d x d t . \tag{2.18}
\end{align*}
$$

Let us deal with the term $\int_{\Gamma}\left(\partial v_{n}(x, T)\right)^{2} \varphi d x$. From (2.6),

$$
\begin{aligned}
\int_{\Gamma} \mu\left(\partial v_{n}(x, T)\right)^{2} \varphi d x & =\sum_{k=1}^{n} \lambda_{k}\left(\int_{\Gamma} v_{T} v_{k} d x\right)^{2} \\
& \leqslant \sum_{k=1}^{\infty} \lambda_{k}\left(\int_{\Gamma} v_{T} v_{k} d x\right)^{2}=\int_{\Gamma} \mu\left(\partial v_{T}(x)\right)^{2} \varphi d x \\
& \leqslant \bar{\mu} \int_{\Gamma}\left(\partial v_{T}(x)\right)^{2} \varphi d x
\end{aligned}
$$

Then, choosing $\lambda=2 \bar{\mu}^{2}\|\partial \varphi\|_{L^{\infty}(\Gamma)}^{2} /\left(\underline{\varphi}^{2} \underline{\mu}\right)$, we obtain that

$$
\begin{equation*}
\int_{\Gamma} 2 \mu\left(\partial v_{n}(x, s)\right)^{2} \varphi d x+\int_{s}^{T} \int_{\Gamma}\left(\partial_{t} v_{n}\right)^{2} \varphi d x d t \leqslant 2 e^{\lambda T} \bar{\varphi}\left(\|h\|_{L^{2}(\Gamma \times(0, T))}^{2}+\bar{\mu} \int_{\Gamma}\left(\partial v_{T}\right)^{2} d x\right) . \tag{2.19}
\end{equation*}
$$

Estimate of $\partial v_{n}$ in $L^{\infty}\left(0, T ; L^{2}(\Gamma)\right)$ and $\partial_{t} v_{n}$ in $L^{2}(\Gamma \times(0, T))$. From (2.19), it is straightforward to see that

$$
\left\|\partial v_{n}\right\|_{L^{\infty}\left(0, T ; L^{2}(\Gamma)\right)}+\left\|\partial_{t} v_{n}\right\|_{L^{2}(\Gamma \times(0, T))} \leqslant C\left(\|h\|_{L^{2}(\Gamma \times(0, T))}+\left\|\partial v_{T}\right\|_{L^{2}(\Gamma)}\right)
$$

for some constant $C$ depending only on $\Gamma, \mu, T$ and $\varphi$.
Estimate of $\partial^{2} v_{n}$ in $L^{2}(\Gamma \times(0, T))$. Finally, using the PDE in (2.1), we can see that $\partial^{2} v_{n}$ belongs to $L^{2}(\Gamma \times(0, T))$ and is bounded by $C\left(\|h\|_{L^{2}(\Gamma \times(0, T))}+\left\|v_{T}\right\|_{V}\right)$, hence $v_{n}$ is bounded in $L^{2}\left(0, T ; H^{2}(\Gamma)\right)$ by the same quantity. The Kirchhoff conditions (which boil down to Neumann conditions at $\partial \Gamma$ ) are therefore satisfied in a strong sense for almost all $t$.

Using [19, Theorem 3.1] (or a similar argument as [7, pages 287-288]), we see that $v$ in $C([0, T] ; V)$.

## 3 The Fokker-Planck equation

This paragraph is devoted to a boundary value problem including a Fokker-Planck equation

$$
\begin{cases}\partial_{t} m-\mu_{\alpha} \partial^{2} m-\partial(b m)=0, & \text { in }\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T), \alpha \in \mathcal{A},  \tag{3.1}\\ \frac{\left.m\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)}{\gamma_{i \alpha}}=\frac{\left.m\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)}{\gamma_{i \beta}}, & t \in(0, T), \alpha, \beta \in \mathcal{A}_{i}, \nu_{i} \in \mathcal{V} \backslash \partial \Gamma, \\ \sum_{\alpha \in \mathcal{A}_{i}} \mu_{\alpha} \partial_{\alpha} m\left(\nu_{i}, t\right)+\left.n_{i \alpha} b\left(\nu_{i}, t\right) m\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=0, & t \in(0, T), \nu_{i} \in \mathcal{V}, \\ m(x, 0)=m_{0}(x), & x \in \Gamma,\end{cases}
$$

where $b \in P C(\Gamma \times[0, T])$ and $m_{0} \in L^{2}(\Gamma)$.
Definition 3.1. A weak solution of (3.1) is a function $m \in L^{2}(0, T ; W) \cap C\left([0, T] ; L^{2}(\Gamma)\right)$ such that $\partial_{t} m \in L^{2}\left(0, T ; V^{\prime}\right)$ and

$$
\left\{\begin{array}{l}
\left\langle\partial_{t} m, v\right\rangle_{V^{\prime}, V}+\mathscr{A}(m, v)=0 \quad \text { for all } v \in V \text { and a.e. } t \in(0, T),  \tag{3.2}\\
m(\cdot, 0)=m_{0},
\end{array}\right.
$$

where $\mathscr{A}: W \times V \rightarrow \mathbb{R}$ is the bilinear form

$$
\mathscr{A}(v, w)=\int_{\Gamma} \mu \partial m \partial v d x+\int_{\Gamma} b m \partial v d x .
$$

Using similar arguments as in Section 2, in particular a Galerkin method, we obtain the following result, the proof of which is omitted.

Theorem 3.2. If $b \in L^{\infty}(\Gamma \times(0, T))$ and $m_{0} \in L^{2}(\Gamma)$, there exists a unique function $m \in$ $L^{2}(0, T ; W) \cap C\left([0, T] ; L^{2}(\Gamma)\right)$ such that $\partial_{t} m \in L^{2}\left(0, T ; V^{\prime}\right)$ and (3.2). Moreover, there exists a constant $C$ which depends on $\left(\mu_{\alpha}\right)_{\alpha \in \mathcal{A}},\|b\|_{\infty}, T$ and $\varphi$, such that

$$
\begin{equation*}
\|m\|_{L^{2}(0, T ; W)}+\|m\|_{L^{\infty}\left(0, T ; L^{2}(\Gamma)\right)}+\left\|\partial_{t} m\right\|_{L^{2}\left(0, T ; V^{\prime}\right)} \leqslant C\left\|m_{0}\right\|_{L^{2}(\Gamma)} . \tag{3.3}
\end{equation*}
$$

Remark 3.3. If $m_{0} \in \mathcal{M}$, which will be the case when solving the MFG system (1.25), then $m(\cdot, t) \in \mathcal{M}$ for all $t \in[0, T]$. Indeed, we use $v \equiv 1 \in V$ as a test-function for (3.1). Since $\partial v=0$, integrating (3.2) from 0 to $t$, we get $\int_{0}^{t} \int_{\Gamma} \partial_{t} m(x, s) d x d s=0$. This implies that

$$
\int_{\Gamma} m(x, t) d x=\int_{\Gamma} m_{0}(x) d x=1, \quad \text { for all } t \in(0, T] .
$$

Setting $m^{-}=-\mathbb{1}_{\{m<0\}} m$, we can also use $v=\varphi^{-1} m^{-} e^{-\lambda t}$ as a test-function for $\lambda \in \mathbb{R}_{+}$. Indeed, the latter function belongs to $L^{2}(0, T ; V)$. Taking $\lambda$ large enough and using similar arguments as for the energy estimate (3.3) yield that $m^{-}=0$, i.e., $m \geqslant 0$.

We end this section by stating a stability result, which will be useful in the proof of the main Theorem.

Lemma 3.4. Let $m_{0 \varepsilon}, b_{\varepsilon}$ be sequences of functions satisfying

$$
m_{0 \varepsilon} \longrightarrow m_{0} \text { in } L^{2}(\Gamma), \quad b_{\varepsilon} \longrightarrow b \text { in } L^{2}(\Gamma \times(0, T)),
$$

and for some positive number $K$ independent of $\varepsilon,\|b\|_{L^{\infty}(\Gamma \times(0, T))} \leqslant K,\left\|b_{\varepsilon}\right\|_{L^{\infty}(\Gamma \times(0, T))} \leqslant K$.
Let $m_{\varepsilon}$ (respectively $m$ ) be the solution of (3.2) corresponding to the datum $m_{0 \varepsilon}$ (resp. $m_{0}$ ) and the coefficient $b_{\varepsilon}\left(\right.$ resp. b). The sequence $\left(m_{\varepsilon}\right)$ converges to $m$ in $L^{2}(0, T ; W) \cap L^{\infty}\left(0, T ; L^{2}(\Gamma)\right)$, and the sequence $\left(\partial_{t} m_{\varepsilon}\right)$ converges to $\left(\partial_{t} m\right)$ in $L^{2}\left(0, T ; V^{\prime}\right)$.
Proof of Lemma 3.4. Taking $\left(m_{\varepsilon}-m\right) e^{-\lambda t} \varphi^{-1}$ as a test-function in the versions of (3.2) satisfied by $m_{\varepsilon}$ and $m$, subtracting, we obtain that

$$
\begin{align*}
& \int_{\Gamma}\left[\frac{1}{2} \partial_{t}\left(\left(m_{\varepsilon}-m\right)^{2} e^{-\lambda t}\right)+\frac{\lambda}{2}\left(m_{\varepsilon}-m\right)^{2} e^{-\lambda t}\right] \varphi^{-1} d x+\int_{\Gamma} \mu\left(\partial\left(m_{\varepsilon}-m\right)\right)^{2} e^{-\lambda t} \varphi^{-1} d x \\
& +\int_{\Gamma} \mu\left(m_{\varepsilon}-m\right) \partial\left(m_{\varepsilon}-m\right) e^{-\lambda t} \partial\left(\varphi^{-1}\right) d x+\int_{\Gamma}\left(b_{\varepsilon} m_{\varepsilon}-b m\right) \partial\left(m_{\varepsilon}-m\right) e^{-\lambda t} \varphi^{-1} d x \\
& +\int_{\Gamma}\left(b_{\varepsilon} m_{\varepsilon}-b m\right)\left(m_{\varepsilon}-m\right) e^{-\lambda t} \partial\left(\varphi^{-1}\right) d x=0 . \tag{3.4}
\end{align*}
$$

There exists a positive constant $K$ such that $\left\|b_{\varepsilon}\right\|_{\infty},\|b\|_{\infty} \leqslant K$ for all $\varepsilon$. Hence, there exists a positive constant $C$ (in fact it varies from one line to the other in what follows) such that

$$
\begin{aligned}
& \int_{\Gamma}\left[\frac{1}{2} \partial_{t}\left(\left(m_{\varepsilon}-m\right)^{2} e^{-\lambda t}\right)+\frac{\lambda}{2}\left(m_{\varepsilon}-m\right)^{2} e^{-\lambda t}\right] \varphi^{-1} d x+\int_{\Gamma} \mu\left(\partial\left(m_{\varepsilon}-m\right)\right)^{2} e^{-\lambda t} \varphi^{-1} d x \\
\leqslant & C \int_{\Gamma}\left(\left|m_{\varepsilon}-m\right|^{2}+\left|m_{\varepsilon}-m\right|\left|\partial\left(m_{\varepsilon}-m\right)\right|+|m|\left|b_{\varepsilon}-b\right|\left(\left|\partial\left(m_{\varepsilon}-m\right)\right|+\left|m_{\varepsilon}-m\right|\right)\right) e^{-\lambda t} \varphi^{-1} d x \\
\leqslant & C \int_{\Gamma}\left(\left|m_{\varepsilon}-m\right|^{2}+\left|b_{\varepsilon}-b\right|^{2} m^{2}\right) e^{-\lambda t} \varphi^{-1} d x+\int_{\Gamma} \frac{\mu}{2}\left(\partial\left(m_{\varepsilon}-m\right)\right)^{2} e^{-\lambda t} \varphi^{-1} d x .
\end{aligned}
$$

The assumptions on the coefficents $b_{\varepsilon}$ and $b$ imply in fact that $b_{\varepsilon} \rightarrow b$ in $L^{p}(\Gamma \times(0, T))$ for all $1 \leqslant p<\infty$. On the other hand, we know that $m \in L^{q}(\Gamma \times(0, T))$ for all $1 \leqslant q<\infty$. From the latter observation with $p=q=4$, we see that the quantity $\int_{0}^{T} \int_{\Gamma}\left(\left|b_{\varepsilon}-b\right|^{2} m^{2}\right) e^{-\lambda t} \varphi^{-1} d x d t$ tends to 0 as $\varepsilon \rightarrow 0$ uniformly in $\lambda>0$. We write

$$
\int_{0}^{T} \int_{\Gamma}\left(\left|b_{\varepsilon}-b\right|^{2} m^{2}\right) e^{-\lambda t} \varphi^{-1} d x d t=o_{\varepsilon}(1)
$$

Choosing $\lambda$ large enough and integrating the latter inequality from 0 to $t \in[0, T]$, we obtain

$$
\left\|m_{\varepsilon}-m\right\|_{L^{2}(0, T ; W)}+\left\|m_{\varepsilon}-m\right\|_{L^{\infty}\left(0, T ; L^{2}(\Gamma)\right)} \leqslant o_{\varepsilon}(1)+C\left\|m_{0 \varepsilon}-m_{0}\right\|_{L^{2}(\Gamma)} .
$$

Subtracting the two versions of (3.2) and using the latter estimate also yields

$$
\left\|\partial_{t} m_{\varepsilon}-\partial_{t} m\right\|_{L^{2}\left(0, T ; V^{\prime}\right)} \leqslant o_{\varepsilon}(1)+C\left\|m_{0 \varepsilon}-m_{0}\right\|_{L^{2}(\Gamma)},
$$

which achieves the proof.

