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TIPS OF TONGUES IN THE DOUBLE STANDARD FAMILY

KUNTAL BANERJEE, XAVIER BUFF, JORDI CANELA, AND ADAM EPSTEIN

ABSTRACT. We answer a question raised by Misiurewicz and Rodrigues concerning the family of degree 2 circle maps $F_\lambda : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ defined by

$$F_\lambda(x) := 2x + a + \frac{b}{\pi} \sin(2\pi x) \quad \text{with} \quad \lambda := (a, b) \in \mathbb{R}/\mathbb{Z} \times (0, 1).$$

We prove that if $F_\lambda^{\circ n} - \text{id}$ has a zero of multiplicity 3 in \mathbb{R}/\mathbb{Z} , then there is a system of local coordinates $(\alpha, \beta) : W \rightarrow \mathbb{R}^2$ defined in a neighborhood W of λ , such that $\alpha(\lambda) = \beta(\lambda) = 0$ and $F_\mu^{\circ n} - \text{id}$ has a multiple zero with $\mu \in W$ if and only if $\beta^3(\mu) = \alpha^2(\mu)$. This shows that the tips of tongues are regular cusps.

INTRODUCTION

Following Misiurewicz and Rodrigues [MR07], we consider the family of circle maps $F_\lambda : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ defined by

$$F_\lambda(x) := 2x + a + \frac{b}{\pi} \sin(2\pi x) \quad \text{with} \quad \lambda := (a, b) \in \mathbb{R}/\mathbb{Z} \times [0, 1].$$

If $b \in [0, 1/2)$, then $F_\lambda : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ is expanding and all periodic cycles of F_λ in \mathbb{R}/\mathbb{Z} are repelling. If $b \in [1/2, 1]$, it may happen that $F_\lambda : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ has a non-repelling cycle. The multiplier of such a cycle belongs to $[0, 1]$. There is at most one such cycle. Connected components of the open sets of parameters $\lambda \in (a, b) \in \mathbb{R}/\mathbb{Z} \times [0, 1]$ for which F_λ has an attracting cycle are called *tongues* (see [MR07] and [D10]). The period of the attracting cycle remains constant in each tongue, and is called the period of the tongue

Let T be a tongue of period $p \geq 1$. The boundary of T consists of two smooth curves which are graphs with respect to b and intersect tangentially at the tip $\lambda_T \in \mathbb{R}/\mathbb{Z} \times (0, 1)$ (see [MR07, MR08] and Figure 1). If $\lambda \in \partial T$ then F_λ has a cycle of period p and multiplier 1. On the one hand, if $\lambda \in \partial T \setminus \{\lambda_T\}$, then points of the cycle are double zeros of $F_\lambda^{\circ p} - \text{id}$. On the other hand, points of the cycle are triple zeros of $F_{\lambda_T}^{\circ p} - \text{id}$.

There is a unique tongue of period 1. Misiurewicz and Rodrigues [MR07] proved that the order of contact of its two boundary curves at the tip is $1/2$. In [MR08] they asked whether this property holds for all tongues of the family F_λ . In this article, we answer positively to this question. More precisely, we prove that near the tip of any tongue, the two boundary curves form an ordinary cusp.

Theorem 1. *Assume $F_\lambda^{\circ n} - \text{id}$ has a zero of multiplicity 3 in \mathbb{R}/\mathbb{Z} . Then there is a system of local coordinates $(\alpha, \beta) : W \rightarrow \mathbb{R}^2$ defined a neighborhood W of λ*

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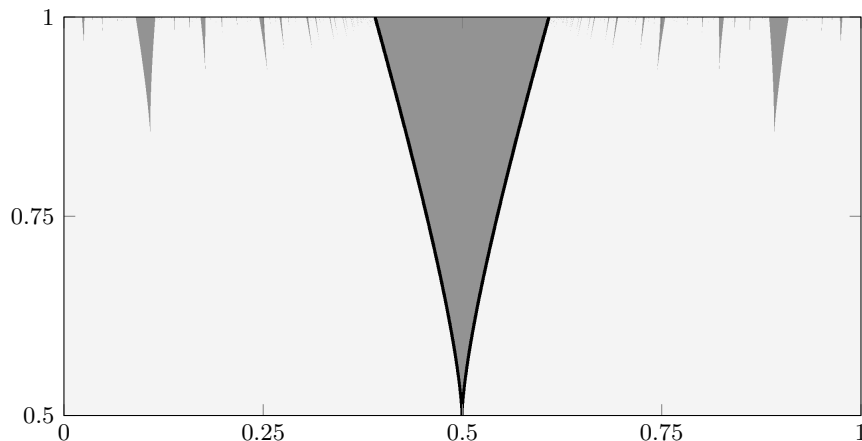


FIGURE 1. The tongues of the family F_λ . The horizontal axis corresponds to the parameter a and the vertical axis to b . We draw in black the boundary of the tongue of period 1.

in $\mathbb{R}/\mathbb{Z} \times (0, 1)$, such that $\alpha(\lambda) = \beta(\lambda) = 0$ and $F_\mu^{\circ n} - \text{id}$ has a multiple zero with $\mu \in W$ if and only if $\beta^3(\mu) = \alpha^2(\mu)$.

Our proof relies on a transversality result due to Adam Epstein for families of finite type analytic maps, which itself relies on an injectivity result of a linear map acting on an appropriate space of quadratic differentials. In §1, we prove that the maps F_λ are finite type analytic maps. In §2, we define the functions α and β . In §3, we identify the derivatives of those functions at λ . In §4, we state and prove the injectivity result. In §5, we prove that (α, β) is a system of local coordinates.

Some classical results on quadratic differentials are collected in Appendix A.

NOTATION

If U is a complex manifold, we denote by TU the tangent bundle of U and for $z \in U$, we denote by $T_z U$ the tangent space to U at z . If $\phi : U \rightarrow \mathbb{C}$ is a holomorphic function, we denote by $d\phi : TU \rightarrow \mathbb{C}$ the exterior derivative of ϕ (this is a holomorphic 1-form on U). If $F : U \rightarrow V$ is a holomorphic map between complex manifolds U and V , we denote by $DF : TU \rightarrow TV$ the bundle map $T_z U \ni v \mapsto D_z F(v) \in T_{F(z)} V$.

Assume $f : U \rightarrow V$ is a holomorphic map between Riemann surfaces. If ω is a holomorphic 1-form on V , then $f^* \omega := \omega \circ Df$ is a holomorphic 1-form on U . If ϑ is a holomorphic vector field on U , then there is a meromorphic vector field $f^* \vartheta$ on U satisfying $Df \circ f^* \vartheta = \vartheta \circ f$.

We will consider various holomorphic families $t \mapsto \gamma_t$ defined near 0 in \mathbb{C} . We will employ the notation

$$\gamma := \gamma_0 \quad \text{and} \quad \dot{\gamma} := \left. \frac{d\gamma_t}{dt} \right|_{t=0}.$$

1. FINITE TYPE ANALYTIC MAPS

The notion of finite type analytic maps originates in [E]. Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be an analytic map of complex 1-manifolds, possibly disconnected. An open set $V \subseteq \mathbb{Y}$ is *evenly covered* by f if $f|_U : U \rightarrow V$ is a homeomorphism for each component U of $f^{-1}(V)$; we say that $y \in \mathbb{Y}$ is a *regular value* for f if some neighborhood $V \ni y$ is evenly covered, and a *singular value* for f otherwise. Note that the set \mathcal{S}_f of singular values is closed. Recall that $x \in \mathbb{X}$ is a *critical point* if the derivative of f at x vanishes, and then $f(x) \in \mathbb{Y}$ is a *critical value*. We say that $y \in \mathbb{Y}$ is an *asymptotic value* if f approaches y along some path tending to infinity relative to \mathbb{X} . It follows from elementary covering space theory that the critical values together with the asymptotic values form a dense subset of \mathcal{S}_f . In particular, every isolated point of \mathcal{S}_f is a critical or asymptotic value.

