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OPTIMAL PERIODIC CONTROL FOR SCALAR DYNAMICS UNDER INTEGRAL CONSTRAINT ON THE INPUT

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ABSTRACT. This paper studies a periodic optimal control problem governed by a one-dimensional system, linear with respect to the control u , under an integral constraint on u . We give conditions for which the value of the cost function at steady state with a constant control \bar{u} can be improved by considering periodic control u with average value equal to \bar{u} . This leads to the so-called “over-yielding” met in several applications. With the use of the Pontryagin Maximum Principle, we provide the optimal synthesis of periodic strategies under the integral constraint. The results are illustrated on a single population model in order to study the effect of periodic inputs on the utility of the stock of resource.

KEY-WORDS. Optimal Control, Pontryagin Maximum Principle, Periodic solutions, Over-yielding.

1. Introduction. In many applications, the control of dynamical models allows to drive the state x of a system to an operating point, typically a steady state \bar{x} which is an equilibrium point of the dynamics under a constant control \bar{u} . When a criterion of performance is associated with the state of the system, it may happen that a periodic trajectory near the steady state gives a better averaged performance than at steady state. But such a gain in the performance could be at the price of higher effort (or cost) on the control variable. The objective of the present work is to investigate the possibility of improving the performance of a steady state with periodic solutions, while keeping the same control effort over each time period. We consider in this work that this effort is measured by the integral of the control $u(\cdot)$ over a period. Keeping the same effort consists then in imposing that the averaged control over a periodic solution is equal to the control \bar{u} at steady state. For this purpose, we formulate an optimal control problem over periodic solutions, under an integral constraint on the control. Periodic optimal control has already been investigated in the literature, mainly under the consideration that solutions are sought near a steady state optimizing the criterion among stationary solutions. In particular, the so-called π -criterion characterizes the existence of “best” periods. It consists first in determining an optimal steady state among constant controls, and then in checking on a linear-quadratic approximation if there exists a frequency of a periodic signal near the nominal constant one that could improve the cost (see [6, 5]). For instance, in [2, 3, 15], this method has been applied on the chemostat model, and it has been shown that its productivity can be improved with a periodic control when

there is a delay in the dynamics. However, there are relatively few theoretical works about global optimality of periodic controls (apart from [18] for the characterization of the value function under quite strong assumptions). Most of the existing works deal with local necessary conditions ([8, 12]), second order conditions ([7, 21, 13]) or approximations techniques ([11, 1, 4]). In [4] for instance, a local analysis is conducted in the context of age-structured system showing how to improve locally the cost function by considering periodic controls versus constant ones (but no integral constraint on the control is considered). It has to be underlined that, in our approach, we do not have to consider that the steady state is optimizing the criterion among all stationary solutions of the system (the optimal steady-state control does not necessarily satisfy the integral constraint). To our knowledge, integral constraint on the control has not been yet considered in problems of determining optimal periodic trajectories. Therefore, our objective is some what different than what has been described above.

In applications for which the control variable is a flow rate of matter (such as in continuously fed reactors for instance [14]), this constraint amounts to consider that a given quantity of matter is available for each period of time, and the problem is then to determine how to deliver this matter during this period (*i.e.*, at a constant flow rate or not?), maintaining a periodic operation over the future times and maximizing the production or the quality of a product over each period. The present problem has been mainly motivated by the modelling of exploited populations of stock (or density) x , see, *e.g.*, [10], for which the control variable u is the harvesting effort (for instance the number of fishermen boats on a lake). In our setting, for a given steady state \bar{x} and its associated constant control \bar{u} , we consider the set of T -periodic trajectories with periodic controls having \bar{u} as average. We say that a *over-yielding* occurs when the averaged utility of the stock $x(\cdot)$ of a T -periodic solution is larger than the utility of the stock \bar{x} . Let us finally mention [16, 17] where periodic inputs are studied in the context of population biology and fisheries management, but with different objectives (no optimization and no such integral constraint are considered).

To our knowledge, this problem has not been yet addressed theoretically in the literature. From a mathematical view point, the integral constraint on the input brings two main difficulties:

1. the existence of non-constant periodic trajectories with a control satisfying the integral constraint,
2. the characterization of an optimal control under both constraints of periodicity of the trajectory and the integral constraint on the input,

that we propose to tackle here for scalar dynamics in general framework.

The paper is organized as follows. In Section 2, we formulate the problem and give a precise definition of over-yielding. We then provide assumptions on the dynamics and the cost function that guarantee or prevent over-yielding. In particular, we show that convexity is playing an important role. In section 3, we synthesize optimal periodic controls (in particular non constant ones) improving the cost function compared to steady-state (see Theorem 3.6). In Section 4, we show how to relax the assumptions of Section 2 that are required on an invariant domain (a, b) of the dynamics, when these ones are fulfilled only in a neighborhood of \bar{x} . This leads us to give a result similar to the one of Section 3 but for restrictive values of the period T . Finally, we illustrate the results of Section 3-4 in Section 5 in the context of sustainable resource management (see, *e.g.*, [10]). We study the impact

on the stock of non-constant periodic inputs (harvesting efforts) but with the same average value, and determine the worst-case scenarios with respect to a given utility of the stock.

2. Existence of over-yielding. Given two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ of class C^1 , we consider the control system

$$\dot{x} = f(x) + ug(x), \quad (1)$$

where u is a control variable taking values in $[-1, 1]$. We suppose that the system satisfies the following hypotheses:

(H1) There exists $(a, b) \in \mathbb{R}^2$ with $a < b$ such that g is positive on the interval $I := (a, b)$ with

$$f(a) - g(a) = 0 \quad \text{and} \quad f(b) + g(b) = 0.$$

(H2) One has $f - g < 0$ and $f + g > 0$ on I .

Remark 1. Hypothesis (H1) implies that the interval I is invariant by (1) whereas Hypothesis (H2) is related to controllability properties of (1) (that will be used in the next section for the synthesis of non-constant periodic trajectories). In the rest of the paper, we shall consider initial conditions in I only.

We define for $x \in I$ the function

$$\psi(x) := -\frac{f(x)}{g(x)}.$$

Notice that Hypotheses (H1)-(H2) imply that one has $\psi(I) \subset [-1, 1]$. Therefore, for any $\bar{x} \in I$, the control value \bar{u} defined as $\bar{u} := \psi(\bar{x})$ is such that $\bar{u} \in [-1, 1]$. Note that any such point \bar{x} is an equilibrium of (1) for the constant control $u = \bar{u}$. Throughout the paper, we fix a point $\bar{x} \in I$ as a nominal steady state. In the sequel, we shall consider T -periodic solutions of (1), where $T \in \mathbb{R}_+^*$, with a T -periodic control u that satisfies the integral constraint

$$\frac{1}{T} \int_0^T u(t) dt = \bar{u}. \quad (2)$$

We then define the set \mathcal{U}_T of admissible controls as

$$\mathcal{U}_T := \{u : [0, +\infty) \rightarrow [-1, 1] \text{ s.t. } u \text{ is meas., } T\text{-periodic and fulfills (2)}\}. \quad (3)$$

One has the following property.

Lemma 2.1. *Under Hypothesis (H1), any T -periodic solution x of (1) in I with $u \in \mathcal{U}_T$ fulfills the property*

$$\int_0^T (\psi(x(t)) - \psi(\bar{x})) dt = 0. \quad (4)$$

Proof. On the interval I , the function g is positive and from equation (1), we get

$$\int_0^T \frac{\dot{x}(t)}{g(x(t))} dt = - \int_0^T \psi(x(t)) dt + \int_0^T u(t) dt.$$

Define the function

$$h(x) := \int_{\bar{x}}^x \frac{d\xi}{g(\xi)}, \quad x \in I,$$

together with the function $t \mapsto y(t) := h(x(t))$ for $t \in [0, T]$. For any control function u that fulfills the constraint (2), one then has

$$y(T) - y(0) = - \int_0^T (\psi(x(t)) - \bar{u}) dt,$$

where $\bar{u} = \psi(\bar{x})$. For any T -periodic solution x in I , y is also T -periodic and one obtains the property (4). \square

We now require the following hypothesis on \bar{x} .

(\bar{H}) The function ψ satisfies the property.