## 4 The Hamilton-Jacobi equation

This section is devoted to the following boundary value problem including a Hamilton-Jacobi equation

$$
\begin{cases}-\partial_{t} v-\mu_{\alpha} \partial^{2} v+H(x, \partial v)=f, & \text { in }\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T), \alpha \in \mathcal{A},  \tag{4.1}\\ \left.v\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=\left.v\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right) & t \in(0, T), \alpha, \beta \in \mathcal{A}_{i}, \nu_{i} \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha} \partial_{\alpha} v\left(\nu_{i}, t\right)=0, & t \in(0, T), \nu_{i} \in \mathcal{V}, \\ v(x, T)=v_{T}(x), & x \in \Gamma,\end{cases}
$$

where $f \in L^{2}(\Gamma \times(0, T)), v_{T} \in V$ and the Hamiltonian $H: \Gamma \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the running assumptions (H).

Definition 4.1. For $f \in L^{2}(\Gamma \times(0, T))$ and $v_{T} \in V$, a weak solution of (4.1) is a function $v \in L^{2}\left(0, T ; H^{2}(\Gamma)\right) \cap C([0, T] ; V)$ such that $\partial_{t} v \in L^{2}(\Gamma \times(0, T))$ and

$$
\left\{\begin{array}{l}
\int_{\Gamma}\left(-\partial_{t} v w+\mu \partial v \partial w+H(x, \partial v) w\right) d x=\int_{\Gamma} f w d x \quad \text { for all } w \in W, \text { a.a. } t \in(0, T),  \tag{4.2}\\
v(x, T)=v_{T}(x)
\end{array}\right.
$$

We start by proving existence and uniqueness of a weak solution for (4.1). Next, further regularity for the solution will be obtained under stronger assumptions.

### 4.1 Existence and uniqueness for the Hamilton-Jacobi equation

Theorem 4.2. Under the running assumptions (H), if $f \in L^{2}(\Gamma \times(0, T))$, then the boundary value problem (4.1) has a unique weak solution.

Uniqueness is a direct consequence of the following proposition.
Proposition 4.3. (Comparison principle) Under the same assumptions as in Theorem 4.2, let $v$ and $\hat{v}$ be respectively weak sub- and super-solution of (4.1), i.e., $v, \hat{v} \in L^{2}\left(0, T ; H^{2}(\Gamma)\right), \partial_{t} v, \partial_{t} \hat{v} \in$ $L^{2}(\Gamma \times(0, T))$ such that

$$
\left\{\begin{array}{l}
\int_{\Gamma}\left(-\partial_{t} v w+\mu \partial v \partial w+H(x, \partial v) w\right) d x \leqslant \int_{\Gamma} f w d x, \\
\int_{\Gamma}\left(-\partial_{t} \hat{v} w+\mu \partial \hat{v} \partial w+H(x, \partial \hat{v}) w\right) d x \geqslant \int_{\Gamma} f w d x, \\
v(x, T) \leqslant v_{T}(x) \leqslant \hat{v}(x, T) \quad \text { for a.a. } x \in \Gamma .
\end{array}\right.
$$

Then $v \leqslant \hat{v}$ in $\Gamma \times(0, T)$.
Proof of Proposition 4.3. Setting $\bar{v}=v-\hat{v}$, we have, for all $w \in W$ such that $w \geqslant 0$ and for a.a $t \in(0, T)$ :

$$
\int_{\Gamma}-\partial_{t} \bar{v} w+\mu \partial \bar{v} \partial w+(H(x, \partial v)-H(x, \partial \hat{v})) w d x \leqslant 0
$$

and $\bar{v}(x, T) \leqslant 0$ for all $x \in \Gamma$. Set $\bar{v}^{+}=\bar{v} \mathbb{1}_{\{\bar{v}>0\}}$ and $w=\bar{v}^{+} e^{\lambda t} \varphi$. We have

$$
\begin{aligned}
& -\int_{\Gamma} \partial_{t}\left(\frac{\left(\bar{v}^{+}\right)^{2}}{2} e^{\lambda t}\right) \varphi d x+\int_{\Gamma} \frac{\lambda}{2}\left(\bar{v}^{+}\right)^{2} e^{\lambda t} \varphi d x+\int_{\Gamma} \mu \partial \bar{v}^{+} \partial\left(\bar{v}^{+} \varphi\right) e^{\lambda t} d x \\
& +\int_{\Gamma}[H(x, \partial v)-H(x, \partial \hat{v})] \bar{v}^{+} \varphi e^{\lambda t} d x=0 .
\end{aligned}
$$

Integrating from 0 to $T$, we get

$$
\begin{aligned}
& \int_{\Gamma}\left(\frac{\bar{v}^{+}(0)^{2}}{2}-\frac{\bar{v}^{+}(T)^{2}}{2} e^{\lambda T}\right) \varphi d x+\int_{0}^{T} \int_{\Gamma} \frac{\lambda}{2}\left(\bar{v}^{+}\right)^{2} e^{\lambda t} \varphi d x d t \\
& +\int_{0}^{T} \int_{\Gamma} \mu\left(\partial \bar{v}^{+}\right)^{2} \varphi e^{\lambda t} d x d t+\int_{0}^{T} \int_{\Gamma} \mu \partial \bar{v}^{+} \bar{v}^{+} \partial \varphi e^{\lambda t} d x d t \\
& +\int_{0}^{T} \int_{\Gamma}[H(x, \partial v)-H(x, \partial \hat{v})] \bar{v}^{+} \varphi e^{\lambda t} d x d t=0 .
\end{aligned}
$$

From (1.32), $|H(x, \partial v)-H(x, \partial \hat{v})| \leqslant C_{0}|\partial \bar{v}|$. Hence, since $\bar{v}^{+}(T)=0$ and $|\partial \bar{v}| \bar{v}^{+}=\left|\partial \bar{v}^{+}\right| \bar{v}^{+}$ almost everywhere, we get

$$
\begin{equation*}
\int_{0}^{T} \int_{\Gamma}\left(\frac{\lambda}{2}\left(\bar{v}^{+}\right)^{2}+\mu\left(\partial \bar{v}^{+}\right)^{2}\right) e^{\lambda t} \varphi d x d t-\int_{0}^{T} \int_{\Gamma}\left(\mu|\partial \varphi|+C_{0} \varphi\right)\left|\partial \bar{v}^{+}\right| \bar{v}^{+} e^{\lambda t} d x d t \leqslant 0 \tag{4.3}
\end{equation*}
$$

For $\lambda$ large enough, the first term in the left hand side is not smaller than the second term. This implies that $\bar{v}^{+}=0$.

Now we prove Theorem 4.2. We start with a bounded Hamiltonian $H$.

Proof of existence in Theorem 4.2 when $H$ is bounded by $C_{H}$. Take $\bar{v} \in L^{2}(0, T ; V)$ and $f \in L^{2}(\Gamma \times(0, T))$. From Theorem 2.4 and Theorem 2.5 with $h=f-H(x, \partial \bar{v})$ and $v_{T} \in V$, the following boundary value problem

$$
\begin{cases}-\partial_{t} v-\mu_{\alpha} \partial^{2} v=f-H(x, \partial \bar{v}), & \text { in }\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T), \alpha \in \mathcal{A},  \tag{4.4}\\ \left.v\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=\left.v\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right), & t \in(0, T), \alpha, \beta \in \mathcal{A}_{i}, \nu_{i} \in \mathcal{A}_{i}, \\ \sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha} \partial_{\alpha} v\left(\nu_{i}, t\right)=0, & t \in \times(0, T), \nu_{i} \in \mathcal{V}, \\ v(x, T)=v_{T}(x), & x \in \Gamma,\end{cases}
$$

has a unique weak solution $v \in L^{2}\left(0, T ; H^{2}(\Gamma)\right) \cap C([0, T] ; V) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right)$. This allows us to define the map $T$ :

$$
\begin{aligned}
T: L^{2}(0, T ; V) & \longrightarrow L^{2}(0, T ; V), \\
\bar{v} & \longmapsto v .
\end{aligned}
$$

From (1.32), $\bar{v} \longmapsto H(x, \partial \bar{v})$ is continuous from $L^{2}(0, T ; V)$ into $L^{2}(\Gamma \times(0, T))$. Using again Theorem 2.5, we have that $T$ is continuous from $L^{2}(0, T ; V)$ to $L^{2}(0, T ; V)$. Moreover, there exists a constant $C$ depending only on $C_{H}, \Gamma,\left(\mu_{\alpha}\right)_{\alpha \in \mathcal{A}}, f, T, \varphi$ and $v_{T}$ such that

$$
\begin{equation*}
\left\|\partial_{t} v\right\|_{L^{2}(\Gamma \times(0, T))}+\|v\|_{L^{2}\left(0, T ; H^{2}(\Gamma)\right)} \leqslant C . \tag{4.5}
\end{equation*}
$$

Therefore, from Aubin-Lions theorem (see Lemma A.1), we obtain that $T\left(L^{2}(0, T ; V)\right.$ ) is relatively compact in $L^{2}(0, T ; V)$. By Schauder fixed point theorem, see [10, Corollary 11.2], $T$ admits a fixed point which is a weak solution of (4.1).