An analytic map $f : \mathbb{X} \rightarrow \mathbb{Y}$ of complex 1-manifolds is of *finite type* if

- f is nowhere locally constant,
- f has no isolated removable singularities,
- \mathbb{Y} is a finite union of compact Riemann surfaces, and
- \mathcal{S}_f is finite.

If \mathbb{Y} is connected, we define $\deg f$ as the finite or infinite number $\text{card}(f^{-1}(y))$ which is independent of $y \in \mathbb{Y} \setminus \mathcal{S}_f$. When $f : \mathbb{X} \rightarrow \mathbb{Y}$ is a finite type analytic map with $\mathbb{X} \subseteq \mathbb{Y}$, we say that f is a finite type analytic map on \mathbb{Y} .

We first prove that the maps F_λ extend to finite type analytic maps.

1.1. Preliminaries. Set $\mathbb{T} := \mathbb{C}/\mathbb{Z}$ and $\Lambda := \mathbb{T} \times \mathbb{C}^*$. Let $F : \Lambda \times \mathbb{T} \rightarrow \mathbb{T}$ be the holomorphic map defined by

$$F(\lambda, z) = 2z + a + \frac{b}{\pi} \sin(2\pi z) \quad \text{with} \quad \lambda := (a, b) \in \Lambda.$$

For $\lambda \in \Lambda$, let $F_\lambda : \mathbb{T} \rightarrow \mathbb{T}$ be the holomorphic map defined by

$$F_\lambda(z) := F(\lambda, z).$$

It will be convenient to consider the global coordinate $\mathbb{T} \ni z \mapsto w := e^{2\pi i z} \in \mathbb{C}^*$. Note that $w : \mathbb{T} \rightarrow \mathbb{C}^*$ is an isomorphism. Thus, adding two points denoted $z = +i\infty$ (or $w = 0$) and $z = -i\infty$ (or $w = \infty$), \mathbb{T} may be compactified into a Riemann surface $\widehat{\mathbb{T}}$ isomorphic to the Riemann sphere.

We will prove that for $\lambda \in \Lambda$, the map $F_\lambda : \mathbb{T} \rightarrow \widehat{\mathbb{T}}$ is a finite type analytic map on $\widehat{\mathbb{T}}$.

1.2. The singular set. Fix $\lambda := (a, b) \in \Lambda$ and set $f := F_{a,b} : \mathbb{T} \rightarrow \widehat{\mathbb{T}}$. Note that

$$w \circ f = e^{2\pi i a} w^2 e^{b(w-1/w)}$$

and

$$f^*(dw) = e^{2\pi i a} e^{b(w-1/w)} (bw^2 + 2w + b) dw.$$

In particular, f has two critical points counting multiplicities: the solutions of $bw^2 + 2w + b = 0$, i.e., the points $c^\pm \in \mathbb{T}$ such that

$$w(c^\pm) = \frac{-1 \pm \sqrt{1 - b^2}}{b}.$$

If $b \neq 1$, those are simple critical points of f . We denote by $\mathcal{C}_f = \{c^+, c^-\} \subset \mathbb{T}$ the set of critical points of f and by $\mathcal{V}_f := f(\mathcal{C}_f) \subset \mathbb{T}$ the set of critical values of f .

Lemma 2. *The singular set \mathcal{S}_f is equal to $\mathcal{V}_f \cup \{\pm i\infty\}$.*

Proof. We already identified the set of critical values of f . Note that $\pm i\infty$ are singular values since those points are omitted values. It is therefore enough to show that f does not have any asymptotic value in \mathbb{T} .

If $v \in \mathbb{T}$ is an asymptotic value, then there exists a curve $\gamma : [0, 1) \rightarrow \mathbb{T}$, such that $\gamma(t) \rightarrow \pm i\infty$ and $f \circ \gamma(t) \rightarrow v$ as $t \rightarrow 1$. We assume that $\gamma(t) \rightarrow +i\infty$. The proof for the case $\gamma(t) \rightarrow -i\infty$ is analogous.

It is convenient to lift via the canonical covering $\pi : \mathbb{C} \rightarrow \mathbb{T} := \mathbb{C}/\mathbb{Z}$. Choose $A \in \mathbb{C}$ such that $\pi(A) = a$. Let $\tilde{f} : \mathbb{C} \rightarrow \mathbb{C}$ be defined by

$$\tilde{f}(Z) = 2Z + A + \frac{b}{\pi} \sin(2\pi Z) \quad \text{so that} \quad \pi \circ \tilde{f} = f \circ \pi.$$

Let $\Gamma : [0, 1) \rightarrow \mathbb{C}$ be a lift of $\gamma : [0, 1) \rightarrow \mathbb{T}$, i.e., satisfying $\pi \circ \Gamma = \gamma$. Then, $\tilde{f} \circ \Gamma$ is a lift of $f \circ \gamma$, thus $\tilde{f} \circ \Gamma(t)$ converges in \mathbb{C} as $t \rightarrow 1$.

Set $X := \operatorname{Re}(\Gamma) : [0, 1) \rightarrow \mathbb{R}$ and $Y := \operatorname{Im}(\Gamma) : [0, 1) \rightarrow \mathbb{R}$. Then,

$$\tilde{f} \circ \Gamma = 2(X + iY) + A + \frac{b}{\pi} \sin(2\pi\Gamma), \quad \sin(2\pi\Gamma) = \frac{e^{-2\pi Y} e^{2\pi i X} - e^{2\pi Y} e^{-2\pi i X}}{2i}$$

and $Y(t) \rightarrow +\infty$ as $t \rightarrow 1$. It follows that as $t \rightarrow 1$,

$$\tilde{f} \circ \Gamma(t) \sim 2X(t) - \frac{b}{4\pi i} e^{2\pi Y(t)} e^{-2\pi i X(t)}.$$

We can distinguish 2 cases. If there exists a sequence $\{t_k\}_{k \in \mathbb{N}}$ converging to 1 with $\{X(t_k)\}_{k \in \mathbb{N}}$ bounded, then

$$|\tilde{f} \circ \Gamma(t_k)| \sim \frac{b}{4\pi} e^{2\pi Y(t_k)} \xrightarrow[k \rightarrow +\infty]{} +\infty.$$

Otherwise, $X(t) \rightarrow \pm\infty$ as $t \rightarrow 1$ and there exists a sequence $\{t_k\}_{k \in \mathbb{N}}$ converging to 1 with $X(t_k) \in \mathbb{Z}$ for all $k \in \mathbb{N}$, so that

$$\tilde{f} \circ \Gamma(t_k) \sim 2X(t_k) + i \frac{b}{4\pi} e^{2\pi Y(t_k)} \xrightarrow[k \rightarrow +\infty]{} \infty.$$

In both cases, the sequence $\{\tilde{f} \circ \Gamma(t_k)\}_{k \in \mathbb{N}}$ cannot converge in \mathbb{C} . □

Corollary 3. *The map $f : \mathbb{T} \rightarrow \widehat{\mathbb{T}}$ is a finite type analytic map on $\widehat{\mathbb{T}}$. More precisely, $f : \mathbb{T} \setminus f^{-1}(\mathcal{V}_f) \rightarrow \mathbb{T} \setminus \mathcal{V}_f$ is a covering map.*

2. SPLITTING TRIPLE ZEROS

In the remainder of the article, we fix a parameter $\lambda := (a, b) \in \mathbb{R}/\mathbb{Z} \times (0, 1)$ such that $F_\lambda^{\circ n} - \operatorname{id}$ has a triple zero $x \in \mathbb{R}/\mathbb{Z}$. We set $f := F_\lambda : \mathbb{T} \rightarrow \mathbb{T}$. The point x is periodic for f with period p dividing n . For $k \geq 0$, we set $x_k := f^{\circ k}(x)$ and we denote by $\langle x \rangle := \{x_0, x_1, \dots, x_{p-1}\}$ the cycle of x .