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in I \setminus \{\bar{x}\}.$$

This hypothesis is related with the asymptotic stability of \bar{x} for the dynamics (1) in I with the constant control \bar{u} (as we shall see in Lemma (2.2), see (5)). For applications, it sounds also reasonable that the given steady state \bar{x} is a stable equilibrium of the system under constant control.

For convenience, we denote by $t \mapsto x(t, u, x_0)$ the solution of (1) with $u \in \mathcal{U}_T$ and taking the value $x_0 \in I$ at time 0. In the following, we shall consider T -periodic solutions with the initial condition $x(0) = \bar{x}$ (i.e., that are such that $x(T, u, \bar{x}) = \bar{x}$ for $u \in \mathcal{U}_T$). We first show that Hypothesis (\bar{H}) guarantees the existence of non-constant such solutions.

Lemma 2.2. *Under Hypotheses (H1)-(\bar{H}), there exist non-constant T -periodic solutions of (1) with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$, for any $T > 0$.*

Proof. Consider the constant control $u = \bar{u}$ and its associated dynamics in I

$$\dot{x} = \bar{f}(x) := g(x)(\bar{u} - \psi(x)) = g(x)(\psi(\bar{x}) - \psi(x)). \quad (5)$$

As the function g is positive on I , Hypothesis (\bar{H}) implies that one has $\bar{f} < 0$ on (\bar{x}, b) , and $\bar{f} > 0$ on (a, \bar{x}) . Therefore, one has the properties

$$\begin{aligned} x_0 \in (\bar{x}, b) &\Rightarrow x(T, \bar{u}, x_0) < x_0, \\ x_0 \in (a, \bar{x}) &\Rightarrow x(T, \bar{u}, x_0) > x_0. \end{aligned} \quad (6)$$

Consider now any bounded T -periodic measurable function $v : [0, +\infty) \rightarrow [-1, 1]$ satisfying

$$\int_0^T v(t) dt = 0,$$

and the control function

$$u_\varepsilon(t) := \bar{u} + \varepsilon v(t),$$

where $\varepsilon \in \mathbb{R}$. Clearly, u_ε satisfies the constraint (2) and for ε small enough, one has $u_\varepsilon(t) \in [-1, 1]$ for any $t \geq 0$. Define then the function

$$\theta(x_0, \varepsilon) := x(T, u_\varepsilon, x_0) - x_0,$$

for $(x_0, \varepsilon) \in I \times \mathbb{R}$. By the Theorem of continuous dependency of the solutions of ordinary differential equations w.r.t. initial conditions and parameters (see for instance [19]), θ is a continuous function. From (6), we deduce that

$$\begin{aligned} x_0 \in (\bar{x}, b) &\Rightarrow \theta(x_0, 0) < 0, \\ x_0 \in (a, \bar{x}) &\Rightarrow \theta(x_0, 0) > 0, \end{aligned}$$

and by continuity of θ , there exists $\varepsilon \neq 0$, $x_0^+ \in (\bar{x}, b)$ and $x_0^- \in (a, \bar{x})$ such that $\theta(x_0^+, \varepsilon) < 0$ and $\theta(x_0^-, \varepsilon) > 0$. By the Mean Value Theorem, we deduce the

existence of $x_0 \in (x_0^-, x_0^+)$ such that $\theta(x_0, \varepsilon) = 0$, that is, the existence of a T -periodic solution x of (1) with a non-constant control u that satisfies the constraint (2). From Lemma 2.1, such solution satisfies

$$\int_0^T (\psi(x(t)) - \psi(\bar{x})) dt = 0,$$

which implies that the map $t \mapsto \psi(x(t)) - \psi(\bar{x})$ cannot be of constant sign on $[0, T]$. Hypothesis (H) implies that $x(t) - \bar{x}$ has to change its sign. Therefore there exists $\bar{t} \in (0, T)$ with $x(\bar{t}) = \bar{x}$ in such a way that the control function \tilde{u} defined by $t \mapsto \tilde{u}(t) := u(t + \bar{t})$ guarantees to have $x(T, \tilde{u}, \bar{x}) = \bar{x}$. \square

Now, let $\ell : \mathbb{R} \rightarrow \mathbb{R}$ be a function of class C^1 and consider the cost function

$$J_T(u) := \frac{1}{T} \int_0^T \ell(x_u(t)) dt, \quad (7)$$

where x_u is the unique solution of (1) such that $x_u(0) = \bar{x}$, associated with a control $u \in \mathcal{U}_T$. Our aim in this work is to address the question of finding a periodic trajectory with $x(0) = \bar{x}$ that has a lower cost than the constant \bar{x} , with a (T -periodic) control of mean value \bar{u} . For this purpose, we introduce the following terminology.

Definition 2.3. Given $T > 0$, we say that (1) exhibits an *over-yielding* for the cost (7) if there exists a T -periodic solution x of (1) with $x(0) = \bar{x}$ associated with a control $u \in \mathcal{U}_T$ such that $J_T(u) < \ell(\bar{x})$.

Moreover, we aim to characterize in the next section the strategies realizing the minimum of the criterion (7) among such controls. The possibility of having an over-yielding relies on specific assumptions on the cost function and the dynamics, that we now introduce.

(H3) The function $\ell : I \rightarrow \mathbb{R}$ is increasing and the function $\gamma := \psi \circ \ell^{-1}$ is strictly convex increasing over $\ell(I)$.

Remark 2. Hypothesis (H3) implies Hypothesis (H). Therefore, by Lemma 2.2, there exist T -periodic solutions x of (1) with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$, that are different of the constant solution \bar{x} , when (H1)-(H2)-(H3) are fulfilled. Hypothesis (H3) also implies that ψ is increasing.

Proposition 2.1. If (H1) and (H3) hold true, any non-constant T -periodic solution x of (1) with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$ satisfies $J_T(u) < \ell(\bar{x})$.

Proof. Consider a T -periodic solution x with $x(0) = \bar{x}$ associated with a control in \mathcal{U}_T . From Lemma 2.1, equality (4) is satisfied and we deduce

$$\int_0^T (\gamma(\ell(x(t))) - \gamma(\ell(\bar{x}))) dt = 0.$$

For a non-constant solution, we find by Jensen's inequality

$$\gamma\left(\frac{1}{T} \int_0^T \ell(x(t)) dt\right) < \frac{1}{T} \int_0^T \gamma(\ell(x(t))) dt = \gamma(\ell(\bar{x})).$$

Since γ is increasing over $\ell(I)$ with, we obtain

$$J_T(u) = \frac{1}{T} \int_0^T \ell(x(t)) dt < \ell(\bar{x}).$$

□

Remark 3. (i) The result of Proposition 2.1 applies in the simple case where $\ell(x) = x$ and ψ is strictly convex and increasing over I .
(ii) If ψ is strictly convex and increasing over I and ℓ is strictly concave increasing over I , the result of Proposition 2.1 also holds true (by a similar reasoning).

We now provide sufficient conditions for preventing any over-yielding.

(H4) There exists a continuous function $\bar{\psi}$ such that

- (i) $\bar{\psi} \geq \psi$ on I with $\bar{\psi}(\bar{x}) = \psi(\bar{x})$,
- (ii) the function $\bar{\gamma} := \bar{\psi} \circ \ell^{-1}$ is concave increasing on $\ell(I)$.

Proposition 2.2. *If (H1) and (H4) hold true then no over-yielding is possible.*

Proof. We suppose by contradiction that there exists a periodic solution x associated with a control $u \in \mathcal{U}_T$ such that

$$J_T(u) = \frac{1}{T} \int_0^T \ell(x(t)) dt < \ell(\bar{x}),$$

The function $\bar{\gamma}$ being increasing on $\ell(I)$, we have

$$\bar{\gamma}\left(\frac{1}{T} \int_0^T \ell(x(t)) dt\right) < \bar{\gamma}(\ell(\bar{x})) = \bar{\psi}(\bar{x}) = \psi(\bar{x}). \quad (8)$$

Using Jensen's inequality for $\bar{\gamma}$, we can write

$$\bar{\gamma}\left(\frac{1}{T} \int_0^T \ell(x(t)) dt\right) \geq \frac{1}{T} \int_0^T \bar{\gamma}(\ell(x(t))) dt. \quad (9)$$

As one has $\bar{\psi} = \bar{\gamma} \circ \ell \geq \psi$ over I , we get

$$\frac{1}{T} \int_0^T \bar{\gamma}(\ell(x(t))) dt \geq \frac{1}{T} \int_0^T \psi(x(t)) dt. \quad (10)$$

Combining inequalities (8), (9), (10), we obtain

$$\psi(\bar{x}) > \frac{1}{T} \int_0^T \psi(x(t)) dt,$$

which is a contradiction with the equality (4) given by Lemma 2.1. □

Remark 4. (i) Thanks to the previous proposition, if $\ell(x) = x$ for $x \in \mathbb{R}$ and ψ is strictly concave, then no over-yielding is possible. In the same way, if ℓ is increasing on I and γ strictly concave increasing over $\ell(I)$, then the same conclusion follows.
(ii) Under hypotheses (H1)-(H3), we say that an over yielding is systematic (which means that it exists for any $T > 0$, see Proposition 2.1).