Proof of existence in Theorem 4.2 in the general case. Now we truncate the Hamiltonian as follows

$$
H_{n}(x, p)= \begin{cases}H(x, p) & \text { if }|p| \leqslant n \\ H\left(x, \frac{p}{|p|} n\right) & \text { if }|p|>n\end{cases}
$$

From the previous proof for bounded Hamiltonians, for all $n$, there exists a solution $v_{n} \in$ $L^{2}\left(0, T ; H^{2}(\Gamma)\right) \cap C([0, T] ; V) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right)$ of (4.1), where $H$ is replaced by $H_{n}$. We propose to send $n$ to $+\infty$ and to show a subsequence of $\left\{v_{n}\right\}$ converges to a solution of (4.1). Hence, we need some uniform estimates for $\left\{v_{n}\right\}$. As in the proof of Proposition 4.3, using $-v_{n} e^{\lambda t} \varphi$ as a test-function, integrating from 0 to $T$ and noticing that $H$ is sublinear, see (1.31), we obtain

$$
\begin{aligned}
& \int_{\Gamma}\left[\frac{v_{n}^{2}(x, 0)}{2}-\frac{v_{n}^{2}(x, T)}{2} e^{\lambda T}\right] \varphi d x+\int_{0}^{T} \int_{\Gamma}\left[\frac{\lambda}{2} v_{n}^{2} e^{\lambda t} \varphi+\mu\left|\partial v_{n}\right|^{2} e^{\lambda t} \varphi+\mu \partial v_{n} v_{n} e^{\lambda t} \partial \varphi\right] d x d t \\
= & -\int_{0}^{T} \int_{\Gamma} H_{n}\left(x, \partial v_{n}\right) v_{n} e^{\lambda t} \varphi d x d t+\int_{0}^{T} \int_{\Gamma} f v_{n} e^{\lambda t} \varphi d x d t \\
\leqslant & C_{0} \int_{0}^{T} \int_{\Gamma}\left(1+\left|\partial v_{n}\right|\right)\left|v_{n}\right| e^{\lambda t} \varphi d x d t+\frac{1}{2} \int_{0}^{T} \int_{\Gamma} f^{2} e^{\lambda t} \varphi d x d t+\frac{1}{2} \int_{0}^{T} \int_{\Gamma} v_{n}^{2} e^{\lambda t} \varphi d x d t .
\end{aligned}
$$

In the following lines, the constant $C$ above will vary from line to line and will depend only on $\left(\mu_{\alpha}\right)_{\alpha \in \mathcal{A}}, C_{H}, T$ and $\varphi$. Taking $\lambda$ large enough leads to the following estimate:

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{2}(0, T ; V)} \leqslant C\left(\|f\|_{L^{2}\left(0, T ; L^{2}(\Gamma)\right)}+\left\|v_{T}\right\|_{L^{2}(\Gamma)}+1\right) \tag{4.6}
\end{equation*}
$$

and thus, from (1.31) again, we also obtain

$$
\begin{aligned}
\int_{0}^{T} \int_{\Gamma}\left|H_{n}\left(x, \partial v_{n}\right)\right|^{2} d x d t & \leqslant \int_{0}^{T} \int_{\Gamma} C_{0}^{2}\left(\left|\partial v_{n}\right|+1\right)^{2} d x d t \leqslant \int_{0}^{T} \int_{\Gamma} 2 C_{0}^{2}\left(\left|\partial v_{n}\right|^{2}+1\right) d x d t \\
& \leqslant C\left(\|f\|_{L^{2}\left(0, T ; L^{2}(\Gamma)\right)}^{2}+\left\|v_{T}\right\|_{L^{2}(\Gamma)}^{2}+1\right)
\end{aligned}
$$

Therefore, $\left\{H_{n}\left(x, \partial v_{n}\right)-f\right\}$ is uniformly bounded in $L^{2}\left(0, T ; L^{2}(\Gamma)\right)$. From Theorem 2.5 , we obtain that $\left(v_{n}\right)_{n \in \mathbb{N}}$ is uniformly bounded in $L^{2}\left(0, T ; H^{2}(\Gamma)\right) \cap C([0, T] ; V) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right)$. By the Aubin-Lions theorem (see Lemma A.1), $\left(v_{n}\right)_{n}$ is relatively compact in $L^{2}(0, T ; V)$ (and bounded in $C([0, T] ; V))$. Hence, up to the extraction of a subsequence, there exists $v \in$ $L^{2}(0, T ; V) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right)$ such that

$$
\begin{equation*}
v_{n} \rightarrow v, \quad \text { in } L^{2}(0, T ; V) \quad(\text { strongly }), \quad \partial_{t} v_{n} \rightharpoonup \partial_{t} v, \quad \text { in } L^{2}(\Gamma \times(0, T)) \quad \text { (weakly). } \tag{4.7}
\end{equation*}
$$

Hence, $H_{n}\left(x, \partial v_{n}\right) \rightarrow H(x, \partial v)$ a.e. in $\Gamma \times(0, T)$. Note also that we can apply Lebesgue dominated convergence theorem to $H_{n}\left(x, \partial v_{n}\right)$ because $H_{n}\left(x, \partial v_{n}\right) \leqslant H\left(x, \partial v_{n}\right) \leqslant C_{0}\left(1+\left|\partial v_{n}\right|\right)$. Therefore, $H_{n}\left(x, \partial v_{n}\right) \rightarrow H(x, \partial v)$ in $L^{2}(\Gamma \times(0, T))$. Thus, it is possible to pass to the limit in the weak formulation satisfied by $v_{n}$ and obtain that for all $w \in W, \chi \in C_{c}(0, T)$,

$$
\int_{0}^{T} \chi(t)\left(-\int_{\Gamma} \partial_{t} v w d x+\int_{\Gamma} \partial v \partial w d x+\int_{\Gamma} H(x, \partial v) w d x\right) d t=\int_{0}^{T} \chi(t)\left(\int_{\Gamma} f w d x\right) d t
$$

Therefore, $v$ satisfies the first line in (4.2).
From Theorem 2.4, $v_{n}(T)=v_{T}$ for all $n$. Since for all $\alpha \in \mathcal{A},\left(v_{n}\right)_{n}$ tends to $v$ in $L^{2}\left(\Gamma_{\alpha} \times\right.$ $(0, T))$ strongly and in $W^{1,2}\left(\Gamma_{\alpha} \times(0, T)\right)$ weakly, $\left.v_{n}\right|_{\Gamma_{\alpha} \times\{t=T\}}$ converges to $\left.v\right|_{\Gamma_{\alpha} \times\{t=T\}}$ in $L^{2}\left(\Gamma_{\alpha}\right)$ strongly. Passing to the limit in the latter identity, we get the second condition in (4.2). We have proven that $v$ is a weak solution of (4.1).

We end the section with a stability result for the Hamilton-Jacobi equation.
Lemma 4.4. Let $\left(v_{T \varepsilon}\right)_{\varepsilon},\left(f_{\varepsilon}\right)_{\varepsilon}$ be sequences of functions satisfying

$$
v_{T \varepsilon} \longrightarrow v_{T} \text { in } V, \quad f_{\varepsilon} \longrightarrow f \text { in } L^{2}(\Gamma \times(0, T))
$$

Let $v_{\varepsilon}$ be the weak solution of (4.1) with data $v_{T \varepsilon}, f_{\varepsilon}$, then $\left(v_{\varepsilon}\right)_{\varepsilon}$ converges in $L^{2}\left(0, T ; H^{2}(\Gamma)\right) \cap$ $C([0, T] ; V) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right)$ to the weak solution $v$ of (4.1) with data $v_{T}, f$.

Proof of Lemma 4.4. Subtracting the two PDEs for $v_{\varepsilon}$ and $v$, multiplying by $\left(v_{\varepsilon}-v\right) e^{\lambda t} \varphi^{-1}$, taking the integral on $\Gamma \times(0, T)$ and using similar computations as in the proof of Proposition 4.3, we obtain

$$
\left\|v_{\varepsilon}-v\right\|_{L^{2}(0, T ; V)} \leqslant C\left(\left\|f_{\varepsilon}-f\right\|_{L^{2}(\Gamma \times(0, T))}+\left\|v_{T \varepsilon}-v_{T}\right\|_{L^{2}(\Gamma)}\right)
$$

for $\lambda$ large enough and $C$ independent of $\varepsilon$. This proves the convergence of $v_{\epsilon}$ to $v$ in $L^{2}(0, T ; V)$. Then, the convergence in $L^{2}\left(0, T ; H^{2}(\Gamma)\right) \cap C([0, T] ; V) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right)$ results from the assumption that $H$ is Lipschitz with respect to its second argument, and from stability results for the linear boundary value problem (2.1) which are obtained with similar arguments as in the proof of Theorem 2.5.

### 4.2 Regularity for the Hamilton-Jacobi equation

In this section, we prove further regularity for the solution of (4.1).
Theorem 4.5. We suppose that the assumptions of Theorem 4.2 hold and that, in addition, $v_{T} \in H^{2}(\Gamma)$ satisfies the Kirchhoff conditions given by the third equation in (4.1), $f \in P C(\Gamma \times$ $[0, T]) \cap L^{2}\left(0, T ; H_{b}^{1}(\Omega)\right)$ and $\partial_{t} f \in L^{2}\left(0, T ; H_{b}^{1}(\Gamma)\right)$.

Then, the unique solution $v$ of (4.1) satisfies $v \in L^{2}\left(0, T ; H^{3}(\Gamma)\right)$ and $\partial_{t} v \in L^{2}\left(0, T ; H^{1}(\Gamma)\right)$. Moreover, there exists a constant $C$ depending only on $\left\|v_{T}\right\|_{H^{2}(\Gamma)},\left(\mu_{\alpha}\right)_{\alpha \in \mathcal{A}}, H$ and $f$ such that

$$
\begin{equation*}
\|v\|_{L^{2}\left(0, T ; H^{3}(\Gamma)\right)}+\left\|\partial_{t} v\right\|_{L^{2}\left(0, T ; H^{1}(\Gamma)\right)} \leqslant C . \tag{4.8}
\end{equation*}
$$

If, in addition, there exists $\eta \in(0,1)$ such that $v_{T} \in C^{2+\eta}(\Gamma)$ then there exists $\tau \in(0,1)$ such $v \in C^{2+\tau, 1+\frac{\tau}{2}}(\Gamma \times[0, T])$, and $v$ is a classical solution of (4.1).

The main idea to prove Theorem 4.5 is to differentiate (4.1) with respect to the space variable and to prove some regularity properties for the derived equation. Let us explain formally our method. Assuming the solution $v$ of (4.1) is in $C^{2,1}(\Gamma \times(0, T))$ and taking the space-derivative of (4.1) on $\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T)$, we have

$$
-\partial_{t} \partial v-\mu_{\alpha} \partial^{3} v+\partial(H(x, \partial v))=\partial f .
$$

Therefore, $u=\partial v$ satisfies the following PDE

$$
-\partial_{t} u-\mu_{\alpha} \partial^{2} u+\partial(H(x, u))=\partial f,
$$

with terminal condition $u(x, T)=\partial v_{T}(x)$. From the Kirchhoff conditions in (4.1) and Remark 1.1, we obtain a condition for $u$ of Dirichlet type, namely

$$
\left.\sum_{\alpha \in \mathcal{A}_{i}} \mu_{\alpha} \gamma_{i \alpha} n_{i \alpha} u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=0, \quad t \in(0, T), \nu_{i} \in \mathcal{V}
$$

Note that the latter condition is an homogeneous Dirichlet condition at the boundary vertices of $\Gamma$.

Now, by extending continuously the PDEs in (4.1) until the vertex $\nu_{i}$ in the branchs $\Gamma_{\alpha}$ and $\Gamma_{\beta}, \alpha, \beta \in \mathcal{A}_{i}$, and using the continuity condition in (4.1), one gets
$-\left.\mu_{\alpha} \partial^{2} v\right|_{\Gamma_{\alpha}}+H^{\alpha}\left(\nu_{i},\left.\partial v\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)-\left.f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=-\left.\mu_{\beta} \partial^{2} v\right|_{\Gamma_{\beta}}+H^{\beta}\left(\nu_{i},\left.\partial v\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)\right)-\left.f\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)$.
This gives a second transmission condition for $u$ at $\nu_{i} \in \mathcal{V} \backslash \partial \Gamma$ of Robin type, namely

$$
\begin{align*}
& \left.\mu_{\alpha} \partial u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)-H^{\alpha}\left(\nu_{i},\left.u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)+\left.f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) \\
= & \left.\mu_{\beta} \partial u\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)-H^{\beta}\left(\nu_{i},\left.u\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)\right)+\left.f\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right), \tag{4.9}
\end{align*}
$$

which is equivalent to

$$
\begin{align*}
& \mu_{\alpha} n_{i \alpha} \partial_{\alpha} u\left(\nu_{i}, t\right)-H^{\alpha}\left(\nu_{i},\left.u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)+\left.f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)  \tag{4.10}\\
= & \mu_{\beta} n_{i \beta} \partial_{\beta} u\left(\nu_{i}, t\right)-H^{\beta}\left(\nu_{i},\left.u\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)\right)+\left.f\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right) .
\end{align*}
$$

Hence, we shall study the following nonlinear boundary value problem for $u=\partial v$,

$$
\begin{cases}-\partial_{t} u-\mu_{\alpha} \partial^{2} u+\partial(H(x, u))=\partial f(x, t), & (x, t) \in\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T), \alpha \in \mathcal{A},  \tag{4.11}\\ \left.\sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha} n_{i \alpha} u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=0, & t \in(0, T), \nu_{i} \in \mathcal{V}, \\ \mu_{\alpha} n_{i \alpha} \partial_{\alpha} u\left(\nu_{i}, t\right)-H^{\alpha}\left(\nu_{i},\left.u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)+\left.f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) & \\ \quad=\mu_{\beta} n_{i \beta} \partial_{\beta} u\left(\nu_{i}, t\right)-H^{\beta}\left(\nu_{i},\left.u\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)\right)+\left.f\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right), & t \in(0, T), \alpha, \beta \in \mathcal{A}_{i}, \nu_{i} \in \mathcal{V} \backslash \partial \Gamma, \\ u(x, T)=u_{T}(x), & x \in \Gamma,\end{cases}
$$

where $\partial f \in L^{2}(\Gamma \times(0, T))$ and $u_{T} \in F$ defined in (4.12) below. Theorem 4.5 will follow by choosing $u_{T}=\partial v_{T}$.