Since $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ preserves the orientation, the multiplier of $f^{\circ p}$ at x is necessarily 1 and there is a local coordinate $\zeta : (\mathbb{T}, x) \rightarrow (\mathbb{C}, 0)$ vanishing at x satisfying

$$(1) \quad \zeta(\bar{z}) = \bar{\zeta}(z) \quad \text{and} \quad \zeta \circ f^{\circ p} = \zeta + \zeta^3 + \mathcal{O}(\zeta^5).$$

According to the Weierstrass Preparation Theorem, there exist a neighborhood $W_1 \subset \Lambda$ of λ , a neighborhood $W_2 \subset \mathbb{C}$ of 0 and analytic functions $A : W_1 \rightarrow \mathbb{C}$, $B : W_1 \rightarrow \mathbb{C}$, $C : W_1 \rightarrow \mathbb{C}$ and $g : W_1 \times W_2 \rightarrow \mathbb{C}$ such that for $\mu \in W_1$,

$$(2) \quad \zeta \circ F_\mu^{\circ p} - \zeta = P_\mu(\zeta) \cdot g(\mu, \zeta)$$

with

$$(3) \quad A(\lambda) = B(\lambda) = C(\lambda) = 0, \quad g(\lambda, \zeta) = 1 + \mathcal{O}(\zeta^2)$$

and

$$(4) \quad P_\mu(\zeta) := A(\mu) + B(\mu)\zeta + C(\mu)\zeta^2 + \zeta^3.$$

The polynomial P_λ has a zero of multiplicity 3 at 0, and as μ varies in W_1 , this zero splits in three zeros (counting multiplicities) of P_μ . When $\mu \in \mathbb{R}/\mathbb{Z} \times (0, 1)$, the map $F_\mu^{\circ n} - \text{id}$ commutes with $z \mapsto \bar{z}$, so that the polynomial P_μ has real coefficients. For such a parameter μ , a multiple zero of P_μ is necessarily real.

For any $\mu \in \Lambda$, the function $\zeta \circ F_\mu^{\circ p} - \zeta$ vanishes at the periodic points of F_μ of period dividing p , and so, divides $\zeta \circ F_\mu^{\circ n} - \zeta$ which vanishes at the periodic points of period dividing n . In addition, if $n = mp$, then $\zeta \circ f^{\circ n} - \zeta = m\zeta^3 + \mathcal{O}(\zeta^5)$. So, there is an analytic function $h : W_1 \times W_2 \rightarrow \mathbb{C}$ such that for $\mu \in W_1$,

$$\zeta \circ F_\mu^{\circ n} - \zeta = P_\mu(\zeta) \cdot h(\mu, \zeta) \quad \text{with} \quad h(\lambda, \zeta) = m + \mathcal{O}(\zeta^2)$$

Since f only has two critical points in \mathbb{T} , it has a single non-repelling cycle, that is, the cycle $\langle x \rangle$. All other cycles of f in \mathbb{R}/\mathbb{Z} are repelling. Shrinking W_1 is necessary, it follows that for $\mu \in W_1$, the function $\zeta \circ F_\mu^{\circ n} - \zeta$ has a multiple zero in \mathbb{R}/\mathbb{Z} if and only if the polynomial P_μ has a multiple zero in \mathbb{R}/\mathbb{Z} . According to the previous discussion, this is the case if and only if P_μ has a multiple zero.

Let $\alpha : W_1 \rightarrow \mathbb{C}$ and $\beta : W_1 \rightarrow \mathbb{C}$ be defined by

$$\alpha := \frac{C^3}{27} - \frac{BC}{6} + \frac{A}{2} \quad \text{and} \quad \beta := \frac{C^2}{9} - \frac{B}{3}.$$

Then,

$$\text{discriminant}(P_\mu) = 108\beta^3(\mu) - 108\alpha^2(\mu).$$

So, if $\mu \in W_1$, the polynomial P_μ has a multiple zero if and only if $\beta^3(\mu) = \alpha^2(\mu)$.

In order to prove Theorem 1, it is therefore enough to show that (α, β) is a system of local coordinates near λ . For this purpose, we shall show that the restrictions of $d\alpha$ and $d\beta$ to $T_\lambda\Lambda$ are linearly independent. Since A , B and C vanish at λ ,

$$d\alpha|_{T_\lambda\Lambda} = \frac{1}{2}dA|_{T_\lambda\Lambda} \quad \text{and} \quad d\beta|_{T_\lambda\Lambda} = -\frac{1}{3}dB|_{T_\lambda\Lambda}.$$

It is therefore enough to show that the forms $dA|_{T_\lambda\Lambda}$ and $dB|_{T_\lambda\Lambda}$ are linearly independent.

3. IDENTIFYING THE DERIVATIVES

Here, we identify $dA(v)$ and $dB(v)$ for $v \in T_\lambda\Lambda$. First, to each $v \in T_\lambda\Lambda$, we shall associate a meromorphic vector field ϑ_v on \mathbb{T} having simple poles along $\mathcal{C}_f \cup \{\pm i\infty\}$, such that for all $z \in \mathbb{T} \setminus \mathcal{C}_f$,

$$Df \circ \vartheta_v(z) := D_{\lambda, z}F(v, 0).$$

Second, for $k \in [1, p]$, let $\zeta_k : (\mathbb{T}, x_k) \rightarrow (\mathbb{C}, 0)$ be the local coordinate vanishing at x_k defined by

$$\zeta_k := \zeta \circ f^{\circ(p-k)}.$$

Our identification goes as follows.

Proposition 4. *Let q_A and q_B be quadratic differentials, defined and meromorphic near $\langle x \rangle$, such that $q_A - (d\zeta_k)^2/\zeta_k$ and $q_B - (d\zeta_k)^2/\zeta_k^2$ are holomorphic at x_k for all $k \in [1, p]$. Then, for all $v \in T_\lambda\Lambda$,*

$$dA(v) = \sum_{k=1}^p \text{residue}(q_A \otimes \vartheta_v, x_k) \quad \text{and} \quad dB(v) = \sum_{k=1}^p \text{residue}(q_B \otimes \vartheta_v, x_k).$$

In the remaining parts of this section we prove Proposition 4.

3.1. Meromorphic vector fields. Assume $v \in T_\lambda\Lambda$ and $z \in \mathbb{T} \setminus \mathcal{C}_f$. Then, the derivative $D_z f : T_z\mathbb{T} \rightarrow T_{f(z)}\mathbb{T}$ is an isomorphism and $D_{\lambda,z}F(v, 0) \in T_{f(z)}\mathbb{T}$. Let ϑ_v be the vector field defined on $\mathbb{T} \setminus \mathcal{C}_f$ by

$$\vartheta_v(z) := (D_z f)^{-1}(D_{\lambda,z}F(v, 0)) \in T_z\mathbb{T}.$$

Lemma 5. *For all $v \in T_\lambda\Lambda$, the vector field ϑ_v is holomorphic on $\mathbb{T} \setminus \mathcal{C}_f$, meromorphic on $\widehat{\mathbb{T}}$, vanishes at $z = \pm i\infty$ and has at worst simple poles along \mathcal{C}_f .*

Proof. The map $v \mapsto \vartheta_v$ is linear. So, it is enough to prove the result for $v_a := d/da$ and $v_b := d/db$. We have

$$\vartheta_{v_a} = \frac{2\pi i e^{2\pi i a} w^2 e^{b(w-1/w)}}{e^{2\pi i a} e^{b(w-1/w)} (bw^2 + 2w + b)} \frac{d}{dw} = \frac{2\pi i w^2}{bw^2 + 2w + b} \frac{d}{dw}$$

and

$$\vartheta_{v_b} = \frac{e^{2\pi i a} w^2 (w - 1/w) e^{b(w-1/w)}}{e^{2\pi i a} e^{b(w-1/w)} (bw^2 + 2w + b)} \frac{d}{dw} = \frac{w^3 - w}{bw^2 + 2w + b} \frac{d}{dw}$$