3. Determination of optimal periodic solutions. In this Section, we assume that Hypotheses (H1)-(H2)-(H3) hold true, so that we know that over-yielding is possible (actually, it is systematic according to Proposition 2.1). For a given $T > 0$, we shall say that a solution x of (1) is *T-admissible* if it is T -periodic with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$. We reformulate the control constraint (2) by considering the augmented dynamics

$$\begin{cases} \dot{x} &= f(x) + ug(x), \\ \dot{y} &= u, \end{cases} \quad (11)$$

together with the boundary conditions:

$$(x(0), y(0)) = (\bar{x}, 0) \quad \text{and} \quad (x(T), y(T)) = (\bar{x}, \bar{u}T). \quad (12)$$

The optimal control problem can be then stated as follows

$$\inf_{u \in \mathcal{U}} \int_0^T \ell(x(t)) dt \quad \text{s.t. } (x, y) \text{ satisfies (11) - (12),} \quad (13)$$

where \mathcal{U} denotes the set of measurable control functions u over $[0, T]$ taking values in $[-1, 1]$. Note that Problem (13) admits a solution by classical existence results. Indeed, hypotheses (H1)-(H2)-(H3) imply that there exist trajectories of (11) satisfying (12). Since the system is affine w.r.t. the control and ℓ is continuous, the existence of an optimal control follows by Filippov's existence theorem [9].

3.1. Application of the Pontryagin Maximum Principle. We derive necessary optimality conditions using the Pontryagin Maximum Principle [20]. Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the Hamiltonian associated with (13) :

$$H = H(x, y, \lambda_x, \lambda_y, \lambda_0, u) = \lambda_0 \ell(x) + \lambda_x f(x) + u(\lambda_x g(x) + \lambda_y),$$

where $\lambda := (\lambda_x, \lambda_y)$ denotes the adjoint vector. Let $u \in \mathcal{U}$ be an optimal control and (x, y) a solution of (11)-(12) associated with u . Then, there exists a scalar $\lambda_0 \leq 0$ and an absolutely continuous map $\lambda : [0, T] \rightarrow \mathbb{R}^2$ satisfying the adjoint equation

$$\begin{cases} \dot{\lambda}_x = -\lambda_0 \ell'(x(t)) - \lambda_x (f'(x(t)) + u(t)g'(x(t))), \\ \dot{\lambda}_y = 0, \end{cases} \quad (14)$$

for a.e. $t \in [0, T]$. Moreover, $(\lambda_0, \lambda) \neq 0$ and the Hamiltonian condition writes

$$u(t) \in \arg \max_{\omega \in [-1, 1]} H(x(t), \lambda(t), \lambda_0, \omega) \quad \text{a.e. } t \in [0, T]. \quad (15)$$

Since the dynamics is affine w.r.t. u , the switching function

$$t \mapsto \phi(t) := \lambda_x(t)g(x(t)) + \lambda_y,$$

provides the following expression of the control u (thanks to (15)):

$$\begin{cases} \phi(t) > 0 & \Rightarrow u(t) = 1, \\ \phi(t) < 0 & \Rightarrow u(t) = -1, \\ \phi(t) = 0 & \Rightarrow u(t) \in [-1, 1]. \end{cases} \quad (16)$$

Moreover, if we differentiate ϕ w.r.t t , we find that for $t \in [0, T]$

$$\dot{\phi}(t) = \lambda_x(t)[f(x(t))g'(x(t)) - f'(x(t))g(x(t))] - \lambda_0 \ell'(x(t))g(x(t)).$$

An extremal trajectory is a quadruple $(x, \lambda, \lambda_0, u)$ where (x, λ) satisfies the state-adjoint equations and u the Hamiltonian condition (15). We recall that a singular arc occurs if ϕ vanishes on some time interval $[t_1, t_2]$ with $t_1 < t_2$, and a switching time $t_s \in (0, T)$ is such that an extremal control u is non-constant in any neighborhood of t_s (which implies that $\phi(t_s) = 0$). It is also worth to mention that from Hypothesis (H2), when $\phi > 0$, resp. $\phi < 0$, then x is increasing, resp. decreasing.

Lemma 3.1. *Under Hypotheses (H1)-(H2)-(H3), there is no abnormal extremal trajectory, i.e., $\lambda_0 \neq 0$.*

Proof. If $\lambda_0 = 0$, then λ_x cannot vanish from the adjoint equation. Otherwise λ_x would be zero over $[0, T]$ and the switching function would be constant equal to λ_y . Since λ_y cannot be simultaneously equal to 0, ϕ would be of constant sign over $[0, T]$ implying that $u = 1$ or $u = -1$ over $[0, T]$ and a contradiction with the periodicity

of $x(\cdot)$ (recall that $f + g > 0$ and $f - g < 0$ over I). As a consequence, λ_x is of constant sign. Now, since $\lambda_0 = 0$, one has

$$\dot{\phi}(t) = \lambda_x(t)g(x(t))^2\psi'(x(t)), \quad t \in [0, T].$$

We deduce that $\dot{\phi}$ is of constant sign (recall that $\psi' > 0$), hence ϕ is monotone. Consequently, the extremal trajectory has at most one switching point. Thus, one has $x(t) > \bar{x}$ for any time $t \in (0, T)$ implying a contradiction with (4). If $x(t) < \bar{x}$ for any time $t \in (0, T)$, we conclude in the same way. \square

Without any loss of generality, we may assume that $\lambda_0 = -1$.

Remark 5. Considering T -periodic optimal solutions in I without requiring the initial condition $x(0) = \bar{x}$, but only $x(T) = x(0)$ provides the transversality condition $\lambda_x(T) = \lambda_x(0)$. However, Lemma 2.1 and Hypothesis (H3) (or simply (\bar{H})) imply that any T -periodic optimal solution $x(\cdot)$ in I has to pass by \bar{x} . Therefore, we can impose $x(0) = \bar{x}$ without any loss of generality, and deduce that $\lambda_x(\cdot)$ is necessarily T -periodic (even though we shall not use this property in the following).

3.2. Properties of switching times. Let us denote by x_m and x_M the minimum and maximum on $[0, T]$ of a T -admissible solution x . Note that for any time $t \in (0, T)$ such that $x(t) \in \{x_m, x_M\}$, then one has $\phi(t) = 0$. Indeed, otherwise one would have $\phi(t) > 0$ or $\phi(t) < 0$. Suppose for instance that $\phi(t) > 0$. From (16), the control u would be equal to 1 in a neighborhood of t , and thus, from (H2), we would have a contradiction with the fact that x_M is the maximum of x . We proceed in the same way if $\phi(t) < 0$.