In order to define the weak solutions of (4.11), we need the following subspaces of $H_{b}^{1}(\Gamma)$.
Definition 4.6. We define the Sobolev spaces

$$
\begin{align*}
& F:=\left\{u \in H_{b}^{1}(\Gamma) \text { and }\left.\sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha} n_{i \alpha} u\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)=0 \text { for all } \nu_{i} \in \mathcal{V}\right\},  \tag{4.12}\\
& E:=\left\{\mathrm{e} \in H_{b}^{1}(\Gamma) \text { and }\left.\sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha} \mathrm{e}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)=0 \text { for all } \nu_{i} \in \mathcal{V}\right\} . \tag{4.13}
\end{align*}
$$

Definition 4.7. Let the function $\psi$ be defined as follows:

$$
\left\{\begin{array}{l}
\psi_{\alpha} \text { is affine on }\left(0, \ell_{\alpha}\right)  \tag{4.14}\\
\left.\psi\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)=\mu_{\alpha} \gamma_{i \alpha}, \text { if } \nu_{i} \in \mathcal{V} \backslash \partial \Gamma, \alpha \in \mathcal{A}_{i}, \\
\psi \text { is constant on the edges } \Gamma_{\alpha} \text { which touch the boundary of } \Gamma .
\end{array}\right.
$$

Note that $\psi$ is positive and bounded. The map $f \longmapsto f \psi$ is an isomorphism from $F$ onto $E$.
Definition 4.8. A weak solution of (4.11) is a function $u \in L^{2}(0, T ; F)$ such that $\partial_{t} u \in$ $L^{2}\left(0, T ; E^{\prime}\right)$, and
$\left\{\begin{array}{l}-\left\langle\partial_{t} u, \mathrm{e}\right\rangle_{E^{\prime}, E}+\int_{\Gamma}(\mu \partial u \partial \mathrm{e}-(H(x, u)) \partial \mathrm{e}) d x=-\int_{\Gamma} f \partial \mathrm{e} d x, \text { for all } e \in E, \text { a.a } t \in(0, T), \\ u(\cdot, T)=u_{T} .\end{array}\right.$
Remark 4.9. Note that if $u$ is regular enough, then (4.15) can also be written

$$
\begin{align*}
& -\left\langle\partial_{t} u, \mathrm{e}\right\rangle_{E^{\prime}, E}+\int_{\Gamma}(\mu \partial u \partial \mathrm{e}+\partial(H(x, u)) \mathrm{e}) d x-\left.\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha}\left[H^{\alpha}\left(\nu_{i},\left.u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)-\left.f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right] \mathrm{e}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right) \\
= & \int_{\Gamma}(\partial f) \mathrm{e} d x \quad \text { for all } e \in E, \text { a.a } t \in(0, T) . \tag{4.16}
\end{align*}
$$

Remark 4.10. To explain formally the definition of weak solutions, let us use $\mathrm{e} \in E$ as a testfunction in the PDE in (4.11). After an integration by parts, we get

$$
\int_{\Gamma}\left(-\partial_{t} u \mathrm{e}+\mu \partial u \partial \mathrm{e}+\partial(H(x, u)) \mathrm{e}\right) d x-\left.\left.\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha} \mu_{\alpha} \partial u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) \mathrm{e}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)=\int_{\Gamma}(\partial f) \mathrm{e} d x,
$$

where $n_{i \alpha}$ is defined in (1.6). On the one hand, from the second transmission condition, there exists a function $c_{i}:(0, T) \rightarrow \mathbb{R}$ such that $\left.\mu_{\alpha} \partial u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)-H^{\alpha}\left(\nu_{i},\left.u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)+\left.f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=c_{i}(t)$ for all $\alpha \in \mathcal{A}_{i}$. It follows that

$$
\begin{aligned}
& -\left.\left.\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha} \mu_{\alpha} \partial u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) \mathrm{e}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right) \\
= & -\left.\sum_{i \in I} c_{i}(t) \sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha} \mathrm{e}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)+\left.\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha}\left[-H^{\alpha}\left(\nu_{i},\left.u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)+\left.f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right] \mathrm{e}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right) \\
= & \left.\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha}\left[-H^{\alpha}\left(\nu_{i},\left.u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)+\left.f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right] \mathrm{e}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right),
\end{aligned}
$$

because $\mathrm{e} \in E$. Then we may use the Remark 4.9 and obtain (4.15).
We start by proving the following result about (4.11) and then give the proof of Theorem 4.5.
Theorem 4.11. Under the running assumptions, if $u_{T} \in F, f \in C(\Gamma \times[0, T]) \cap L^{2}\left(0, T ; H_{b}^{1}(\Gamma)\right.$ and $\partial_{t} f \in L^{2}\left(0, T ; H_{b}^{1}(\Gamma)\right)$, then (4.11) has a unique weak solution $u$. Moreover, there exists a constant $C$ depending only on $\Gamma, T, \psi,\left\|u_{T}\right\|_{F},\|\partial f\|_{L^{2}(\Gamma \times(0, T))},\|f\|_{C(\Gamma \times[0, T])}$ and $\left\|\partial_{t} f\right\|_{L^{2}\left(0, T ; H_{b}^{1}(\Gamma)\right)}$ such that

$$
\begin{equation*}
\|u\|_{L^{2}\left(0, T ; H_{b}^{2}(\Gamma)\right)}+\|u\|_{C([0, T] ; F)}+\left\|\partial_{t} u\right\|_{L^{2}(\Gamma \times(0, T))} \leqslant C . \tag{4.17}
\end{equation*}
$$

Remark 4.12. Theorem 4.11 implies that $u(\cdot, t) \in C^{1}\left(\Gamma_{\alpha}\right)$ for all $\alpha \in \mathcal{A}$ for a.e. $t$. Hence, the transmission conditions for $u$ hold in a classical sense for a.e. $t \in[0, T]$.

We use the Galerkin's method to construct solutions of certain finite-dimension approximations to (4.11).

We notice first that the symmetric bilinear form $\check{\mathscr{B}}(u, v):=\int_{\Gamma} \mu \psi^{-1} \partial u \partial v$ is such that $(u, v) \mapsto$ $(u, v)_{L^{2}(\Gamma)}+\check{\mathscr{B}}(u, v)$ is an inner product in $E$ equivalent to the standard inner product in $E$, namely $(u, v)_{E}=(u, v)_{L^{2}(\Gamma)}+\int_{\Gamma} \partial u \partial v$. Therefore, by standard Fredholm's theory, there exist

- a non decreasing sequence of nonnegative real numbers $\left(\lambda_{k}\right)_{k=1}^{\infty}$, that tends to $+\infty$ as $k \rightarrow \infty$
- A Hilbert basis $\left(\mathrm{e}_{k}\right)_{k=1}^{\infty}$ of $L^{2}(\Gamma)$, which is also a a total sequence of $E$ (and orthogonal if $E$ is endowed with the scalar product $\left.(u, v)_{L^{2}(\Gamma)}+\check{\mathscr{B}}(u, v)\right)$,
such that

$$
\begin{equation*}
\check{\mathscr{B}}\left(\mathrm{e}_{k}, e\right)=\lambda_{k}\left(\mathrm{e}_{k}, e\right)_{L^{2}(\Gamma)}, \quad \text { for all } e \in E . \tag{4.18}
\end{equation*}
$$

Note that

$$
\int_{\Gamma} \mu \partial \mathrm{e}_{k} \partial \mathrm{e}_{\ell} \psi^{-1} d x= \begin{cases}\lambda_{k} & \text { if } k=\ell \\ 0 & \text { if } k \neq \ell\end{cases}
$$

Note also that $\mathrm{e}_{k}$ is a weak solution of

$$
\begin{cases}-\mu_{\alpha} \partial\left(\psi^{-1} \partial \mathrm{e}_{k}\right)=\lambda_{k} \mathrm{e}_{k} & \text { in } \Gamma_{\alpha} \backslash \mathcal{V}, \alpha \in \mathcal{A},  \tag{4.19}\\ \frac{\partial_{\alpha} \mathrm{e}_{k}\left(\nu_{i}\right)}{\gamma_{i \alpha}}=\frac{\partial_{\beta} \mathrm{e}_{k}\left(\nu_{i}\right)}{\gamma_{i \beta}} & \text { for all } \alpha, \beta \in \mathcal{A}_{i}, \\ \left.\sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha} e_{k}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)=0 & \text { if } \nu_{i} \in \mathcal{V}\end{cases}
$$

which implies that $\left.\mathrm{e}_{k}\right|_{\Gamma_{\alpha}} \in C^{2}\left(\Gamma_{\alpha}\right)$ for all $\alpha \in \mathcal{A}$.
Finally, the sequence $\left(\mathrm{f}_{k}\right)_{k=1}^{\infty}$ given by $\mathrm{f}_{k}=\psi^{-1} \mathrm{e}_{k}$ is a total family in $F$ (but is not orthogonal).

Lemma 4.13. Under the assumptions made in Theorem 4.11, for any positive integer $n$, there exist $n$ absolutely continuous functions $y_{k}^{n}:[0, T] \rightarrow \mathbb{R}, k=1, \ldots, n$, and a function $u_{n}$ : $[0, T] \rightarrow L^{2}(\Gamma)$ of the form

$$
\begin{equation*}
u_{n}(t)=\sum_{k=1}^{n} y_{k}^{n}(t) \mathrm{f}_{k}, \tag{4.20}
\end{equation*}
$$

such that for all $k=1, \ldots, n$,

$$
\begin{equation*}
y_{k}^{n}(T)=\int_{\Gamma} u_{T} \mathrm{f}_{k} \psi^{2} d x \tag{4.21}
\end{equation*}
$$

and

$$
\begin{equation*}
-\frac{d}{d t}\left(u_{n}, \mathrm{f}_{k} \psi\right)_{L^{2}(\Gamma)}+\int_{\Gamma}\left(\mu \partial u_{n}-H\left(x, u_{n}\right)\right) \partial\left(\mathrm{f}_{k} \psi\right) d x=-\int_{\Gamma} f \partial\left(\mathrm{f}_{k} \psi\right) d x . \tag{4.22}
\end{equation*}
$$

Proof of Lemma 4.13. The proof follows the same lines as the one of Lemma 2.2 but it is more technical since we obtain a system of nonlinear differential equations. For $n \geqslant 1$, we consider the symmetric $n$ by $n$ matrix $M_{n}$ defined by

$$
\left(M_{n}\right)_{k \ell}=\int_{\Gamma} \mathrm{f}_{k} \mathrm{f}_{\ell} \psi d x
$$

Since $\psi$ is positive and bounded and since $\left(\psi \mathrm{f}_{k}\right)_{k=1}^{\infty}$ is a Hilbert basis of $L^{2}(\Gamma)$, we can check that $M_{n}$ is a positive definite matrix and there exist two constants $c, C$ independent of $n$ such that

$$
\begin{equation*}
c|\xi|^{2} \leqslant \sum_{k, \ell=1}^{n}\left(M_{n}\right)_{k \ell} \xi_{k} \xi_{\ell} \leqslant C|\xi|^{2}, \quad \text { for all } \xi \in \mathbb{R}^{n} \tag{4.23}
\end{equation*}
$$

Looking for $u_{n}$ of the form (4.20) and setting $Y=\left(y_{1}^{n}, \ldots, y_{n}^{n}\right)^{T}, \dot{Y}=\left(\frac{d}{d t} y_{1}^{n}, \ldots, \frac{d}{d t} y_{n}^{n}\right)^{T},(2.7)$ implies that we have to solve the following a system of ODEs:

$$
\left\{\begin{array}{l}
-M_{n} \dot{Y}(t)+B Y(t)+\mathcal{H}(Y)(t)=G(t),  \tag{4.24}\\
Y(T)=\left(\int_{\Gamma} u_{T} \mathrm{f}_{1} \psi^{2} d x, \cdots, \int_{\Gamma} u_{T} \mathrm{f}_{n} \psi^{2} d x\right)^{T}
\end{array}\right.
$$

where

- $\left.B_{k \ell}=\int_{\Gamma} \mu \partial \mathrm{f}_{\ell} \partial\left(\psi_{\mathrm{f}}\right) \mathrm{k}\right) d x$
- $\mathcal{H}_{i}(Y)=-\int_{\Gamma} H\left(x, Y^{T} F\right) \partial\left(\mathrm{f}_{i} \psi\right) d x$ with $F=\left(\mathrm{f}_{1}, \cdots, \mathrm{f}_{n}\right)^{T}$ and $Y^{T} F=\sum_{\ell} y_{\ell}^{n} \mathrm{f}_{\ell}=u_{n}$
- $G_{i}(t)=-\int_{\Gamma} f(x, t) \partial\left(\mathrm{f}_{i} \psi\right) d x$ for all $i \in 1, \cdots, n$.