Those two vector fields have the required properties. \square

Denote by \mathcal{T}_f the space of meromorphic vector fields on $\widehat{\mathbb{T}}$ which are holomorphic on $\mathbb{T} \setminus \mathcal{C}_f$, vanish at $\pm i\infty$ and have at worst simple poles along \mathcal{C}_f . In other words,

$$\mathcal{T}_f := \left\{ \frac{c_3 w^3 + c_2 w^2 + c_1 w}{bw^2 + 2w + b} \frac{d}{dw} ; (c_1, c_2, c_3) \in \mathbb{C}^3 \right\}.$$

Let $\Theta_f : T_\lambda\Lambda \rightarrow \mathcal{T}_f$ be the linear map defined by

$$\Theta_f(v) := \vartheta_v.$$

Let $\tau \in \mathcal{T}_f$ be the radial vector field

$$\tau := w \frac{d}{dw}.$$

Note that $\tau - f^*\tau$ belongs to \mathcal{T}_f . Indeed,

$$\tau - f^*\tau = \frac{bw^3 + w^2 + bw}{bw^2 + 2w + b} \frac{d}{dw} \in \mathcal{T}_f.$$

Lemma 6. *The space \mathcal{T}_f is the direct sum of the image of Θ_f and the line spanned by $\tau - f^*\tau$:*

$$\mathcal{T}_f = \text{Im}(\Theta_f) \oplus \text{Vect}(\tau - f^*\tau).$$

Proof. The dimension of \mathcal{T}_f is 3. Thus, it is enough to show that the three vector fields ϑ_{v_a} , ϑ_{v_b} and $\tau - f^*\tau$ are linearly independent. Equivalently, it is enough to show that the three functions

$$w^2, \quad w^3 - w \quad \text{and} \quad bw^3 + w^2 + bw$$

are linearly independent. This is true since $b \neq 0$. \square

Assume now $v \in T_\lambda\Lambda$ and let $t \mapsto \lambda_t \in \Lambda$ be a curve such that $\dot{\lambda} = v$. Let $t \mapsto f_t$ be the family of maps defined by

$$f_t := F_{\lambda_t} : \mathbb{T} \rightarrow \mathbb{T}.$$

Then, for each $z \in \mathbb{T}$,

$$\dot{f}(z) = D_{\lambda,z}F(v, 0) = D_z f \circ \vartheta_v(z) \quad \text{with} \quad \vartheta_v := \Theta_f(v) \in \mathcal{T}_f.$$

Lemma 7. *For all $k \geq 1$,*

$$\left. \frac{df_t^{\circ k}}{dt} \right|_{t=0} = Df^{\circ k} \circ \vartheta_v^k \quad \text{with} \quad \vartheta_v^k := \vartheta_v + f^*\vartheta_v + \dots + (f^{\circ(k-1)})^*\vartheta_v.$$

Proof. The proof follows from an elementary induction on $k \geq 1$ using the following fact: if $h_t = g_t \circ f_t$ with $\dot{f} = Df \circ \vartheta$ and $\dot{g} = Dg \circ \tau$, then

$$\dot{h} = \dot{g} \circ f + Dg \circ \dot{f} = Dg \circ \tau \circ f + Dg \circ Df \circ \vartheta = Dh \circ (f^*\tau + \vartheta). \quad \square$$

Note that the poles of ϑ_v^n are the critical points of f and their iterated preimages (up to order $n-1$). The two critical points of f are in $\mathbb{T} \setminus \mathbb{R}/\mathbb{Z}$, and so are all their preimages. Therefore, ϑ_v^n is holomorphic in a neighborhood of \mathbb{R}/\mathbb{Z} . In particular, it is holomorphic near the parabolic periodic point $x \in \mathbb{R}/\mathbb{Z}$.

3.2. Polar parts of quadratic differentials. Our identification of the derivatives $dA|_{T_\lambda\Lambda}$ and $dB|_{T_\lambda\Lambda}$ relies on the use of quadratic differentials (see Appendix A for basics regarding quadratic differentials). Recall that $\zeta : (\mathbb{T}, x) \rightarrow (\mathbb{C}, 0)$ is a local coordinate vanishing at x such that

$$\zeta \circ f^{\circ p} = \zeta + \zeta^3 + \mathcal{O}(\zeta^5).$$

We shall use the quadratic differential $(d\zeta)^2/\zeta$ and $(d\zeta)^2/\zeta^2$ which are defined and meromorphic near x in \mathbb{T} .

Following §A.7, if $Z \subset \mathbb{T}$ is a finite set, if q is a quadratic differential, defined and meromorphic near Z and if ϑ is a vector field, defined and meromorphic near Z , we shall use the notation

$$\langle q, \vartheta \rangle_Z := \sum_{z \in Z} \text{residue}(q \otimes \vartheta, z).$$

If q has at worst simple poles along Z and if θ is defined on Z with $\theta(z) \in T_z\mathbb{T}$ for $z \in Z$, we shall use the notation

$$\langle q, \theta \rangle_Z := \langle q, \vartheta \rangle_Z$$

where ϑ is any vector field, defined and holomorphic near Z , with $\vartheta(z) = \theta(z)$ for $z \in Z$. The result does not depend on the choice of extension.

Lemma 8. *For all $v \in T_\lambda\Lambda$,*

$$dA(v) = \left\langle \frac{(d\zeta)^2}{\zeta}, \vartheta_v^p \right\rangle_x \quad \text{and} \quad dB(v) = \left\langle \frac{(d\zeta)^2}{\zeta^2}, \vartheta_v^p \right\rangle_x.$$

Proof. According to Equations (2), (3) and (4),

$$\zeta \circ f_t^{\circ p} - \zeta = (A(\lambda_t) + B(\lambda_t)\zeta + \mathcal{O}(\zeta^2)) \cdot (1 + \mathcal{O}(\zeta^2)).$$

Taking the derivative with respect to t and evaluating at $t = 0$ yields

$$d\zeta \circ Df^{\circ p} \circ \vartheta_v^p = dA(v) + dB(v)\zeta + \mathcal{O}(\zeta^2).$$

According to Equation (1),

$$\zeta \circ f^{\circ p} = \zeta + \mathcal{O}(\zeta^3) \quad \text{so that} \quad d\zeta \circ Df^{\circ p} = (1 + \mathcal{O}(\zeta^2))d\zeta.$$

As a consequence

$$d\zeta(\vartheta_v^p) = d\zeta \circ Df^{\circ p} \circ \vartheta_v^p + \mathcal{O}(\zeta^2) = dA(v) + dB(v)\zeta + \mathcal{O}(\zeta^2).$$

Thus,

$$dA(v) = \text{residue} \left(\frac{d\zeta(\vartheta_v^p)}{\zeta} d\zeta, x \right) = \left\langle \frac{(d\zeta)^2}{\zeta}, \vartheta_v^p \right\rangle_x$$

and similarly

$$dB(v) = \text{residue} \left(\frac{d\zeta(\vartheta_v^p)}{\zeta^2} d\zeta, x \right) = \left\langle \frac{(d\zeta)^2}{\zeta^2}, \vartheta_v^p \right\rangle_x. \quad \square$$

Rather than working near x with the vector field ϑ_v^p , it will be convenient to work along the cycle $\langle x \rangle$ with the vector field ϑ_v . Recall that for $k \in [1, p]$, the local coordinate $\zeta_k : (\mathbb{T}, x_k) \rightarrow (\mathbb{C}, 0)$ vanishes at x_k and is defined by

$$\zeta_k := \zeta \circ f^{\circ(p-k)}.$$

Lemma 9. *For all $k \in \mathbb{Z}/p\mathbb{Z}$,*

$$f^* \left(\frac{(d\zeta_{k+1})^2}{\zeta_{k+1}} \right) - \frac{(d\zeta_k)^2}{\zeta_k} \quad \text{and} \quad f^* \left(\frac{(d\zeta_{k+1})^2}{\zeta_{k+1}} \right) - \frac{(d\zeta_k)^2}{\zeta_k}$$

are holomorphic near x_k .

