Finite Horizon Mean Field Games on Networks

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March 6, 2019

Abstract

We consider finite horizon stochastic mean field games in which the state space is a network. They are described by a system coupling a backward in time Hamilton-Jacobi-Bellman equation and a forward in time Fokker-Planck equation. The value function u is continuous and satisfies general Kirchhoff conditions at the vertices. The density m of the distribution of states satisfies dual transmission conditions: in particular, m is generally discontinuous across the vertices, and the values of m on each side of the vertices satisfy special compatibility conditions. The stress is put on the case when the Hamiltonian is Lipschitz continuous. Existence and uniqueness are proven.

1 Introduction and main results

This work is the continuation of [2] which was devoted to mean field games on networks in the case of an infinite time horizon. The topic of mean field games (MFGs for short) is more and more investigated since the pioneering works [14, 15, 16] of Lasry and Lions: it aims at studying the asymptotic behavior of stochastic differential games (Nash equilibria) as the number N of agents tends to infinity. We refer to [2] for a more extended discussion on MFGs and for additional references on the analysis of the system of PDEs that stem from the model when there is no common noise.

A network (or a graph) is a set of items, referred to as vertices (or nodes or crosspoints), with connections between them referred to as edges. In the recent years, there has been an increasing interest in the investigation of dynamical systems and differential equations on networks, in particular in connection with problems of data transmission and traffic management. The literature on optimal control in which the state variable takes its values on a network is recent: deterministic control problems and related Hamilton-Jacobi equations were studied in [1, 3, 11, 12, 20, 21]. Stochastic processes on networks and related Kirchhoff conditions at the vertices were studied in [8, 9].

The present work is devoted to finite horizon stochastic mean field games (MFGs) taking place on networks. The most important difficulty will be to deal with the transition conditions at the vertices. The latter are obtained from the theory of stochastic control in [9, 8], see Section 1.3 below. In [6], the first article on MFGs on networks, Camilli and Marchi consider a particular type of Kirchhoff condition at the vertices for the value function: this condition comes from an assumption which can be informally stated as follows: consider a vertex ν of the

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network and assume that it is the intersection of p edges $\Gamma_1, \dots, \Gamma_p$, ; if, at time τ, the controlled stochastic process X_t associated to a given agent hits ν, then the probability that X_{τ^+} belongs to Γ_i is proportional to the diffusion coefficient in Γ_i. Under this assumption, it can be seen that the density of the distribution of states is continuous at the vertices of the network. In the present work, the above mentioned assumption is not made any longer. Therefore, it will be seen below that the value function satisfies more general Kirchhoff conditions, and accordingly, that the density of the distribution of states is no longer continuous at the vertices; the continuity condition is then replaced by suitable compatibility conditions on the jumps across the vertices. A complete study of the system of differential equations arising in infinite horizon mean field games on networks with at most quadratic Hamiltonians and very general coupling costs has been supplied in a previous work, see [2].

In the present work, we focus on a more basic case, namely finite horizon MFG with globally Lipschitz Hamiltonian with rather strong assumptions on the coupling cost. This will allow us to concentrate on the difficulties induced by the Kirchhoff conditions. Therefore, this work should be seen as a first and necessary step in order to deal with more difficult situations, for example with quadratic or subquadratic Hamiltonians. We believe that treating such cases will be possible by combining the results contained in the present work with methods that can be found in [22, 23], see also [5, 13, 17] for references on Hamilton-Jacobi equations.

After obtaining the transmission conditions at the vertices for both the value function and the density, we shall prove existence and uniqueness of weak solutions of the uncoupled Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck (FP) equations (in suitable space-time Sobolev spaces), and regularity results.

The present work is organized as follows: the remainder of Section 1 is devoted to setting the problem and obtaining the system of partial differential equations and the transmission conditions at the vertices. Section 2 contains useful results on a modified heat equation in the network with general Kirchhoff conditions. Section 3 is devoted to the Fokker-Planck equation. Weak solutions are defined by using a special pair of Sobolev spaces of functions defined on the network referred to as V and W below. Section 4 is devoted to the HJB equation supplemented with the Kirchhoff conditions: it addresses the main difficulty of the work, consisting of obtaining regularity results for the weak solution (note that, to the best of our knowledge, such results for networks and general Kirchhoff conditions do not exist in the literature). Finally, the proofs of the main results of existence and uniqueness for the MFG system of partial differential equations are completed in Section 5.

1.1 Networks and function spaces

1.1.1 The geometry

A bounded network Γ (or a bounded connected graph) is a connected subset of $\mathbb{R}^n$ made of a finite number of bounded non-intersecting straight segments, referred to as edges, which connect nodes referred to as vertices. The finite collection of vertices and the finite set of closed edges are respectively denoted by $\mathcal{V} := \{\nu_i, i \in I\}$ and $\mathcal{E} := \{\Gamma_\alpha, \alpha \in \mathcal{A}\}$, where I and A are finite sets of indices contained in $\mathbb{N}$. We assume that for $\alpha, \beta \in \mathcal{A}$, if $\alpha \neq \beta$, then $\Gamma_\alpha \cap \Gamma_\beta$ is either empty or made of a single vertex. The length of Γ_α is denoted by ℓ_α. Given $\nu_i \in \mathcal{V}$, the set of indices of edges that are adjacent to the vertex ν_i is denoted by $\mathcal{A}_i = \{\alpha \in \mathcal{A} : \nu_i \in \Gamma_\alpha\}$. A vertex ν_i is named a *boundary vertex* if $\sharp(\mathcal{A}_i) = 1$, otherwise it is named a *transition vertex*. The set containing all the boundary vertices is named the *boundary* of the network and is denoted by $\partial\Gamma$ hereafter.

The edges $\Gamma_\alpha \in \mathcal{E}$ are oriented in an arbitrary manner. In most of what follows, we shall make the following arbitrary choice that an edge $\Gamma_\alpha \in \mathcal{E}$ connecting two vertices ν_i and ν_j, with $i < j$ is oriented from ν_i toward ν_j: this induces a natural parametrization $\pi_\alpha : [0, \ell_\alpha] \to \Gamma_\alpha = [\nu_i, \nu_j]$:

$$\pi_\alpha(y) = (\ell_\alpha - y)\nu_i + y\nu_j \quad \text{for } y \in [0, \ell_\alpha]. \tag{1.1}$$

For a function $v : \Gamma \to \mathbb{R}$ and $\alpha \in \mathcal{A}$, we define $v_\alpha : (0, \ell_\alpha) \to \mathbb{R}$ by

$$v_\alpha(y) := v \circ \pi_\alpha(y), \qquad \text{for all } y \in (0, \ell_\alpha).$$

The function v_α is a priori defined only in $(0, \ell_\alpha)$. When it is possible, we extend it by continuity at the boundary by setting

$$v_\alpha(0) := \lim_{y \to 0^+} v_\alpha(y) \text{ and } v_\alpha(\ell_\alpha) := \lim_{y \to \ell_\alpha^-} v_\alpha(y).$$

In that latter case, we can define

$$v|_{\Gamma_\alpha}(x) = \begin{cases} v_\alpha\left(\pi_\alpha^{-1}(x)\right), & \text{if } x \in \Gamma_\alpha \backslash \mathcal{V}, \\ v_\alpha(0) = \lim_{y \to 0^+} v_\alpha(y), & \text{if } x = \nu_i, \\ v_\alpha(\ell_\alpha) = \lim_{y \to \ell_\alpha^-} v_\alpha(y), & \text{if } x = \nu_j. \end{cases} \tag{1.2}$$

Notice that $v|_{\Gamma_\alpha}$ does not coincide with the original function v at the vertices in general when v is not continuous.

Remark 1.1. In what precedes, the edges have been arbitrarily oriented from the vertex with the smaller index toward the vertex with the larger one. Other choices are of course possible. In particular, by possibly dividing a single edge into two, adding thereby new artificial vertices, it is always possible to assume that for all vertices $\nu_i \in \mathcal{V}$,

$$\text{either } \pi_\alpha(0) = \nu_i, \text{ for all } \alpha \in \mathcal{A}_i \text{ or } \pi_\alpha(\ell_\alpha) = \nu_i, \text{ for all } \alpha \in \mathcal{A}_i. \tag{1.3}$$

This idea was used by Von Below in [24]: some edges of Γ are cut into two by adding artificial vertices so that the new oriented network $\overline{\Gamma}$ has the property (1.3), see Figure 1 for an example.

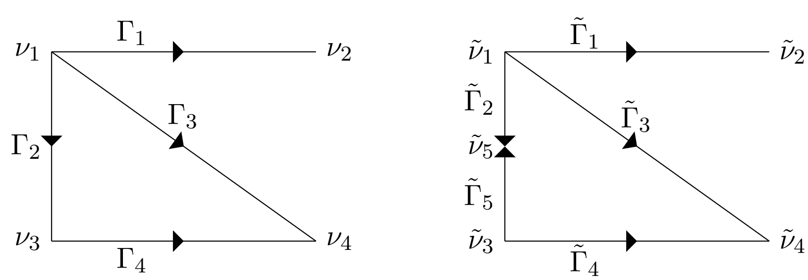

Figure 1: Left: the network Γ in which the edges are oriented toward the vertex with larger index (4 vertices and 4 edges). Right: a new network $\tilde{\Gamma}$ obtained by adding an artificial vertex (5 vertices and 5 edges): the oriented edges sharing a given vertex ν either have all their starting point equal ν, or have all their terminal point equal ν.

1.1.2 Function spaces related to the space variable

The set of continuous functions on Γ is denoted by $C(\Gamma)$ and we set

$$PC(\Gamma) = \left\{ v : \Gamma \to \mathbb{R} \ : \ \text{for all } \alpha \in \mathcal{A}, \left| \begin{array}{l} v_\alpha \in C(0, \ell_\alpha) \\ v_\alpha \text{ can be extended by continuity to } [0, \ell_\alpha]. \end{array} \right. \right\}.$$

By the definition of piecewise continuous functions $v \in PC(\Gamma)$, for all $\alpha \in \mathcal{A}$, it is possible to define $v|_{\Gamma_\alpha}$ by (1.2) and we have $v|_{\Gamma_\alpha} \in C(\Gamma_\alpha)$, $v_\alpha \in C([0, \ell_\alpha])$.

For $m \in \mathbb{N}$, the space of m-times continuously differentiable functions on Γ is defined by

$$C^m(\Gamma) := \{ v \in C(\Gamma) : v_\alpha \in C^m([0, \ell_\alpha]) \text{ for all } \alpha \in \mathcal{A} \}.$$

Notice that $v \in C^m(\Gamma)$ is assumed to be continuous on Γ, and that its restriction $v_{|\Gamma_\alpha}$ to each edge Γ_α belongs to $C^m(\Gamma_\alpha)$. The space $C^m(\Gamma)$ is endowed with the norm $\|v\|_{C^m(\Gamma)} := \sum_{\alpha \in \mathcal{A}} \sum_{k \leqslant m} \left\| \partial^k v_\alpha \right\|_{L^\infty(0,\ell_\alpha)}$. For $\sigma \in (0,1)$, the space $C^{m,\sigma}(\Gamma)$, contains the functions $v \in C^m(\Gamma)$ such that $\partial^m v_\alpha \in C^{0,\sigma}([0, \ell_\alpha])$ for all $\alpha \in \mathcal{A}$; it is endowed with the norm

$$\|v\|_{C^{m,\sigma}(\Gamma)} := \|v\|_{C^m(\Gamma)} + \sup_{\alpha \in \mathcal{A}} \sup_{\substack{y \neq z \\ y,z \in [0,\ell_\alpha]}} \frac{|\partial^m v_\alpha(y) - \partial^m v_\alpha(z)|}{|y - z|^\sigma}.$$

For a positive integer m and a function $v \in C^m(\Gamma)$, we set for $k \leqslant m$,

$$\partial^k v(x) = \partial^k v_\alpha \left(\pi_\alpha^{-1}(x) \right) \text{ if } x \in \Gamma_\alpha \backslash \mathcal{V}. \tag{1.4}$$

For a vertex ν, we define $\partial_\alpha v(\nu)$ as the *outward* directional derivative of $v|_{\Gamma_\alpha}$ at ν as follows:

$$\partial_\alpha v(\nu) := \begin{cases} \lim\limits_{h \to 0^+} \dfrac{v_\alpha(0) - v_\alpha(h)}{h}, & \text{if } \nu = \pi_\alpha(0), \\ \lim\limits_{h \to 0^+} \dfrac{v_\alpha(\ell_\alpha) - v_\alpha(\ell_\alpha - h)}{h}, & \text{if } \nu = \pi_\alpha(\ell_\alpha). \end{cases} \tag{1.5}$$

For all $i \in I$ and $\alpha \in \mathcal{A}_i$, setting

$$n_{i\alpha} = \begin{cases} 1 & \text{if } \nu_i = \pi_\alpha(\ell_\alpha), \\ -1 & \text{if } \nu_i = \pi_\alpha(0), \end{cases} \tag{1.6}$$

we have

$$\partial_\alpha v(\nu_i) = n_{i\alpha} \, \partial v|_{\Gamma_\alpha}(\nu_i) = n_{i\alpha} \, \partial v_\alpha(\pi_\alpha^{-1}(\nu_i)). \tag{1.7}$$

Remark 1.2. Changing the orientation of the edge does not change the value of $\partial_\alpha v(\nu)$ in (1.5).

We say that v is Lebesgue-integrable on Γ_α if v_α is Lebesgue-integrable on $(0, \ell_\alpha)$. In this case, for all $x_1, x_2 \in \Gamma_\alpha$,

$$\int_{[x_1,x_2]} v(x)\, dx := \int_{\pi_\alpha^{-1}(x_1)}^{\pi_\alpha^{-1}(x_2)} v_\alpha(y)\, dy. \tag{1.8}$$

When v is Lebesgue-integrable on Γ_α for all $\alpha \in \mathcal{A}$, we say that v is Lebesgue-integrable on Γ and we define

$$\int_\Gamma v(x)\, dx := \sum_{\alpha \in \mathcal{A}} \int_0^{\ell_\alpha} v_\alpha(y)\, dy.$$

The space $L^p(\Gamma) = \{v : v|_{\Gamma_\alpha} \in L^p(\Gamma_\alpha) \text{ for all } \alpha \in \mathcal{A}\}$, $p \in [1, \infty]$, is endowed with the norm $\|v\|_{L^p(\Gamma)} := \left(\sum_{\alpha\in\mathcal{A}} \|v_\alpha\|^p_{L^p(0,\ell_\alpha)}\right)^{\frac{1}{p}}$ if $1 \leqslant p < \infty$, and $\max_{\alpha\in\mathcal{A}} \|v_\alpha\|_{L^\infty(0,\ell_\alpha)}$ if $p = +\infty$. We shall also need to deal with functions on Γ whose restrictions to the edges are weakly-differentiable: we shall use the same notations for the weak derivatives.

Definition 1.3. For any integer $s \geqslant 1$ and any real number $p \geqslant 1$, the Sobolev space $W_b^{s,p}(\Gamma)$ is defined as follows

$$W_b^{s,p}(\Gamma) := \{v : \Gamma \to \mathbb{R} \text{ s.t. } v_\alpha \in W^{s,p}(0, \ell_\alpha) \text{ for all } \alpha \in \mathcal{A}\},$$

and endowed with the norm

$$\|v\|_{W_b^{s,p}(\Gamma)} = \left(\sum_{k=1}^{s} \sum_{\alpha\in\mathcal{A}} \left\|\partial^k v_\alpha\right\|^p_{L^p(0,\ell_\alpha)} + \|v\|^p_{L^p(\Gamma)}\right)^{\frac{1}{p}}.$$

For $s \in \mathbb{N}\backslash\{0\}$, we also set $H_b^s(\Gamma) = W_b^{s,2}(\Gamma)$ and $H^s(\Gamma) = C(\Gamma) \cap H_b^s(\Gamma)$.

Finally, when dealing with probability distributions in mean field games, we will often use the set $\mathcal{M}$ of probability densities, i.e., $m \in L^1(\Gamma)$, $m \geqslant 0$ and $\int_\Gamma m(x)dx = 1$.

1.1.3 Some space-time function spaces

The space of continuous real valued functions on $\Gamma \times [0, T]$ is denoted by $C(\Gamma \times [0, T])$.

Let $PC(\Gamma \times [0, T])$ be the space of the functions $v : \Gamma \times [0, T] \to \mathbb{R}$ such that

1. for all $t \in [0, T]$, $v(\cdot, t)$ belongs to $PC(\Gamma)$
2. for all $\alpha \in \mathcal{A}$, $v|_{\Gamma_\alpha \times [0,T]}$ is continuous on $\Gamma_\alpha \times [0, T]$;

For a function $v \in PC(\Gamma \times [0, T])$, $\alpha \in \mathcal{A}$, we set $v_\alpha(y, t) = v|_{\Gamma_\alpha\times[0,t]}(\pi_\alpha(y), t)$ for all $(y, t) \in [0, \ell_\alpha] \times [0, T]$.

For two nonnegative integers m and n, let $C^{m,n}(\Gamma \times [0, T])$ be the space of continuous real valued functions v on $\Gamma \times [0, T]$ such that for all $\alpha \in \mathcal{A}$, $v|_{\Gamma_\alpha\times[0,T]} \in C^{m,n}(\Gamma_\alpha \times [0, T])$. For $\sigma \in (0, 1)$, $\tau \in (0, 1)$, we define in the same manner $C^{m+\sigma,n+\tau}(\Gamma \times [0, T])$

Useful results on continuous and compact embeddings of space-time function spaces are given in Appendix A.

1.2 A class of stochastic processes on Γ

After rescaling the edges, it may be assumed that $\ell_\alpha = 1$ for all $\alpha \in \mathcal{A}$. Let $\mu_\alpha, \alpha \in \mathcal{A}$ and $p_{i\alpha}, i \in I, \alpha \in \mathcal{A}_i$ be positive constants such that $\sum_{\alpha\in\mathcal{A}_i} p_{i\alpha} = 1$. Consider also a real valued function $a \in PC(\Gamma \times [0, T])$, such that, for all $\alpha \in \mathcal{A}$, $t \in [0, T]$, $a|_{\Gamma_\alpha}(\cdot, t)$ belongs to $C^1(\Gamma_\alpha)$.

As in Remark 1.1, we make the assumption (1.3) by possibly adding artificial nodes: if ν_i is such an artificial node, then $\sharp(\mathcal{A}_i) = 2$, and we assume that $p_{i\alpha} = 1/2$ for $\alpha \in \mathcal{A}_i$. The diffusion parameter μ has the same value on the two sides of an artificial vertex. Similarly, the function a does not have jumps across an artificial vertex.