Since the matrix $M$ is invertible and the function $\mathcal{H}$ is Lipschitz continuous by (1.32), the system (4.24) has a unique global solution. This ends the proof of the lemma.

We start by giving some estimates for the approximation $u_{n}$.
Lemma 4.14. Under the assumptions made in Theorem 4.11, there exists a constant $C$ depending only on $\Gamma, T, \psi,\left\|u_{T}\right\|_{F},\|\partial f\|_{L^{2}(\Gamma \times(0, T))}\|f\|_{C(\Gamma \times[0, T])}$ and $\left\|\partial_{t} f\right\|_{L^{2}\left(0, T ; H_{b}^{1}(\Gamma)\right)}$ such that

$$
\left\|u_{n}\right\|_{L^{\infty}(0, T ; F)}+\left\|u_{n}\right\|_{L^{2}\left(0, T ; H_{b}^{2}(\Gamma)\right)}+\left\|\partial_{t} u_{n}\right\|_{L^{2}(\Gamma \times(0, T))} \leqslant C .
$$

Proof of Lemma 4.14. We divide the proof into two steps:
Step 1: Uniform estimates of $u_{n}$ in $L^{\infty}\left(0, T ; L^{2}(\Gamma)\right), L^{2}(0, T ; F)$ and $W^{1,2}\left(0, T ; E^{\prime}\right)$. Multiplying (4.22) by $y_{k}^{n}(t) \mathrm{f}_{k} e^{\lambda t} \psi$ where $\lambda$ is a positive constant to be chosen later, summing for $k=1, \ldots, n$ and using (4.20), we get

$$
-\int_{\Gamma} \partial_{t} u_{n} u_{n} e^{\lambda t} \psi d x+\int_{\Gamma}\left(\mu \partial u_{n}-H\left(x, u_{n}\right)\right) \partial\left(u_{n} e^{\lambda t} \psi\right) d x=-\int_{\Gamma} f \partial\left(u_{n} \psi e^{\lambda t}\right) d x .
$$

In the following lines, $C$ will be a constant that may vary from lines to lines. Since $H$ satisfies (1.31) and $f$ is bounded, there exists a constant $C$ such that

$$
\begin{align*}
& -\int_{\Gamma}\left[\partial_{t}\left(\frac{u_{n}^{2}}{2} e^{\lambda t}\right)-\frac{\lambda}{2} u_{n}^{2} e^{\lambda t}\right] \psi d x+\int_{\Gamma} \mu\left|\partial u_{n}\right|^{2} e^{\lambda t} \psi d x-C \int_{\Gamma}\left|u_{n}\right|\left(\left|u_{n}\right|+\left|\partial u_{n}\right|\right) e^{\lambda t} d x \\
\leqslant & C \int_{\Gamma}\left(\left|u_{n}\right|+\left|\partial u_{n}\right|\right) e^{\lambda t} d x \tag{4.25}
\end{align*}
$$

The desired estimate on $u_{n}$ is obtained from the previous inequality in a similar way as in the proof of Lemma 2.3, by taking $\lambda$ large enough.

Step 2: Uniform estimates of $u_{n}$ in $L^{\infty}(0, T ; F) \cap L^{2}\left(0, T ; H_{b}^{2}(\Gamma)\right)$ and of $\partial_{t} u_{n}$ in $L^{2}(\Gamma \times$ $(0, T))$. Multiplying (4.22) by $\partial_{t} y_{k}^{n}(t) \mathrm{f}_{k} e^{\lambda t} \psi$ where $\lambda$ is a positive constant to be chosen later, integrating by part the term containing $H$ and $f$ (all the integration by parts are justified) summing for $k=1, \ldots, n$ and using (4.20), we obtain that

$$
\begin{align*}
& -\int_{\Gamma}\left(\partial_{t} u_{n}\right)^{2} e^{\lambda t} \psi d x+\int_{\Gamma} \mu \partial u_{n} \partial\left(\partial_{t} u_{n} e^{\lambda t} \psi\right) d x+\int_{\Gamma} \partial\left(H\left(x, u_{n}\right)\right) \partial_{t} u_{n} e^{\lambda t} \psi d x  \tag{4.26}\\
& -\left.\left.\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha}\left[H^{\alpha}\left(\nu_{i},\left.u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)-\left.f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right] \partial_{t} u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) \psi\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right) e^{\lambda t}=\int_{\Gamma} \partial f \partial_{t} u_{n} \psi e^{\lambda t} d x .
\end{align*}
$$

Note that from (1.32) and (1.33),

$$
\begin{equation*}
\left|\partial\left(H\left(x, u_{n}\right)\right)\right| \leqslant C_{0}\left(1+\left|u_{n}\right|+\left|\partial u_{n}\right|\right) \tag{4.27}
\end{equation*}
$$

so, from Step 1, this function is bounded in $L^{2}(\Gamma \times(0, T))$ by a constant. Moreover,

$$
\begin{equation*}
\int_{s}^{T} \int_{\Gamma} \partial f \partial_{t} u_{n} \psi e^{\lambda t} d x d t \leqslant C\left(\int_{s}^{T} \int_{\Gamma}(\partial f)^{2} e^{\lambda t} d x d t\right)^{\frac{1}{2}}\left(\int_{s}^{T} \int_{\Gamma}\left(\partial_{t} u_{n}\right)^{2} e^{\lambda t} \psi d x d t\right)^{\frac{1}{2}} \tag{4.28}
\end{equation*}
$$

and we can also estimate the term $\int_{\Gamma} \mu \partial u_{n} \partial\left(\partial_{t} u_{n} e^{\lambda t} \psi\right) d x$ as in the proof of Theorem 2.5. Therefore, the only new difficulty with respect to the proof of Theorem 2.5 consists of obtaining a bound for the term

$$
\left.\left.\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha}\left[H^{\alpha}\left(\nu_{i},\left.u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)-\left.f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right] \partial_{t} u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) e^{\lambda t} \psi\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right) .
$$

Let $\mathcal{J}_{i \alpha}(p)$ be the primitive function of $p \mapsto H^{\alpha}\left(\nu_{i}, p\right)$ such that $\mathcal{J}_{i \alpha}(0)=0$ :

$$
\left.H^{\alpha}\left(\nu_{i},\left.u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, s\right)\right) \partial_{t} u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, s\right)=\frac{d}{d t} \mathcal{J}_{i \alpha}\left(\left.u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, s\right)\right) .
$$

We can then write

$$
\begin{aligned}
& -\int_{s}^{T}\left(\left.\left.n_{i \alpha} H^{\alpha}\left(\nu_{i},\left.u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right) \partial_{t} u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) e^{\lambda t} \psi\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)\right) d t \\
= & \left.n_{i \alpha} \psi\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)\left(-\mathcal{J}_{i \alpha}\left(\left.u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, T\right)\right) e^{\lambda T}+\mathcal{J}_{i \alpha}\left(\left.u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, s\right)\right) e^{\lambda s}+\lambda \int_{s}^{T} \mathcal{J}_{i \alpha}\left(\left.u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right) e^{\lambda t} d t\right) .
\end{aligned}
$$

Since $H^{\alpha}(x, \cdot)$ is sublinear, see $(1.31),\left|\mathcal{J}_{i \alpha}(p)\right|$ is subquadratic, i.e., $\left|\mathcal{J}_{i \alpha}(p)\right| \leqslant C\left(1+p^{2}\right)$, for a constant $C$ independent of $\alpha$ and $i$. This implies that

$$
\begin{aligned}
& \left|\int_{s}^{T}\left(\left.\left.n_{i \alpha} H^{\alpha}\left(\nu_{i},\left.u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right) \partial_{t} u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) e^{\lambda t} \psi\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)\right) d t\right| \\
\leqslant & C\left(e^{\lambda T}+\left.u_{n}^{2}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, T\right) e^{\lambda T}+\left.u_{n}^{2}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, s\right) e^{\lambda s}\right)+C \lambda \int_{0}^{T}\left(1+\left.u_{n}^{2}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right) e^{\lambda t} d t
\end{aligned}
$$

Note that, from Step 1 and the stability of the trace, $\lambda \int_{s}^{T}\left(1+\left.u_{n}^{2}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right) e^{\lambda t} d t \leqslant C \lambda e^{\lambda T}$. To summarize

$$
\begin{align*}
& \left|\int_{s}^{T}\left(\left.\left.n_{i \alpha} H^{\alpha}\left(\nu_{i},\left.u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right) \partial_{t} u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) e^{\lambda t} \psi\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)\right) d t\right|  \tag{4.29}\\
\leqslant & C\left(\left.u_{n}^{2}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, T\right) e^{\lambda T}+\left.u_{n}^{2}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, s\right)^{2} e^{\lambda s}\right)+\tilde{C}(\lambda)
\end{align*}
$$

Similarly, using the fact that $f \in C(\Gamma \times[0, T])$ and $\left.\partial_{t} f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, \cdot\right) \in L^{2}(0, T)$, and integrating by part, we see that

$$
\begin{aligned}
& \left.\left|\int_{s}^{T} f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) \partial_{t} u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) e^{\lambda t} d t \mid \\
= & \left|\left(\left.f\right|_{\Gamma_{\alpha}} u_{n}\right)\right|_{\Gamma_{\alpha}}\left(\nu_{i}, T\right) e^{\lambda T}-\left.\left(\left.f\right|_{\Gamma_{\alpha}} u_{n}\right)\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) e^{\lambda s}-\left.\int_{s}^{T}\left(\left.\lambda f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)+\left.\partial_{t} f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right) u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) e^{\lambda t} d t \mid \\
\leqslant & C\left(\left|u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, T\right)\left|e^{\lambda T}+\left|u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, s\right)\right| e^{\lambda s}+\lambda \int_{s}^{T}\left|u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) \mid e^{\lambda t} d t\right) \\
& +\left.\frac{1}{2} \int_{s}^{T} u_{n}^{2}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) e^{\lambda t} d t+\frac{1}{2} \int_{s}^{T}\left(\left.\partial_{t} f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)^{2} e^{\lambda t} d t
\end{aligned}
$$

From Step 1 and the assumptions on $f$, the last three terms in the right hand side of the latter estimate are bounded by a constant depending on $\lambda$, but not on $n$. To summarize,

$$
\begin{equation*}
\left.\left|\int_{s}^{T} f\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) \partial_{t} u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) e^{\lambda t} d t \mid \leqslant C\left(\left|u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, T\right)\left|e^{\lambda T}+\left|u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, s\right)\right| e^{\lambda s}\right)+\tilde{C}(\lambda) \tag{4.30}
\end{equation*}
$$

To conclude from (4.29) and (4.30), we use the following estimates

$$
\left\{\begin{array}{l}
\left|u_{n}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right) \mid \leqslant C\left(\int_{\Gamma_{\alpha}}\left|u_{n}(x, t)\right| d x+\int_{\Gamma_{\alpha}}\left|\partial u_{n}(x, t)\right| d x\right)  \tag{4.31}\\
u_{n}^{2} \mid \Gamma_{\alpha}\left(\nu_{i}, t\right) \leqslant C\left(\int_{\Gamma_{\alpha}} u_{n}^{2}(x, t) d x+\int_{\Gamma_{\alpha}}\left|u_{n} \partial u_{n}(x, t)\right| d x\right)
\end{array}\right.
$$

for $t=s$ and $t=T$.
Then proceeding as in the proof of Theorem 2.5 and combining (4.26), (4.27), (4.28), (4.29) and (4.30) with (4.31), we find the desired estimates by taking $\lambda$ large enough.