Proof. If $k \in [1, p-1]$, then $\zeta_k = \zeta_{k+1} \circ f$, so that

$$f^* \left(\frac{(d\zeta_{k+1})^2}{\zeta_{k+1}} \right) = \frac{(d\zeta_k)^2}{\zeta_k} \quad \text{and} \quad f^* \left(\frac{(d\zeta_{k+1})^2}{\zeta_{k+1}^2} \right) = \frac{(d\zeta_k)^2}{\zeta_k^2}.$$

If $k = p$, then $\zeta_p = \zeta$ and $\zeta_1 \circ f = \zeta \circ f^{\circ p} = (1 + \mathcal{O}(\zeta_p^2))\zeta_p$. As a consequence, $f^*(d\zeta_1) = (1 + \mathcal{O}(\zeta_p^2))d\zeta_p$,

$$f^* \left(\frac{(d\zeta_1)^2}{\zeta_1} \right) = (1 + \mathcal{O}(\zeta_p^2)) \frac{(d\zeta_p)^2}{\zeta_p} \quad \text{and} \quad f^* \left(\frac{(d\zeta_1)^2}{\zeta_1^2} \right) = (1 + \mathcal{O}(\zeta_p^2)) \frac{(d\zeta_p)^2}{\zeta_p^2}. \quad \square$$

Proof of Proposition 4. Recall that $\zeta_p = \zeta$. According to the previous lemma, for all $k \in \mathbb{Z}/p\mathbb{Z}$,

$$(f^{\circ k})^* \frac{(d\zeta_k)^2}{\zeta_k} - \frac{(d\zeta)^2}{\zeta}$$

is holomorphic near x . By assumption, $q_A - (d\zeta_k)^2/\zeta_k$ is holomorphic at x_k . It follows that $(f^{\circ k})^* q_A - (d\zeta)^2/\zeta$ is holomorphic near x .

Since $(f^{\circ k})^* \vartheta_v$ is holomorphic near x , we therefore have

$$\left\langle \frac{(d\zeta)^2}{\zeta}, (f^{\circ k})^* \vartheta_v \right\rangle_x = \langle (f^{\circ k})^* q_A, (f^{\circ k})^* \vartheta_v \rangle_x = \langle q_A, \vartheta_v \rangle_{x_k}.$$

As a consequence

$$dA(v) = \left\langle \frac{(d\zeta)^2}{\zeta}, \sum_{k=0}^{p-1} (f^{\circ k})^* \vartheta_v \right\rangle_x = \sum_{k=0}^{p-1} \langle q_A, \vartheta_v \rangle_{x_k} = \langle q_A, \vartheta_v \rangle_{\langle x \rangle}.$$

This proves Proposition 4 for dA . The proof for dB is similar. \square

4. INJECTIVITY OF ∇_f

In order to prove Theorem 1, we need to use the global properties of the map f . Up to now, we only used the local properties near the cycle. For this purpose, it is important that the quadratic differentials q_A and q_B which appear in Proposition 4 are globally meromorphic on $\widehat{\mathbb{T}}$. Here, we define such quadratic differentials q_A and q_B and we prove that the linear map

$$\nabla_f := \text{id} - f_*$$

is well defined and injective on $\text{Vect}(q_A, q_B)$.

4.1. A space of quadratic differentials. Denote by $\mathcal{Q}(\mathbb{T})$ the space of meromorphic quadratic differentials on $\widehat{\mathbb{T}}$ which have at worst simple poles at $z = \pm i\infty$. Given $Z \subset \mathbb{T}$, denote by $\mathcal{Q}(\mathbb{T}; Z) \subset \mathcal{Q}(\mathbb{T})$ the subspace of quadratic differentials which are holomorphic outside Z . Finally, we denote by $\mathcal{Q}^1(\mathbb{T}; Z) \subset \mathcal{Q}(\mathbb{T}; Z)$ the subspace of quadratic differentials having at worst simple poles.

Lemma 10. *Any polar part of quadratic differential along $\langle x \rangle$ may be realized as the polar part of a quadratic differential in $\mathcal{Q}(\mathbb{T}; \langle x \rangle \cup \{c^+\})$ having at worst a simple pole at c^+ .*

Proof. For all $k \in [0, p-1]$, the quadratic differentials

$$\frac{(dw)^2}{(w - w(x_k))(w - w(c^+))w}, \quad \frac{(dw)^2}{(w - w(x_k))^2 w} \quad \text{and} \quad \frac{(dw)^2}{(w - w(x_k))^j} \quad \text{for } j \geq 3$$

belong to $\mathcal{Q}(\mathbb{T}; \langle x \rangle \cup \{c^+\})$. The first has a simple pole at x_k , the second has a double pole at x_k , and the third has a pole of order $j \geq 3$ at x_k . Thus, they generate the space of polar parts at x_k . \square

From now on, we assume that $q_A \in \mathcal{Q}(\mathbb{T}; \langle x \rangle \cup \{c^+\})$ and $q_B \in \mathcal{Q}(\mathbb{T}; \langle x \rangle \cup \{c^+\})$ have at worst simple poles at c^+ and that $q_A - (d\zeta_k)^2/\zeta_k$ and $q_B - (d\zeta_k)^2/\zeta_k^2$ are holomorphic at x_k for all $k \in [0, p-1]$. We set

$$\mathcal{Q}_f := \text{Vect}(q_A, q_B).$$

4.2. Pairing quadratic differentials in \mathcal{Q}_f with vector fields in \mathcal{T}_f . Recall that

$$\tau := w \frac{d}{dw} \quad \text{and} \quad \tau - f^*(\tau) \in \mathcal{T}_f.$$

Lemma 11. *For all $q \in \mathcal{Q}_f$,*

$$\langle q, \tau - f^*\tau \rangle_{\langle x \rangle} = 0.$$

Proof. Assume $q \in \mathcal{Q}_f$. According to Lemma 9, $q - f^*q$ is holomorphic near $\langle x \rangle$. Since τ is also holomorphic near $\langle x \rangle$, and since f is a local isomorphism near $\langle x \rangle$,

$$\langle q, f^*\tau \rangle_{\langle x \rangle} = \langle f^*q, f^*\tau \rangle_{\langle x \rangle} = \langle q, \tau \rangle_{\langle x \rangle}. \quad \square$$

4.3. Pushing forward quadratic differentials in \mathcal{Q}_f . According to Corollary 3, $f : \mathbb{T} \setminus f^{-1}(\mathcal{V}_f) \rightarrow \mathbb{T} \setminus \mathcal{V}_f$ is a covering map. Here, we show that for all $q \in \mathcal{Q}$, the following series defines a meromorphic quadratic differential on $\widehat{\mathbb{T}}$:

$$(5) \quad f_*q = \sum_{g \text{ inverse branch of } f} g^*q.$$

The (minor) difficulty is that the degree of the covering map is not finite, and that q may fail to be integrable on $\widehat{\mathbb{T}}$ since it may have multiple poles along $\langle x \rangle$. So, we cannot apply directly the results presented in Appendix A. The reason why the series in Equation (5) converges is that q is locally integrable near the essential singularities of f , i.e., the points $\pm i\infty$.

Lemma 12. *If $q \in \mathcal{Q}_f$, the series in Equation (5) converges locally uniformly in $\mathbb{T} \setminus (\mathcal{V}_f \cup \langle x \rangle)$. Its sum f_*q is a meromorphic quadratic differential on $\widehat{\mathbb{T}}$.*

Proof. Assume $q \in \mathcal{Q}_f$. Let $V \subset \mathbb{T} \setminus (\mathcal{V}_f \cup \langle x \rangle)$ be compactly contained in $\widehat{\mathbb{T}} \setminus \langle x \rangle$. Then, $U := f^{-1}(V)$ is compactly contained in $\widehat{\mathbb{T}} \setminus \langle x \rangle$. In particular, q is integrable on U . In addition, $f : U \rightarrow V$ is a covering map. It follows that the series in Equation (5) converges uniformly on V and that f_*q is integrable on V . This shows f_*q is holomorphic on $\mathbb{T} \setminus (\mathcal{V}_f \cup \langle x \rangle)$ and has at worst simple poles at $\pm i\infty$ and on \mathcal{V}_f .