Consider a Brownian motion (W_t) defined on the real line. Following Freidlin and Sheu ([8]), we know that there exists a unique Markov process on Γ with continuous sample paths that can be written (X_t, α_t) where $X_t \in \Gamma_{\alpha_t}$ (if $X_t = \nu_i$, $i \in I$, α_t is arbitrarily chosen as the smallest index in $\mathcal{A}_i$) such that, defining the process $x_t = \pi_{\alpha_t}(X_t)$ with values in $[0, 1]$,

- we have
$$dx_t = \sqrt{2\mu_{\alpha_t}} dW_t + a_{\alpha_t}(x_t,t)dt + d\ell_{i,t} + dh_{i,t}, \tag{1.9}$$
- $\ell_{i,t}$ is continuous non-decreasing process (measurable with respect to the σ-field generated by (X_t, α_t)) which increases only when $X_t = \nu_i$ and $x_t = 0$,
- $h_{i,t}$ is continuous non-increasing process (measurable with respect to the σ-field generated by (X_t, α_t)) which decreases only when $X_t = \nu_i$ and $x_t = 1$,

and for all function $v \in C^{2,1}(\Gamma \times [0,T])$ such that
$$\sum_{\alpha \in \mathcal{A}_i} p_{i\alpha} \partial_\alpha v(\nu_i, t) = 0, \quad \text{for all } i \in I, t \in [0,T], \tag{1.10}$$
the process
$$M_t = v(X_t,t) - \int_0^t \Big(\partial_t v(X_s,s) + \mu_{\alpha_s} \partial^2 v(X_s,s) + a|_{\Gamma_{\alpha_s}}(X_s,s)\, \partial v(X_s,s) \Big) ds \tag{1.11}$$
is a martingale, i.e.,
$$\mathbb{E}(M_t | X_s) = M_s, \quad \text{for all } 0 \leqslant s < t \leqslant T. \tag{1.12}$$
For what follows, it will be convenient to set
$$D := \left\{ u \in C^2(\Gamma) : \sum_{\alpha \in \mathcal{A}_i} p_{i\alpha} \partial_\alpha u(\nu_i) = 0, \quad \text{for all } i \in I \right\}. \tag{1.13}$$
Remark 1.4. Note that in (1.10), the condition at boundary vertices boils down to a Neumann condition.

Remark 1.5. The assumption that all the edges have unit length is not restrictive, because we can always rescale the constants μ_α and the piecewise continuous function a.

The goal is to derive the boundary value problem satisfied by the law of the stochastic process X_t. Since the derivation here is formal, we assume that the law of the stochastic process X_t is a measure which is absolutely continuous with respect to the Lebesgue measure on Γ and regular enough so that the following computations make sense. Let $m(x,t)$ be its density. We have
$$\mathbb{E}[v(X_t,t)] = \int_\Gamma v(x,t)\, m(x,t)\, dx, \quad \text{for all } v \in PC(\Gamma \times [0,T]). \tag{1.14}$$
Consider $u \in C^{2,1}(\Gamma \times [0,T])$ such that for all $t \in [0,T]$, $u(\cdot,t) \in D$. Then, from (1.11)-(1.12), we see that
$$\mathbb{E}[u(X_t,t)] = \mathbb{E}[u(X_0,0)] + \mathbb{E}\left[\int_0^t \Big(\partial_t u(X_s,s) + \mu_{\alpha_s} \partial^2 u(X_s,s) + a|_{\Gamma_{\alpha_s}}(X_s,s)\, \partial u(X_s,s) \Big) ds \right]. \tag{1.15}$$
Using (1.14) and taking the time-derivative of each member of (1.15), we obtain
$$\int_\Gamma \partial_t(um)(x,t)dx = \mathbb{E}\Big(\partial_t u(X_t,t) + \mu_{\alpha_s} \partial^2 u(X_t,t) + a|_{\Gamma_{\alpha_s}}(X_t,t)\, \partial u(X_t,t) \Big).$$
Using again (1.14), we get
$$\int_\Gamma \big(\mu \partial^2 u(x,t) + a(x,t) \partial u(x,t) \big)\, m(x,t) dx = \int_\Gamma u(x,t) \partial_t m(x,t) dx. \tag{1.16}$$

By integration by parts, recalling (1.3), we get

$$\begin{aligned}0 \;=\;& \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \big(\partial_t m(x,t) - \mu_\alpha \partial^2 m(x,t) + \partial(am)(x,t)\big)\, u(x,t)dx \\ & - \sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} \left[n_{i\alpha} a|_{\Gamma_\alpha}(\nu_i,t) m|_{\Gamma_\alpha}(\nu_i,t) - \mu_\alpha \partial_\alpha m(\nu_i,t)\right] u|_{\Gamma_\alpha}(\nu_i,t) \\ & - \sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha m|_{\Gamma_\alpha}(\nu_i,t)\partial_\alpha u(\nu_i,t), \end{aligned} \tag{1.17}$$

where $n_{i\alpha}$ is defined in (1.6).

We choose first, for every $\alpha \in \mathcal{A}$, a smooth function u which is compactly supported in $(\Gamma_\alpha\backslash\mathcal{V}) \times [0,T]$. Hence $u|_{\Gamma_\beta}(\nu_i,t) = 0$ and $\partial_\beta u(\nu_i,t) = 0$ for all $i \in I, \beta \in \mathcal{A}_i$. Notice that $u(\cdot,t) \in D$. It follows that m satisfies

$$\big(\partial_t m - \mu_\alpha \partial^2 m + \partial\,(ma)\big)\,(x,t) = 0, \quad \text{for } x \in \Gamma_\alpha\backslash\mathcal{V},\ t\in(0,T),\ \alpha\in\mathcal{A}. \tag{1.18}$$

For a smooth function $\chi : [0,T] \to \mathbb{R}$ compactly supported in $(0,T)$, we may choose for every $i \in I$, a smooth function u such that $u(\nu_j,t) = \chi(t)\delta_{i,j}$ for all $t \in [0,T]$, $j\in I$ and $\partial_\alpha u(\nu_j,t) = 0$ for all $t \in [0,T]$, $j \in I$ and $\alpha \in \mathcal{A}_j$, we infer a condition for m at the vertices,

$$\sum_{\alpha\in\mathcal{A}_i} n_{i\alpha} a|_{\Gamma_\alpha}(\nu_i,t) m|_{\Gamma_\alpha}(\nu_i,t) - \mu_\alpha\partial_\alpha m(\nu_i,t) = 0 \qquad \text{for all } i\in I,\ t\in(0,T). \tag{1.19}$$

This condition is called a transmission condition if ν_i is a transition vertex and reduces to a Robin boundary condition when ν_i is a boundary vertex.

Finally, for a smooth function $\chi : [0,T] \to \mathbb{R}$ compactly supported in $(0,T)$, for every transition vertex $\nu_i \in \mathcal{V}\backslash\partial\Gamma$ and $\alpha,\beta\in\mathcal{A}_i$, we choose u such that

- $u(\cdot,t) \in D$
- $\partial_\alpha u(\nu_i,t) = \chi(t)/p_{i\alpha}$, $\partial_\beta u(\nu_i) = -\chi(t)/p_{i\beta}$, $\partial_\gamma u(\nu_i) = 0$ if $\gamma \in \mathcal{A}_i\backslash\{\alpha,\beta\}$
- The directional derivatives of u at the vertices $\nu \neq \nu_i$ are 0.

Using such a test-function in (1.17) yields a jump condition for m,

$$\frac{m|_{\Gamma_\alpha}(\nu_i,t)}{\gamma_{i\alpha}} = \frac{m|_{\Gamma_\beta}(\nu_i,t)}{\gamma_{i\beta}}, \quad \text{for all } \alpha,\beta\in\mathcal{A}_i, \nu_i\in\mathcal{V}, t\in(0,T),$$

in which

$$\gamma_{i\alpha} = \frac{p_{i\alpha}}{\mu_\alpha}, \quad \text{for all } i\in I, \alpha\in\mathcal{A}_i. \tag{1.20}$$

Summarizing, we get the following boundary value problem for m (recall that the coefficients $n_{i\alpha}$ are defined in (1.6)):

$$\begin{cases} \partial_t m - \mu_\alpha\partial^2 m + \partial\,(ma) = 0, & (x,t)\in(\Gamma_\alpha\backslash\mathcal{V})\times(0,T),\ \alpha\in\mathcal{A}, \\ \displaystyle\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha\partial_\alpha m\,(\nu_i,t) - n_{i\alpha} a|_{\Gamma_\alpha}(\nu_i) m|_{\Gamma_\alpha}(\nu_i,t) = 0, & t\in(0,T), \nu_i\in\mathcal{V}, \\ \dfrac{m|_{\Gamma_\alpha}(\nu_i,t)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i,t)}{\gamma_{i\beta}}, & t\in(0,T),\ \alpha,\beta\in\mathcal{A}_i,\ \nu_i\in\mathcal{V}, \\ m(x,0) = m_0(x), & x\in\Gamma. \end{cases} \tag{1.21}$$

1.3 Formal derivation of the MFG system on Γ

Here we aim at obtaining the MFG system of forward-backward partial differential equations on the network, at least formally. The assumptions that we are going to make below on the optimal control problem are a little restrictive, for two reasons: first, we wish to avoid some technicalities linked to the measurability of the control process; second, the assumptions on the costs must be consistent with the assumptions that we shall make on the Hamiltonian, see Section 1.4.2 below. In particular, we shall impose that the Hamiltonian is globally Lipschitz continuous. More general and difficult cases, e.g., quadratic Hamiltonians, will be the subject of a future work.

Consider a continuum of indistinguishable agents moving on the network Γ. The state of a representative agent at time t is a time-continuous controlled stochastic process X_t as defined in Section 1.2, where the control is the drift a_t, supposed to be in the form $a_t = a(X_t, t)$. Let $m(\cdot, t)$ be the probability measure on Γ that describes the distribution of states at time t.

For a representative agent, the optimal control problem is of the form

$$v(x,t) = \inf_{a_s} \mathbb{E}_{xt} \left[\int_t^T \left(L(X_s, a_s) + \mathscr{V}[m(\cdot,t)](X_s) \right) ds + v_T(X_T) \right], \tag{1.22}$$

where $\mathbb{E}_{xt}$ stands for the expectation conditioned by the event $X_t = x$.

We discuss the ingredients appearing in (1.22):

- We assume that the control is in a feeback form $a_t = a(X_t, t)$ where the function a, defined on $\Gamma \times [0,T]$, is sufficiently regular in the edges of the network. Then, almost surely if $X_t \in \Gamma_\alpha \backslash \mathcal{V}$,
$$d\pi_\alpha^{-1}(X_t) = a_\alpha(\pi_\alpha^{-1}(X_t), t)dt + \sqrt{2\mu_\alpha} dW_t.$$
An informal way to describe the behavior of the process at the vertices is as follows: if X_t hits $\nu_i \in \mathcal{V}$, then it enters Γ_α, $\alpha \in \mathcal{A}_i$ with probability $p_{i\alpha} > 0$ ($p_{i\alpha}$ was introduced in Section 1.2). We assume that there is an optimal feedback law $a^\star$.

- We assume that for all $\alpha \in \mathcal{A}$, a_α maps $[0, \ell_\alpha] \times [0,T]$ to a compact interval $A_\alpha = [\underline{a_\alpha}, \overline{a_\alpha}]$.

- The contribution of the control to the running cost involves the Lagrangian L, i.e., a real valued function defined on $\cup_{\alpha \in \mathcal{A}} (\Gamma_\alpha \backslash \mathcal{V} \times A_\alpha)$. If $x \in \Gamma_\alpha \backslash \mathcal{V}$ and $a \in A_\alpha$, then $L(x,a) = L_\alpha(\pi_\alpha^{-1}(x), a)$, where L_α is a continuous real valued function defined on $[0, \ell_\alpha] \times A_\alpha$. We assume that $L_\alpha(x, \cdot)$ is strictly convex on A_α.

- The contribution of the distribution of states to the running ccost involves the coupling cost operator, which can either be nonlocal, i.e., $\mathscr{V} : \mathcal{P}(\Gamma) \to \mathcal{C}^2(\Gamma)$ (where $\mathcal{P}(\Gamma)$ is the set of Borel probability measures on Γ), or local, i.e., $\mathscr{V}[m](x) = F(m(x))$ for a continuous function $F : \mathbb{R}^+ \to \mathbb{R}$, assuming that m is absolutely continuous with respect to the Lebesgue measure and identifying with its density.

- The last term is the terminal cost v_T, which depends only on the state variable for simplicity.

Under suitable additional assumptions, Ito calculus as in [8, 9] and the dynamic programming principle lead to the following HJB equation on Γ, more precisely the following boundary value problem

$$\begin{cases} -\partial_t v - \mu_\alpha \partial^2 v + H(x, \partial v) = \mathscr{V}[m(\cdot,t)](x), & \text{in } (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i, t) = 0, & \text{if } (\nu_i, t) \in \mathcal{V} \times (0,T), \\ v|_{\Gamma_\alpha}(\nu_i, t) = v|_{\Gamma_\beta}(\nu_i, t) & \text{for all } \nu_i \in \mathcal{V}, t \in (0,T)\, \alpha, \beta \in \mathcal{A}_i, \\ v(x,T) = v_T(x) & \text{in } \Gamma. \end{cases} \tag{1.23}$$

We refer to [14, 15, 16] for the interpretation of the value function v. Let us comment the different equations in (1.23):

1. The first equation is a HJB equation the Hamiltonian H of which is a real valued function defined on $(\cup_{\alpha \in \mathcal{A}} \Gamma_\alpha \backslash \mathcal{V}) \times \mathbb{R}$ given by

$$H(x,p) = \sup_{a \in A_\alpha} \left\{ -ap - L_\alpha \left(\pi_\alpha^{-1}(x), a \right) \right\} \quad \text{for } x \in \Gamma_\alpha \backslash \mathcal{V} \text{ and } p \in \mathbb{R}. \tag{1.24}$$

   We assume that L is such that the Hamiltonians $H|_{\Gamma_\alpha \times \mathbb{R}}$ are Lipschitz continuous with respect to p and C^1.

2. The second equation in (1.23) is a Kirchhoff transmission condition (or Neumann boundary condition if $\nu_i \in \partial\Gamma$); it is the consequence of the assumption on the behavior of X_s at vertices. It involves the positive constants $\gamma_{i\alpha}$ defined in (1.20).

3. The third condition means in particular that v is continuous at the vertices.

4. The fourth condition is a terminal condition for the backward in time HJB equation.

If (1.23) has a smooth solution, then it provides a feedback law for the optimal control problem, i.e.,

$$a^\star(x,t) = -\partial_p H(x, \partial v(x,t)).$$

At the MFG equilibrium, m is the density of the invariant measure associated with the optimal feedback law, so, according to Section 1.2, it satisfies (1.21), where a is replaced by $a^\star = -\partial_p H(x, \partial v(x,t))$. We end up with the following system:

$$\begin{cases} -\partial_t v - \mu_\alpha \partial^2 v + H(x, \partial v) = \mathscr{V}[m(\cdot,t)](x), & (x,t) \in (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \alpha \in \mathcal{A}, \\ \partial_t m - \mu_\alpha \partial^2 m - \partial(m \partial_p H(x, \partial v)) = 0, & (x,t) \in (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i, t) = 0, & (\nu_i, t) \in \mathcal{V} \times (0,T), \\ \sum_{\alpha \in \mathcal{A}_i} \mu_\alpha \partial_\alpha m(\nu_i, t) + n_{i\alpha} \partial_p H^\alpha(\nu_i, \partial v|_{\Gamma_\alpha}(\nu_i, t))\, m|_{\Gamma_\alpha}(\nu_i, t) = 0, & (\nu_i, t) \in \mathcal{V} \times (0,T), \\ v|_{\Gamma_\alpha}(\nu_i, t) = v|_{\Gamma_\beta}(\nu_i, t), \ \dfrac{m|_{\Gamma_\alpha}(\nu_i, t)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i, t)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i, (\nu_i, t) \in \mathcal{V} \times (0,T), \\ v(x,T) = v_T(x), \ m(x,0) = m_0(x) & x \in \Gamma, \end{cases} \tag{1.25}$$

where $H^\alpha := H|_{\Gamma_\alpha \times \mathbb{R}}$. At a vertex ν_i, $i \in I$, the transmission conditions for both v and m consist of $d_{\nu_i} = \sharp(\mathcal{A}_i)$ linear relations, which is the appropriate number of relations to have a well posed problem. If $\nu_i \in \partial\Gamma$, there is of course only one Neumann like condition for v and for m.

1.4 Assumptions and main results

Before giving the precise definition of solutions of the MFG system (1.25) and stating our result, we need to introduce some suitable functions spaces.

1.4.1 Function spaces related to the Kirchhoff conditions

The following function spaces will be the key ingredients in order to build weak solutions of (1.25).

Definition 1.6. We define two Sobolev spaces: $V := H^1(\Gamma)$, and

$$W := \left\{ w : \Gamma \to \mathbb{R} : \ w \in H_b^1(\Gamma) \ \text{ and } \ \frac{w|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \frac{w|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}} \text{ for all } i \in I, \ \alpha, \beta \in \mathcal{A}_i \right\}, \tag{1.26}$$

which is a subspace of $H_b^1(\Gamma)$.

Definition 1.7. Let the function $\varphi \in W$ be defined as follows:

$$\begin{cases} \varphi_\alpha \text{ is affine on } (0, \ell_\alpha), \\ \varphi|_{\Gamma_\alpha}(\nu_i) = \gamma_{i\alpha}, \ \text{ if } \alpha \in \mathcal{A}_i, \\ \varphi \text{ is constant on the edges } \Gamma_\alpha \text{ which touch the boundary of } \Gamma. \end{cases} \tag{1.27}$$

Note that φ is positive and bounded. We set $\overline{\varphi} = \max_\Gamma \varphi$, $\underline{\varphi} = \min_\Gamma \varphi$.

Remark 1.8. One can see that $v \in V \longmapsto v\varphi$ is an isomorphism from V onto W and $w \in W \longmapsto w\varphi^{-1}$ is the inverse isomorphism.

Definition 1.9. Let the function space $\mathcal{W} \subset W$ be defined as follows:

$$\mathcal{W} := \left\{ m : \Gamma \to \mathbb{R} : m_\alpha \in C^1([0, \ell_\alpha]) \ \text{ and } \ \frac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \frac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}} \text{ for all } i \in I, \alpha, \beta \in \mathcal{A}_i \right\}. \tag{1.28}$$

1.4.2 Running assumptions (H)

(Diffusion constants) $(\mu_\alpha)_{\alpha\in\mathcal{A}}$ is a family of positive numbers.

(Jump coefficients) $(\gamma_{i\alpha})_{\alpha\in\mathcal{A}_i}$ is a family of positive numbers such that $\displaystyle\sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha = 1$.

(Hamiltonian) The Hamiltonian H is defined by the collection $H^\alpha := H|_{\Gamma_\alpha\times\mathbb{R}}$, $\alpha \in \mathcal{A}$: we assume that

$$H^\alpha \in C^1(\Gamma_\alpha \times \mathbb{R}), \tag{1.29}$$

$$H^\alpha(x, \cdot) \text{ is convex in } p, \qquad \text{for any } x \in \Gamma_\alpha, \tag{1.30}$$

$$H^\alpha(x, p) \leqslant C_0(|p| + 1), \qquad \text{for any } (x, p) \in \Gamma_\alpha \times \mathbb{R}, \tag{1.31}$$

$$|\partial_p H^\alpha(x, p)| \leqslant C_0, \qquad \text{for any } (x, p) \in \Gamma_\alpha \times \mathbb{R}, \tag{1.32}$$

$$|\partial_x H^\alpha(x, p)| \leqslant C_0(|p| + 1), \qquad \text{for any } (x, p) \in \Gamma_\alpha \times \mathbb{R}, \tag{1.33}$$

for a constant C_0 independent of α.