Proposition 3.1. *Under Hypotheses (H1)-(H2)-(H3), any extremal satisfies the following properties.*

1. *At any switching time $t_s \in (0, T)$, one has $x(t_s) \in \{x_m, x_M\}$.*
2. *It has no singular arc.*

Proof. Let t_1, t_2 in $[0, T]$ be such that $x(t_1) = x_m$ and $x(t_2) = x_M$ with x_m, x_M in I . We deduce that $\lambda_x(t_1)g(x_m) = \lambda_x(t_2)g(x_M) = -\lambda_y$. Now, since H is conserved along any extremal trajectory (see for instance [9]), one has

$$H = -\ell(x_M) - \lambda_y \frac{f(x_M)}{g(x_M)} = -\ell(x_m) - \lambda_y \frac{f(x_m)}{g(x_m)},$$

implying that (recall that $\gamma = \psi \circ \ell^{-1}$)

$$\frac{1}{\lambda_y} = \frac{\psi(x_M) - \psi(x_m)}{\ell(x_M) - \ell(x_m)} = \frac{\gamma(\ell(x_M)) - \gamma(\ell(x_m))}{\ell(x_M) - \ell(x_m)}, \quad (17)$$

As γ is increasing over $\ell(I)$, one has $\lambda_y > 0$. Suppose now that t_s is a switching time such that $x(t_s) \in (x_m, x_M)$. Using a similar computation as above, we find that

$$\frac{1}{\lambda_y} = \frac{\psi(x_M) - \psi(x(t_s))}{\ell(x_M) - \ell(x(t_s))} = \frac{\gamma(\ell(x_M)) - \gamma(\ell(x(t_s)))}{\ell(x_M) - \ell(x(t_s))}. \quad (18)$$

Since γ and ℓ are respectively strictly convex and increasing on $[x_m, x_M]$, (17) and (18) imply a contradiction, thus $x(t_s) \in \{x_m, x_M\}$ as was to be proved.

Suppose now by a contradiction that there exists a time interval $[t_1, t_2]$ such that $\phi(t) = \dot{\phi}(t) = 0$ for $t \in [t_1, t_2]$. Combining $\phi = \dot{\phi} = 0$ over $[t_1, t_2]$, one finds that

$$\begin{aligned} & \ell'(x(t)) - \lambda_y \psi'(x(t)) = 0, \quad \forall t \in [t_1, t_2], \\ \Rightarrow & 1 - \lambda_y \gamma'(\ell(x(t))) = 0, \quad \forall t \in [t_1, t_2] \text{ (recall that } \psi = \gamma \circ \ell), \\ \Rightarrow & \frac{1}{\lambda_y} = \gamma'(\ell(x(t))), \quad \forall t \in [t_1, t_2], \end{aligned}$$

Now, since the extremities of the singular arc t_1 and t_2 must be switching times, one must have $x(t_1), x(t_2) \in \{x_m, x_M\}$. Suppose for instance that $x(t_1) = x_m$. One then gets $\frac{1}{\lambda_y} = \gamma'(\ell(x_m))$ which is a contradiction with (17) (since γ is strictly convex) and similarly at $t = t_2$. This completes the proof. \square

At this stage, we have thus proved that optimal trajectories are of bang-bang type (*i.e.*, they are concatenations of arcs with $u = \pm 1$) such that at each switching time t_s one has $x(t_s) \in \{x_m, x_M\}$. One can show that the number of switching times is finite (by doing a similar reasoning as for the exclusion of singular arcs).

Moreover, this number is necessarily even. Indeed, let $x(\cdot)$ be a T -admissible solution of (1) associated with a control $u \in \mathcal{U}_T$ having $2n + 1$ switching times over $[0, T]$ with $n > 0$. Note that $n = 0$ is impossible since the map $t \mapsto x(t) - \bar{x}$ has to change its sign over $[0, T]$. Hence, it has to be equal at least once to \bar{x} on the interval $(0, T)$, say at a time \bar{t} , to satisfy (4) which invalidates $n = 0$. Finally, observe that the sign of $\dot{x}(0^+)$ and $\dot{x}(T^-)$, with an odd number of switches, are necessarily distinct. It follows that the sign of $\dot{x}(T^-)$ and $\dot{x}(T^+)$ are also distinct. From the initial condition $x(\bar{t}) = \bar{x}$, the T -periodic solution over $(\bar{t}, T + \bar{t})$ switches then at time $t = T$, which belongs to the interval $(\bar{t}, T + \bar{t})$. Since $x(T) = \bar{x}$, we have a contradiction with point 1 of Proposition 3.1. Hence the number of switches is even.

We focus now on extremal trajectories with two switches.

3.3. Trajectories with two switches. For a given $T > 0$, we consider trajectories $t \mapsto x(t)$ solutions of (1) on $[0, T]$ with $x(0) = \bar{x}$ and associated with a control u defined by two switching times t_1, t_2 with $0 < t_1 < t_2 < T$:

$$u(t) = \begin{cases} 1, & t \in [0, t_1), \\ -1, & t \in [t_1, t_2), \\ 1, & t \in [t_2, T]. \end{cases} \quad (19)$$

These trajectories, that we shall call $B_+B_-B_+$ trajectories, will play an important role in the following. Note that under Hypotheses (H1)-(H2) a $B_+B_-B_+$ trajectory is characterized uniquely by its maximal and minimal values $x_M = x(t_1)$ and $x_m = x(t_2)$ in I . For convenience, we define on the interval I the function

$$\eta(x) := \frac{1}{f(x) + g(x)} - \frac{1}{f(x) - g(x)}.$$

From Hypothesis (H2), note that η is C^1 and positive function on I .

Lemma 3.2. *Under Hypotheses (H1)-(H2), if a $B_+B_-B_+$ trajectory is T -periodic, then the pair (x_m, x_M) satisfies*

$$\int_{x_m}^{x_M} \eta(x) dx = T. \quad (20)$$

Moreover, if the corresponding control satisfies (2) then the pair (x_m, x_M) satisfies

$$\int_{x_m}^{x_M} \eta(x) \psi(x) dx = \bar{u}T. \quad (21)$$

Proof. For $t \in [0, t_1) \cup [t_2, T)$, one has $\dot{x} = f(x) + g(x) > 0$ and one can write

$$t_1 = \int_{\bar{x}}^{x_M} \frac{dx}{f(x) + g(x)}, \quad T - t_2 = \int_{x_m}^{x(T)} \frac{dx}{f(x) + g(x)}.$$

Similarly for $t \in [t_1, t_2)$, one has $\dot{x} = f(x) - g(x) < 0$ and

$$t_2 - t_1 = - \int_{x_m}^{x_M} \frac{dx}{f(x) - g(x)}.$$

One then obtains

$$T = \int_{x_m}^{x(T)} \frac{dx}{f(x) + g(x)} - \int_{x_m}^{x_M} \frac{dx}{f(x) - g(x)} + \int_{\bar{x}}^{x_M} \frac{dx}{f(x) + g(x)},$$

and for a T -periodic solution, $x(T) = \bar{x}$ gives exactly the property (20). Proceeding with the same decomposition of the interval $[0, T]$, one can write

$$\int_0^T u(t) dt = \int_{\bar{x}}^{x_M} \frac{dx}{f(x) + g(x)} - \int_{x_M}^{x_m} \frac{dx}{f(x) - g(x)} + \int_{x_m}^{\bar{x}} \frac{dx}{f(x) + g(x)},$$

which gives the quality

$$\int_{x_m}^{x_M} \left(\frac{1}{f(x) + g(x)} + \frac{1}{f(x) - g(x)} \right) dx = \bar{u}T,$$

when u fulfills (2). Finally, notice that one has

$$\frac{1}{f(x) + g(x)} + \frac{1}{f(x) - g(x)} = \eta(x) \psi(x),$$

for $x \in I$, and thus property (21) is satisfied. \square

We first analyze the possibilities of satisfying the integral condition (20).

Lemma 3.3. *Under Hypotheses (H1)-(H2), for any $T > 0$ there exists a unique function $\beta_T : [a, b] \mapsto [a, b]$ that satisfies $\beta_T(\alpha) > \alpha$ for any $\alpha \in I$ and*

$$\int_{\alpha}^{\beta_T(\alpha)} \eta(x) dx = T, \quad \alpha \in I.$$

Moreover β_T is of class C^1 , increasing and bijective from $[a, b]$ to $[a, b]$.