Let us end the proof by proving (4.31). The function $\phi=\left.u_{n}\right|_{\Gamma_{\alpha}}(\cdot, t)$ is in $H^{1}\left(\Gamma_{\alpha}\right)$. By the continuous embedding $H^{1}\left(\Gamma_{\alpha}\right) \hookrightarrow C\left(\Gamma_{\alpha}\right)$, we can define $\phi$ in the pointwise sense (and even at two endpoints of any edges, see (1.2)). For all $\alpha \in \mathcal{A}$ and $x, y \in \Gamma_{\alpha}$, we have $\phi(x)=$ $\phi(y)+\int_{[y, x]} \partial \phi(\xi) d \xi$. It follows
$\left|\Gamma_{\alpha}\right| \phi(x)=\int_{\Gamma_{\alpha}} \phi(x) d y=\int_{\Gamma_{\alpha}} \phi(y) d y+\int_{\Gamma_{\alpha}} \int_{[y, x]} \partial \phi(\xi) d \xi d y \leqslant \int_{\Gamma_{\alpha}}|\phi(\xi)| d \xi+\left|\Gamma_{\alpha}\right| \int_{\Gamma_{\alpha}}|\partial \phi(\xi)| d \xi$,
which gives the first estimate setting $x=\nu_{i}$. The second estimate is obtained in the same way replacing $\phi$ by $\phi^{2}$ and using the fact that $W^{1,1}\left(\Gamma_{\alpha}\right) s s$ is continuously imbedded in $C\left(\Gamma_{\alpha}\right)$.

Proof of Theorem 4.11. From Lemma 4.14, up to the extraction of a subsequence, there exists $u \in L^{2}\left(0, T ; H_{b}^{2}(\Gamma)\right) \cap W^{1,2}(\Gamma \times(0, T))$ such that

$$
\begin{cases}u_{n} \rightharpoonup u, & \text { in } L^{2}\left(0, T ; F \cap H_{b}^{2}(\Gamma)\right),  \tag{4.32}\\ \partial_{t} u_{n} \rightharpoonup \partial_{t} u, & \text { in } L^{2}(\Gamma \times(0, T))\end{cases}
$$

Moreover, by Aubin-Lions Theorem (see Lemma A.1),

$$
L^{2}\left(0, T ; F \cap H_{b}^{2}(\Gamma)\right) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right) \stackrel{\text { compact }}{\hookrightarrow} L^{2}(0, T ; F),
$$

so up to the extraction of a subsequence, we may assume that $u_{n} \rightarrow u$ in $L^{2}(0, T ; F)$ and almost everywhere. Moreover, from the compactness of the trace operator from $W^{1,2}\left(\Gamma_{\alpha} \times(0, T)\right)$ to $L^{2}\left(\partial \Gamma_{\alpha} \times(0, T)\right),\left.\left.u_{n}\right|_{\partial \Gamma_{\alpha} \times(0, T)} \rightarrow u\right|_{\partial \Gamma_{\alpha} \times(0, T)}$ in $L^{2}\left(\partial \Gamma_{\alpha} \times(0, T)\right)$ and for almost every $t \in(0, T)$. Similarly, $\left.\left.u_{n}\right|_{\Gamma_{\alpha} \times\{t=T\}} \rightarrow u\right|_{\Gamma_{\alpha} \times\{t=T\}}$ in $L^{2}\left(\Gamma_{\alpha}\right)$ and almost everywhere in $\Gamma_{\alpha}$. Then, using the Lipschitz continuity of $H$ with respect to its second argument, and similar arguments as in the proof of Theorem 2.4, we obtain the existence of a solution of (4.11) satisfying (4.17) by letting $n \rightarrow+\infty$. Since $H^{2}\left(\Gamma_{\alpha}\right) \subset C^{1+\sigma}\left(\Gamma_{\alpha}\right)$ for some $\sigma \in(0,1 / 2), u(\cdot, t) \in C^{1+\sigma}\left(\Gamma_{\alpha}\right)$ for all $\alpha \in \mathcal{A}$ and a.a. $t$.

Finally, the proof of uniqueness is a consequence of the energy estimate (4.17) for $u$.
Next, we want to prove that, if $u$ is the solution of (4.11) and $v$ is the solution (4.1), then $\partial u=v$. It means that we have to define a primitive function on the network $\Gamma$.

Definition 4.15. Let $x \in \Gamma_{\alpha_{0}}=\left[\nu_{i_{0}}, \nu_{i_{1}}\right]$ and $y \in \Gamma_{\alpha_{m}}=\left[\nu_{i_{m}}, \nu_{i_{m+1}}\right]$. We denote the set of paths joining from $x$ to $y$ by $\overrightarrow{x y}$. More precisely, if $\mathcal{L} \in \overrightarrow{x y}$, we can write $\mathcal{L}$ under the form

$$
\mathcal{L}=x \rightarrow \nu_{i_{1}} \rightarrow \nu_{i_{2}} \rightarrow \ldots \rightarrow \nu_{i_{m}} \rightarrow y,
$$

with $\nu_{i_{k}} \in \mathcal{V}$ and $\left[\nu_{i_{k}}, \nu_{i_{k+1}}\right]=\Gamma_{\alpha_{k}}$. The integral of a function $\phi$ on $\mathcal{L}$ is defined by

$$
\begin{equation*}
\int_{\mathcal{L}} \phi(\xi) d \xi=\int_{\left[x, \nu_{i_{1}}\right]} \phi(\xi) d \xi+\sum_{k=1}^{m} \int_{\left[\nu_{i_{k}}, \nu_{i_{k+1}}\right]} \phi(\xi) d \xi+\int_{\left[\nu_{i_{m}}, y\right]} \phi(\xi) d \xi \tag{4.33}
\end{equation*}
$$

recalling that the integrals on a segment are defined in (1.8).
Lemma 4.16. Let $u$ be the unique solution of (4.11) with $u_{T}=\partial v_{T}$. Then for all $x, y \in \Gamma$ and a.e. $t \in[0, T]$,

$$
\int_{\mathcal{L}_{1}} u(\zeta, t) d \zeta=\int_{\mathcal{L}_{2}} u(\zeta, t) d \zeta, \quad \text { for all } \mathcal{L}_{1}, \mathcal{L}_{2} \in \overrightarrow{x y}
$$

This means that the integral of $u$ from $x$ to $y$ does not depend on the path. Hence, for any $\mathcal{L} \in \overrightarrow{x y}$, we can define

$$
\int_{\overrightarrow{x y}} u(\zeta, t) d \zeta:=\int_{\mathcal{L}} u(\zeta, t) d \zeta .
$$

Proof of Lemma 4.16. First, it is sufficient to prove $\int_{\mathcal{L}} u(\zeta, t) d \zeta=0$ for all $\mathcal{L} \in \overrightarrow{x x}$. Secondly, if a given edge is browsed twice in opposite senses, the two related contributions to the integral sum to zero. It follows that, without loss of generality, we only need to consider loops in $\overrightarrow{x x}$ such that all the complete edges that it contains are browsed once only. It is also easy to see that we can focus on the case when $x \in \mathcal{V}$. To summarize, we only need to prove that

$$
\int_{\mathcal{L}} u(\zeta, t) d \zeta=0
$$

when $\nu_{i_{0}} \in \mathcal{V} \backslash \partial \Gamma$ and $\mathcal{L}=\nu_{i_{0}} \rightarrow \nu_{i_{1}} \rightarrow \ldots \rightarrow \nu_{i_{m}} \rightarrow \nu_{i_{0}}$, where $\nu_{i_{k}} \neq \nu_{i_{\ell}}$ for $k \neq l$.
The following conditions

1. e $\left.\right|_{\Gamma_{\alpha}}=0$ on each edge $\Gamma_{\alpha}$ not contained in $\mathcal{L}$
2. for all $k=0, \ldots m-1$, $\left.\mathrm{e}\right|_{\Gamma_{\alpha_{k}}}=1_{i_{k}<i_{k+1}}-1_{i_{k}>i_{k+1}}$ if $\Gamma_{\alpha_{k}}$ is the edge joining $\nu_{i_{k}}$ and $\nu_{i_{k+1}}$
3. $\left.\mathrm{e}\right|_{\Gamma_{\alpha_{m}}}=1_{i_{m}<i_{0}}-1_{i_{m}>i_{0}}$ if $\Gamma_{\alpha_{m}}$ is the edge joining $\nu_{i_{m}}$ and $\nu_{i_{0}}$
define a unique function $e \in E$ which takes at most two values on $\mathcal{L}$, namely $\pm 1$.
From Definition 4.15, we have

$$
\begin{aligned}
\frac{d}{d t} \int_{\mathcal{L}} u(\zeta, t) d t & =\sum_{k=0}^{m} \frac{d}{d t} \int_{\left[\nu_{i_{k}}, \nu_{i_{k+1}}\right]} u(\zeta, t) d \zeta+\frac{d}{d t} \int_{\left[\nu_{i_{m}}, \nu_{i_{0}}\right]} u(\zeta, t) d \zeta \\
& =\frac{d}{d t} \int_{\Gamma} u(\zeta, t) \mathrm{e}(\zeta) d \zeta=\int_{\Gamma} \partial_{t} u(\zeta, t) \mathrm{e}(\zeta) d \zeta
\end{aligned}
$$

Then, using Definition 4.8, Remark 4.9 and Remark 4.10 yields that

$$
\begin{aligned}
& \frac{d}{d t} \int_{\mathcal{L}} u(\zeta, t) d \zeta \\
= & \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_{\alpha}}\left[-\mu_{\alpha} \partial^{2} u(\zeta, t)+\partial H(\zeta, u(\zeta, t))-\partial f(\zeta, t)\right] \mathrm{e}(\zeta) d \zeta \\
= & \sum_{k=0}^{m} \int_{\Gamma_{\alpha_{k}}}\left[-\mu_{\alpha_{k}} \partial^{2} u(\zeta, t)+\partial H(\zeta, u(\zeta, t))-\partial f(\zeta, t)\right] \mathrm{e}(\zeta) d \zeta \\
= & \left.\sum_{k=0}^{m} \mathrm{e}\right|_{\Gamma_{\alpha_{k}}}\left(\nu_{i}\right)\binom{n_{i_{k+1} \alpha_{k}}\left(-\left.\mu_{\alpha_{k}} \partial u\right|_{\Gamma_{\alpha_{k}}}\left(\nu_{i_{k+1}}, t\right)+H^{\alpha}\left(\nu_{i_{k+1}},\left.u\right|_{\Gamma_{\alpha_{k}}}\left(\nu_{i_{k+1}}, t\right)\right)-f\left(\nu_{i_{k+1}}, t\right)\right)}{+n_{i_{k} \alpha_{k}}\left(-\left.\mu_{\alpha_{k}} \partial u\right|_{\Gamma_{\alpha_{k}}}\left(\nu_{i_{k}}, t\right)+H^{\alpha}\left(\nu_{i_{k}},\left.u\right|_{\Gamma_{\alpha_{k}}}\left(\nu_{i_{k}}, t\right)\right)-f\left(\nu_{i_{k}}, t\right)\right)}
\end{aligned}
$$

where we have set $i_{m+1}=i_{0}$. Now using (4.9) (which is satisfied for a.e. $t$ from the regularity of $u$ ) and the fact that $\mathrm{e} \in E$, we conclude that

$$
\begin{equation*}
\frac{d}{d t} \int_{\mathcal{L}} u(\zeta, t) d \zeta=0 \tag{4.34}
\end{equation*}
$$

Hence

$$
\int_{\mathcal{L}} u(\zeta, t) d \zeta=\int_{\mathcal{L}} u(\zeta, T) d \zeta=\int_{\mathcal{L}} u_{T}(\zeta) d \zeta=\int_{\mathcal{L}} \partial v_{T}(\zeta) d \zeta=0
$$

where the last identity comes from the assumption that $v_{T} \in V$ (the continuity of $v_{T}$ ).