(Coupling operator) We assume that $\mathscr{V}$ is a continuous map from $L^2(\Gamma)$ to $L^2(\Gamma)$, such that for all $m \in L^2(\Gamma)$,

$$\|\mathscr{V}[m]\|_{L^2(\Gamma)} \leqslant C(\|m\|_{L^2(\Gamma)} + 1). \tag{1.34}$$

Note that such an assumption is satisfied by local operators of the form $\mathscr{V}[m](x) = F(m(x))$ where F is a Lipschitz-continuous function.

(Initial and terminal data) $m_0 \in L^2(\Gamma) \cap \mathcal{M}$ and $v_T \in H^1(\Gamma)$.

The above set of assumptions, referred to as (H), will be the running assumptions hereafter. We will use the following notation: $\underline{\mu} := \min_{\alpha\in\mathcal{A}} \mu_\alpha > 0$ and $\overline{\mu} := \max_{\alpha\in\mathcal{A}} \mu_\alpha$.

1.4.3 Strictly increasing coupling

We will also say that the coupling $\mathscr{V}$ is strictly increasing if, for any $m_1, m_2 \in \mathcal{M} \cap L^2(\Gamma)$,

$$\int_\Gamma (m_1 - m_2)(\mathscr{V}[m_1] - \mathscr{V}[m_2])dx \geqslant 0$$

and equality implies $m_1 = m_2$.

1.4.4 Stronger assumptions on the coupling operator

We will sometimes need to strengthen the assumptions on the coupling operator, namely that $\mathscr{V}$ has the following smoothing properties:

$\mathscr{V}$ maps the topological dual of W to $H^1_b(\Gamma)$; more precisely, $\mathscr{V}$ defines a Lipschitz map from W' to $H^1_b(\Gamma)$.

Note that such an assumption is not satisfied by local operators.

1.4.5 Definition of solutions and main result

Definition 1.10. (solutions of the MFG system) A weak solution of the Mean Field Games system (1.25) is a pair (v, m) such that

$$\begin{aligned}
&v \in L^2\left(0,T;H^2(\Gamma)\right) \cap C([0,T];V),\ \partial_t v \in L^2\left(0,T;L^2(\Gamma)\right),\\
&m \in L^2(0,T;W) \cap C((0,T];L^2(\Gamma)\cap\mathcal{M}),\ \partial_t m \in L^2\left(0,T;V'\right),
\end{aligned}$$

v satisfies

$$\begin{cases}
-\displaystyle\sum_{\alpha\in\mathcal{A}}\int_{\Gamma_\alpha}\left[\partial_t v(x,t)\,\mathsf{w}(x) + \mu_\alpha \partial v(x,t)\,\partial\mathsf{w}(x) + H(x,\partial v(x,t))\,\mathsf{w}(x)\right]dx\\
\qquad\qquad = \displaystyle\int_\Gamma \mathscr{V}[m(\cdot,t)](x)\mathsf{w}(x)\,dx, \quad \text{for all } \mathsf{w}\in W, \text{ a.e. } t\in(0,T),\\
v(x,T) = v_T(x) \quad \text{for a.e. } x\in\Gamma,
\end{cases}$$

and m satisfies

$$\begin{cases}
\displaystyle\sum_{\alpha\in\mathcal{A}}\int_{\Gamma_\alpha}\left[\partial_t m(x,t)\,\mathsf{v}(x)\,dx + \mu_\alpha \partial m(x,t)\,\partial\mathsf{v}(x) + \partial_p H(x,\partial v(x,t))\,m(x,t)\,\partial\mathsf{v}(x)\right]dx\\
\qquad\qquad = 0, \quad \text{for all } \mathsf{v}\in V, \text{ a.e. } t\in(0,T),\\
m(x,0) = m_0(x) \quad \text{for a.e. } x\in\Gamma,
\end{cases}$$

where V and W are introduced in Definition 1.6.

We are ready to state the main result:

Theorem 1.11. *Under assumptions (H),*

(i) (Existence) There exists a weak solution (v, m) of (1.25).

(ii) (Uniqueness) If $\mathscr{V}$ is strictly increasing (see 1.4.3), then the solution is unique.

(iii) (Regularity) If $\mathscr{V}$ satisfies furthermore the stronger assumptions made in Section 1.4.4 and if $v_T \in C^{2+\eta}(\Gamma) \cap D$ for some $\eta \in (0,1)$ (D is given in (1.13)), then $v \in C^{2,1}(\Gamma \times [0,T])$. Moreover, if for all $\alpha \in \mathcal{A}$, $\partial_p H^\alpha(x,p)$ is a Lipschitz function defined in $\Gamma_\alpha \times \mathbb{R}$, and if $m_0 \in W$, then $m \in C([0,T];W) \cap W^{1,2}(0,T;L^2(\Gamma)) \cap L^2(0,T;H_b^2(\Gamma))$.

2 Preliminary: a modified heat equation on the network with general Kirchhoff conditions

This section contains results on the solvability of some linear boundary value problems with terminal condition, that will be useful in what follows. Consider

$$\begin{cases} -\partial_t v - \mu_\alpha \partial^2 v = h, & \text{in } (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i, t) = v|_{\Gamma_\beta}(\nu_i, t), & t \in (0,T)\, \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i, t) = 0, & t \in (0,T), \nu_i \in \mathcal{V}, \\ v(x,T) = v_T(x), & x \in \Gamma, \end{cases} \tag{2.1}$$

where $h \in L^2(0,T;W')$ and $v_T \in L^2(\Gamma)$.

Definition 2.1. If $v_T \in L^2(\Gamma)$ and $h \in L^2(0,T;W')$, a weak solution of (2.1) is a function $v \in L^2(0,T;V) \cap C([0,T];L^2(\Gamma))$ such that $\partial_t v \in L^2(0,T;W')$ and

$$\begin{cases} -\langle \partial_t v(t), w \rangle_{W',W} + \mathscr{B}(v(\cdot,t), w) = \langle h(t), w \rangle_{W',W} & \text{for all } w \in W \text{ and a.e. } t \in (0,T), \\ v(x,T) = v_T(x), \end{cases} \tag{2.2}$$

where $\mathscr{B} : V \times W \to \mathbb{R}$ is the bilinear form defined as follows:

$$\mathscr{B}(v,w) := \int_\Gamma \mu \partial v \partial w dx = \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v \partial w dx.$$

We use the Galerkin's method (see [7]), i.e., we construct solutions of some finite-dimensional approximations to (2.1).

Recall that φ has been defined in Definition 1.7. We notice first that the symmetric bilinear form $\widehat{\mathscr{B}}(u,v) := \int_\Gamma \mu \varphi \partial u \partial v$ is such that $(u,v) \mapsto (u,v)_{L^2(\Gamma)} + \widehat{\mathscr{B}}(u,v)$ is an inner product in V equivalent to the standard inner product in V, namely $(u,v)_V = (u,v)_{L^2(\Gamma)} + \int_\Gamma \partial u \partial v$. Therefore, by standard Fredholm's theory, there exist

- a non decreasing sequence of nonnegative real numbers $(\lambda_k)_{k=1}^\infty$, that tends to $+\infty$ as $k \to \infty$
- A Hilbert basis $(\mathsf{v}_k)_{k=1}^\infty$ of $L^2(\Gamma)$, which is also a a total sequence of V (and orthogonal if V is endowed with the scalar product $(u,v)_{L^2(\Gamma)} + \widehat{\mathscr{B}}(u,v)$),

such that

$$\widehat{\mathscr{B}}(\mathsf{v}_k, v) = \lambda_k(\mathsf{v}_k, v)_{L^2(\Gamma)} \quad \text{for all } v \in V. \tag{2.3}$$

Note that

$$\int_\Gamma \mu \partial \mathsf{v}_k \partial \mathsf{v}_\ell \varphi dx = \begin{cases} \lambda_k & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell. \end{cases}$$

Note also that v_k is a weak solution of

$$\begin{cases} -\mu_\alpha \partial (\varphi \partial \mathsf{v}_k) = \lambda_k \mathsf{v}_k, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \mathsf{v}_k|_{\Gamma_\alpha}(\nu_i) = \mathsf{v}_k|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, \\ \sum_{\alpha \in \mathcal{A}} \gamma_{i\alpha} \mu_\alpha \partial_\alpha \mathsf{v}_k(\nu_i) = 0, & \nu_i \in \mathcal{V}, \end{cases} \tag{2.4}$$

which implies that $\mathsf{v}_k \in C^2(\Gamma)$.

Finally, by Remark 1.8, the sequence $(\varphi \mathsf{v}_k)_{k=1}^\infty$ is a total family in W (but is not orthogonal if W is endowed with the standard inner product).

Lemma 2.2. *For any positive integer n, there exist n absolutely continuous functions y_k^n : $[0,T] \to \mathbb{R}$, $k = 1, \dots, n$, and a function $v_n : [0,T] \to L^2(\Gamma)$ of the form*

$$v_n(x,t) = \sum_{k=1}^n y_k^n(t) \mathsf{v}_k(x), \tag{2.5}$$

such that

$$y_k^n(T) = \int_\Gamma v_T \mathsf{v}_k dx, \quad \textit{for } k = 1, \dots, n, \tag{2.6}$$

and

$$-\frac{d}{dt}(v_n, \mathsf{v}_k\varphi)_{L^2(\Gamma)} + \mathscr{B}(v_n, \mathsf{v}_k\varphi) = \langle h(t), \mathsf{v}_k\varphi \rangle, \quad \textit{for a.a. } t \in (0,T), \textit{ for all } k = 1, \dots, n. \tag{2.7}$$

Proof of Lemma 2.2. For $n \geqslant 1$, we consider the symmetric n by n matrix M_n defined by

$$(M_n)_{k\ell} = \int_\Gamma \mathsf{v}_k \mathsf{v}_\ell \varphi dx.$$

Since φ is positive and $(\mathsf{v}_k)_{k=1}^\infty$ is a Hilbert basis of $L^2(\Gamma)$, we can check that M_n is a positive definite matrix and there exist two constants c, C independent of n such that

$$c|\xi|^2 \leqslant \sum_{k,\ell=1}^n (M_n)_{k\ell} \xi_k \xi_\ell \leqslant C|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2.8}$$

Looking for v_n of the form (2.5), and setting $Y = (y_1^n, \dots, y_n^n)^T$, $\dot{Y} = \left(\frac{d}{dt}y_1^n, \dots, \frac{d}{dt}y_n^n\right)^T$, (2.7) implies that we have to solve the following system of differential equations

$$-M_n \dot{Y} + BY = F_n, \quad Y(T) = \left(\int_\Gamma v_T \mathsf{v}_1, \dots, \int_\Gamma v_T \mathsf{v}_n\right)^T,$$

where $B_{k\ell} = \mathscr{B}(\mathsf{v}_\ell, \mathsf{v}_k\varphi)$ and $F_n(t) = (\langle h(t), \mathsf{v}_1\varphi \rangle, \dots, \langle h(t), \mathsf{v}_n\varphi \rangle)^T$. Since the matrix M_n is invertible, the ODE system has a unique absolutely continuous solution. The lemma is proved. ∎

We propose to send n to $+\infty$ and show that a subsequence of $\{v_n\}$ converges to a solution of (2.1). Hence, we need some uniform estimates for $\{v_n\}$.

Lemma 2.3. *There exists a constant C depending only on Γ, $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, T and φ such that*

$$\|v_n\|_{L^\infty(0,T;L^2(\Gamma))} + \|v_n\|_{L^2(0,T;V)} + \|\partial_t v_n\|_{L^2(0,T;W')} \leqslant C\left(\|h\|_{L^2(0,T;W')} + \|v_T\|_{L^2(\Gamma)}\right).$$

Proof of Lemma 2.3. Multiplying (2.7) by $y_k^n(t)\,e^{\lambda t}$ for a positive constant λ to be chosen later, summing for $k=1,\dots,n$ and using the formula (2.5) for v_n, we get

$$-\int_\Gamma \partial_t v_n v_n e^{\lambda t}\varphi dx + \int_\Gamma \mu\partial v_n \partial\left(v_n e^{\lambda t}\varphi\right)dx = e^{\lambda t}\langle h(t), v_n\varphi\rangle_{W',W},$$

and

$$-\int_\Gamma\left[\partial_t\left(\frac{v_n^2}{2}e^{\lambda t}\right) - \frac{\lambda}{2}v_n^2 e^{\lambda t}\right]\varphi dx + \int_\Gamma \mu\left(\partial v_n\right)^2 e^{\lambda t}\varphi dx + \int_\Gamma \mu\partial v_n v_n e^{\lambda t}\partial\varphi dx = e^{\lambda t}\langle h(t), v_n\varphi\rangle.$$

Integrating both sides from s to T, we obtain

$$\begin{aligned}
&\int_\Gamma\left(\frac{v_n^2(x,s)}{2}e^{\lambda s} - \frac{v_n^2(x,T)}{2}e^{\lambda T}\right)\varphi dx + \frac{\lambda}{2}\int_s^T\int_\Gamma v_n^2 e^{\lambda t}\varphi dxdt \\
&+\int_s^T\int_\Gamma \mu(\partial v_n)^2 e^{\lambda t}\varphi dxdt + \int_s^T\int_\Gamma \mu\partial v_n v_n e^{\lambda t}\partial\varphi dxdt \\
= &\int_s^T e^{\lambda t}\langle h(t), v_n(t)\varphi\rangle dt \\
\leqslant\ & C\int_s^T e^{\lambda t}\|h(t)\|_{W'}\|v_n(t)\|_V dt \\
\leqslant\ & \frac{1}{2}\int_s^T\int_\Gamma \mu\left((\partial v_n)^2 + v_n^2\right)e^{\lambda t}\varphi dxdt + \frac{C^2}{2\underline{\mu}}\int_s^T e^{\lambda t}\|h(t)\|^2_{W'}dt,
\end{aligned}$$

where C is positive constant depending on φ, because of Remark 1.8. Therefore,

$$\begin{aligned}
&e^{\lambda s}\int_\Gamma \frac{v_n^2(x,s)}{2}\varphi dx + \frac{1}{4}\int_s^T\int_\Gamma \mu(\partial v_n)^2\varphi e^{\lambda t}dxdt + \left(\frac{\lambda}{2} - \frac{\overline{\mu}}{2} - \overline{\mu}\frac{\|\partial\varphi\|^2_{L^\infty(\Gamma)}}{\underline{\varphi}^2}\right)\int_s^T\int_\Gamma v_n^2 e^{\lambda t}\varphi dxdt \\
\leqslant\ & e^{\lambda T}\int_\Gamma \frac{v_n^2(x,T)}{2}\varphi dx + \frac{C^2}{2\underline{\mu}}e^{\lambda T}\int_s^T \|h(t)\|^2_{W'}dt.
\end{aligned}$$

Choosing $\lambda \geqslant 1/2 + \overline{\mu} + 2\overline{\mu}||\partial\varphi||^2_{L^\infty(\Gamma)}/\underline{\varphi}^2$ and noticing that $\int_\Gamma v_n^2(x,T)\,\varphi dx$ is bounded by $\overline{\varphi}\int_\Gamma v_T^2 dx$ from (2.6), it follows that

$$\begin{aligned}
&\int_\Gamma v_n^2(x,s)\varphi dx + \int_s^T\int_\Gamma v_n^2\varphi dxdt + \int_s^T\int_\Gamma \mu(\partial v_n)^2\varphi dxdt \\
\leqslant\ & 2e^{\lambda T}\left(\frac{C^2}{\underline{\mu}}\|h\|^2_{L^2(0,T;W')} + \overline{\varphi}\int_\Gamma v_T^2 dx\right).
\end{aligned} \tag{2.9}$$

Estimate of v_n in $L^\infty\left(0,T;L^2(\Gamma)\right)$ and $L^2(0,T;V)$. From (2.9), it is straightforward to see that

$$\|v_n\|_{L^\infty(0,T;L^2(\Gamma))} + \|v_n\|_{L^2(0,T;V)} \leqslant C\left(\|h\|_{L^2(0,T;W')} + \|v_T\|_{L^2(\Gamma)}\right) \tag{2.10}$$

for some constant C depending only on $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, φ and T.

Estimate $\partial_t v_n$ in $L^2(0,T;W')$. Consider the closed subspace G_1 of W defined by $G_1 = \left\{w \in W : \int_\Gamma \mathsf{v}_k w dx = 0 \text{ for all } k \leqslant n\right\}$. It has a finite co-dimension equal to n. Consider also the n-dimensional subspace $G_2 = \operatorname{span}\{\mathsf{v}_1\varphi, \dots, \mathsf{v}_n\varphi\}$ of W. The invertibility of the matrix M_n introduced in the proof of Lemma 2.2 implies that $G_1 \cap G_2 = \{0\}$. This implies that $W = G_1 \oplus G_2$. For $w \in W$, we can write w of the form $w = w_n + \hat{w}_n$, where $w_n \in G_2$ and $\hat{w}_n \in G_1$. Hence, for a.e. $t \in [0,T]$, from (2.5) and (2.7), one gets

$$\langle \partial_t v_n(t), w\rangle_{W',W} = \frac{d}{dt}\left(\int_\Gamma v_n w dx\right) = \frac{d}{dt}\left(\int_\Gamma v_n w_n dx\right) = -\langle h(t), w_n\rangle_{W',W} + \int_\Gamma \mu \partial v_n \partial w_n dx. \tag{2.11}$$

Since there exists a constant C independent of n such that $\|w_n\|_W \leqslant C\|w\|_W$, it follows that

$$\|\partial_t v_n(t)\|_{W'} \leqslant C\left(\|h(t)\|_{W'} + \overline{\mu}\,\|v_n(t)\|_V\right),$$

for almost every t, and therefore, from (2.10), we obtain

$$\|\partial_t v_n(t)\|^2_{L^2(0,T;W')} \leqslant C\left(\|h\|^2_{L^2(0,T;W')} + \|v_T\|^2_{L^2(\Gamma)}\right),$$

for a constant C independent of n. □

Theorem 2.4. *There exists a unique solution v of (2.1), which satisfies*

$$\|v\|_{L^\infty(0,T;L^2(\Gamma))} + \|v\|_{L^2(0,T;V)} + \|\partial_t v\|_{L^2(0,T;W')} \leqslant C\left(\|h\|_{L^2(0,T;W')} + \|v_T\|_{L^2(\Gamma)}\right), \tag{2.12}$$

where C is a constant that depends only on Γ, $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, T and φ.