Proof. The function $f + g$ is of class C^1 and positive on I with $(f + g)(b) = 0$. Thus, it is easy to see that $K_+ := -\min_{x \in [a, b]} (f + g)'(x) > 0$. It follows that one has the inequality $(f + g)(x) \leq K_+(b - x)$ for any $x \in I$. As the function η satisfies

$$\eta(x) > \frac{1}{f(x) + g(x)} \geq \frac{1}{K_+(b - x)} > 0, \quad x \in I,$$

one deduces that the map

$$\chi : (\xi_-, \xi_+) \mapsto \chi(\xi_-, \xi_+) := \int_{\xi_-}^{\xi_+} \eta(x) dx,$$

is such that for any $\alpha \in I$, $\chi(\alpha, \cdot)$ is of class C^1 , increasing with $\chi(\alpha, \alpha) = 0$ and $\chi(\alpha, b) = +\infty$. By the Implicit Function Theorem, there exists a unique map

$\beta_T : I \mapsto I$ of class C^1 , such that $\chi(\alpha, \beta_T(\alpha)) = T$ for any $\alpha \in I$. Moreover, one has

$$\beta'_T(\alpha) = \frac{\eta(\alpha)}{\eta(\beta_T(\alpha))} > 0, \quad \alpha \in I.$$

The function β_T is thus increasing, and then admits limits at the points a_+ and b_- . Therefore one has $\beta_T(a_+) := \lim_{\alpha \rightarrow a_+} \beta_T(\alpha) \geq a$ and $\beta_T(b_-) := \lim_{\alpha \rightarrow b_-} \beta_T(\alpha) \leq b$ that verify $\chi(a, \beta_T(a_+)) = T$ and $\chi(b, \beta_T(b_-)) = T$, since χ is continuous. As previously, $f - g < 0$ on I with $(f - g)(a) = 0$ implies that $K_- := -\min_{x \in [a, b]} (f - g)'(x) > 0$. It follows that $(f - g)(x) \geq K_-(a - x)$ for any $x \in I$. Thus, we deduce that

$$\eta(x) > -\frac{1}{f(x) - g(x)} \geq \frac{1}{K_-(x - a)} > 0, \quad x \in I.$$

If $\beta_T(a_+) > a$, one should then have $\chi(a, \beta_T(a_+)) = +\infty$ which is not possible since one has $\chi(\alpha, \beta_T(\alpha)) = T$ for any $\alpha \in I$. So, one has $\beta_T(a_+) = a$. As the function η is positive on I , one also has $\beta_T(\alpha) > \alpha$ for any $\alpha \in I$, and we deduce that $\beta_T(b_-) = b$. This proves that β_T can be extended to a one-to-one mapping from $[a, b]$ to $[a, b]$. \square

We are now ready to show that there exists a unique $B_+B_-B_+$ trajectory that satisfies both integral conditions (20) and (21).

Proposition 3.2. *Under Hypotheses (H1)-(H2)-(\bar{H}), there exists a unique pair $(x_m, x_M) \in I^2$ satisfying (20)-(21), and one has $x_m < \bar{x} < x_M$.*

Proof. From Lemma 3.3, condition (20) implies to have $x_M = \beta_T(x_m)$. We thus have simply to show the uniqueness of x_m for the condition (21) to be fulfilled. Consider the function $F : [a, b] \rightarrow \mathbb{R}$ defined by

$$F(\alpha) := \int_{\alpha}^{\beta_T(\alpha)} \eta(x)(\psi(x) - \psi(\bar{x})) dx, \quad (22)$$

and notice that conditions (20) and (21) are both satisfied exactly when $F(x_m) = 0$. From Hypothesis (\bar{H}) and the properties satisfied by the function β_T (see Lemma 3.3), one has $F(\alpha) > 0$ for any $\alpha \in [\bar{x}, b]$, and $F(\alpha) < 0$ for any $\alpha \in (a, \beta_T^{-1}(\bar{x})]$. By the Mean Value Theorem, there exists $x_m \in (\beta_T^{-1}(\bar{x}), \bar{x})$ such that $F(x_m) = 0$. Moreover, one has

$$F'(\alpha) = \eta(\beta_T(\alpha))(\psi(\beta_T(\alpha)) - \psi(\bar{x}))\beta'_T(\alpha) - \eta(\alpha)(\psi(\alpha) - \psi(\bar{x})).$$

As β_T is increasing and ψ satisfies (\bar{H}), we obtain $F'(\alpha) > 0$ for any $\alpha < \bar{x}$ with $\beta_T(\alpha) > \bar{x}$, that is exactly for $\alpha \in (\beta_T^{-1}(\bar{x}), \bar{x})$, and we conclude about the existence and uniqueness of x_m, x_M in I , with $x_m < \bar{x}$ and $x_M > \bar{x}$. \square

Remark 6. The existence and uniqueness of a T -admissible $B_+B_-B_+$ trajectory is a straightforward consequence of Lemma 3.2 and Proposition 3.2. Indeed, under Hypotheses (H1)-(H2)-(\bar{H}), Proposition 3.2 allows to uniquely define a pair (x_m, x_M) satisfying (20)-(21). Consider now a solution $x(\cdot)$ of (1) such that $x(0) = \bar{x}$ which is such that $u = 1$ until $x(\cdot)$ reaches x_M , say at a time t_1 and then $u = -1$ from t_1 until the first time $t_2 > t_1$ such that $x(t_2) = x_m$, and finally $u = 1$ until $x(\cdot)$ reaches \bar{x} . For any $T > 0$, this construction defines a unique $B_+B_-B_+$ trajectory that is T -admissible, thanks to (20)-(21).

It is also worth to mention that x_m and x_M depend on the period T . In the next Lemma, we provide properties of x_m and x_M as functions of T .

Lemma 3.4. *Under Hypotheses (H1)-(H2)-(\bar{H}), the functions $T \mapsto x_m(T)$ and $T \mapsto x_M(T)$ are continuously differentiable, and respectively decreasing and increasing. Moreover, one has*

$$\lim_{T \rightarrow +\infty} x_m(T) = a \quad \text{and} \quad \lim_{T \rightarrow +\infty} x_M(T) = b. \quad (23)$$

Proof. For each $T > 0$, we know from Proposition 3.2 that there exists a unique pair $(x_m(T), x_M(T)) \in I^2$ satisfying (20)-(21). By the Implicit Function Theorem, x_m and x_M are continuously differentiable w.r.t. T . Let us denote by x_m' , x_M' the derivatives of x_m and x_M w.r.t. T . Differentiating (20)-(21) w.r.t. T then yields

$$\underbrace{\begin{bmatrix} \eta(x_M(T)) & -\eta(x_m(T)) \\ \eta(x_M(T))\psi(x_M(T)) & -\eta(x_m(T))\psi(x_m(T)) \end{bmatrix}}_{X(T)} \begin{bmatrix} x_M'(T) \\ x_m'(T) \end{bmatrix} = \begin{bmatrix} 1 \\ \psi(\bar{x}) \end{bmatrix},$$

where $\det(X(T)) := \eta(x_M(T))\eta(x_m(T))(\psi(x_M(T)) - \psi(x_m(T))) > 0$. Then $x_M'(T)$, $x_m'(T)$ are given by the expressions

$$\begin{cases} x_M'(T) &= \frac{\eta(x_m(T))(\psi(\bar{x}) - \psi(x_m(T)))}{\det(X(T))} > 0, \\ x_m'(T) &= \frac{\eta(x_M(T))(\psi(\bar{x}) - \psi(x_M(T)))}{\det(X(T))} < 0. \end{cases}$$

From (20) and (21), one has

$$\frac{T}{2}(\bar{u} + 1) = \int_{x_m(T)}^{x_M(T)} \frac{dx}{f(x) + g(x)} < \int_a^{x_M(T)} \frac{dx}{f(x) + g(x)}.$$

Taking the limit when T tends to $+\infty$ in both side of this inequality, one obtains $\lim_{T \rightarrow +\infty} x_M(T) = b$. Similarly one can prove that $\lim_{T \rightarrow +\infty} x_m(T) = a$. \square

3.4. Optimal solutions. According to Proposition 3.2, for any $T > 0$, we have seen that there is a unique $B_+B_-B_+$ trajectory $\hat{x}_T(\cdot)$ that is T -admissible, generated by a control that we shall denote \hat{u}_T . Moreover, there exists a unique $\bar{t} \in (0, T)$ such that $\hat{x}_T(\bar{t}) = \bar{x}$. Therefore, there are exactly two T -admissible solutions $\hat{x}_T(\cdot)$, $\check{x}_T(\cdot)$ with two switches, given by \hat{u}_T and \check{u}_T with

$$\check{u}_T(t) := \hat{u}_T(t + \bar{t}), \quad t \geq 0,$$

which have the same cost. Similarly, we denote by $B_-B_+B_-$ the trajectory \check{x}_T . We now study the monotonicity of the cost $J_T(\hat{u}_T)$ with respect to T . This property is crucial for the optimal synthesis (Theorem 3.6) and relies on the convexity assumptions on the data.