Lemma 4.17. If $u_{T}=\partial v_{T} \in F$, then the weak solution $u$ of (4.11) satisfies $u=\partial v$ where $v$ is the unique solution of (4.1).

Proof of Lemma 4.17. For simplicity, we write the proof in the case when $\partial \Gamma \neq \varnothing$. The proof is similar in the other case.

Let us fix some vertex $\nu_{k} \in \partial \Gamma$. From standard regularity results for Hamilton-Jacobi equation with homogeneous Neumann condition, we know that that there exists $\omega$, a closed neighborhood of $\left\{\nu_{k}\right\}$ in $\Gamma$ made of a single straigt line segment and containing no other vertices of $\Gamma$ than $\nu_{k}$, such that $\left.v\right|_{\omega \times(0, T)} \in L^{2}\left(0, T ; H^{3}(\omega)\right) \cap C\left([0, T] ; H^{2}(\omega) \cap W^{1,2}\left(0, T ; H^{1}(\omega)\right)\right.$. Hence, $v$ satisfies the Hamilton-Jacobi equation at almost every point of $\omega \times(0, T)$. Moreover the equation

$$
\begin{equation*}
\partial_{t} v\left(\nu_{k}, t\right)+\mu \partial^{2} v\left(\nu_{k}, t\right)-H\left(\nu_{k}, 0\right)+f\left(\nu_{k}, t\right)=0 \tag{4.35}
\end{equation*}
$$

holds for almost every $t \in(0, T)$ and in $L^{2}(0, T)$.
For every $x \in \Gamma$ and $t \in[0, T]$, we define

$$
\begin{equation*}
\hat{v}(x, t)=v\left(\nu_{k}, t\right)+\int_{\overrightarrow{\nu_{k} x}} u(\zeta, t) d \zeta . \tag{4.36}
\end{equation*}
$$

Remark 4.18. If $\partial \Gamma=\varnothing$, then the proof should be modified by replacing $\nu_{k}$ by a point $\nu \in \Gamma \backslash \mathcal{V}$ and by using local regularity results for the HJB equation in (4.1).

We claim that $\hat{v}$ is a solution of (4.1).
First, $\hat{v}(\cdot, t)$ is continuous on $\Gamma$. Indeed, $\hat{v}(y, t)-\hat{v}(x, t)=\int_{\overrightarrow{x y}} u(\zeta, t) d \zeta$. On the other hand, $u \in C([0, T] ; F) \subset L^{\infty}(\Gamma \times[0, T])$. It follows that $|\hat{v}(y, t)-\hat{v}(x, t)| \leqslant\|u\|_{L^{\infty}(\Gamma \times[0, T])} \operatorname{dist}(x, y)$ which implies that $\hat{v}(\cdot, t)$ is continuous on $\Gamma$.

Next, from the terminal conditions for $u$,

$$
\hat{v}(x, T)=v\left(\nu_{k}, T\right)+\int_{\overrightarrow{\nu_{k} x}} u(\zeta, T) d \zeta=v_{T}\left(\nu_{k}\right)+\int_{\overrightarrow{\nu_{k} x}} \partial v_{T}(\zeta) d \zeta=v_{T}(x),
$$

where the last identity follows from the continuity of $v_{T}$ on $\Gamma$.
Let us check the Kirchhoff condition for $\hat{v}$. Take $\nu_{i} \in \mathcal{V}$ and $\alpha \in \mathcal{A}_{i}$. From (1.7), for a.e. $t \in(0, T), \partial_{\alpha} \hat{v}\left(\nu_{i}, t\right)=\left.n_{i \alpha} \partial \hat{v}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)$ and from (4.36), $\left.\partial \hat{v}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=\left.u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)$. Since $u(\cdot, t) \in F$, we get

$$
\sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha} \partial_{\alpha} \hat{v}\left(\nu_{i}, t\right)=\left.\sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha} n_{i \alpha} u\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=0,
$$

which is exactly the Kirchhoff condition for $\hat{v}$ at $\nu_{i}$.
There remains to prove $\hat{v}$ solves the Hamilton-Jacobi equation in $\Gamma \backslash \mathcal{V}$ : Take $x \in \Gamma_{\alpha} \backslash \mathcal{V}$ for some $\alpha \in \mathcal{A}$ and consider a path $\overrightarrow{\nu_{k} \vec{x}} \ni \mathcal{L}=\nu_{i_{0}} \rightarrow \cdots \rightarrow \nu_{i_{m}} \rightarrow x$, where $i_{0}=k$ and $\nu_{i_{m}} \in \Gamma_{\alpha}$. Let $\nu_{i_{m+1}}$ be the other endpoint of $\Gamma_{\alpha}$. We proceed as in the proof of Lemma 4.16: the following conditions

1. e $\left.\right|_{\Gamma_{\alpha}}=0$ on each edge $\Gamma_{\alpha}$ not contained in $\mathcal{L}$
2. for all $j=0, \ldots m, \mathrm{e}_{\Gamma_{j}}=1_{i_{j}<i_{j+1}}-1_{i_{j}>i_{j+1}}$ if $\Gamma_{j}$ is the edge joining $\nu_{i_{j}}$ and $\nu_{i_{j+1}}$
define a unique piecewise constant function e which takes at most two values on $\mathcal{L}$, namely $\pm 1$. Note that e does not belong to $E$ because $\mathrm{e}\left(\nu_{k}\right) \neq 0$, but that e satisfies $\left.\sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha} \mathrm{e}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right)=$ 0 for all $\nu_{i} \in \mathcal{V} \backslash \partial \Gamma$.

Using this function, a similar computation as in the proof of Lemma 4.16 implies that, for almost every $t \in(0, T)$,

$$
\begin{aligned}
\partial_{t} \hat{v}(x, t)-\partial_{t} v\left(\nu_{k}, t\right)= & -\left.\mu_{\alpha} \partial u\right|_{\Gamma_{\alpha}}(x, t)+H\left(x,\left.u\right|_{\Gamma_{\alpha}}(x, t)\right)-f(x, t) \\
& +\mu_{\alpha} \partial_{2} v\left(\nu_{k}, t\right)-H\left(\nu_{k}, 0\right)+f\left(\nu_{k}, t\right) .
\end{aligned}
$$

Then, using (4.35) and the fact that $\partial \hat{v}=u$, the latter identity yields that for almost every $(x, t) \in(0, T) \times \Gamma$,

$$
\partial_{t} \hat{v}(x, t)+\mu_{\alpha} \partial^{2} \hat{v}(x, t)-H(x, \partial \hat{v}(x, t))+f(x, t)=0 .
$$

We have proven that $\hat{v}$ is a solution of (4.1). Since $v$ is the unique solution of (4.1), we conclude that $v=\hat{v}$ and $\partial v=u$.

We are now ready to give the proof of Theorem 4.5.
Proof of Theorem 4.5. Since $\partial v=u$ by Lemma 4.17 and $u$ satisfies (4.17) by Theorem 4.11, we obtain that $v \in L^{2}\left(0, T ; H^{3}(\Gamma)\right)$ and $\partial_{t} v \in L^{2}\left(0, T ; H^{1}(\Gamma)\right)$ and (4.8) holds.

Therefore, using an interpolation result combined with Sobolev embeddings, see [4] or Lemma A. 2 in the Appendix, $v \in C^{1+\sigma, \sigma / 2}(\Gamma \times[0, T])$ for some $0<\sigma<1$.

Finally, we know that since $f \in W^{1,2}\left(0, T, H_{b}^{1}(\Gamma)\right),\left.f\right|_{\Gamma_{\alpha} \times[0, T]} \in C^{\eta, \eta}\left(\Gamma_{\alpha} \times[0, T]\right)$ for all $\eta \in(0,1 / 2)$. If $f \in C^{\eta, \frac{\eta}{2}}\left(\Gamma_{\alpha} \times[0, T]\right)$ for some $\eta \in(0,1 / 2)$, we claim that $v \in C^{2,1}(\Gamma \times[0, T])$. This is a direct consequence of a theorem of Von Below, see the main theorem in [24], for the (modified) heat equation

$$
\begin{equation*}
-\partial_{t} w-\mu_{\alpha} \partial^{2} w=g(x, t) \quad \text { in }\left(\Gamma_{\alpha} \backslash \mathcal{V}\right) \times(0, T) \tag{4.37}
\end{equation*}
$$

with the same Kirchhoff conditions as in (4.1): Note that if the terminal Cauchy condition for $w$ is $w(\cdot, t=T)=v_{T}$ and if $g=f-H(x, \partial v)$, then $w=v$. Now $g=f-H(x, \partial v) \in$ $C^{\tau, \frac{\tau}{2}}\left(\Gamma_{\alpha} \times[0, T]\right)$, where $1 / 2>\tau=\min (\sigma, \eta)>0$. Using the result in [24], we obtain that $v=w \in C^{2+\tau, 1+\tau / 2}\left(\Gamma_{\alpha} \times[0, T]\right)$, then that $v$ is a classical solution of (4.1).

## 5 Existence, uniqueness and regularity for the MFG system (Proof of Theorem 1.11)

Proof of existence in Theorem 1.11. Given $m_{0}$ and $v_{T}$, let us construct the map $T: L^{2}(0, T ; V) \rightarrow$ $L^{2}(0, T ; V)$ as follows.

Given $v \in L^{2}(0, T ; V)$, we first define $m$ as the weak solution of (3.1) with initial data $m_{0}$ and $b=H_{p}(x, \partial v)$. We know that $m \in L^{2}(0, T ; W) \cap C\left([0, T] ; L^{2}(\Gamma)\right) \cap W^{1,2}\left(0, T ; V^{\prime}\right)$.

We claim that if $v_{n} \rightarrow v$ in $L^{2}(0, T ; V)$ then $H_{p}\left(\cdot, \partial v_{n}\right)$ tends to $H_{p}(\cdot, \partial v)$ in $L^{2}(\Gamma \times(0, T))$. To prove the claim, we argue by contradiction: assume that there exist a positive number $\epsilon$ and a subsequence $v_{\phi(n)}$ such that $\left\|H_{p}\left(\cdot, \partial v_{\phi(n)}\right)-H_{p}(\cdot, \partial v)\right\|_{L^{2}(\Gamma \times(0, T))}>\epsilon$. Then since $\partial v_{\phi(n)}$ tends to $\partial v$ in $L^{2}(\Gamma \times(0, T))$, we can extract another subsequence $v_{\psi(n)}$ from $v_{\phi(n)}$ such that $\partial v_{\psi(n)}$ tends to $\partial v$ almost every where in $\Gamma \times(0, T)$. From the continuity of $H_{p}$, we deduce that $H_{p}\left(\cdot, \partial v_{\psi(n)}\right)$ tends to $H_{p}(\cdot, \partial v)$ almost everywhere in $\Gamma \times(0, T)$. Since there exists a positive constant $C_{0}$ such that $\left\|H_{p}\left(\cdot, \partial v_{\psi(n)}\right)\right\|_{\infty} \leqslant C_{0},\left\|H_{p}(\cdot, \partial v)\right\|_{\infty} \leqslant C_{0}$, Lebesgue dominated convergence theorem ensures that $H_{p}\left(\cdot, \partial v_{\psi(n)}\right)$ tends to $H_{p}(\cdot, \partial v)$ in $L^{2}(\Gamma \times(0, T))$, which is the desired contradiction.