Proof of Theorem 2.4. From Lemma 2.3, the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;V)$ and the sequence $(\partial_t v_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;W')$. Hence, up to the extraction of a subsequence, there exists a function v such that $v \in L^2(0,T;V)$, $\partial_t v \in L^2(0,T;W')$ and

$$\begin{cases} v_n \rightharpoonup v & \text{weakly in } L^2(0,T;V), \\ \partial_t v_n \rightharpoonup \partial_t v & \text{weakly in } L^2(0,T;W'). \end{cases} \tag{2.13}$$

Fix an integer N and choose a function $\overline{v} \in C^1([0,T];V)$ having the form

$$\overline{v}(t) = \sum_{k=1}^N d_k(t)\,\mathsf{v}_k, \tag{2.14}$$

where $d_1, \dots, d_N$ are given real valued C^1 functions defined in $[0,T]$. For all $n \geqslant N$, multiplying (2.7) by $d_k(t)$, summing for $k = 1, \dots, n$ and integrating over $(0,T)$ leads to

$$-\int_0^T \int_\Gamma \partial_t v_n \overline{v}\varphi dx dt + \int_0^T \int_\Gamma \mu \partial v_n \partial(\overline{v}\varphi)\, dx dt = \int_0^T \langle h, \overline{v}\varphi\rangle dt. \tag{2.15}$$

Letting $n \to +\infty$, we obtain from (2.13) that

$$-\int_0^T \langle \partial_t v, \overline{v}\varphi\rangle dt + \int_0^T \int_\Gamma \mu \partial v \partial(\overline{v}\varphi)\, dx dt = \int_0^T \langle h, \overline{v}\varphi\rangle dt. \tag{2.16}$$

Since the functions of the form (2.14) are dense in $L^2(0,T;V)$, (2.16) holds for all test function $\overline{v} \in L^2(0,T;V)$. Recalling the isomorphism $\overline{v} \in V \mapsto \overline{v}\varphi \in W$ (see Remark 1.8), we obtain that, for all $w \in W$ and $\psi \in C_c^1(0,T)$,

$$-\int_0^T \langle \partial_t v, w\rangle \psi dt + \int_0^T \int_\Gamma \mu \partial v \partial w \psi dx dt = \int_0^T \langle h, w\rangle \psi dt.$$

This implies that, for a.e. $t \in (0,T)$,

$$-\langle \partial_t v, w\rangle + \mathscr{B}(v,w) = \langle h, w\rangle \quad \text{for all } w \in W.$$

Using [19, Theorem 3.1] (or the same argument as in [7, pages 287-288]), we see that $v \in C([0,T]; L^2_\varphi(\Gamma))$, where $L^2_\varphi(\Gamma) = \{w : \Gamma \to \mathbb{R} \ : \ \int_\Gamma w^2 \varphi dx < +\infty\}$, and since φ is bounded from below and from above by positive numbers, $L^2_\varphi(\Gamma) = L^2(\Gamma)$ with equivalent norms. Moreover,

$$\max_{0 \leqslant t \leqslant T} \|v(\cdot,t)\|_{L^2(\Gamma)} \leqslant C \left(\|\partial_t v\|_{L^2(0,T;W')} + \|v\|_{L^2(0,T;V)} \right).$$

We are now going to prove $v(T) = v_T$. For all $\overline{v} \in C^1([0,T];V)$ of the form (2.14) and such that $\overline{v}(0) = 0$, we deduce from (2.15) and (2.16) that

$$\begin{aligned} &-\int_0^T \int_\Gamma \partial_t \overline{v} v_n \varphi dx dt - \int_\Gamma \overline{v}(T) v_n(T) \varphi dx + \int_0^T \int_\Gamma \mu \partial v_n \partial (\overline{v}\varphi) dx dt \\ = &-\int_0^T \int_\Gamma \partial_t \overline{v} v \varphi dx dt - \int_\Gamma \overline{v}(T) v(T) \varphi dx + \int_0^T \int_\Gamma \mu \partial v \partial (\overline{v}\varphi) dx dt. \end{aligned}$$

We know that $v_n(T) \to v_T$ in $L^2(\Gamma)$. Then, using (2.13), we obtain

$$\int_\Gamma \overline{v}(T) v_T \varphi dx = \int_\Gamma \overline{v}(T) v(T) \varphi dx.$$

Since the functions of the form $\sum_{k=1}^N d_k(T) \mathsf{v}_k$ are dense in $L^2(\Gamma)$, we conclude that $v(T) = v_T$.

In order to prove the energy estimate (2.12), we use $ve^{\lambda t}\varphi$ as a test function in (2.2) and apply similar arguments as in the proof of Lemma 2.3 for λ large enough, we get (2.12).

Finally, if $h = 0$ and $v_T = 0$, by the energy estimate for v in (2.12), we deduce that $v = 0$. Uniqueness is proved. □

Theorem 2.5. *If $v_T \in V$ and $h \in L^2(\Gamma \times (0,T))$, then the unique solution v of (2.1) satisfies $v \in L^2(0,T;H^2(\Gamma)) \cap C([0,T];V)$ and $\partial_t v \in L^2(0,T;L^2(\Gamma))$. Moreover,*

$$\|v\|_{L^\infty(0,T;V)} + \|v\|_{L^2(0,T;H^2(\Gamma))} + \|\partial_t v\|_{L^2(0,T;L^2(\Gamma))} \leqslant C \left(\|h\|_{L^2(0,T;L^2(\Gamma))} + \|v_T\|_V \right), \tag{2.17}$$

for a positive constant C that depends only on Γ, $(\mu_\alpha)_{\alpha \in \mathcal{A}}$, T and φ.

Proof of Theorem 2.5. It is enough to prove estimate (2.17) for v_n.

Multiplying (2.7) by $-\frac{d}{dt} y_k^n$, summing for $k = 1, \ldots, n$ and using (2.5) leads to

$$\int_\Gamma (\partial_t v_n)^2 \varphi dx - \int_\Gamma \mu \partial v_n \partial (\partial_t v_n \varphi) dx = -\int_\Gamma h \partial_t v_n \varphi dx,$$

hence

$$\int_\Gamma (\partial_t v_n)^2 \varphi dx - \int_\Gamma \mu \partial_t \frac{(\partial v_n)^2}{2} \varphi dx - \int_\Gamma \mu \partial v_n \partial_t v_n \partial \varphi dx = -\int_\Gamma h \partial_t v_n \varphi dx.$$

Multiplying by $e^{\lambda t}$ where λ will chosen later, and taking the integral from s to T, we obtain

$$\begin{aligned}
& \int_s^T \int_\Gamma (\partial_t v_n)^2 e^{\lambda t}\varphi dxdt - \int_\Gamma \frac{\mu}{2}\left[(\partial v_n(T))^2 e^{\lambda T} - (\partial v_n(s))^2 e^{\lambda s}\right]\varphi dx \\
& +\lambda \int_s^T \int_\Gamma \frac{\mu}{2}(\partial v_n)^2 e^{\lambda t}\varphi dxdt - \int_s^T \int_\Gamma \mu \partial v_n \partial_t v_n e^{\lambda t}\partial\varphi dxdt \\
= & -\int_s^T \int_\Gamma h\partial_t v_n e^{\lambda t}\varphi dxdt \\
\leqslant & \ \frac{1}{2}\int_s^T \int_\Gamma h^2 e^{\lambda t}\varphi dxdt + \frac{1}{2}\int_s^T \int_\Gamma (\partial_t v_n)^2 e^{\lambda t}\varphi dxdt. \qquad (2.18)
\end{aligned}$$

Let us deal with the term $\int_\Gamma (\partial v_n(x,T))^2\varphi dx$. From (2.6),

$$\begin{aligned}
\int_\Gamma \mu(\partial v_n(x,T))^2\varphi dx &= \sum_{k=1}^{n} \lambda_k \left(\int_\Gamma v_T \mathsf{v}_k dx\right)^2 \\
&\leqslant \sum_{k=1}^{\infty} \lambda_k \left(\int_\Gamma v_T \mathsf{v}_k dx\right)^2 = \int_\Gamma \mu(\partial v_T(x))^2\varphi dx \\
&\leqslant \overline{\mu}\int_\Gamma (\partial v_T(x))^2\varphi dx.
\end{aligned}$$

Then, choosing $\lambda = 2\overline{\mu}^2 \|\partial\varphi\|^2_{L^\infty(\Gamma)}/(\underline{\varphi}^2\underline{\mu})$, we obtain that

$$\int_\Gamma 2\mu(\partial v_n(x,s))^2\varphi dx + \int_s^T \int_\Gamma (\partial_t v_n)^2\varphi dxdt \leqslant 2e^{\lambda T}\overline{\varphi}\left(\|h\|^2_{L^2(\Gamma\times(0,T))} + \overline{\mu}\int_\Gamma (\partial v_T)^2 dx\right). \qquad (2.19)$$

Estimate of ∂v_n in $L^\infty\left(0,T;L^2(\Gamma)\right)$ and $\partial_t v_n$ in $L^2(\Gamma\times(0,T))$. From (2.19), it is straightforward to see that

$$\|\partial v_n\|_{L^\infty(0,T;L^2(\Gamma))} + \|\partial_t v_n\|_{L^2(\Gamma\times(0,T))} \leqslant C\left(\|h\|_{L^2(\Gamma\times(0,T))} + \|\partial v_T\|_{L^2(\Gamma)}\right)$$

for some constant C depending only on Γ, μ, T and φ.

Estimate of $\partial^2 v_n$ in $L^2(\Gamma\times(0,T))$. Finally, using the PDE in (2.1), we can see that $\partial^2 v_n$ belongs to $L^2(\Gamma\times(0,T))$ and is bounded by $C\left(\|h\|_{L^2(\Gamma\times(0,T))} + \|v_T\|_V\right)$, hence v_n is bounded in $L^2\left(0,T;H^2(\Gamma)\right)$ by the same quantity. The Kirchhoff conditions (which boil down to Neumann conditions at $\partial\Gamma$) are therefore satisfied in a strong sense for almost all t.

Using [19, Theorem 3.1] (or a similar argument as [7, pages 287-288]), we see that v in $C([0,T];V)$.

□

3 The Fokker-Planck equation

This paragraph is devoted to a boundary value problem including a Fokker-Planck equation

$$\begin{cases}
\partial_t m - \mu_\alpha \partial^2 m - \partial(bm) = 0, & \text{in } (\Gamma_\alpha\backslash\mathcal{V})\times(0,T),\ \alpha\in\mathcal{A}, \\
\dfrac{m|_{\Gamma_\alpha}(\nu_i,t)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i,t)}{\gamma_{i\beta}}, & t\in(0,T),\ \alpha,\beta\in\mathcal{A}_i,\ \nu_i\in\mathcal{V}\backslash\partial\Gamma, \\
\sum\limits_{\alpha\in\mathcal{A}_i} \mu_\alpha\partial_\alpha m(\nu_i,t) + n_{i\alpha} b(\nu_i,t)\, m|_{\Gamma_\alpha}(\nu_i,t) = 0, & t\in(0,T),\ \nu_i\in\mathcal{V}, \\
m(x,0) = m_0(x), & x\in\Gamma,
\end{cases} \qquad (3.1)$$

where $b \in PC(\Gamma \times [0,T])$ and $m_0 \in L^2(\Gamma)$.

Definition 3.1. A weak solution of (3.1) is a function $m \in L^2(0,T;W) \cap C([0,T];L^2(\Gamma))$ such that $\partial_t m \in L^2(0,T;V')$ and

$$\begin{cases} \langle \partial_t m, v\rangle_{V',V} + \mathscr{A}(m,v) = 0 & \text{for all } v \in V \text{ and a.e. } t \in (0,T), \\ m(\cdot,0) = m_0, \end{cases} \tag{3.2}$$

where $\mathscr{A} : W \times V \to \mathbb{R}$ is the bilinear form

$$\mathscr{A}(v,w) = \int_\Gamma \mu \partial m \partial v dx + \int_\Gamma b m \partial v dx.$$

Using similar arguments as in Section 2, in particular a Galerkin method, we obtain the following result, the proof of which is omitted.

Theorem 3.2. *If $b \in L^\infty(\Gamma \times (0,T))$ and $m_0 \in L^2(\Gamma)$, there exists a unique function $m \in L^2(0,T;W) \cap C([0,T];L^2(\Gamma))$ such that $\partial_t m \in L^2(0,T;V')$ and (3.2). Moreover, there exists a constant C which depends on $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, $\|b\|_\infty$, T and φ, such that*

$$\|m\|_{L^2(0,T;W)} + \|m\|_{L^\infty(0,T;L^2(\Gamma))} + \|\partial_t m\|_{L^2(0,T;V')} \leqslant C \|m_0\|_{L^2(\Gamma)}. \tag{3.3}$$

Remark 3.3. If $m_0 \in \mathcal{M}$, which will be the case when solving the MFG system (1.25), then $m(\cdot,t) \in \mathcal{M}$ for all $t \in [0,T]$. Indeed, we use $v \equiv 1 \in V$ as a test-function for (3.1). Since $\partial v = 0$, integrating (3.2) from 0 to t, we get $\int_0^t \int_\Gamma \partial_t m(x,s) dx ds = 0$. This implies that

$$\int_\Gamma m(x,t)dx = \int_\Gamma m_0(x)dx = 1, \qquad \text{for all } t \in (0,T].$$

Setting $m^- = -\mathbb{1}_{\{m<0\}} m$, we can also use $v = \varphi^{-1} m^- e^{-\lambda t}$ as a test-function for $\lambda \in \mathbb{R}_+$. Indeed, the latter function belongs to $L^2(0,T;V)$. Taking λ large enough and using similar arguments as for the energy estimate (3.3) yield that $m^- = 0$, i.e., $m \geqslant 0$.

We end this section by stating a stability result, which will be useful in the proof of the main Theorem.

Lemma 3.4. *Let $m_{0\varepsilon}, b_\varepsilon$ be sequences of functions satisfying*

$$m_{0\varepsilon} \longrightarrow m_0 \text{ in } L^2(\Gamma), \qquad b_\varepsilon \longrightarrow b \text{ in } L^2(\Gamma \times (0,T)),$$

and for some positive number K independent of ε, $\|b\|_{L^\infty(\Gamma\times(0,T))} \leqslant K$, $\|b_\varepsilon\|_{L^\infty(\Gamma\times(0,T))} \leqslant K$. Let m_ε (respectively m) be the solution of (3.2) corresponding to the datum $m_{0\varepsilon}$ (resp. m_0) and the coefficient b_ε (resp. b). The sequence (m_ε) converges to m in $L^2(0,T;W) \cap L^\infty(0,T;L^2(\Gamma))$, and the sequence $(\partial_t m_\varepsilon)$ converges to $(\partial_t m)$ in $L^2(0,T;V')$.

Proof of Lemma 3.4. Taking $(m_\varepsilon - m) e^{-\lambda t} \varphi^{-1}$ as a test-function in the versions of (3.2) satisfied by m_ε and m, subtracting, we obtain that

$$\begin{aligned} &\int_\Gamma \left[\frac{1}{2} \partial_t \left((m_\varepsilon - m)^2 e^{-\lambda t} \right) + \frac{\lambda}{2} (m_\varepsilon - m)^2 e^{-\lambda t} \right] \varphi^{-1} dx + \int_\Gamma \mu (\partial (m_\varepsilon - m))^2 e^{-\lambda t} \varphi^{-1} dx \\ &+ \int_\Gamma \mu (m_\varepsilon - m) \partial (m_\varepsilon - m) e^{-\lambda t} \partial(\varphi^{-1}) dx + \int_\Gamma (b_\varepsilon m_\varepsilon - bm) \partial (m_\varepsilon - m) e^{-\lambda t} \varphi^{-1} dx \\ &+ \int_\Gamma (b_\varepsilon m_\varepsilon - bm)(m_\varepsilon - m) e^{-\lambda t} \partial(\varphi^{-1}) dx = 0. \end{aligned} \tag{3.4}$$

There exists a positive constant K such that $\|b_\varepsilon\|_\infty, \|b\|_\infty \leqslant K$ for all ε. Hence, there exists a positive constant C (in fact it varies from one line to the other in what follows) such that

$$\begin{aligned}
&\int_\Gamma \left[\frac{1}{2}\partial_t \left((m_\varepsilon - m)^2 e^{-\lambda t}\right) + \frac{\lambda}{2}(m_\varepsilon - m)^2 e^{-\lambda t}\right]\varphi^{-1}dx + \int_\Gamma \mu(\partial(m_\varepsilon - m))^2 e^{-\lambda t}\varphi^{-1}dx \\
\leqslant\ & C\int_\Gamma \left(|m_\varepsilon - m|^2 + |m_\varepsilon - m|\,|\partial(m_\varepsilon - m)| + |m|\,|b_\varepsilon - b|\,(|\partial(m_\varepsilon - m)| + |m_\varepsilon - m|)\right) e^{-\lambda t}\varphi^{-1}dx \\
\leqslant\ & C\int_\Gamma \left(|m_\varepsilon - m|^2 + |b_\varepsilon - b|^2 m^2\right) e^{-\lambda t}\varphi^{-1}dx + \int_\Gamma \frac{\mu}{2}(\partial(m_\varepsilon - m))^2 e^{-\lambda t}\varphi^{-1}dx.
\end{aligned}$$

The assumptions on the coefficents b_ε and b imply in fact that $b_\varepsilon \to b$ in $L^p(\Gamma \times (0,T))$ for all $1 \leqslant p < \infty$. On the other hand, we know that $m \in L^q(\Gamma \times (0,T))$ for all $1 \leqslant q < \infty$. From the latter observation with $p = q = 4$, we see that the quantity $\int_0^T \int_\Gamma \left(|b_\varepsilon - b|^2 m^2\right) e^{-\lambda t}\varphi^{-1}dxdt$ tends to 0 as $\varepsilon \to 0$ uniformly in $\lambda > 0$. We write

$$\int_0^T \int_\Gamma \left(|b_\varepsilon - b|^2 m^2\right) e^{-\lambda t}\varphi^{-1}dxdt = o_\varepsilon(1).$$

Choosing λ large enough and integrating the latter inequality from 0 to $t \in [0,T]$, we obtain

$$\|m_\varepsilon - m\|_{L^2(0,T;W)} + \|m_\varepsilon - m\|_{L^\infty(0,T;L^2(\Gamma))} \leqslant o_\varepsilon(1) + C\,\|m_{0\varepsilon} - m_0\|_{L^2(\Gamma)}.$$

Subtracting the two versions of (3.2) and using the latter estimate also yields

$$\|\partial_t m_\varepsilon - \partial_t m\|_{L^2(0,T;V')} \leqslant o_\varepsilon(1) + C\,\|m_{0\varepsilon} - m_0\|_{L^2(\Gamma)},$$

which achieves the proof. □

4 The Hamilton-Jacobi equation

This section is devoted to the following boundary value problem including a Hamilton-Jacobi equation

$$\begin{cases}
-\partial_t v - \mu_\alpha \partial^2 v + H(x, \partial v) = f, & \text{in } (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T),\ \alpha \in \mathcal{A}, \\
v|_{\Gamma_\alpha}(\nu_i, t) = v|_{\Gamma_\beta}(\nu_i, t) & t \in (0,T),\ \alpha, \beta \in \mathcal{A}_i,\ \nu_i \in \mathcal{V}, \\
\sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i, t) = 0, & t \in (0,T),\ \nu_i \in \mathcal{V}, \\
v(x,T) = v_T(x), & x \in \Gamma,
\end{cases} \tag{4.1}$$

where $f \in L^2(\Gamma \times (0,T))$, $v_T \in V$ and the Hamiltonian $H : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfies the running assumptions (H).

Definition 4.1. For $f \in L^2(\Gamma \times (0,T))$ and $v_T \in V$, a weak solution of (4.1) is a function $v \in L^2\left(0,T;H^2(\Gamma)\right) \cap C([0,T];V)$ such that $\partial_t v \in L^2(\Gamma \times (0,T))$ and

$$\begin{cases}
\displaystyle\int_\Gamma (-\partial_t v w + \mu \partial v \partial w + H(x, \partial v) w)\, dx = \int_\Gamma f w dx \quad \text{for all } w \in W, \text{ a.a. } t \in (0,T), \\
v(x,T) = v_T(x).
\end{cases} \tag{4.2}$$

We start by proving existence and uniqueness of a weak solution for (4.1). Next, further regularity for the solution will be obtained under stronger assumptions.