To see that f_*q is meromorphic near x_k , $k \in [1, p]$, let $V \subset \mathbb{T} \setminus \mathcal{V}_f$ be a topological disk containing x_k . Then, $U := f^{-1}(V)$ is the disjoint union of a topological disk U' containing x_{k-1} and an open set U'' compactly contained in $\widehat{\mathbb{T}} \setminus \langle x \rangle$. Then, $(f|_{U''})_*q$ is holomorphic. In addition, $f : U' \rightarrow U$ is an isomorphism so that $(f|_{U'})_*q -$ and thus $f_*q = (f|_{U'})_*q + (f|_{U''})_*q -$ is meromorphic near x_k . \square

We may therefore consider the linear map

$$\nabla_f := \text{id} - f_* : \mathcal{Q}_f \rightarrow \mathcal{Q}(\mathbb{T}).$$

It will be convenient to set

$$Y := \{c^+\} \cup \mathcal{V}_f.$$

Lemma 13. *We have the inclusion*

$$\nabla_f(\mathcal{Q}_f) \subseteq \mathcal{Q}^1(\mathbb{T}; Y).$$

Proof. Assume $q \in \mathcal{Q}_f$. As mentioned in the proof of the previous lemma, f_*q is holomorphic on $\mathbb{T} \setminus (\mathcal{V}_f \cup \langle x \rangle)$ and has at worst simple poles at $\pm i\infty$ and on \mathcal{V}_f . In addition, for $k \in [1, p]$, the polar part of f_*q at x_k is equal to the polar part of g^*q where g is the inverse branch of f sending x_k to x_{k-1} . According to Lemma 9, $q - f_*q$ is therefore holomorphic near $\langle x \rangle$. It follows that $q - f_*q \in \mathcal{Q}^1(\mathbb{T}; Y)$. \square

4.4. Injectivity of ∇_f . An observation due to Adam Epstein is that the linear map ∇_f is injective on \mathcal{Q}_f , and that this is the key to the proof of Theorem 1.

Proposition 14. *The linear map $\nabla_f : \mathcal{Q}_f \rightarrow \mathcal{Q}(\mathbb{T})$ is injective.*

Proof. We must prove that $f_*q \neq q$ for $q \in \mathcal{Q}_f \setminus \{0\}$. If q were integrable on \mathbb{T} , the result would follow immediately from Proposition 21, since we would have $\|f_*q\|_{L^1(\mathbb{T})} < \|q\|_{L^1(\mathbb{T})}$. Since q may have double poles near $\langle x \rangle$, it may fail to be integrable on \mathbb{T} . In that case, we may proceed as follows.

Assume $q \in \mathcal{Q}_f \setminus \{0\}$. For $\varepsilon > 0$ small, let $V_\varepsilon \subset \mathbb{T}$ be the union of topological disks

$$V_\varepsilon := \bigcup_{k=1}^p \{|\zeta_k| < \varepsilon\}.$$

Set $U_\varepsilon := f^{-1}(V_\varepsilon) \subset \mathbb{T}$. Then, $\langle x \rangle \subset U_\varepsilon$ and so, q is integrable on $\mathbb{T} \setminus U_\varepsilon$. As a consequence,

$$\|f_*q\|_{L^1(\mathbb{T} \setminus V_\varepsilon)} < \|q\|_{L^1(\mathbb{T} \setminus U_\varepsilon)}.$$

Similarly, for $\varepsilon' < \varepsilon$,

$$\|f_*q\|_{L^1(V_\varepsilon \setminus V_{\varepsilon'})} < \|q\|_{L^1(U_\varepsilon \setminus U_{\varepsilon'})}.$$

As a consequence, the function

$$\varepsilon \mapsto \|q\|_{L^1(\mathbb{T} \setminus U_\varepsilon)} - \|f_*q\|_{L^1(\mathbb{T} \setminus V_\varepsilon)}$$

is positive and decreasing. In particular, it has a positive limit. Note that

$$\|q\|_{L^1(\mathbb{T} \setminus U_\varepsilon)} - \|q\|_{L^1(\mathbb{T} \setminus V_\varepsilon)} = \|q\|_{L^1(V_\varepsilon \setminus U_\varepsilon)} - \|q\|_{L^1(U_\varepsilon \setminus V_\varepsilon)} \leq \|q\|_{L^1(V_\varepsilon \setminus U_\varepsilon)}.$$

We deduce from the following lemma that $f_*q \neq q$. \square

Lemma 15. *For any $q \in \mathcal{Q}_f$,*

$$\lim_{\varepsilon \rightarrow 0} \|q\|_{L^1(V_\varepsilon \setminus U_\varepsilon)} = 0.$$

Proof. Since $\zeta \circ f^{op} = \zeta + \mathcal{O}(\zeta^3)$, there is a constant κ_1 such that

$$|\zeta \circ f^{op}| \geq |\zeta| - \kappa_1 |\zeta|^3.$$

Since q has at worst a double pole at x , there is a constant κ_2 such that for $|\zeta|$ small enough

$$|q| \leq \kappa_2 \frac{|\mathrm{d}\zeta|^2}{|\zeta|^2}.$$

Note that for $\varepsilon > 0$ small enough,

$$V_\varepsilon \setminus U_\varepsilon = \{|\zeta| < \varepsilon\} \setminus \{|\zeta \circ f^{op}| < \varepsilon\} \subset \{\varepsilon - \kappa_1 \varepsilon^3 \leq |\zeta| < \varepsilon\}.$$

Thus,

$$0 \leq \|q\|_{L^1(V_\varepsilon \setminus U_\varepsilon)} \leq \int_0^{2\pi} \int_{\varepsilon - \kappa_1 \varepsilon^3}^\varepsilon \kappa_2 \frac{r \mathrm{d}r \mathrm{d}t}{r^2} = 2\pi \kappa_2 \ln \frac{1}{1 - \kappa_1 \varepsilon^2} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad \square$$

5. LINEAR INDEPENDENCE

We may now complete the proof that $\mathrm{d}A|_{\mathbb{T}_{\lambda\Lambda}}$ and $\mathrm{d}B|_{\mathbb{T}_{\lambda\Lambda}}$ are linearly independent. According to Proposition 4, for all $v \in \mathbb{T}_{\lambda\Lambda}$,

$$\mathrm{d}A(v) = \langle q_A, \Theta_f(v) \rangle_{\langle x \rangle} \quad \text{and} \quad \mathrm{d}B(v) = \langle q_B, \Theta_f(v) \rangle_{\langle x \rangle}.$$

According to Lemma 6,

$$\mathcal{T}_f = \mathrm{Im}(\Theta_f) \oplus \mathrm{Vect}(\tau - f^*\tau).$$

Showing that $\mathrm{d}A|_{\mathbb{T}_{\lambda\Lambda}}$ and $\mathrm{d}B|_{\mathbb{T}_{\lambda\Lambda}}$ are linearly independent therefore amounts to proving that for all $q \in \mathcal{Q}_f \setminus \{0\}$, there exists $\vartheta \in \mathcal{T}_f$ such that $\langle q, \vartheta \rangle_{\langle x \rangle} \neq 0$.

5.1. Guiding vector fields. Set $Z := \mathcal{C}_f \cup \mathcal{V}_f$ and denote by $\mathbb{T}Z$ the space of maps $\xi : Z \rightarrow \mathbb{T}\mathbb{T}$ satisfying $\xi(z) \in T_z\mathbb{T}$ for all $z \in Z$.