Lemma 3.5. *Under Hypotheses (H1)-(H2)-(H3), one has*

$$S > T > 0 \Rightarrow J_S(\hat{u}_S) < J_T(\hat{u}_T).$$

Proof. Following (19), we denote by t_1 and t_2 the two successive instants of $(0, T)$ for which one has $\hat{u}_T = +1$ over $[0, t_1) \cup [t_2, T]$ and $\hat{u}_T = -1$ over $[t_1, t_2)$. In the same way, we define s_1, s_2 as the two successive instants of $(0, S)$ such that one has $\hat{u}_S = +1$ over $[0, s_1) \cup [s_2, S]$ and $\hat{u}_S = -1$ over $[s_1, s_2)$. Let us also denote by x, y the solutions of (1) corresponding to \hat{u}_T and \hat{u}_S respectively and set $x_M := x(t_1)$, $x_m := x(t_2)$, $y_M := y(s_1)$, $y_m := y(s_2)$.

From Lemma 3.4, one has $x_M < y_M$, $x_m > y_m$, $t_1 < s_1$, and $t_2 < s_2$. So, we introduce a E defined by

$$E := \{s \in [0, S] ; y(s) > x_M \text{ or } y(s) < x_m\},$$

together with a function $\varphi : [0, T] \rightarrow [0, S] \setminus E$ by

$$\varphi(t) := \begin{cases} t & \text{if } t \in [0, t_1), \\ t + \delta_1 & \text{if } t \in [t_1, t_2), \\ t + \delta_1 + \delta_2 & \text{if } t \in [t_2, T], \end{cases}$$

where δ_1 , resp. δ_2 is the time spent by y over x , resp. below x . They are given by

$$\delta_1 := \text{meas}(\{s \in [0, S] ; y(s) > x_M\}), \quad \delta_2 := \text{meas}(\{s \in [0, S] ; y(s) < x_m\}).$$

By construction one has $x(t) = y(\varphi(t))$, for $t \in [0, T]$ and φ is bijective, thus $\text{meas}(E) = S - T$. Moreover, for any monotonic function $\rho : I \rightarrow \mathbb{R}$ one has

$$\int_0^T \rho(x(t)) dt = \int_0^T \rho(y(\varphi(t))) dt = \int_{[0, S] \setminus E} \rho(y(s)) ds,$$

by considering the change of variable $s = \varphi(t)$. We then get

$$\int_0^T \ell(x(t)) dt = \int_{[0, S] \setminus E} \ell(y(s)) ds, \quad (24)$$

and

$$\int_0^T \gamma(\ell(x(t))) dt = \int_{[0, S] \setminus E} \gamma(\ell(y(s))) ds.$$

As both controls \hat{u}_T and \hat{u}_S satisfy the constraint (4), one has

$$\frac{1}{T} \int_0^T \gamma(\ell(x(t))) dt = \frac{1}{S} \int_0^S \gamma(\ell(y(s))) ds = \bar{u},$$

which implies

$$\frac{1}{S - T} \int_E \gamma(\ell(y(s))) ds = \bar{u}. \quad (25)$$

Let us now consider a function $\hat{\gamma} : [\ell(y_m), \ell(y_M)] \rightarrow \mathbb{R}$ defined by

$$\hat{\gamma}(\xi) := \begin{cases} \gamma(\ell(x_m)) + \frac{\gamma(\ell(x_M)) - \gamma(\ell(x_m))}{\ell(x_M) - \ell(x_m)} (\xi - \ell(x_m)) & \text{for } \xi \in [\ell(x_m), \ell(x_M)], \\ \gamma(\xi) & \text{otherwise,} \end{cases}$$

(see Fig. 1). First, note that $\hat{\gamma}$ is convex increasing and satisfies

$$\hat{\gamma}(\xi) > \gamma(\xi) \text{ for } \xi \in (\ell(x_m), \ell(x_M)). \quad (26)$$

As one has $\gamma = \hat{\gamma}$ in $[\ell(y_m), \ell(y_M)] \setminus [\ell(x_m), \ell(x_M)]$, we also have, thanks to (25),

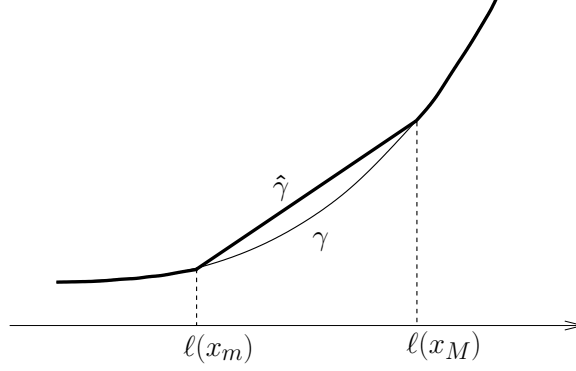
$$\frac{1}{S - T} \int_E \hat{\gamma}(\ell(y(s))) ds = \bar{u}.$$

By Jensen's inequality, we obtain

$$\frac{1}{S - T} \int_E \ell(y(s)) ds \leq \hat{\gamma}^{-1}(\bar{u}). \quad (27)$$

Now, since $\hat{\gamma}$ is affine over $[\ell(x_m), \ell(x_M)]$, one obtains

$$\hat{\gamma} \left(\frac{1}{T} \int_0^T \ell(x(t)) dt \right) = \frac{1}{T} \int_0^T \hat{\gamma}(\ell(x(t))) dt > \frac{1}{T} \int_0^T \gamma(\ell(x(t))) dt = \bar{u},$$

FIGURE 1. Functions $\gamma = \psi \circ \ell^{-1}$ and $\hat{\gamma}$ defined above.

using the fact that $x(t) \in [x_m, x_M]$ for $t \in [0, T]$, (26) and (4). Therefore, one has

$$\frac{1}{T} \int_0^T \ell(x(t)) dt > \hat{\gamma}^{-1}(\bar{u}). \quad (28)$$

We get by (24), (27) and (28)

$$\begin{aligned} \frac{1}{S} \int_0^S \ell(y(s)) ds &= \frac{1}{S} \int_E \ell(y(s)) ds + \frac{1}{S} \int_{[0, S] \setminus E} \ell(y(s)) ds \\ &\leq \frac{S-T}{S} \hat{\gamma}^{-1}(\bar{u}) + \frac{1}{S} \int_0^T \ell(x(t)) dt \\ &< \frac{S-T}{S} \frac{1}{T} \int_0^T \ell(x(t)) dt + \frac{T}{S} \frac{1}{T} \int_0^T \ell(x(t)) dt \\ &= \frac{1}{T} \int_0^T \ell(x(t)) dt, \end{aligned}$$

which concludes the proof. \square

We now give our main result.

Theorem 3.6. *Assume that Hypotheses (H1)-(H2)-(H3) are fulfilled. Then, for any $T > 0$, there are two optimal solutions of (13) given by the controls \hat{u}_T and \check{u}_T .*

Proof. Since (H3) implies (\bar{H}) , Proposition 3.2 gives the uniqueness of a T -admissible $B_+ B_- B_+$ trajectory (see Remark 6), which amounts to state that there are exactly two extremals with two switches (corresponding to $n = 1$), given by the controls $\hat{u}_T(\cdot)$ and $\check{u}_T(\cdot)$. Recall that they have same cost because $\check{u}_T(\cdot)$ is obtained by a time translation of $\hat{u}_T(\cdot)$.

Now, Proposition 3.1 shows that an optimal trajectory consists in $2n$ (with $n \geq 1$) switches, that occur exactly at the maximal and minimal values. It should be noted that any such trajectory with $2n$ switches ($n \geq 1$) is $\frac{T}{n}$ -periodic. By construction, an extremal has to cross \bar{x} after its two first switches, say at $\bar{t} > 0$. From $t = \bar{t}$, the control alternates the same values $+1$ and -1 and switching points occur at exactly the same values of $x(\cdot)$, namely x_M and x_m . Therefore, using Cauchy-Lipschitz's Theorem, one gets $x(t) = x(t + \bar{t})$ for any $t \in [0, \bar{t}]$ and successively on the intervals $[\bar{t}, 2\bar{t}], \dots, [(n-1)\bar{t}, n\bar{t}]$. Therefore $x(\cdot)$ is \bar{t} -periodic with $x(n\bar{t}) = x(T) = \bar{x}$, thus $\bar{t} = T/n$. We deduce that an extremal with $2n$ switches is T/n -periodic. To conclude,

suppose that an optimal trajectory has $2n$ switches with $n > 1$. Its cost is then equal to $J(\hat{u}_{T/n})$. Applying Lemma 3.5 with T and T/n gives $J(\hat{u}_T) < J(\hat{u}_{T/n})$, which proves that the optimal solution is achieved for $n = 1$ (i.e., with two switches). \square

An interesting consequence of Lemma 3.5 is the monotonicity property of the cost function evaluated at the optimal solution as a function of T .