4.1 Existence and uniqueness for the Hamilton-Jacobi equation

Theorem 4.2. *Under the running assumptions (H), if $f \in L^2(\Gamma \times (0,T))$, then the boundary value problem (4.1) has a unique weak solution.*

Uniqueness is a direct consequence of the following proposition.

Proposition 4.3. *(Comparison principle) Under the same assumptions as in Theorem 4.2, let v and $\hat{v}$ be respectively weak sub- and super-solution of (4.1), i.e., $v, \hat{v} \in L^2\left(0,T; H^2(\Gamma)\right), \partial_t v, \partial_t \hat{v} \in L^2(\Gamma \times (0,T))$ such that*

$$\begin{cases} \displaystyle\int_\Gamma (-\partial_t v w + \mu \partial v \partial w + H(x, \partial v)\, w)\, dx \leqslant \int_\Gamma f w dx, \\ \displaystyle\int_\Gamma (-\partial_t \hat{v} w + \mu \partial \hat{v} \partial w + H(x, \partial \hat{v})\, w)\, dx \geqslant \int_\Gamma f w dx, \\ v(x,T) \leqslant v_T(x) \leqslant \hat{v}(x,T) \quad \text{for a.a. } x \in \Gamma. \end{cases} \quad \text{for all } w \in W,\ w \geqslant 0,\ \text{a.a. } t \in (0,T),$$

Then $v \leqslant \hat{v}$ in $\Gamma \times (0,T)$.

Proof of Proposition 4.3. Setting $\overline{v} = v - \hat{v}$, we have, for all $w \in W$ such that $w \geqslant 0$ and for a.a $t \in (0,T)$:

$$\int_\Gamma -\partial_t \overline{v} w + \mu \partial \overline{v} \partial w + (H(x,\partial v) - H(x, \partial \hat{v}))\, w dx \leqslant 0,$$

and $\overline{v}(x,T) \leqslant 0$ for all $x \in \Gamma$. Set $\overline{v}^+ = \overline{v}\, \mathbb{1}_{\{\overline{v}>0\}}$ and $w = \overline{v}^+ e^{\lambda t} \varphi$. We have

$$\begin{aligned} &-\int_\Gamma \partial_t \left(\frac{(\overline{v}^+)^2}{2} e^{\lambda t} \right) \varphi dx + \int_\Gamma \frac{\lambda}{2} (\overline{v}^+)^2 e^{\lambda t} \varphi dx + \int_\Gamma \mu \partial \overline{v}^+ \partial(\overline{v}^+ \varphi) e^{\lambda t} dx \\ &+ \int_\Gamma [H(x,\partial v) - H(x,\partial \hat{v})]\, \overline{v}^+ \varphi e^{\lambda t} dx = 0. \end{aligned}$$

Integrating from 0 to T, we get

$$\begin{aligned} &\int_\Gamma \left(\frac{\overline{v}^+(0)^2}{2} - \frac{\overline{v}^+(T)^2}{2} e^{\lambda T} \right) \varphi dx + \int_0^T \int_\Gamma \frac{\lambda}{2} (\overline{v}^+)^2 e^{\lambda t} \varphi dx dt \\ &+ \int_0^T \int_\Gamma \mu (\partial \overline{v}^+)^2 \varphi e^{\lambda t} dx dt + \int_0^T \int_\Gamma \mu \partial \overline{v}^+ \overline{v}^+ \partial \varphi e^{\lambda t} dx dt \\ &+ \int_0^T \int_\Gamma [H(x,\partial v) - H(x,\partial \hat{v})]\, \overline{v}^+ \varphi e^{\lambda t} dx dt = 0. \end{aligned}$$

From (1.32), $|H(x,\partial v) - H(x,\partial \hat{v})| \leqslant C_0 |\partial \overline{v}|$. Hence, since $\overline{v}^+(T) = 0$ and $|\partial \overline{v}| \overline{v}^+ = |\partial \overline{v}^+| \overline{v}^+$ almost everywhere, we get

$$\int_0^T \int_\Gamma \left(\frac{\lambda}{2} (\overline{v}^+)^2 + \mu (\partial \overline{v}^+)^2 \right) e^{\lambda t} \varphi dx dt - \int_0^T \int_\Gamma (\mu |\partial \varphi| + C_0 \varphi)\, |\partial \overline{v}^+| \overline{v}^+ e^{\lambda t} dx dt \leqslant 0. \tag{4.3}$$

For λ large enough, the first term in the left hand side is not smaller than the second term. This implies that $\overline{v}^+ = 0$. $\square$

Now we prove Theorem 4.2. We start with a bounded Hamiltonian H.

Proof of existence in Theorem 4.2 when H is bounded by C_H. Take $\overline{v} \in L^2(0,T;V)$ and $f \in L^2(\Gamma \times (0,T))$. From Theorem 2.4 and Theorem 2.5 with $h = f - H(x, \partial\overline{v})$ and $v_T \in V$, the following boundary value problem

$$\begin{cases} -\partial_t v - \mu_\alpha \partial^2 v = f - H(x, \partial\overline{v}), & \text{in } (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \ \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i, t) = v|_{\Gamma_\beta}(\nu_i, t), & t \in (0,T), \ \alpha, \beta \in \mathcal{A}_i, \ \nu_i \in \mathcal{A}_i, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i, t) = 0, & t \in \times (0,T), \ \nu_i \in \mathcal{V}, \\ v(x,T) = v_T(x), & x \in \Gamma, \end{cases} \tag{4.4}$$

has a unique weak solution $v \in L^2\left(0,T; H^2(\Gamma)\right) \cap C([0,T];V) \cap W^{1,2}\left(0,T;L^2(\Gamma)\right)$. This allows us to define the map T:

$$T : L^2(0,T;V) \longrightarrow L^2(0,T;V), \qquad \overline{v} \longmapsto v.$$

From (1.32), $\overline{v} \longmapsto H(x, \partial\overline{v})$ is continuous from $L^2(0,T;V)$ into $L^2(\Gamma \times (0,T))$. Using again Theorem 2.5, we have that T is continuous from $L^2(0,T;V)$ to $L^2(0,T;V)$. Moreover, there exists a constant C depending only on C_H,Γ,$(\mu_\alpha)_{\alpha\in\mathcal{A}}$, f, T, φ and v_T such that

$$\|\partial_t v\|_{L^2(\Gamma\times(0,T))} + \|v\|_{L^2(0,T;H^2(\Gamma))} \leqslant C. \tag{4.5}$$

Therefore, from Aubin-Lions theorem (see Lemma A.1), we obtain that $T\left(L^2(0,T;V)\right)$ is relatively compact in $L^2(0,T;V)$. By Schauder fixed point theorem, see [10, Corollary 11.2], T admits a fixed point which is a weak solution of (4.1). $\square$

Proof of existence in Theorem 4.2 in the general case. Now we truncate the Hamiltonian as follows

$$H_n(x,p) = \begin{cases} H(x,p) & \text{if } |p| \leqslant n, \\ H\left(x, \dfrac{p}{|p|} n\right) & \text{if } |p| > n. \end{cases}$$

From the previous proof for bounded Hamiltonians, for all n, there exists a solution $v_n \in L^2\left(0,T;H^2(\Gamma)\right) \cap C([0,T];V) \cap W^{1,2}\left(0,T;L^2(\Gamma)\right)$ of (4.1), where H is replaced by H_n. We propose to send n to $+\infty$ and to show a subsequence of $\{v_n\}$ converges to a solution of (4.1). Hence, we need some uniform estimates for $\{v_n\}$. As in the proof of Proposition 4.3, using $-v_n e^{\lambda t}\varphi$ as a test-function, integrating from 0 to T and noticing that H is sublinear, see (1.31), we obtain

$$\begin{aligned} & \int_\Gamma \left[\frac{v_n^2(x,0)}{2} - \frac{v_n^2(x,T)}{2} e^{\lambda T}\right] \varphi dx + \int_0^T \int_\Gamma \left[\frac{\lambda}{2} v_n^2 e^{\lambda t}\varphi + \mu |\partial v_n|^2 e^{\lambda t}\varphi + \mu \partial v_n v_n e^{\lambda t} \partial\varphi\right] dxdt \\ = \ & -\int_0^T \int_\Gamma H_n(x, \partial v_n) v_n e^{\lambda t}\varphi dxdt + \int_0^T \int_\Gamma f v_n e^{\lambda t}\varphi dxdt \\ \leqslant \ & C_0 \int_0^T \int_\Gamma (1 + |\partial v_n|) |v_n| e^{\lambda t}\varphi dxdt + \frac{1}{2}\int_0^T \int_\Gamma f^2 e^{\lambda t}\varphi dxdt + \frac{1}{2}\int_0^T \int_\Gamma v_n^2 e^{\lambda t}\varphi dxdt. \end{aligned}$$

In the following lines, the constant C above will vary from line to line and will depend only on $(\mu_\alpha)_{\alpha\in\mathcal{A}}, C_H, T$ and φ. Taking λ large enough leads to the following estimate:

$$\|v_n\|_{L^2(0,T;V)} \leqslant C\left(\|f\|_{L^2(0,T;L^2(\Gamma))} + \|v_T\|_{L^2(\Gamma)} + 1\right), \tag{4.6}$$

and thus, from (1.31) again, we also obtain

$$\int_0^T \int_\Gamma |H_n(x, \partial v_n)|^2 \, dxdt \leqslant \int_0^T \int_\Gamma C_0^2 (|\partial v_n| + 1)^2 \, dxdt \leqslant \int_0^T \int_\Gamma 2C_0^2 \left(|\partial v_n|^2 + 1 \right) dxdt$$
$$\leqslant C \left(\|f\|^2_{L^2(0,T;L^2(\Gamma))} + \|v_T\|^2_{L^2(\Gamma)} + 1 \right).$$

Therefore, $\{H_n(x, \partial v_n) - f\}$ is uniformly bounded in $L^2\left(0, T; L^2(\Gamma)\right)$. From Theorem 2.5, we obtain that $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^2\left(0, T; H^2(\Gamma)\right) \cap C([0,T]; V) \cap W^{1,2}\left(0, T; L^2(\Gamma)\right)$. By the Aubin-Lions theorem (see Lemma A.1), $(v_n)_n$ is relatively compact in $L^2(0, T; V)$ (and bounded in $C([0,T]; V)$). Hence, up to the extraction of a subsequence, there exists $v \in L^2(0, T; V) \cap W^{1,2}\left(0, T; L^2(\Gamma)\right)$ such that

$$v_n \to v, \quad \text{in } L^2(0, T; V) \quad \text{(strongly)}, \qquad \partial_t v_n \rightharpoonup \partial_t v, \quad \text{in } L^2(\Gamma \times (0, T)) \quad \text{(weakly)}. \tag{4.7}$$

Hence, $H_n(x, \partial v_n) \to H(x, \partial v)$ a.e. in $\Gamma \times (0, T)$. Note also that we can apply Lebesgue dominated convergence theorem to $H_n(x, \partial v_n)$ because $H_n(x, \partial v_n) \leqslant H(x, \partial v_n) \leqslant C_0(1 + |\partial v_n|)$. Therefore, $H_n(x, \partial v_n) \to H(x, \partial v)$ in $L^2(\Gamma \times (0, T))$. Thus, it is possible to pass to the limit in the weak formulation satisfied by v_n and obtain that for all $w \in W$, $\chi \in C_c(0, T)$,

$$\int_0^T \chi(t) \left(-\int_\Gamma \partial_t v w dx + \int_\Gamma \partial v \partial w dx + \int_\Gamma H(x, \partial v) \, w dx \right) dt = \int_0^T \chi(t) \left(\int_\Gamma f w dx \right) dt.$$

Therefore, v satisfies the first line in (4.2).

From Theorem 2.4, $v_n(T) = v_T$ for all n. Since for all $\alpha \in \mathcal{A}$, $(v_n)_n$ tends to v in $L^2(\Gamma_\alpha \times (0, T))$ strongly and in $W^{1,2}(\Gamma_\alpha \times (0, T))$ weakly, $v_n|_{\Gamma_\alpha \times \{t=T\}}$ converges to $v|_{\Gamma_\alpha \times \{t=T\}}$ in $L^2(\Gamma_\alpha)$ strongly. Passing to the limit in the latter identity, we get the second condition in (4.2). We have proven that v is a weak solution of (4.1). $\square$

We end the section with a stability result for the Hamilton-Jacobi equation.

Lemma 4.4. *Let $(v_{T\varepsilon})_\varepsilon, (f_\varepsilon)_\varepsilon$ be sequences of functions satisfying*

$$v_{T\varepsilon} \longrightarrow v_T \text{ in } V, \qquad f_\varepsilon \longrightarrow f \text{ in } L^2(\Gamma \times (0, T)).$$

Let v_ε be the weak solution of (4.1) with data $v_{T\varepsilon}, f_\varepsilon$, then $(v_\varepsilon)_\varepsilon$ converges in $L^2\left(0, T; H^2(\Gamma)\right) \cap C([0,T]; V) \cap W^{1,2}\left(0, T; L^2(\Gamma)\right)$ to the weak solution v of (4.1) with data v_T, f.

Proof of Lemma 4.4. Subtracting the two PDEs for v_ε and v, multiplying by $(v_\varepsilon - v) e^{\lambda t} \varphi^{-1}$, taking the integral on $\Gamma \times (0, T)$ and using similar computations as in the proof of Proposition 4.3, we obtain

$$\|v_\varepsilon - v\|_{L^2(0,T;V)} \leqslant C \left(\|f_\varepsilon - f\|_{L^2(\Gamma \times (0,T))} + \|v_{T\varepsilon} - v_T\|_{L^2(\Gamma)} \right),$$

for λ large enough and C independent of ε. This proves the convergence of v_ϵ to v in $L^2(0, T; V)$. Then, the convergence in $L^2\left(0, T; H^2(\Gamma)\right) \cap C([0,T]; V) \cap W^{1,2}\left(0, T; L^2(\Gamma)\right)$ results from the assumption that H is Lipschitz with respect to its second argument, and from stability results for the linear boundary value problem (2.1) which are obtained with similar arguments as in the proof of Theorem 2.5. $\square$

4.2 Regularity for the Hamilton-Jacobi equation

In this section, we prove further regularity for the solution of (4.1).

Theorem 4.5. *We suppose that the assumptions of Theorem 4.2 hold and that, in addition, $v_T \in H^2(\Gamma)$ satisfies the Kirchhoff conditions given by the third equation in (4.1), $f \in PC(\Gamma \times [0,T]) \cap L^2(0,T;H_b^1(\Omega))$ and $\partial_t f \in L^2(0,T;H_b^1(\Gamma))$.*

Then, the unique solution v of (4.1) satisfies $v \in L^2\left(0,T;H^3\left(\Gamma\right)\right)$ and $\partial_t v \in L^2\left(0,T;H^1\left(\Gamma\right)\right)$. Moreover, there exists a constant C depending only on $\|v_T\|_{H^2(\Gamma)}$, $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, H and f such that

$$\|v\|_{L^2(0,T;H^3(\Gamma))} + \|\partial_t v\|_{L^2(0,T;H^1(\Gamma))} \leqslant C. \tag{4.8}$$

If, in addition, there exists $\eta \in (0,1)$ such that $v_T \in C^{2+\eta}(\Gamma)$ then there exists $\tau \in (0,1)$ such $v \in C^{2+\tau,1+\frac{\tau}{2}}(\Gamma \times [0,T])$, and v is a classical solution of (4.1).

The main idea to prove Theorem 4.5 is to differentiate (4.1) with respect to the space variable and to prove some regularity properties for the derived equation. Let us explain formally our method. Assuming the solution v of (4.1) is in $C^{2,1}(\Gamma \times (0,T))$ and taking the space-derivative of (4.1) on $(\Gamma_\alpha \backslash \mathcal{V}) \times (0,T)$, we have

$$-\partial_t \partial v - \mu_\alpha \partial^3 v + \partial\left(H(x,\partial v)\right) = \partial f.$$

Therefore, $u = \partial v$ satisfies the following PDE

$$-\partial_t u - \mu_\alpha \partial^2 u + \partial\left(H\left(x,u\right)\right) = \partial f,$$

with terminal condition $u(x,T) = \partial v_T(x)$. From the Kirchhoff conditions in (4.1) and Remark 1.1, we obtain a condition for u of Dirichlet type, namely

$$\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha \gamma_{i\alpha} n_{i\alpha} u|_{\Gamma_\alpha}(\nu_i,t) = 0, \quad t \in (0,T),\ \nu_i \in \mathcal{V}.$$

Note that the latter condition is an homogeneous Dirichlet condition at the boundary vertices of Γ.

Now, by extending continuously the PDEs in (4.1) until the vertex ν_i in the branchs Γ_α and Γ_β, $\alpha,\beta \in \mathcal{A}_i$, and using the continuity condition in (4.1), one gets

$$-\mu_\alpha \partial^2 v|_{\Gamma_\alpha} + H^\alpha\left(\nu_i, \partial v|_{\Gamma_\alpha}(\nu_i,t)\right) - f|_{\Gamma_\alpha}(\nu_i,t) = -\mu_\beta \partial^2 v|_{\Gamma_\beta} + H^\beta(\nu_i, \partial v|_{\Gamma_\beta}(\nu_i,t)) - f|_{\Gamma_\beta}(\nu_i,t).$$

This gives a second transmission condition for u at $\nu_i \in \mathcal{V}\backslash\partial\Gamma$ of Robin type, namely

$$\begin{aligned} &\mu_\alpha \partial u|_{\Gamma_\alpha}\left(\nu_i,t\right) - H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}\left(\nu_i,t\right) \\ =&\mu_\beta \partial u|_{\Gamma_\beta}\left(\nu_i,t\right) - H^\beta(\nu_i, u|_{\Gamma_\beta}(\nu_i,t)) + f|_{\Gamma_\beta}(\nu_i,t), \end{aligned} \tag{4.9}$$

which is equivalent to

$$\begin{aligned} &\mu_\alpha n_{i\alpha} \partial_\alpha u\left(\nu_i,t\right) - H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}\left(\nu_i,t\right) \\ =&\mu_\beta n_{i\beta} \partial_\beta u\left(\nu_i,t\right) - H^\beta(\nu_i, u|_{\Gamma_\beta}(\nu_i,t)) + f|_{\Gamma_\beta}(\nu_i,t). \end{aligned} \tag{4.10}$$

Hence, we shall study the following nonlinear boundary value problem for $u = \partial v$,

$$\begin{cases} -\partial_t u - \mu_\alpha \partial^2 u + \partial \left(H\left(x,u\right)\right) = \partial f\left(x,t\right), & (x,t) \in \left(\Gamma_\alpha \backslash \mathcal{V}\right) \times (0,T), \ \alpha \in \mathcal{A}, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha n_{i\alpha} u|_{\Gamma_\alpha}\left(\nu_i, t\right) = 0, & t \in (0,T), \ \nu_i \in \mathcal{V}, \\ \mu_\alpha n_{i\alpha} \partial_\alpha u\left(\nu_i, t\right) - H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}\left(\nu_i, t\right) & \\ \quad = \mu_\beta n_{i\beta} \partial_\beta u\left(\nu_i, t\right) - H^\beta(\nu_i, u|_{\Gamma_\beta}(\nu_i,t)) + f|_{\Gamma_\beta}\left(\nu_i, t\right), & t \in (0,T), \ \alpha, \beta \in \mathcal{A}_i, \ \nu_i \in \mathcal{V} \backslash \partial\Gamma, \\ u\left(x,T\right) = u_T\left(x\right), & x \in \Gamma, \end{cases} \tag{4.11}$$

where $\partial f \in L^2\left(\Gamma \times (0,T)\right)$ and $u_T \in F$ defined in (4.12) below. Theorem 4.5 will follow by choosing $u_T = \partial v_T$.