Lemma 16. *For any $\vartheta \in \mathcal{T}_f$, there exists a unique $\xi_\vartheta \in \mathbb{T}Z$ such that for any vector field ξ , defined and holomorphic near Z with $\xi(z) = \xi_\vartheta(z)$ for $z \in Z$, the vector field $\vartheta + \xi - f^*\xi$ is holomorphic and vanishes along \mathcal{C}_f .*

Proof. Fix $\vartheta \in \mathcal{T}_f$. Let us first prove the uniqueness of $\xi_\vartheta \in \mathbb{T}Z$. Assume ξ_1 and ξ_2 are two vector fields, defined and holomorphic near Z , such that $\vartheta + \xi_1 - f^*\xi_1$ and $\vartheta + \xi_2 - f^*\xi_2$ are holomorphic near \mathcal{C}_f . Then, $(\xi_1 - \xi_2) - f^*(\xi_1 - \xi_2)$ is holomorphic and vanishes along \mathcal{C}_f . As a consequence, $Df \circ (\xi_1 - \xi_2) - (\xi_1 - \xi_2) \circ f$ vanishes on \mathcal{C}_f . Since $Df \circ (\xi_1 - \xi_2)$ vanishes on \mathcal{C}_f , this forces $\xi_1 - \xi_2$ to vanish on \mathcal{V}_f . In that case, $f^*(\xi_1 - \xi_2)$ vanishes on \mathcal{C}_f and so, $\xi_1 - \xi_2$ vanishes on \mathcal{C}_f . This shows the uniqueness of $\xi_\vartheta \in \mathbb{T}Z$.

This also proves that if $\vartheta + \xi - f^*\xi$ is holomorphic and vanishes along \mathcal{C}_f for some vector field ξ , defined and holomorphic near Z with $\xi(z) = \xi_\vartheta(z)$ for $z \in Z$, then $\vartheta + \xi - f^*\xi$ is holomorphic and vanishes along \mathcal{C}_f for any vector field ξ , defined and holomorphic near Z with $\xi(z) = \xi_\vartheta(z)$ for $z \in Z$.

Let us now prove the existence of $\xi_\vartheta \in \mathbb{T}Z$. Note that $Df \circ \vartheta$ is a map from $\mathbb{T} \setminus \mathcal{C}_f$ to the tangent bundle $\mathbb{T}\mathbb{T}$. Note that it is not a vector field since for $z \in \mathbb{T}$, the vector $Df \circ \vartheta(z)$ belongs to $T_{f(z)}\mathbb{T}$. However, since ϑ has at worst simple poles along \mathcal{C}_f and since Df vanishes on \mathcal{C}_f , the map $Df \circ \vartheta$ extends holomorphically to \mathbb{T} . Set

$$\xi_\vartheta(f(c^\pm)) := Df \circ \vartheta(c^\pm).$$

Next, let ξ be any vector field, defined and holomorphic near \mathcal{V}_f , coinciding with ξ_ϑ on \mathcal{V}_f . Then, $\vartheta - f^*\xi$ is holomorphic near \mathcal{C}_f and we may set

$$\xi_\vartheta(c^\pm) := (\vartheta - f^*\xi)(c^\pm). \quad \square$$

Recall that $Y := \{c^+\} \cup \mathcal{V}_f \subset Z$. It will be convenient to consider the linear map $\Xi_f : \mathcal{T}_f \rightarrow \mathbb{T}Y$ defined by

$$\Xi_f(\vartheta) = \xi_\vartheta|_Y.$$

Lemma 17. *The map $\Xi_f : \mathcal{T}_f \rightarrow \mathbb{T}Y$ is an isomorphism.*

Proof. Since the dimensions of \mathcal{T}_f and $\mathbb{T}Y$ are both equal to three, it is enough to show that the map is injective. Assume $\vartheta \in \mathcal{T}_f$ and ξ_ϑ vanishes on $\{c^+\} \cup \mathcal{V}_f$. Let ξ be a vector field, defined and holomorphic near Z , which coincides with ξ_ϑ on Z . We may assume that ξ identically vanishes near $\{c^+\} \cup \mathcal{V}_f$. Then, $\vartheta + f^*\xi - \xi = \vartheta - \xi$ is holomorphic and vanishes on \mathcal{C}_f . This shows that ϑ is holomorphic near \mathcal{C}_f and vanishes at c^+ . As a consequence, ϑ is globally holomorphic on $\widehat{\mathbb{T}}$, and vanishes at three points: c^+ , $+i\infty$ and $-i\infty$. So, $\vartheta = 0$. \square

5.2. From the cycle to the critical set. We may now transfer the local computations done near the cycle $\langle x \rangle$ to local computations done near the critical set Y .

Lemma 18. *For all $\vartheta \in \mathcal{T}_f$ and all $q \in \mathcal{Q}_f$,*

$$\langle q, \vartheta \rangle_{\langle x \rangle} = \langle \nabla_f q, \Xi_f(\vartheta) \rangle_Y.$$

Proof. Let ξ be a vector field, defined and holomorphic near Z , coinciding with $\xi_\vartheta := \Xi_f(\vartheta)$ on Z . Then, $\vartheta + \xi - f^*\xi$ is holomorphic near \mathcal{C}_f . In addition, since $\nabla_f q$ is holomorphic near c^- ,

$$\begin{aligned} \langle \nabla_f q, \xi_\vartheta \rangle_Y &= \langle q - f_*q, \xi \rangle_Z = \langle q, \xi \rangle_{\mathcal{C}_f} - \langle f_*q, \xi \rangle_{\mathcal{V}_f} \\ &= \langle q, \xi \rangle_{\mathcal{C}_f} - \langle q, f^*\xi \rangle_{\mathcal{C}_f} \\ &= \langle q, -\vartheta \rangle_{\mathcal{C}_f} = \langle q, \vartheta \rangle_{\langle x \rangle}. \end{aligned}$$

In the second line, we used the fact that the only poles of $q \otimes f^*\xi$ in $f^{-1}(\mathcal{V}_f)$ belong to \mathcal{C}_f . For the last equality, we used the fact that $q \otimes \vartheta$ is a globally meromorphic 1-form on $\widehat{\mathbb{T}}$, whose poles are contained in $\mathcal{C}_f \cup \langle x \rangle$, and that the sum of all residues of a globally meromorphic 1-form on a compact Riemann surface is 0. \square

5.3. Completion of the proof. Assume by contradiction that $dA|_{\mathbb{T}_{\lambda\Lambda}}$ and $dB|_{\mathbb{T}_{\lambda\Lambda}}$ are not linearly independent. Then, there is a $q \in \mathcal{Q}_f \setminus \{0\}$ such that for all $\vartheta \in \mathcal{T}_f$,

$$0 = \langle q, \vartheta \rangle_{\langle x \rangle} = \langle \nabla_f q, \Xi_f(\vartheta) \rangle_Y.$$

According to Lemma 17, the map $\Xi_f : \mathcal{T}_f \rightarrow \text{TY}$ is an isomorphism. In particular, it is surjective. It follows that for all $\xi \in \text{TY}$,

$$\langle \nabla_f q, \xi \rangle_Y = 0.$$

As a consequence, $\nabla_f q$ is holomorphic near Y and thus, has at most three simple poles at c^- , $+\infty$ and $-\infty$. A non zero quadratic differential on $\widehat{\mathbb{T}}$ has at least four poles, counting multiplicities. Thus, $\nabla_f q = 0$.

According to Proposition 14, the map $\nabla_f : \mathcal{Q}_f \rightarrow \mathcal{Q}^1(\mathbb{T}; Y)$ is injective. It follows that $q = 0$. Contradiction.

This completes the proof of Theorem 1.

APPENDIX A. QUADRATIC DIFFERENTIALS

A.1. Meromorphic quadratic differentials. A *quadratic differential* on a Riemann surface U is a section of the square of the cotangent bundle $\text{T}^*U \otimes \text{T}^*U$. We shall usually think of a quadratic differential q as a field of quadratic forms. In particular, if ϑ is a vector field on U and ϕ is a function on U , then $q(\vartheta)$ is a function on U and $q(\phi\vartheta) = \phi^2 q(\vartheta)$.

If $\zeta : U \rightarrow \mathbb{C}$ is a coordinate, we shall use the notation $(d\zeta)^2 = d\zeta \otimes d\zeta$ - not be confused with 1-form $d(\zeta^2)$. Then, a quadratic differential q on U is of the form $q = \phi (d\zeta)^2$ for some function ϕ . We say that q is meromorphic on U if ϕ is meromorphic on U . In that case, the order of q at a point $x \in U$ is $\text{ord}_x q := \text{ord}_x \phi$, i.e., 0 if ϕ is holomorphic and does not vanish at x , $k \geq 1$ if ϕ has a zero of multiplicity k at x , and $-k \leq -1$ if ϕ has a pole of multiplicity k at x .