Corollary 3.1. *The optimal criterion $T \mapsto J_T(\hat{u}_T)$ is decreasing w.r.t. T .*

4. Relaxing the assumptions for local over-yielding. The previous sections have shown the crucial role played by the monotonicity property of the function ψ and the convexity of the function γ on the interval I (see Hypotheses (\bar{H}) and $(H3)$). In the present section, we consider situations for which these conditions are not fulfilled on the whole interval I but only in a neighborhood of \bar{x} . Typically, there could exist other values of \bar{x} satisfying $\psi(\bar{x}) = \bar{u}$ (Hypothesis (\bar{H}) is thus not fulfilled on I) or γ could be only locally convex in a neighborhood of \bar{x} (Hypothesis $(H3)$ is thus not fulfilled on I). The idea is then to restrict the values of the period T for characterizing (periodic) optimal solutions remaining in a neighborhood of \bar{x} (and presenting over-yielding). Therefore, we expect to no longer have a systematic over-yielding (see remark 7 and Example 5.2.2 as an illustration).

We first revisit Proposition 3.2 as follows.

Proposition 4.1. *Assume that Hypotheses $(H1)$ – $(H2)$ are fulfilled with $\psi'(\bar{x}) > 0$. Then there exists $T_{max} > 0$ such that for any $T \in (0, T_{max})$, there exists a unique $(x_m, x_M) \in I^2$ that verify (20) and (21) with*

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in [x_m, x_M] \setminus \{\bar{x}\}. \quad (29)$$

Proof. Consider a sub-interval $J := (\tilde{a}, \tilde{b}) \subset I$ with $\tilde{a} < \bar{x} < \tilde{b}$ such that the property

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in J \setminus \{\bar{x}\}. \quad (30)$$

is fulfilled (as ψ' is strictly positive at \bar{x} , we know that such an interval exists). Let us then consider the function \tilde{f} defined on the interval $[a, b]$ by

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in J, \\ -g(x) \left(\frac{\psi(\tilde{a}) + 1}{\tilde{a} - a} (x - a) - 1 \right) & \text{if } x \in [a, \tilde{a}], \\ -g(x) \left(\frac{1 - \psi(\tilde{b})}{b - \tilde{b}} (b - x) + 1 \right) & \text{if } x \in [\tilde{b}, b]. \end{cases}$$

Clearly, the pair (\tilde{f}, g) satisfies Hypotheses $(H1)$ – $(H2)$ – (\bar{H}) . The function \tilde{f} is not C^1 but Lipschitz continuous, but one can easily check that Lemma 3.3, Proposition 3.2 and Lemma 3.4 are still valid with f merely Lipschitz continuous. This gives the existence and uniqueness of x_m and x_M that verify (20) and (21) for the pair (\tilde{f}, g) and any $T > 0$. As $T \mapsto x_m(T)$ and $T \mapsto x_M(T)$ are respectively decreasing and increasing w.r.t. T (recall Lemma 3.4), there exists $\tilde{T} > 0$ such that $x_m(\tilde{T}) = \tilde{a}$ or $x_M(\tilde{T}) = \tilde{b}$. As f coincides with \tilde{f} on $[\tilde{a}, \tilde{b}]$, we conclude that x_m, x_M are the unique numbers that verify (20) and (21) on $[\tilde{a}, \tilde{b}]$ for the pair (f, g) and any $T \leq \tilde{T}$. This can be done for any sub-interval J that verifies condition (30). We then consider T_{max} as the supremum of \tilde{T} for all such sub-intervals J . \square

Given $T < T_{max}$, one may wonder if is enough to require Hypothesis (H3) to be fulfilled on $[x_m, x_M]$ (instead of I) to obtain the optimality of the controls \hat{u}_T , \check{u}_T as in Theorem 3.6. However, there could exist extremal trajectories taking values outside the interval $[x_m, x_M]$, without requiring additional assumption on the function ψ outside this set.

For this purpose, we consider the two controls u^- and u^+ defined by one switching time $t^- \in (0, T)$ (for u^-) and $t^+ \in (0, T)$ (for u^+) as

$$u^-(t) = \begin{cases} -1, & t \in [0, t^-), \\ 1, & t \in [t^-, T], \end{cases}$$

$$u^+(t) = \begin{cases} 1, & t \in [0, t^+), \\ -1, & t \in [t^+, T], \end{cases}$$

such that the corresponding trajectories $x(\cdot, u^-, \bar{x})$ and $x(\cdot, u^+, \bar{x})$ are T -periodic (see Fig. 2). Let us then define $x_T^- \in \mathbb{R}$, $x_T^+ \in \mathbb{R}$ as

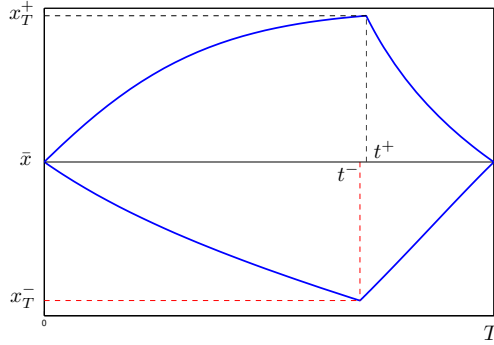


FIGURE 2. T -periodic solutions $x(\cdot, u^-, \bar{x})$ and $x(\cdot, u^+, \bar{x})$.

$$\begin{cases} x_T^- &:= x(t^+, u^+, \bar{x}), \\ x_T^+ &:= x(t^-, u^+, \bar{x}). \end{cases} \quad (31)$$

One can check that under Hypotheses (H1)-(H2), any T -periodic solution $x(\cdot)$ of (1) with $x(0) = \bar{x}$ and control u taking values in $[-1, 1]$ verifies

$$x(t) \in [x_T^-, x_T^+], \quad \forall t \in [0, T]. \quad (32)$$

Indeed, by comparison of solutions of scalar ODEs over $[0, t^+]$, one obtains (since $u^+(t) = 1$ on $[0, t^+]$ and $f + ug \leq f + g$, $u \in [-1, 1]$):

$$x(t) \leq x(t, u^+, \bar{x}), \quad \forall t \in [0, t^+].$$

Furthermore, over the time interval $[t^+, T]$, the same reasoning for the backward dynamics yields (since $u^+(t) = -1$ on $[t^+, T]$ and $-(f + ug) \leq -(f - g)$, $u \in [-1, 1]$):

$$x(t) \leq x(t, u^+, x_T^+), \quad \forall t \in [t^+, T].$$

It follows that

$$x(t) \leq x(t, u^+, \bar{x}), \quad \forall t \in [0, T].$$

By a similar argumentation with the control u^- in place of u^+ , one concludes that

$$x(t, u^-, \bar{x}) \leq x(t) \leq x(t, u^+, \bar{x}), \quad \forall t \in [0, T],$$

which completes the proof of Property (32). It can also be observed that one has $x_T^- < \bar{x} < x_T^+$ and $(x_T^-, x_T^+) \rightarrow (\bar{x}, \bar{x})$ when $T \rightarrow 0$.

We give now a result requiring the condition (29) to be fulfilled on $[x_T^-, x_T^+]$, which guarantees that any optimal solution is in the interval $[x_m, x_M]$.