In order to define the weak solutions of (4.11), we need the following subspaces of $H^1_b\left(\Gamma\right)$.

Definition 4.6. We define the Sobolev spaces

$$F := \left\{ u \in H^1_b\left(\Gamma\right) \text{ and } \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha n_{i\alpha} u|_{\Gamma_\alpha}\left(\nu_i\right) = 0 \text{ for all } \nu_i \in \mathcal{V} \right\}, \tag{4.12}$$

$$E := \left\{ \mathsf{e} \in H^1_b\left(\Gamma\right) \text{ and } \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \mathsf{e}|_{\Gamma_\alpha}\left(\nu_i\right) = 0 \text{ for all } \nu_i \in \mathcal{V} \right\}. \tag{4.13}$$

Definition 4.7. Let the function ψ be defined as follows:

$$\begin{cases} \psi_\alpha \text{ is affine on } (0, \ell_\alpha), \\ \psi|_{\Gamma_\alpha}\left(\nu_i\right) = \mu_\alpha \gamma_{i\alpha}, \text{ if } \nu_i \in \mathcal{V} \backslash \partial\Gamma, \ \alpha \in \mathcal{A}_i, \\ \psi \text{ is constant on the edges } \Gamma_\alpha \text{ which touch the boundary of } \Gamma. \end{cases} \tag{4.14}$$

Note that ψ is positive and bounded. The map $f \longmapsto f\psi$ is an isomorphism from F onto E.

Definition 4.8. A weak solution of (4.11) is a function $u \in L^2\left(0,T;F\right)$ such that $\partial_t u \in L^2\left(0,T;E'\right)$, and

$$\begin{cases} -\langle \partial_t u, \mathsf{e} \rangle_{E',E} + \displaystyle\int_\Gamma \Big(\mu \partial u \partial \mathsf{e} - \left(H\left(x,u\right)\right) \partial \mathsf{e} \Big) dx = -\int_\Gamma f \partial \mathsf{e} dx, & \text{for all } e \in E, \ \text{a.a } t \in (0,T), \\ u\left(\cdot, T\right) = u_T. \end{cases} \tag{4.15}$$

Remark 4.9. Note that if u is regular enough, then (4.15) can also be written

$$- \left\langle \partial_t u, \mathsf{e} \right\rangle_{E',E} + \int_\Gamma \Big(\mu \partial u \partial \mathsf{e} + \partial\left(H\left(x,u\right)\right) \mathsf{e} \Big) dx - \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \left[H^\alpha\left(\nu_i, u|_{\Gamma_\alpha}\left(\nu_i, t\right)\right) - f|_{\Gamma_\alpha}\left(\nu_i, t\right) \right] \mathsf{e}|_{\Gamma_\alpha}\left(\nu_i\right)$$
$$= \int_\Gamma (\partial f) \mathsf{e} dx \qquad \text{for all } e \in E, \ \text{a.a } t \in (0,T). \tag{4.16}$$

Remark 4.10. To explain formally the definition of weak solutions, let us use $\mathsf{e} \in E$ as a test-function in the PDE in (4.11). After an integration by parts, we get

$$\int_\Gamma \left(-\partial_t u \mathsf{e} + \mu \partial u \partial \mathsf{e} + \partial\left(H\left(x,u\right)\right) \mathsf{e} \right) dx - \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \mu_\alpha \partial u|_{\Gamma_\alpha}(\nu_i, t) \mathsf{e}|_{\Gamma_\alpha}(\nu_i) = \int_\Gamma (\partial f) \mathsf{e} dx,$$

where $n_{i\alpha}$ is defined in (1.6). On the one hand, from the second transmission condition, there exists a function $c_i : (0,T) \to \mathbb{R}$ such that $\mu_\alpha \partial u|_{\Gamma_\alpha}(\nu_i,t) - H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}(\nu_i,t) = c_i(t)$ for all $\alpha \in \mathcal{A}_i$. It follows that

$$\begin{aligned}
& -\sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} n_{i\alpha}\mu_\alpha \partial u|_{\Gamma_\alpha}(\nu_i,t)\mathsf{e}|_{\Gamma_\alpha}(\nu_i) \\
= \quad & -\sum_{i\in I} c_i(t)\sum_{\alpha\in\mathcal{A}_i} n_{i\alpha}\mathsf{e}|_{\Gamma_\alpha}(\nu_i) + \sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} n_{i\alpha}\left[-H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}(\nu_i,t)\right]\mathsf{e}|_{\Gamma_\alpha}(\nu_i) \\
= \quad & \sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} n_{i\alpha}\left[-H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}(\nu_i,t)\right]\mathsf{e}|_{\Gamma_\alpha}(\nu_i),
\end{aligned}$$

because $\mathsf{e} \in E$. Then we may use the Remark 4.9 and obtain (4.15).

We start by proving the following result about (4.11) and then give the proof of Theorem 4.5.

Theorem 4.11. *Under the running assumptions, if $u_T \in F$, $f \in C(\Gamma \times [0,T]) \cap L^2(0,T;H_b^1(\Gamma)$ and $\partial_t f \in L^2(0,T;H_b^1(\Gamma))$, then (4.11) has a unique weak solution u. Moreover, there exists a constant C depending only on Γ, T, ψ, $\|u_T\|_F$, $\|\partial f\|_{L^2(\Gamma\times(0,T))}$, $\|f\|_{C(\Gamma\times[0,T])}$ and $\|\partial_t f\|_{L^2(0,T;H_b^1(\Gamma))}$ such that*

$$\|u\|_{L^2(0,T;H_b^2(\Gamma))} + \|u\|_{C([0,T];F)} + \|\partial_t u\|_{L^2(\Gamma\times(0,T))} \leqslant C. \tag{4.17}$$

Remark 4.12. Theorem 4.11 implies that $u(\cdot,t) \in C^1(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}$ for a.e. t. Hence, the transmission conditions for u hold in a classical sense for a.e. $t \in [0,T]$.

We use the Galerkin's method to construct solutions of certain finite-dimension approximations to (4.11).

We notice first that the symmetric bilinear form $\breve{\mathscr{B}}(u,v) := \int_\Gamma \mu\psi^{-1}\partial u \partial v$ is such that $(u,v) \mapsto (u,v)_{L^2(\Gamma)} + \breve{\mathscr{B}}(u,v)$ is an inner product in E equivalent to the standard inner product in E, namely $(u,v)_E = (u,v)_{L^2(\Gamma)} + \int_\Gamma \partial u \partial v$. Therefore, by standard Fredholm's theory, there exist

- a non decreasing sequence of nonnegative real numbers $(\lambda_k)_{k=1}^\infty$, that tends to $+\infty$ as $k \to \infty$
- A Hilbert basis $(\mathsf{e}_k)_{k=1}^\infty$ of $L^2(\Gamma)$, which is also a a total sequence of E (and orthogonal if E is endowed with the scalar product $(u,v)_{L^2(\Gamma)} + \breve{\mathscr{B}}(u,v)$),

such that

$$\breve{\mathscr{B}}(\mathsf{e}_k, e) = \lambda_k(\mathsf{e}_k, e)_{L^2(\Gamma)}, \quad \text{for all } e \in E. \tag{4.18}$$

Note that

$$\int_\Gamma \mu \partial \mathsf{e}_k \partial \mathsf{e}_\ell \psi^{-1} dx = \begin{cases} \lambda_k & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell. \end{cases}$$

Note also that e_k is a weak solution of

$$\begin{cases} -\mu_\alpha \partial\left(\psi^{-1}\partial \mathsf{e}_k\right) = \lambda_k \mathsf{e}_k & \text{in } \Gamma_\alpha\backslash\mathcal{V}, \alpha \in \mathcal{A}, \\ \dfrac{\partial_\alpha \mathsf{e}_k(\nu_i)}{\gamma_{i\alpha}} = \dfrac{\partial_\beta \mathsf{e}_k(\nu_i)}{\gamma_{i\beta}} & \text{for all } \alpha,\beta \in \mathcal{A}_i, \\ \sum_{\alpha\in\mathcal{A}_i} n_{i\alpha} e_k|_{\Gamma_\alpha}(\nu_i) = 0 & \text{if } \nu_i \in \mathcal{V}. \end{cases} \tag{4.19}$$

which implies that $\mathsf{e}_k|_{\Gamma_\alpha} \in C^2(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}$.

Finally, the sequence $(\mathsf{f}_k)_{k=1}^\infty$ given by $\mathsf{f}_k = \psi^{-1}\mathsf{e}_k$ is a total family in F (but is not orthogonal).

Lemma 4.13. *Under the assumptions made in Theorem 4.11, for any positive integer n, there exist n absolutely continuous functions $y_k^n : [0,T] \to \mathbb{R}$, $k = 1,\dots,n$, and a function u_n : $[0,T] \to L^2(\Gamma)$ of the form*

$$u_n(t) = \sum_{k=1}^{n} y_k^n(t)\, \mathsf{f}_k, \tag{4.20}$$

such that for all $k = 1,\dots,n$,

$$y_k^n(T) = \int_\Gamma u_T \mathsf{f}_k \psi^2 dx, \tag{4.21}$$

and

$$-\frac{d}{dt}(u_n, \mathsf{f}_k\psi)_{L^2(\Gamma)} + \int_\Gamma \Big(\mu \partial u_n - H(x, u_n)\Big) \partial(\mathsf{f}_k \psi)\, dx = -\int_\Gamma f \partial(\mathsf{f}_k\psi) dx. \tag{4.22}$$

Proof of Lemma 4.13. The proof follows the same lines as the one of Lemma 2.2 but it is more technical since we obtain a system of nonlinear differential equations. For $n \geqslant 1$, we consider the symmetric n by n matrix M_n defined by

$$(M_n)_{k\ell} = \int_\Gamma \mathsf{f}_k \mathsf{f}_\ell \psi dx.$$

Since ψ is positive and bounded and since $(\psi \mathsf{f}_k)_{k=1}^\infty$ is a Hilbert basis of $L^2(\Gamma)$, we can check that M_n is a positive definite matrix and there exist two constants c, C independent of n such that

$$c\,|\xi|^2 \leqslant \sum_{k,\ell=1}^{n} (M_n)_{k\ell}\, \xi_k \xi_\ell \leqslant C\,|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{4.23}$$

Looking for u_n of the form (4.20) and setting $Y = (y_1^n, \dots, y_n^n)^T$, $\dot{Y} = \left(\frac{d}{dt}y_1^n, \dots, \frac{d}{dt}y_n^n\right)^T$, (2.7) implies that we have to solve the following a system of ODEs:

$$\begin{cases} -M_n \dot{Y}(t) + BY(t) + \mathcal{H}(Y)(t) = G(t), & t \in [0,T] \\ Y(T) = \left(\displaystyle\int_\Gamma u_T \mathsf{f}_1 \psi^2 dx, \cdots, \int_\Gamma u_T \mathsf{f}_n \psi^2 dx\right)^T, \end{cases} \tag{4.24}$$

where

- $B_{k\ell} = \int_\Gamma \mu \partial \mathsf{f}_\ell \partial(\psi \mathsf{f}_k) dx$
- $\mathcal{H}_i(Y) = -\int_\Gamma H(x, Y^T F)\partial(\mathsf{f}_i\psi)dx$ with $F = (\mathsf{f}_1, \cdots, \mathsf{f}_n)^T$ and $Y^T F = \sum_\ell y_\ell^n \mathsf{f}_\ell = u_n$
- $G_i(t) = -\int_\Gamma f(x,t)\partial(\mathsf{f}_i\psi)dx$ for all $i \in 1,\cdots,n$.

Since the matrix M is invertible and the function $\mathcal{H}$ is Lipschitz continuous by (1.32), the system (4.24) has a unique global solution. This ends the proof of the lemma. $\square$

We start by giving some estimates for the approximation u_n.

Lemma 4.14. *Under the assumptions made in Theorem 4.11, there exists a constant C depending only on Γ, T, ψ, $\|u_T\|_F$, $\|\partial f\|_{L^2(\Gamma\times(0,T))}$ $\|f\|_{C(\Gamma\times[0,T])}$ and $\|\partial_t f\|_{L^2(0,T;H_b^1(\Gamma))}$ such that*

$$\|u_n\|_{L^\infty(0,T;F)} + \|u_n\|_{L^2\left(0,T;H_b^2(\Gamma)\right)} + \|\partial_t u_n\|_{L^2(\Gamma\times(0,T))} \leqslant C.$$

Proof of Lemma 4.14. We divide the proof into two steps:

Step 1: Uniform estimates of u_n in $L^\infty(0,T;L^2(\Gamma))$, $L^2(0,T;F)$ and $W^{1,2}(0,T;E')$. Multiplying (4.22) by $y_k^n(t)\,\mathsf{f}_k e^{\lambda t}\psi$ where λ is a positive constant to be chosen later, summing for $k=1,\dots,n$ and using (4.20), we get

$$-\int_\Gamma \partial_t u_n u_n e^{\lambda t}\psi dx + \int_\Gamma \Big(\mu\partial u_n - H(x,u_n)\Big)\partial\left(u_n e^{\lambda t}\psi\right)dx = -\int_\Gamma f\partial(u_n\psi e^{\lambda t})dx.$$

In the following lines, C will be a constant that may vary from lines to lines. Since H satisfies (1.31) and f is bounded, there exists a constant C such that

$$\begin{aligned}&-\int_\Gamma\left[\partial_t\left(\frac{u_n^2}{2}e^{\lambda t}\right)-\frac{\lambda}{2}u_n^2e^{\lambda t}\right]\psi dx+\int_\Gamma\mu\left|\partial u_n\right|^2e^{\lambda t}\psi dx - C\int_\Gamma |u_n|\left(|u_n|+|\partial u_n|\right)e^{\lambda t}dx\\ \leqslant& C\int_\Gamma(|u_n|+|\partial u_n|)e^{\lambda t}dx. \end{aligned}\tag{4.25}$$

The desired estimate on u_n is obtained from the previous inequality in a similar way as in the proof of Lemma 2.3, by taking λ large enough.

Step 2: Uniform estimates of u_n in $L^\infty(0,T;F)\cap L^2(0,T;H_b^2(\Gamma))$ and of $\partial_t u_n$ in $L^2(\Gamma\times(0,T))$. Multiplying (4.22) by $\partial_t y_k^n(t)\,\mathsf{f}_k e^{\lambda t}\psi$ where λ is a positive constant to be chosen later, integrating by part the term containing H and f (all the integration by parts are justified) summing for $k=1,\dots,n$ and using (4.20), we obtain that

$$\begin{aligned}&-\int_\Gamma(\partial_t u_n)^2e^{\lambda t}\psi dx+\int_\Gamma\mu\partial u_n\partial\left(\partial_t u_n e^{\lambda t}\psi\right)dx+\int_\Gamma\partial\left(H\left(x,u_n\right)\right)\partial_t u_n e^{\lambda t}\psi dx\\ &-\sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i}n_{i\alpha}\left[H^\alpha(\nu_i,u_n|_{\Gamma_\alpha}(\nu_i,t))-f|_{\Gamma_\alpha}(\nu_i,t)\right]\partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\,\psi|_{\Gamma_\alpha}(\nu_i)\,e^{\lambda t}=\int_\Gamma\partial f\partial_t u_n\psi e^{\lambda t}dx.\end{aligned}\tag{4.26}$$

Note that from (1.32) and (1.33),

$$\left|\partial\left(H\left(x,u_n\right)\right)\right|\leqslant C_0(1+|u_n|+|\partial u_n|)\tag{4.27}$$

so, from Step 1, this function is bounded in $L^2(\Gamma\times(0,T))$ by a constant. Moreover,

$$\int_s^T\int_\Gamma\partial f\partial_t u_n\psi e^{\lambda t}dxdt\leqslant C\left(\int_s^T\int_\Gamma(\partial f)^2e^{\lambda t}dxdt\right)^{\frac12}\left(\int_s^T\int_\Gamma(\partial_t u_n)^2e^{\lambda t}\psi dxdt\right)^{\frac12},\tag{4.28}$$

and we can also estimate the term $\int_\Gamma\mu\partial u_n\partial\left(\partial_t u_n e^{\lambda t}\psi\right)dx$ as in the proof of Theorem 2.5. Therefore, the only new difficulty with respect to the proof of Theorem 2.5 consists of obtaining a bound for the term

$$\sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i}n_{i\alpha}\left[H^\alpha\left(\nu_i,u_n|_{\Gamma_\alpha}(\nu_i,t)\right)-f|_{\Gamma_\alpha}(\nu_i,t)\right]\partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda t}\psi|_{\Gamma_\alpha}(\nu_i).$$

Let $\mathcal{J}_{i\alpha}(p)$ be the primitive function of $p\mapsto H^\alpha(\nu_i,p)$ such that $\mathcal{J}_{i\alpha}(0)=0$:

$$H^\alpha\left(\nu_i,u_n|_{\Gamma_\alpha}(\nu_i,s)\right)\partial_t u_n|_{\Gamma_\alpha}(\nu_i,s)=\frac{d}{dt}\mathcal{J}_{i\alpha}(u_n|_{\Gamma_\alpha}(\nu_i,s)).$$

We can then write

$$\begin{aligned}&-\int_s^T\Big(n_{i\alpha}H^\alpha\left(\nu_i,u_n|_{\Gamma_\alpha}(\nu_i,t)\right)\partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda t}\psi|_{\Gamma_\alpha}(\nu_i)\Big)dt\\ =&n_{i\alpha}\psi|_{\Gamma_\alpha}(\nu_i)\left(-\mathcal{J}_{i\alpha}\left(u_n|_{\Gamma_\alpha}(\nu_i,T)\right)e^{\lambda T}+\mathcal{J}_{i\alpha}\left(u_n|_{\Gamma_\alpha}(\nu_i,s)\right)e^{\lambda s}+\lambda\int_s^T\mathcal{J}_{i\alpha}\left(u_n|_{\Gamma_\alpha}(\nu_i,t)\right)e^{\lambda t}dt\right).\end{aligned}$$