A.2. Pullback. The derivative $Df : \text{TU} \rightarrow \text{TV}$ of a holomorphic map $f : U \rightarrow V$ naturally induces a pullback map f^* from quadratic differentials on V to quadratic differentials on U :

$$f^*q := q \circ Df.$$

Lemma 19. *If $f : (U, x) \rightarrow (V, y)$ is holomorphic at x , and q is meromorphic at $y = f(x)$, then*

$$2 + \text{ord}_x(f^*q) = \deg_x f \cdot (2 + \text{ord}_y q).$$

Proof. Choose local coordinates $z : (U, x) \rightarrow (\mathbb{C}, 0)$ and $w : (V, y) \rightarrow (\mathbb{C}, 0)$ such that $w \circ f = z^k$, with $k := \deg_x f$. If $q = \phi (dw)^2$, then $f^*q = \phi \circ f \cdot (kz^{k-1}dz)^2$. Thus,

$$2 + \text{ord}_x(f^*q) = 2 + \text{ord}_x(\phi \circ f) + (2k - 2) = 2k + k \cdot \text{ord}_y \phi = k \cdot (2 + \text{ord}_y q). \quad \square$$

A.3. Pushforward for finite degree covering maps. Assume $f : U \rightarrow V$ is a finite degree covering map. If q is a quadratic differential on U , we define a quadratic differential f_*q on V by

$$f_*q := \sum_{g \text{ inverse branch of } f} g^*q.$$

If q is holomorphic on U , then f_*q is holomorphic on V .

Lemma 20. *Assume $U := \widehat{U} \setminus \{x\}$ and $V := \widehat{V} \setminus \{y\}$ are punctured disks, $f : U \rightarrow V$ is a covering map ramifying at x with local degree $\deg_x f$ and q is meromorphic at x . Then, f_*q has at worst a pole at y and*

$$2 + \text{ord}_y(f_*q) \geq \frac{2 + \text{ord}_x q}{\deg_x f}.$$

Proof. The group of deck transformations of $f : U \rightarrow V$ is a cyclic group of order $\deg_x f$. Note that

$$f^*(f_*q) = \sum_{h \text{ deck transformation of } f} h^*q,$$

and $\text{ord}_x h^*q = \text{ord}_x q$ for all deck transformations h , so that

$$\text{ord}_x f^*(f_*q) \geq \text{ord}_x q.$$

Then,

$$2 + \text{ord}_y(f_*q) = \frac{2 + \text{ord}_x f^*(f_*q)}{\deg_x f} \geq \frac{2 + \text{ord}_x q}{\deg_x f}. \quad \square$$

A.4. Integrable quadratic differentials. If q is a quadratic differential on U , we denote by $|q|$ the positive $(1, 1)$ -form on U defined by

$$|q|(\vartheta_1, \vartheta_2) := \frac{1}{2} |q(\vartheta_1 - i\vartheta_2)| - \frac{1}{2} |q(\vartheta_1 + i\vartheta_2)|.$$

If $\zeta : U \rightarrow \mathbb{C}$ is a coordinate and $q = \phi (d\zeta)^2$, then

$$|q| = |\phi| \cdot \frac{i}{2} d\zeta \wedge d\bar{\zeta}.$$

We shall say that q is *integrable* on U if

$$\|q\|_{L^1(U)} := \int_U |q| < \infty.$$

Note that q is integrable in a neighborhood of a pole if and only if the pole is simple. If $f : U \rightarrow V$ is an isomorphism and q is an integrable quadratic differential on V , then f^*q is integrable on U and $\|f^*q\|_{L^1(U)} = \|q\|_{L^1(V)}$.

A.5. Pushforward for infinite degree covering maps. Assume $f : U \rightarrow V$ is an infinite degree covering map. If q is an integrable quadratic differential on U , we may still define

$$f_*q := \sum_{g \text{ inverse branch of } f} g^*q.$$

Indeed, the series converges in L^1_{loc} since if $V' \subset V$ is a topological disk, so that the inverse branches $g : V' \rightarrow U$ of f are defined on V' , and if $U' := f^{-1}(V')$, then

$$\sum \|g^*q\|_{L^1(V')} = \|q\|_{L^1(U')} \leq \|q\|_{L^1(U)}.$$

The limit of a sequence of holomorphic functions converging in L^1_{loc} is itself holomorphic. It follows that if q is holomorphic on U , then f_*q is holomorphic on V .

A.6. The Contraction Principle.

Proposition 21. *Let $f : U \rightarrow V$ be a covering map and let q be a holomorphic integrable quadratic differential on U . Then, $\|f_*q\|_{L^1(V)} \leq \|q\|_{L^1(U)}$ and equality holds if and only if either $q = 0$, or the degree of f is finite and $f^*(f_*q) = \deg(f) \cdot q$.*

Proof. The proof is an immediate application of the triangle inequality: for any topological disk $V' \subset V$, we have

$$\int_{V'} |f_*q| = \int_{V'} \left| \sum g^*q \right| \leq \int_{V'} \sum |g^*q| = \sum \int_{V'} |g^*q| = \int_{f^{-1}(V')} |q|,$$

where the sums range over the inverse branches $g : V' \rightarrow U$ of f . It follows that

$$\int_V |f_*q| \leq \int_{f^{-1}(V)} |q| = \int_U |q|$$

with equality if and only if for all inverse branches g of f , we have $g^*q = \psi_g f_*q$ for some function $\psi_g : V' \rightarrow [0, 1]$ satisfying $\sum_g \psi_g = 1$. Setting $\phi(g(y)) := \psi_g(y)$, we see that $q = \phi f^*(f_*q)$ for some function $\phi : U \rightarrow [0, 1]$. Since q and $f^*(f_*q)$ are holomorphic, either $q = 0$, or the function ϕ is constant, let us say equal to $c \in [0, 1]$. Since $\sum_g \psi_g = 1$, we have that $\deg(f) \cdot c = 1$, which forces the degree of f to be finite with $f^*(f_*q) = \deg(f) \cdot q$. \square

A.7. Pairing quadratic differentials and vector fields. If q is a quadratic differential on U and ϑ is a vector field on U , we may consider the 1-form $q \otimes \vartheta$ defined on U by its action on vector fields τ :

$$q \otimes \vartheta(\tau) = \frac{1}{4}(q(\vartheta + \tau) - q(\vartheta - \tau)).$$

Note that if $q = \phi (d\zeta)^2$ and $\vartheta = \psi d/d\zeta$, then $q \otimes \vartheta = \phi\psi d\zeta$.

If $x \in U$, and if ϑ and q are meromorphic on U , we set

$$\langle q, \vartheta \rangle_x := \text{residue}(q \otimes \vartheta, x).$$

If q has at worst a simple pole at x , then $\langle q, \vartheta \rangle_x$ only depends on $\theta := \vartheta(0)$, and we use the notation

$$\langle q, \theta \rangle_x := \langle q, \vartheta \rangle_x.$$

Lemma 22. *Let $U := \widehat{U} \setminus \{x\}$ and $V := \widehat{V} \setminus \{y\}$ be punctured disks, let $f : U \rightarrow V$ be a covering map ramifying at x , let q be a meromorphic quadratic differential on \widehat{U} and let ϑ be a meromorphic vector field on U . Then*

$$\langle f_*q, \vartheta \rangle_y = \langle q, f^*\vartheta \rangle_x.$$

Proof. Let $\gamma \subset V$ be a loop around y with basepoint a . Then

$$\int_{\gamma} (f_*q) \otimes \vartheta = \sum_g \int_{\gamma \setminus \{a\}} (g^*q) \otimes \vartheta = \sum_g \int_{g(\gamma \setminus \{a\})} q \otimes f^*\vartheta = \int_{f^{-1}(\gamma)} q \otimes f^*\vartheta,$$

where the sum ranges over the inverse branches g of f defined on $\gamma \setminus \{a\}$. \square

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