To summarize, $H_{p}\left(\cdot, \partial v_{n}\right)$ tends to $H_{p}(\cdot, \partial v)$ in $L^{2}(\Gamma \times(0, T))$ on the one hand, and for a positive constant $C_{0},\left\|H_{p}\left(\cdot, \partial v_{n}\right)\right\|_{\infty} \leqslant C_{0},\left\|H_{p}(\cdot, \partial v)\right\|_{\infty} \leqslant C_{0}$. Using Lemma 3.4, we see that $m_{n}$, the weak solution of (3.1) with initial data $m_{0}$ and $b=H_{p}\left(x, \partial v_{n}\right)$ converges to $m$ in $L^{2}(0, T ; W) \cap L^{\infty}\left(0, T ; L^{2}(\Gamma)\right) \cap W^{1,2}\left(0, T ; V^{\prime}\right)$. Hence, the map $v \mapsto m$ is continuous from $L^{2}(0, T ; V)$ to $L^{2}(0, T ; W) \cap L^{\infty}\left(0, T ; L^{2}(\Gamma)\right) \cap W^{1,2}\left(0, T ; V^{\prime}\right)$. Moreover, the a priori estimate (3.3) holds uniformly with respect to $v$.

Then, knowing $m$, we construct $T(v) \equiv \widetilde{v}$ as the unique weak solution of (4.1) with $f(x, t)=$ $\mathscr{V}[m(\cdot, t)](x)$. Note that $m \mapsto f$ is continuous and locally bounded from $L^{2}(\Gamma \times(0, T))$ to
$L^{2}(\Gamma \times(0, T))$. Then Lemma 4.4 ensures that the map $m \rightarrow \tilde{v}$ is continuous from $L^{2}(\Gamma \times(0, T))$ to $L^{2}\left(0, T ; H^{2}(\Gamma)\right) \cap L^{\infty}(0, T ; V) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right)$. From Aubin-Lions theorem, see Lemma A.1, $m \rightarrow \tilde{v}$ maps bounded sets of $L^{2}(\Gamma \times(0, T))$ to relatively compact sets of $L^{2}(0, T ; V)$.

Therefore, the map $T: v \mapsto \tilde{v}$ is continuous from $L^{2}(0, T ; V)$ to $L^{2}(0, T ; V)$ and has a relatively compact image. Finally, we apply Schauder fixed point theorem [10, Corollary 11.2] and conclude that the map $T$ admits a fixed point $v$. We know that $v \in L^{2}\left(0, T ; H^{2}(\Gamma)\right) \cap$ $L^{\infty}(0, T ; V) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right)$ and $m \in L^{2}(0, T ; W) \cap L^{\infty}\left(0, T ; L^{2}(\Gamma)\right) \cap W^{1,2}\left(0, T ; V^{\prime}(\Gamma)\right)$.

Hence, there exists a weak solution $(v, m)$ to the mean field games system (1.25).
Proof of uniqueness in Theorem 1.11. We assume that there exist two solutions ( $v_{1}, m_{1}$ ) and $\left(v_{2}, m_{2}\right)$ of (1.25). We set $\bar{v}=v_{1}-v_{2}$ and $\bar{m}=m_{1}-m_{2}$ and write the system for $\bar{v}, \bar{m}$

$$
\begin{cases}-\partial_{t} \bar{v}-\mu_{\alpha} \partial^{2} \bar{v}+H\left(x, \partial v_{1}\right)-H\left(x, \partial v_{2}\right)-\left(\mathscr{V}\left[m_{1}\right]-\mathscr{V}\left[m_{2}\right]\right)=0, & x \in \Gamma_{\alpha} \backslash \mathcal{V}, t \in(0, T), \\ \partial_{t} \bar{m}-\mu_{\alpha} \partial^{2} \bar{m}-\partial\left(m_{1} \partial_{p} H\left(x, \partial m_{1}\right)-m_{2} \partial_{p} H\left(x, \partial m_{2}\right)\right)=0 & x \in \Gamma_{\alpha} \backslash \mathcal{V}, t \in(0, T), \\ \left.\bar{v}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)=\left.\bar{v}\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right), \frac{\left.\bar{m}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)}{\gamma_{i \alpha}}=\frac{\left.\bar{m}\right|_{\Gamma_{\beta}}\left(\nu_{i}, t\right)}{\gamma_{i \beta}}, & \alpha, \beta \in \mathcal{A}_{i}, \nu_{i} \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_{i}} \gamma_{i \alpha} \mu_{\alpha} \partial_{\alpha} \bar{v}\left(\nu_{i}, t\right)=0, & \nu_{i} \in \mathcal{V}, t \in(0, T), \\ \sum_{\alpha \in \mathcal{A}_{i}} n_{i \alpha}\left[\left.m_{1}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right) \partial_{p} H^{\alpha}\left(\nu_{i},\left.\partial v_{1}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)-\left.m_{2}\right|_{\Gamma_{\alpha}}\left(\nu_{i}\right) \partial_{p} H^{\alpha}\left(\nu_{i},\left.\partial v_{2}\right|_{\Gamma_{\alpha}}\left(\nu_{i}, t\right)\right)\right] & \\ \quad+\sum_{\alpha \in \mathcal{A}_{i}} \mu_{\alpha} \partial_{\alpha} \bar{m}\left(\nu_{i}, t\right)=0, & \nu_{i} \in \mathcal{V}, t \in(0, T), \\ \bar{v}(x, T)=0, \bar{m}(x, 0)=0 . & \end{cases}
$$

Testing by $\bar{m}$ the boundary value problem satified by $\bar{u}$, testing by $\bar{u}$ the boundary value problem satified by $\bar{m}$, subtracting, we obtain

$$
\begin{aligned}
& \int_{0}^{T} \int_{\Gamma}\left(m_{1}-m_{2}\right)\left(\mathscr{V}\left[m_{1}\right]-\mathscr{V}\left[m_{2}\right)\right] d x d t+\int_{0}^{T} \int_{\Gamma} \partial_{t}(\bar{m} \bar{v}) d x d t \\
& +\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_{\alpha}} m_{1}\left[H\left(x, \partial v_{2}\right)-H\left(x, \partial v_{1}\right)-\partial_{p} H\left(x, \partial v_{1}\right) \partial \bar{v}\right] d x \\
& +\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_{\alpha}} m_{2}\left[H\left(x, \partial v_{1}\right)-H\left(x, \partial v_{2}\right)+\partial_{p} H\left(x, \partial v_{1}\right) \partial \bar{v}\right] d x=0 .
\end{aligned}
$$

Since $V$ is strictly monotone, the first sum is nonnegative. Moreover,

$$
\int_{0}^{T} \int_{\Gamma} \partial_{t}(\bar{m} \bar{v}) d x d t=\int_{\Gamma}[\bar{m}(x, T) \bar{v}(x, T)-\bar{m}(x, 0) \bar{v}(x, 0)] d x=0
$$

since $\bar{v}(x, T)=0$ and $\bar{m}(x, 0)=0$. From the convexity of $H$ and the fact that $m_{1}, m_{2}$ are nonnegative, the last two sums are nonnegative. Therefore, all the terms are zero and thanks again to the fact that $\mathscr{V}$ is strictly increasing, we obtain $m_{1}=m_{2}$. From Lemma 4.2, we finally obtain $v_{1}=v_{2}$.

Proof of regularity in Theorem 1.11. We make the stronger assumptions written in Section 1.4.4 on the coupling operator $\mathscr{V}$. We know that $\mathscr{V}[m] \in W^{1,2}\left(0, T, H_{b}^{1}(\Gamma)\right) \cap P C(\Gamma \times[0, T])$. Assuming also that $v_{T} \in V$ and $\partial v_{T} \in F$, we can apply the regularity result in Theorem 4.5: $v \in L^{2}\left(0, T ; H^{3}(\Gamma)\right) \cap W^{1,2}\left(0, T ; H^{1}(\Gamma)\right)$.

Moreover, since $\mathscr{V}[m] \in W^{1,2}\left(0, T, H_{b}^{1}(\Gamma)\right)$, we know that $\left.(\mathscr{V}[m])\right|_{\Gamma_{\alpha} \times[0, T]} \in C^{\sigma, \sigma / 2}\left(\Gamma_{\alpha} \times\right.$ $[0, T])$ for all $0<\sigma<1 / 2$. If $v_{T} \in C^{2+\eta} \cap D$ for some $\eta \in(0,1)$ ( $D$ is defined in (1.13)), then from Theorem 4.5, $v \in C^{2+\tau, 1+\tau / 2}(\Gamma \times[0, T])$ for some $\tau \in(0,1)$ and the boundary value problem for $v$ is satisfied in a classical sense.

In turn, if for all $\alpha \in \mathcal{A}, \partial p H^{\alpha}(x, p)$ is a Lipschitz function defined in $\Gamma_{\alpha} \times \mathbb{R}$, and if $m_{0} \in W$, then we can use the latter regularity of $v$ and arguments similar to those contained in the proof of Theorem 2.5 and prove that $m \in C([0, T] ; W) \cap W^{1,2}\left(0, T ; L^{2}(\Gamma)\right) \cap L^{2}\left(0, T ; H_{b}^{2}(\Gamma)\right)$.

## A Some continuous and compact embeddings

Lemma A.1. (Aubin-Lions Lemma, see [18]) Let $X_{0}, X$ and $X_{1}$ be function spaces, ( $X_{0}$ and $X_{1}$ are reflexive). Suppose that $X_{0}$ is compactly embedded in $X$ and that $X$ is continuously embedded in $X_{1}$. Consider some real numbers $1<p, q<+\infty$. Then the following set

$$
\left\{v:(0, T) \mapsto X_{0}: v \in L^{p}\left(0, T ; X_{0}\right), \partial_{t} v \in L^{q}\left(0, T ; X_{1}\right)\right\}
$$

is compactly embedded in $L^{p}(0, T ; X)$.
Lemma A.2. (Amann, see [4]) Let $\phi:[a, b] \times[0, T] \rightarrow \mathbb{R}$ such that $\phi \in L^{2}\left(0, T ; H^{2}(a, b)\right)$ and $\partial_{t} \phi \in L^{2}\left(0, T ; L^{2}(a, b)\right)$. Then $\phi \in C^{s}\left(0, T ; H^{1}(a, b)\right)$ for some $s \in(0,1 / 2)$.

This result is a consequence of the general result [4, Theorem 1.1] taking into account [4, Remark 7.4]. More precisely, we have

$$
E_{1}:=H^{2}(a, b) \xrightarrow{\text { compact }} E:=H^{1}(a, b) \hookrightarrow E_{0}:=L^{2}(a, b) .
$$

Let $r_{0}=r_{1}=r=2, \sigma_{0}=0, \sigma_{1}=2$ and $\sigma=1$. For any $\nu \in(0,1)$, we define

$$
\frac{1}{r_{\nu}}=\frac{1}{r_{0}}+\frac{1-\nu}{r_{1}}, \quad \sigma_{\nu}:=(1-\nu) s_{0}+\nu s_{1}
$$

This implies that $r_{\nu}=2$ and $\sigma_{\nu}=2 \nu$. Therefore, if $\nu \in(1 / 2,1)$, then the following inequality is satisfied

$$
\sigma-1 / r<\sigma_{\nu}-1 / r_{\nu}<\sigma_{1}-1 / r_{1}
$$

Hence, we infer from [4, Remark 7.4]

$$
E_{1} \hookrightarrow\left(E_{0}, E_{1}\right)_{\nu, 1} \hookrightarrow\left(E_{0} \cdot E_{1}\right)_{\nu, r_{\nu}}=W^{\sigma_{\nu}, r_{\nu}}(a, b) \hookrightarrow E,
$$

where $\left(E_{0}, E_{1}\right)_{\nu, 1},\left(E_{0} . E_{1}\right)_{\nu, r_{\nu}}$ are interpolation spaces. This is precisely the assumption allowing to apply [4, Theorem 1.1], which gives the result of Lemma A.2.

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