Since $H^\alpha(x,\cdot)$ is sublinear, see (1.31), $|\mathcal{J}_{i\alpha}(p)|$ is subquadratic, i.e., $|\mathcal{J}_{i\alpha}(p)| \leqslant C(1+p^2)$, for a constant C independent of α and i. This implies that

$$\begin{aligned}
&\left|\int_s^T \left(n_{i\alpha} H^\alpha\left(\nu_i, u_n|_{\Gamma_\alpha}(\nu_i,t)\right) \partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\, e^{\lambda t} \psi|_{\Gamma_\alpha}(\nu_i)\right) dt\right| \\
\leqslant & C\left(e^{\lambda T} + u_n^2|_{\Gamma_\alpha}(\nu_i,T)\, e^{\lambda T} + u_n^2|_{\Gamma_\alpha}(\nu_i,s)\, e^{\lambda s}\right) + C\lambda \int_0^T \left(1+u_n^2|_{\Gamma_\alpha}(\nu_i,t)\right) e^{\lambda t} dt.
\end{aligned}$$

Note that, from Step 1 and the stability of the trace, $\lambda \int_s^T \left(1+u_n^2|_{\Gamma_\alpha}(\nu_i,t)\right) e^{\lambda t} dt \leqslant C\lambda e^{\lambda T}$. To summarize

$$\begin{aligned}
&\left|\int_s^T \left(n_{i\alpha} H^\alpha\left(\nu_i, u_n|_{\Gamma_\alpha}(\nu_i,t)\right) \partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\, e^{\lambda t} \psi|_{\Gamma_\alpha}(\nu_i)\right) dt\right| \\
\leqslant & C\left(u_n^2|_{\Gamma_\alpha}(\nu_i,T)\, e^{\lambda T} + u_n^2|_{\Gamma_\alpha}(\nu_i,s)^2\, e^{\lambda s}\right) + \tilde{C}(\lambda).
\end{aligned} \tag{4.29}$$

Similarly, using the fact that $f \in C(\Gamma \times [0,T])$ and $\partial_t f|_{\Gamma_\alpha}(\nu_i,\cdot) \in L^2(0,T)$, and integrating by part, we see that

$$\begin{aligned}
& \left|\int_s^T f|_{\Gamma_\alpha}(\nu_i,t)\, \partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\, e^{\lambda t} dt\right| \\
= & \left|(f|_{\Gamma_\alpha} u_n)|_{\Gamma_\alpha}(\nu_i,T)\, e^{\lambda T} - (f|_{\Gamma_\alpha} u_n)|_{\Gamma_\alpha}(\nu_i,t)\, e^{\lambda s} - \int_s^T \left(\lambda f|_{\Gamma_\alpha}(\nu_i,t) + \partial_t f|_{\Gamma_\alpha}(\nu_i,t)\right) u_n|_{\Gamma_\alpha}(\nu_i,t)\, e^{\lambda t} dt\right| \\
\leqslant & C\left(|u_n|_{\Gamma_\alpha}(\nu_i,T)|e^{\lambda T} + |u_n|_{\Gamma_\alpha}(\nu_i,s)|e^{\lambda s} + \lambda \int_s^T |u_n|_{\Gamma_\alpha}(\nu_i,t)|\, e^{\lambda t} dt\right) \\
& + \frac{1}{2}\int_s^T u_n^2|_{\Gamma_\alpha}(\nu_i,t)\, e^{\lambda t} dt + \frac{1}{2}\int_s^T \left(\partial_t f|_{\Gamma_\alpha}(\nu_i,t)\right)^2 e^{\lambda t} dt.
\end{aligned}$$

From Step 1 and the assumptions on f, the last three terms in the right hand side of the latter estimate are bounded by a constant depending on λ, but not on n. To summarize,

$$\left|\int_s^T f|_{\Gamma_\alpha}(\nu_i,t)\, \partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\, e^{\lambda t} dt\right| \leqslant C\left(|u_n|_{\Gamma_\alpha}(\nu_i,T)|e^{\lambda T} + |u_n|_{\Gamma_\alpha}(\nu_i,s)|e^{\lambda s}\right) + \tilde{C}(\lambda). \tag{4.30}$$

To conclude from (4.29) and (4.30), we use the following estimates

$$\begin{cases}
|u_n|_{\Gamma_\alpha}(\nu_i,t)| \leqslant C\left(\displaystyle\int_{\Gamma_\alpha} |u_n(x,t)|\, dx + \int_{\Gamma_\alpha} |\partial u_n(x,t)|\, dx\right), \\
u_n^2|_{\Gamma_\alpha}(\nu_i,t) \leqslant C\left(\displaystyle\int_{\Gamma_\alpha} u_n^2(x,t)\, dx + \int_{\Gamma_\alpha} |u_n \partial u_n(x,t)|\, dx\right),
\end{cases} \tag{4.31}$$

for $t = s$ and $t = T$.

Then proceeding as in the proof of Theorem 2.5 and combining (4.26), (4.27), (4.28), (4.29) and (4.30) with (4.31), we find the desired estimates by taking λ large enough.

Let us end the proof by proving (4.31). The function $\phi = u_n|_{\Gamma_\alpha}(\cdot,t)$ is in $H^1(\Gamma_\alpha)$. By the continuous embedding $H^1(\Gamma_\alpha) \hookrightarrow C(\Gamma_\alpha)$, we can define ϕ in the pointwise sense (and even at two endpoints of any edges, see (1.2)). For all $\alpha \in \mathcal{A}$ and $x,y \in \Gamma_\alpha$, we have $\phi(x) = \phi(y) + \int_{[y,x]} \partial\phi(\xi) d\xi$. It follows

$$|\Gamma_\alpha|\phi(x) = \int_{\Gamma_\alpha} \phi(x) dy = \int_{\Gamma_\alpha} \phi(y) dy + \int_{\Gamma_\alpha}\int_{[y,x]} \partial\phi(\xi) d\xi dy \leqslant \int_{\Gamma_\alpha} |\phi(\xi)| d\xi + |\Gamma_\alpha| \int_{\Gamma_\alpha} |\partial\phi(\xi)| d\xi,$$

which gives the first estimate setting $x = \nu_i$. The second estimate is obtained in the same way replacing ϕ by ϕ^2 and using the fact that $W^{1,1}(\Gamma_\alpha)ss$ is continuously imbedded in $C(\Gamma_\alpha)$.

□

Proof of Theorem 4.11. From Lemma 4.14, up to the extraction of a subsequence, there exists $u \in L^2\left(0,T;H_b^2\left(\Gamma\right)\right) \cap W^{1,2}\left(\Gamma\times(0,T)\right)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } L^2\left(0,T;F\cap H_b^2\left(\Gamma\right)\right), \\ \partial_t u_n \rightharpoonup \partial_t u, & \text{in } L^2\left(\Gamma\times(0,T)\right). \end{cases} \tag{4.32}$$

Moreover, by Aubin-Lions Theorem (see Lemma A.1),

$$L^2\left(0,T;F\cap H_b^2\left(\Gamma\right)\right)\cap W^{1,2}\left(0,T;L^2\left(\Gamma\right)\right) \overset{\text{compact}}{\hookrightarrow} L^2\left(0,T;F\right),$$

so up to the extraction of a subsequence, we may assume that $u_n \to u$ in $L^2(0,T;F)$ and almost everywhere. Moreover, from the compactness of the trace operator from $W^{1,2}(\Gamma_\alpha\times(0,T))$ to $L^2(\partial\Gamma_\alpha\times(0,T))$, $u_n|_{\partial\Gamma_\alpha\times(0,T)} \to u|_{\partial\Gamma_\alpha\times(0,T)}$ in $L^2(\partial\Gamma_\alpha\times(0,T))$ and for almost every $t\in(0,T)$. Similarly, $u_n|_{\Gamma_\alpha\times\{t=T\}} \to u|_{\Gamma_\alpha\times\{t=T\}}$ in $L^2(\Gamma_\alpha)$ and almost everywhere in Γ_α. Then, using the Lipschitz continuity of H with respect to its second argument, and similar arguments as in the proof of Theorem 2.4, we obtain the existence of a solution of (4.11) satisfying (4.17) by letting $n\to+\infty$. Since $H^2(\Gamma_\alpha)\subset C^{1+\sigma}(\Gamma_\alpha)$ for some $\sigma\in(0,1/2)$, $u(\cdot,t)\in C^{1+\sigma}\left(\Gamma_\alpha\right)$ for all $\alpha\in\mathcal{A}$ and a.a. t.

Finally, the proof of uniqueness is a consequence of the energy estimate (4.17) for u. □

Next, we want to prove that, if u is the solution of (4.11) and v is the solution (4.1), then $\partial u = v$. It means that we have to define a primitive function on the network Γ.

Definition 4.15. Let $x\in\Gamma_{\alpha_0} = [\nu_{i_0},\nu_{i_1}]$ and $y\in\Gamma_{\alpha_m} = \left[\nu_{i_m},\nu_{i_{m+1}}\right]$. We denote the set of paths joining from x to y by $\overrightarrow{xy}$. More precisely, if $\mathcal{L}\in\overrightarrow{xy}$, we can write $\mathcal{L}$ under the form

$$\mathcal{L} = x\to\nu_{i_1}\to\nu_{i_2}\to\ldots\to\nu_{i_m}\to y,$$

with $\nu_{i_k}\in\mathcal{V}$ and $\left[\nu_{i_k},\nu_{i_{k+1}}\right] = \Gamma_{\alpha_k}$. The integral of a function ϕ on $\mathcal{L}$ is defined by

$$\int_{\mathcal{L}}\phi\left(\xi\right)d\xi = \int_{[x,\nu_{i_1}]}\phi\left(\xi\right)d\xi + \sum_{k=1}^{m}\int_{[\nu_{i_k},\nu_{i_{k+1}}]}\phi\left(\xi\right)d\xi + \int_{[\nu_{i_m},y]}\phi\left(\xi\right)d\xi, \tag{4.33}$$

recalling that the integrals on a segment are defined in (1.8).

Lemma 4.16. *Let u be the unique solution of (4.11) with $u_T = \partial v_T$. Then for all $x,y\in\Gamma$ and a.e. $t\in[0,T]$,*

$$\int_{\mathcal{L}_1}u\left(\zeta,t\right)d\zeta = \int_{\mathcal{L}_2}u\left(\zeta,t\right)d\zeta,\quad \text{for all } \mathcal{L}_1,\mathcal{L}_2\in\overrightarrow{xy}.$$

This means that the integral of u from x to y does not depend on the path. Hence, for any $\mathcal{L}\in\overrightarrow{xy}$, we can define

$$\int_{\overrightarrow{xy}}u\left(\zeta,t\right)d\zeta := \int_{\mathcal{L}}u\left(\zeta,t\right)d\zeta.$$

Proof of Lemma 4.16. First, it is sufficient to prove $\int_{\mathcal{L}} u(\zeta,t)\,d\zeta = 0$ for all $\mathcal{L} \in \overrightarrow{xx}$. Secondly, if a given edge is browsed twice in opposite senses, the two related contributions to the integral sum to zero. It follows that, without loss of generality, we only need to consider loops in $\overrightarrow{xx}$ such that all the complete edges that it contains are browsed once only. It is also easy to see that we can focus on the case when $x \in \mathcal{V}$. To summarize, we only need to prove that

$$\int_{\mathcal{L}} u(\zeta,t)\,d\zeta = 0$$

when $\nu_{i_0} \in \mathcal{V}\backslash\partial\Gamma$ and $\mathcal{L} = \nu_{i_0} \to \nu_{i_1} \to \ldots \to \nu_{i_m} \to \nu_{i_0}$, where $\nu_{i_k} \neq \nu_{i_\ell}$ for $k \neq l$.

The following conditions

1. $\mathsf{e}|_{\Gamma_\alpha} = 0$ on each edge Γ_α not contained in $\mathcal{L}$
2. for all $k = 0, \ldots m-1$, $\mathsf{e}|_{\Gamma_{\alpha_k}} = 1_{i_k < i_{k+1}} - 1_{i_k > i_{k+1}}$ if Γ_{α_k} is the edge joining ν_{i_k} and $\nu_{i_{k+1}}$
3. $\mathsf{e}|_{\Gamma_{\alpha_m}} = 1_{i_m < i_0} - 1_{i_m > i_0}$ if Γ_{α_m} is the edge joining ν_{i_m} and ν_{i_0}

define a unique function $\mathsf{e} \in E$ which takes at most two values on $\mathcal{L}$, namely ± 1.

From Definition 4.15, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{L}} u(\zeta,t)\,dt &= \sum_{k=0}^{m} \frac{d}{dt}\int_{[\nu_{i_k},\nu_{i_{k+1}}]} u(\zeta,t)\,d\zeta + \frac{d}{dt}\int_{[\nu_{i_m},\nu_{i_0}]} u(\zeta,t)\,d\zeta \\ &= \frac{d}{dt}\int_{\Gamma} u(\zeta,t)\,\mathsf{e}(\zeta)\,d\zeta = \int_{\Gamma} \partial_t u(\zeta,t)\,\mathsf{e}(\zeta)\,d\zeta.\end{aligned}$$

Then, using Definition 4.8, Remark 4.9 and Remark 4.10 yields that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathcal{L}} u(\zeta,t)\,d\zeta \\ =& \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[-\mu_\alpha \partial^2 u(\zeta,t) + \partial H(\zeta, u(\zeta,t)) - \partial f(\zeta,t)\right] \mathsf{e}(\zeta)\,d\zeta \\ =& \sum_{k=0}^{m} \int_{\Gamma_{\alpha_k}} \left[-\mu_{\alpha_k} \partial^2 u(\zeta,t) + \partial H(\zeta, u(\zeta,t)) - \partial f(\zeta,t)\right] \mathsf{e}(\zeta)\,d\zeta \\ =& \sum_{k=0}^{m} \mathsf{e}|_{\Gamma_{\alpha_k}}(\nu_i) \left(\begin{array}{l} n_{i_{k+1}\alpha_k} \left(-\mu_{\alpha_k} \partial u|_{\Gamma_{\alpha_k}}(\nu_{i_{k+1}},t) + H^\alpha\left(\nu_{i_{k+1}}, u|_{\Gamma_{\alpha_k}}(\nu_{i_{k+1}},t)\right) - f(\nu_{i_{k+1}},t)\right) \\ +n_{i_k\alpha_k} \left(-\mu_{\alpha_k} \partial u|_{\Gamma_{\alpha_k}}(\nu_{i_k},t) + H^\alpha\left(\nu_{i_k}, u|_{\Gamma_{\alpha_k}}(\nu_{i_k},t)\right) - f(\nu_{i_k},t)\right) \end{array} \right),\end{aligned}$$

where we have set $i_{m+1} = i_0$. Now using (4.9) (which is satisfied for a.e. t from the regularity of u) and the fact that $\mathsf{e} \in E$, we conclude that

$$\frac{d}{dt}\int_{\mathcal{L}} u(\zeta,t)\,d\zeta = 0. \tag{4.34}$$

Hence

$$\int_{\mathcal{L}} u(\zeta,t)\,d\zeta = \int_{\mathcal{L}} u(\zeta,T)\,d\zeta = \int_{\mathcal{L}} u_T(\zeta)\,d\zeta = \int_{\mathcal{L}} \partial v_T(\zeta)\,d\zeta = 0,$$

where the last identity comes from the assumption that $v_T \in V$ (the continuity of v_T).

$\square$

Lemma 4.17. *If $u_T = \partial v_T \in F$, then the weak solution u of (4.11) satisfies $u = \partial v$ where v is the unique solution of (4.1).*

Proof of Lemma 4.17. For simplicity, we write the proof in the case when $\partial\Gamma \neq \varnothing$. The proof is similar in the other case.

Let us fix some vertex $\nu_k \in \partial\Gamma$. From standard regularity results for Hamilton-Jacobi equation with homogeneous Neumann condition, we know that that there exists ω, a closed neighborhood of $\{\nu_k\}$ in Γ made of a single straigt line segment and containing no other vertices of Γ than ν_k, such that $v|_{\omega\times(0,T)} \in L^2(0,T;H^3(\omega)) \cap C([0,T];H^2(\omega) \cap W^{1,2}(0,T;H^1(\omega))$. Hence, v satisfies the Hamilton-Jacobi equation at almost every point of $\omega \times (0,T)$. Moreover the equation

$$\partial_t v(\nu_k,t) + \mu\partial^2 v(\nu_k,t) - H(\nu_k,0) + f(\nu_k,t) = 0 \tag{4.35}$$

holds for almost every $t \in (0,T)$ and in $L^2(0,T)$.

For every $x \in \Gamma$ and $t \in [0,T]$, we define

$$\hat{v}\,(x,t) = v\,(\nu_k,t) + \int_{\overrightarrow{\nu_k x}} u\,(\zeta,t)\,d\zeta. \tag{4.36}$$

Remark 4.18. If $\partial\Gamma = \varnothing$, then the proof should be modified by replacing ν_k by a point $\nu \in \Gamma\backslash\mathcal{V}$ and by using local regularity results for the HJB equation in (4.1).

We claim that $\hat{v}$ is a solution of (4.1).

First, $\hat{v}\,(\cdot,t)$ is continuous on Γ. Indeed, $\hat{v}(y,t) - \hat{v}(x,t) = \int_{\overrightarrow{xy}} u(\zeta,t)d\zeta$. On the other hand, $u \in C([0,T];F) \subset L^\infty(\Gamma \times [0,T])$. It follows that $|\hat{v}(y,t) - \hat{v}(x,t)| \leqslant ||u||_{L^\infty(\Gamma\times[0,T])}\text{dist}(x,y)$ which implies that $\hat{v}\,(\cdot,t)$ is continuous on Γ.

Next, from the terminal conditions for u,

$$\hat{v}\,(x,T) = v\,(\nu_k,T) + \int_{\overrightarrow{\nu_k x}} u\,(\zeta,T)\,d\zeta = v_T\,(\nu_k) + \int_{\overrightarrow{\nu_k x}} \partial v_T\,(\zeta)\,d\zeta = v_T\,(x)\,,$$

where the last identity follows from the continuity of v_T on Γ.

Let us check the Kirchhoff condition for $\hat{v}$. Take $\nu_i \in \mathcal{V}$ and $\alpha \in \mathcal{A}_i$. From (1.7), for a.e. $t \in (0,T)$, $\partial_\alpha \hat{v}(\nu_i,t) = n_{i\alpha}\partial\hat{v}|_{\Gamma_\alpha}(\nu_i,t)$ and from (4.36), $\partial\hat{v}|_{\Gamma_\alpha}(\nu_i,t) = u|_{\Gamma_\alpha}(\nu_i,t)$. Since $u(\cdot,t) \in F$, we get

$$\sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha\partial_\alpha\hat{v}(\nu_i,t) = \sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha n_{i\alpha} u|_{\Gamma_\alpha}(\nu_i,t) = 0,$$

which is exactly the Kirchhoff condition for $\hat{v}$ at ν_i.

There remains to prove $\hat{v}$ solves the Hamilton-Jacobi equation in $\Gamma\backslash\mathcal{V}$: Take $x \in \Gamma_\alpha\backslash\mathcal{V}$ for some $\alpha \in \mathcal{A}$ and consider a path $\overrightarrow{\nu_k x} \ni \mathcal{L} = \nu_{i_0} \to \cdots \to \nu_{i_m} \to x$, where $i_0 = k$ and $\nu_{i_m} \in \Gamma_\alpha$. Let $\nu_{i_{m+1}}$ be the other endpoint of Γ_α. We proceed as in the proof of Lemma 4.16: the following conditions

1. $\mathsf{e}|_{\Gamma_\alpha} = 0$ on each edge Γ_α not contained in $\mathcal{L}$

2. for all $j = 0, \ldots m$, $\mathsf{e}|_{\Gamma_j} = 1_{i_j<i_{j+1}} - 1_{i_j>i_{j+1}}$ if Γ_j is the edge joining ν_{i_j} and $\nu_{i_{j+1}}$

define a unique piecewise constant function e which takes at most two values on $\mathcal{L}$, namely ± 1. Note that e does not belong to E because $\mathsf{e}(\nu_k) \neq 0$, but that e satisfies $\sum_{\alpha\in\mathcal{A}_i} n_{i\alpha}\mathsf{e}|_{\Gamma_\alpha}(\nu_i) = 0$ for all $\nu_i \in \mathcal{V}\backslash\partial\Gamma$.

Using this function, a similar computation as in the proof of Lemma 4.16 implies that, for almost every $t \in (0,T)$,

$$\begin{aligned}\partial_t\hat{v}(x,t) - \partial_t v(\nu_k,t) = &- \mu_\alpha\partial u|_{\Gamma_\alpha}(x,t) + H\left(x, u|_{\Gamma_\alpha}(x,t)\right) - f\,(x,t)\\ &+ \mu_\alpha\partial_2 v\,(\nu_k,t) - H\,(\nu_k,0) + f\,(\nu_k,t)\,.\end{aligned}$$

Then, using (4.35) and the fact that $\partial \hat{v} = u$, the latter identity yields that for almost every $(x,t) \in (0,T) \times \Gamma$,

$$\partial_t \hat{v}(x,t) + \mu_\alpha \partial^2 \hat{v}(x,t) - H(x, \partial \hat{v}(x,t)) + f(x,t) = 0.$$

We have proven that $\hat{v}$ is a solution of (4.1). Since v is the unique solution of (4.1), we conclude that $v = \hat{v}$ and $\partial v = u$. $\square$

We are now ready to give the proof of Theorem 4.5.

Proof of Theorem 4.5. Since $\partial v = u$ by Lemma 4.17 and u satisfies (4.17) by Theorem 4.11, we obtain that $v \in L^2\left(0,T; H^3(\Gamma)\right)$ and $\partial_t v \in L^2\left(0,T; H^1(\Gamma)\right)$ and (4.8) holds.

Therefore, using an interpolation result combined with Sobolev embeddings, see [4] or Lemma A.2 in the Appendix, $v \in C^{1+\sigma,\sigma/2}(\Gamma \times [0,T])$ for some $0 < \sigma < 1$.

Finally, we know that since $f \in W^{1,2}(0,T,H_b^1(\Gamma))$, $f|_{\Gamma_\alpha \times [0,T]} \in C^{\eta,\eta}(\Gamma_\alpha \times [0,T])$ for all $\eta \in (0,1/2)$. If $f \in C^{\eta,\frac{\eta}{2}}(\Gamma_\alpha \times [0,T])$ for some $\eta \in (0,1/2)$, we claim that $v \in C^{2,1}(\Gamma \times [0,T])$. This is a direct consequence of a theorem of Von Below, see the main theorem in [24], for the (modified) heat equation

$$-\partial_t w - \mu_\alpha \partial^2 w = g(x,t) \quad \text{in } (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \tag{4.37}$$

with the same Kirchhoff conditions as in (4.1): Note that if the terminal Cauchy condition for w is $w(\cdot, t = T) = v_T$ and if $g = f - H(x, \partial v)$, then $w = v$. Now $g = f - H(x,\partial v) \in C^{\tau,\frac{\tau}{2}}(\Gamma_\alpha \times [0,T])$, where $1/2 > \tau = \min(\sigma, \eta) > 0$. Using the result in [24], we obtain that $v = w \in C^{2+\tau,1+\tau/2}(\Gamma_\alpha \times [0,T])$, then that v is a classical solution of (4.1). $\square$

5 Existence, uniqueness and regularity for the MFG system (Proof of Theorem 1.11)

Proof of existence in Theorem 1.11. Given m_0 and v_T, let us construct the map $T : L^2(0,T;V) \to L^2(0,T;V)$ as follows.

Given $v \in L^2(0,T;V)$, we first define m as the weak solution of (3.1) with initial data m_0 and $b = H_p(x, \partial v)$. We know that $m \in L^2(0,T;W) \cap C([0,T]; L^2(\Gamma)) \cap W^{1,2}(0,T;V')$.

We claim that if $v_n \to v$ in $L^2(0,T;V)$ then $H_p(\cdot, \partial v_n)$ tends to $H_p(\cdot, \partial v)$ in $L^2(\Gamma \times (0,T))$. To prove the claim, we argue by contradiction: assume that there exist a positive number ϵ and a subsequence $v_{\phi(n)}$ such that $\|H_p(\cdot, \partial v_{\phi(n)}) - H_p(\cdot, \partial v)\|_{L^2(\Gamma \times (0,T))} > \epsilon$. Then since $\partial v_{\phi(n)}$ tends to ∂v in $L^2(\Gamma \times (0,T))$, we can extract another subsequence $v_{\psi(n)}$ from $v_{\phi(n)}$ such that $\partial v_{\psi(n)}$ tends to ∂v almost every where in $\Gamma \times (0,T)$. From the continuity of H_p, we deduce that $H_p(\cdot, \partial v_{\psi(n)})$ tends to $H_p(\cdot, \partial v)$ almost everywhere in $\Gamma \times (0,T)$. Since there exists a positive constant C_0 such that $\|H_p(\cdot, \partial v_{\psi(n)})\|_\infty \leqslant C_0$, $\|H_p(\cdot, \partial v)\|_\infty \leqslant C_0$, Lebesgue dominated convergence theorem ensures that $H_p(\cdot, \partial v_{\psi(n)})$ tends to $H_p(\cdot, \partial v)$ in $L^2(\Gamma \times (0,T))$, which is the desired contradiction.

To summarize, $H_p(\cdot, \partial v_n)$ tends to $H_p(\cdot, \partial v)$ in $L^2(\Gamma \times (0,T))$ on the one hand, and for a positive constant C_0, $\|H_p(\cdot, \partial v_n)\|_\infty \leqslant C_0$, $\|H_p(\cdot, \partial v)\|_\infty \leqslant C_0$. Using Lemma 3.4, we see that m_n, the weak solution of (3.1) with initial data m_0 and $b = H_p(x, \partial v_n)$ converges to m in $L^2(0,T;W) \cap L^\infty\left(0,T;L^2(\Gamma)\right) \cap W^{1,2}(0,T;V')$. Hence, the map $v \mapsto m$ is continuous from $L^2(0,T;V)$ to $L^2(0,T;W) \cap L^\infty\left(0,T;L^2(\Gamma)\right) \cap W^{1,2}(0,T;V')$. Moreover, the a priori estimate (3.3) holds uniformly with respect to v.

Then, knowing m, we construct $T(v) \equiv \widetilde{v}$ as the unique weak solution of (4.1) with $f(x,t) = \mathscr{V}[m(\cdot,t)](x)$. Note that $m \mapsto f$ is continuous and locally bounded from $L^2(\Gamma \times (0,T))$ to

$L^2(\Gamma\times(0,T))$. Then Lemma 4.4 ensures that the map $m\to\tilde v$ is continuous from $L^2(\Gamma\times(0,T))$ to $L^2\left(0,T;H^2(\Gamma)\right)\cap L^\infty(0,T;V)\cap W^{1,2}(0,T;L^2(\Gamma))$. From Aubin-Lions theorem, see Lemma A.1, $m\to\tilde v$ maps bounded sets of $L^2(\Gamma\times(0,T))$ to relatively compact sets of $L^2(0,T;V)$.

Therefore, the map $T: v\mapsto\tilde v$ is continuous from $L^2(0,T;V)$ to $L^2(0,T;V)$ and has a relatively compact image. Finally, we apply Schauder fixed point theorem [10, Corollary 11.2] and conclude that the map T admits a fixed point v. We know that $v\in L^2\left(0,T;H^2(\Gamma)\right)\cap L^\infty(0,T;V)\cap W^{1,2}(0,T;L^2(\Gamma))$ and $m\in L^2(0,T;W)\cap L^\infty\left(0,T;L^2(\Gamma)\right)\cap W^{1,2}(0,T;V'(\Gamma))$.

Hence, there exists a weak solution (v,m) to the mean field games system (1.25). ☐

Proof of uniqueness in Theorem 1.11. We assume that there exist two solutions (v_1,m_1) and (v_2,m_2) of (1.25). We set $\overline{v}=v_1-v_2$ and $\overline{m}=m_1-m_2$ and write the system for $\overline{v},\overline{m}$

$$\begin{cases}-\partial_t\overline{v}-\mu_\alpha\partial^2\overline{v}+H(x,\partial v_1)-H(x,\partial v_2)-(\mathscr{V}[m_1]-\mathscr{V}[m_2])=0, & x\in\Gamma_\alpha\backslash\mathcal{V},\ t\in(0,T),\\ \partial_t\overline{m}-\mu_\alpha\partial^2\overline{m}-\partial\left(m_1\partial_pH(x,\partial m_1)-m_2\partial_pH(x,\partial m_2)\right)=0 & x\in\Gamma_\alpha\backslash\mathcal{V},\ t\in(0,T),\\ \overline{v}|_{\Gamma_\alpha}(\nu_i,t)=\overline{v}|_{\Gamma_\beta}(\nu_i,t),\ \dfrac{\overline{m}|_{\Gamma_\alpha}(\nu_i,t)}{\gamma_{i\alpha}}=\dfrac{\overline{m}|_{\Gamma_\beta}(\nu_i,t)}{\gamma_{i\beta}}, & \alpha,\beta\in\mathcal{A}_i,\ \nu_i\in\mathcal{V},\\ \displaystyle\sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha\overline{v}(\nu_i,t)=0, & \nu_i\in\mathcal{V},\ t\in(0,T),\\ \displaystyle\sum_{\alpha\in\mathcal{A}_i}n_{i\alpha}\left[m_1|_{\Gamma_\alpha}(\nu_i)\,\partial_pH^\alpha\left(\nu_i,\partial v_1|_{\Gamma_\alpha}(\nu_i,t)\right)-m_2|_{\Gamma_\alpha}(\nu_i)\,\partial_pH^\alpha\left(\nu_i,\partial v_2|_{\Gamma_\alpha}(\nu_i,t)\right)\right] & \\ \quad+\displaystyle\sum_{\alpha\in\mathcal{A}_i}\mu_\alpha\partial_\alpha\overline{m}(\nu_i,t)=0, & \nu_i\in\mathcal{V},\ t\in(0,T),\\ \overline{v}(x,T)=0,\ \overline{m}(x,0)=0.\end{cases}$$

Testing by $\overline{m}$ the boundary value problem satified by $\overline{u}$, testing by $\overline{u}$ the boundary value problem satified by $\overline{m}$, subtracting, we obtain

$$\begin{aligned}&\int_0^T\int_\Gamma(m_1-m_2)\left(\mathscr{V}[m_1]-\mathscr{V}[m_2]\right]dxdt+\int_0^T\int_\Gamma\partial_t\left(\overline{m}\,\overline{v}\right)dxdt\\&+\sum_{\alpha\in\mathcal{A}}\int_{\Gamma_\alpha}m_1\left[H(x,\partial v_2)-H(x,\partial v_1)-\partial_pH(x,\partial v_1)\,\partial\overline{v}\right]dx\\&+\sum_{\alpha\in\mathcal{A}}\int_{\Gamma_\alpha}m_2\left[H(x,\partial v_1)-H(x,\partial v_2)+\partial_pH(x,\partial v_1)\,\partial\overline{v}\right]dx=0.\end{aligned}$$

Since V is strictly monotone, the first sum is nonnegative. Moreover,

$$\int_0^T\int_\Gamma\partial_t\left(\overline{m}\,\overline{v}\right)dxdt=\int_\Gamma[\overline{m}(x,T)\,\overline{v}(x,T)-\overline{m}(x,0)\,\overline{v}(x,0)]dx=0,$$

since $\overline{v}(x,T)=0$ and $\overline{m}(x,0)=0$. From the convexity of H and the fact that m_1,m_2 are nonnegative, the last two sums are nonnegative. Therefore, all the terms are zero and thanks again to the fact that $\mathscr{V}$ is strictly increasing, we obtain $m_1=m_2$. From Lemma 4.2, we finally obtain $v_1=v_2$. ☐

Proof of regularity in Theorem 1.11. We make the stronger assumptions written in Section 1.4.4 on the coupling operator $\mathscr{V}$. We know that $\mathscr{V}[m]\in W^{1,2}(0,T,H_b^1(\Gamma))\cap PC(\Gamma\times[0,T])$. Assuming also that $v_T\in V$ and $\partial v_T\in F$, we can apply the regularity result in Theorem 4.5: $v\in L^2\left(0,T;H^3(\Gamma)\right)\cap W^{1,2}\left(0,T;H^1(\Gamma)\right)$.

Moreover, since $\mathscr{V}[m] \in W^{1,2}(0,T,H_b^1(\Gamma))$, we know that $(\mathscr{V}[m])|_{\Gamma_\alpha\times[0,T]} \in C^{\sigma,\sigma/2}(\Gamma_\alpha \times [0,T])$ for all $0 < \sigma < 1/2$. If $v_T \in C^{2+\eta} \cap D$ for some $\eta \in (0,1)$ (D is defined in (1.13)), then from Theorem 4.5, $v \in C^{2+\tau,1+\tau/2}(\Gamma \times [0,T])$ for some $\tau \in (0,1)$ and the boundary value problem for v is satisfied in a classical sense.

In turn, if for all $\alpha \in \mathcal{A}$, $\partial p H^\alpha(x,p)$ is a Lipschitz function defined in $\Gamma_\alpha \times \mathbb{R}$, and if $m_0 \in W$, then we can use the latter regularity of v and arguments similar to those contained in the proof of Theorem 2.5 and prove that $m \in C([0,T];W) \cap W^{1,2}(0,T;L^2(\Gamma)) \cap L^2(0,T;H_b^2(\Gamma))$. $\square$

A Some continuous and compact embeddings

Lemma A.1. *(Aubin-Lions Lemma, see [18]) Let X_0,X and X_1 be function spaces, (X_0 and X_1 are reflexive). Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Consider some real numbers $1 < p,q < +\infty$. Then the following set*

$$\{v : (0,T) \mapsto X_0 : \ v \in L^p(0,T;X_0),\ \partial_t v \in L^q(0,T;X_1)\}$$

is compactly embedded in $L^p(0,T;X)$.

Lemma A.2. *(Amann, see [4]) Let $\phi : [a,b] \times [0,T] \to \mathbb{R}$ such that $\phi \in L^2(0,T;H^2(a,b))$ and $\partial_t\phi \in L^2(0,T;L^2(a,b))$. Then $\phi \in C^s(0,T;H^1(a,b))$ for some $s \in (0,1/2)$.*

This result is a consequence of the general result [4, Theorem 1.1] taking into account [4, Remark 7.4]. More precisely, we have

$$E_1 := H^2(a,b) \overset{\text{compact}}{\hookrightarrow} E := H^1(a,b) \hookrightarrow E_0 := L^2(a,b).$$

Let $r_0 = r_1 = r = 2$, $\sigma_0 = 0$, $\sigma_1 = 2$ and $\sigma = 1$. For any $\nu \in (0,1)$, we define

$$\frac{1}{r_\nu} = \frac{1}{r_0} + \frac{1-\nu}{r_1}, \quad \sigma_\nu := (1-\nu)s_0 + \nu s_1.$$

This implies that $r_\nu = 2$ and $\sigma_\nu = 2\nu$. Therefore, if $\nu \in (1/2,1)$, then the following inequality is satisfied

$$\sigma - 1/r < \sigma_\nu - 1/r_\nu < \sigma_1 - 1/r_1.$$

Hence, we infer from [4, Remark 7.4]

$$E_1 \hookrightarrow (E_0,E_1)_{\nu,1} \hookrightarrow (E_0.E_1)_{\nu,r_\nu} = W^{\sigma_\nu,r_\nu}(a,b) \hookrightarrow E,$$

where $(E_0,E_1)_{\nu,1}$, $(E_0.E_1)_{\nu,r_\nu}$ are interpolation spaces. This is precisely the assumption allowing to apply [4, Theorem 1.1], which gives the result of Lemma A.2.

Acknowledgment. This work was partially supported by the ANR (Agence Nationale de la Recherche) through MFG project ANR-16-CE40-0015-01. The work of O. Ley and N. Tchou is partially supported by the Centre Henri Lebesgue ANR-11-LABX-0020-01.

References

[1] Y. Achdou, F. Camilli, A. Cutrì, and N. Tchou, Hamilton–Jacobi equations constrained on networks, *NoDEA Nonlinear Differential Equations Appl.* **20** (2013), no. 3, 413–445.

[2] Y. Achdou, M-K. Dao, O. Ley, N. Tchou A Class of Infinite Horizon Mean Field Games on Networks. *To appear in Netw. Heterog. Media.*

[3] Y. Achdou, S. Oudet, and N. Tchou, Hamilton-Jacobi equations for optimal control on junctions and networks, *ESAIM Control Optim. Calc. Var.* **21** (2015), no. 3, 876–899.

[4] H. Amann. Compact embeddings of vector-valued Sobolev and Besov spaces. *Glas. Mat. Ser. III*, 35(55)(1):161–177, 2000.

[5] L. Boccardo, F. Murat, J-P. Puel. Existence results for some quasilinear parabolic equations. *Nonlinear Anal.* 13 (1989), no. 4, 373–392.

[6] F. Camilli and C. Marchi. Stationary mean field games systems defined on networks. *SIAM J. Control Optim.*, 54(2):1085–1103, 2016.

[7] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[8] M. Freidlin and S.-J. Sheu. Diffusion processes on graphs: stochastic differential equations, large deviation principle. *Probab. Theory Related Fields*, 116(2):181–220, 2000.

[9] M. I. Freidlin and A. D. Wentzell. Diffusion processes on graphs and the averaging principle. *Ann. Probab.*, 21(4):2215–2245, 1993.

[10] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[11] C. Imbert, R. Monneau, and H. Zidani, A Hamilton-Jacobi approach to junction problems and application to traffic flows, *ESAIM Control Optim. Calc. Var.* **19** (2013), no. 1, 129–166.

[12] C. Imbert and R. Monneau, Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks, *Ann. Sci. Éc. Norm. Supér.* (4) **50** (2017), no. 2, 357–448.

[13] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968.

[14] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.

[15] J-M. Lasry and P-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.

[16] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.

[17] G.M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[18] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod; Gauthier-Villars, Paris, 1969.

[19] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1.* Dunod, Paris, 1968.

[20] P-L. Lions and P. Souganidis, Viscosity solutions for junctions: well posedness and stability, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **27** (2016), no. 4, 535–545.

[21] P-L. Lions and P. Souganidis, *Well-posedness for multi-dimensional junction problems with Kirchoff-type conditions*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **28** (2017), no. 4, 807–816.

[22] A. Porretta. Weak solutions to Fokker-Planck equations and mean field games. *Arch. Ration. Mech. Anal.*, 216(1):1–62, 2015.

[23] A. Porretta, On the weak theory for mean field games systems, *Boll. U.M.I.* **10** (2017), no. 3, 411–439.

[24] J. von Below. Classical solvability of linear parabolic equations on networks. *J. Differential Equations*, 72(2):316–337, 1988.