Proposition 4.2. *Assume that Hypotheses (H1)-(H2) are fulfilled with $\psi'(\bar{x}) > 0$. Take $T \in (0, T_{max})$ such that*

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in [x_T^-, x_T^+] \setminus \{\bar{x}\} \quad (33)$$

where x_T^- , x_T^+ are defined in (31). Then there exists unique x_m , x_M in $[x_T^-, x_T^+]$ satisfying (20) and (21). If ψ is increasing on $[x_m, x_M]$, then any T -admissible solution $x(\cdot)$ verifies

$$\hat{x} := \max_{t \in [0, T]} x(t) \leq x_M \quad \text{and} \quad \check{x} := \min_{t \in [0, T]} x(t) \geq x_m.$$

Proof. Fix $T \in (0, T_{max})$ that fulfills condition (33). Note that this is possible since ψ is increasing in a neighborhood of \bar{x} and $(x_T^-, x_T^+) \rightarrow (\bar{x}, \bar{x})$ when $T \rightarrow 0$.

According to Proposition 4.1, there exists unique x_m , x_M that verify (20) and (21). Since there exists a T -admissible trajectory taking the values x_m and x_M , one has necessarily

$$x_T^- < x_m < \bar{x} < x_M < x_T^+. \quad (34)$$

Consider now any T -admissible solution x . From the property (32), one has $\hat{x} \leq x_T^+$ and $\check{x} \geq x_T^-$. Moreover, from condition (33) and Lemma 2.1, one has $\hat{x} > \bar{x} > \check{x}$. Let $\hat{t} \in]0, T[$ be such that $x(\hat{t}) = \hat{x}$ and suppose that one has $\hat{x} > x_M$. We can assume, without loss of generality, that $x(t) \geq \bar{x}$ is satisfied for any $t \in [0, \hat{t}]$ (if not, consider $t_0 := \sup\{t < \hat{t} ; x(t) < \bar{x}\}$ and replace $x(\cdot)$ by $x(\cdot + t_0)$). Let $(A, B) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ be defined by

$$A := \int_{\bar{x}}^{\hat{x}} \frac{dx}{f(x) + g(x)} \quad \text{and} \quad B := - \int_{\bar{x}}^{\hat{x}} \frac{dx}{f(x) - g(x)}.$$

It can be observed that A and B are the fastest times for a solution of (1) to reach, respectively, \hat{x} from \bar{x} (with the constant control $u = 1$) and \bar{x} from \hat{x} (with the constant control $u = -1$). Clearly, one has $\hat{t} \geq A$ and $T - \hat{t} > B$.

We construct now a T -periodic solution \tilde{x} of (1) such that $\tilde{x}(0) = \bar{x}$ and associated with a control \tilde{u} defined as follows

$$\tilde{u}(t) = \begin{cases} \bar{u} & \text{if } t \in [0, \hat{t} - A[, \\ 1 & \text{if } t \in [\hat{t} - A, \hat{t} \cup [t^\dagger, T], \\ -1 & \text{if } t \in [\hat{t}, t^\dagger[, \end{cases} \quad (35)$$

where t^\dagger is given by

$$t^\dagger = T - \int_{x^\dagger}^{\bar{x}} \frac{dx}{f(x) + g(x)},$$

and x^\dagger is a solution of $\kappa(x^\dagger) = T - \hat{t}$, the map $\kappa(\cdot)$ being defined by

$$\kappa(\xi) := \int_{\xi}^{\bar{x}} \frac{dx}{f(x) + g(x)} - \int_{\xi}^{\hat{x}} \frac{dx}{f(x) - g(x)}, \quad \xi \in I.$$

By Hypothesis (H2), the function κ is decreasing and one has

$$\kappa(x_m) = \int_{x_m}^{\hat{x}} \eta(x) dx - A > \int_{x_m}^{x_M} \eta(x) dx - \hat{t} = T - \hat{t},$$

and $\kappa(\bar{x}) = B < T - \hat{t}$. Therefore x^\dagger is uniquely defined with $x^\dagger \in (x_m, \bar{x})$. Moreover, one has

$$t^\dagger = \hat{t} - \int_{x^\dagger}^{\hat{x}} \frac{dx}{f(x) - g(x)} \in]\hat{t}, T[.$$

Expression (35) is thus well defined. The solution \tilde{x} is depicted on Fig. 3.

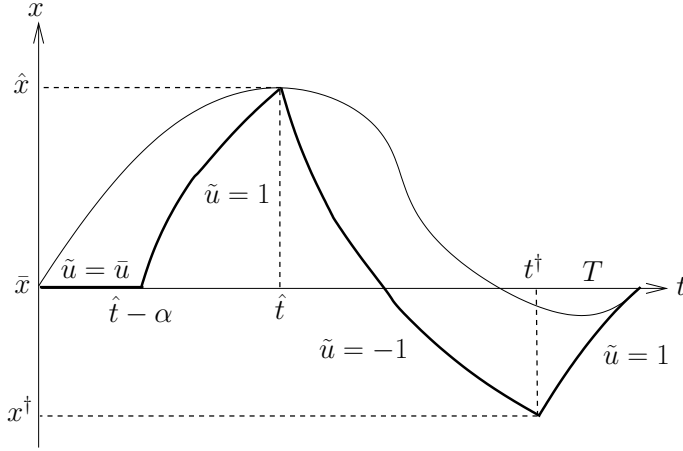


FIGURE 3. The solution \tilde{x} in thick line, x in thin line.

Clearly \tilde{x} reaches \hat{x} at time \hat{t} and it is below the function x on the interval $[0, \hat{t}]$. On the interval $[\hat{t}, t^\dagger]$, \tilde{x} has the fastest descent and therefore stays also below x on this interval. At time $t = t^\dagger$, one has $\tilde{x}(t^\dagger) = x^\dagger$. Finally, the constant control $u = 1$ is the only one that allows to connect x^\dagger at time t^\dagger to \bar{x} at time T . So, any periodic solution has to be above \tilde{x} on $[t^\dagger, T]$. We conclude that one has $x(t) \geq \tilde{x}(t)$ for any $t \in [0, T]$. As $\psi(x) > \psi(\bar{x})$ for $x \in [x_M, \hat{x}]$ and ψ is increasing on $[x_m, x_M]$, and as we have shown that $x(t) > x_m$ for any $t \in [0, T]$, one can write

$$\begin{aligned} \int_0^T (\psi(x(t)) - \psi(\bar{x})) dt &> \int_{\{t \in [0, T] | x(t) \leq x_M\}} (\psi(x(t)) - \psi(\bar{x})) dt \\ &\geq \int_{\{t \in [0, T] | x(t) \leq x_M\}} (\psi(\tilde{x}(t)) - \psi(\bar{x})) dt \\ &= \int_{x^\dagger}^{x_M} (\psi(x) - \psi(\bar{x})) \eta(x) dx. \end{aligned}$$

To conclude, since one has $x^\dagger > x_m$ and $\eta > 0$ on I , one obtains

$$\int_0^T (\psi(x(t)) - \psi(\bar{x})) dt > \int_{x_m}^{x_M} (\psi(x) - \psi(\bar{x})) \eta(x) dx = 0,$$

which is not possible according to Lemma 2.1. We then conclude that the inequality $\hat{x} \leq x_M$ is satisfied. In a similar manner, one can prove the other inequality $\tilde{x} \geq x_m$. \square

For periods $T > 0$ that fulfill conditions of Proposition 4.2, we know that optimal solutions remain in the set $[x_m, x_M]$. We then obtain the same conclusion as Theorem 3.6 when Hypothesis (H3) is fulfilled on the interval $[x_m, x_M]$ only, as stated by the following Theorem.

Theorem 4.1. *Assume that Hypotheses (H1)-(H2) are fulfilled and consider $T > 0$ such that*

- i) $(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0$ for any $x \in [x_T^-, x_T^+] \setminus \{\bar{x}\}$, where x_T^-, x_T^+ are defined in (31),*
- ii) ℓ is increasing on $[x_m, x_M]$ and $\gamma = \psi \circ \ell^{-1}$ is strictly convex increasing on $[\ell(x_m), \ell(x_M)]$, where x_m and x_M are given by Proposition 4.1.*

Then, there are two optimal solutions of (13), given by the controls \hat{u}_T and \check{u}_T .

Proof. First, assumption ii) implies that ψ is increasing on the interval $[x_m, x_M]$. Thanks to i), we know from Proposition 4.2 that any extremal is such that x takes values within the interval $[x_m, x_M]$. With assumption ii) instead of Hypothesis (H3), the reader can easily check that the arguments of Theorem 3.6 apply in the same manner on $[x_m, x_M]$ (instead of the whole interval I), to prove that only the extremals \hat{x} and \check{x} are optimal. \square

Remark 7. When hypotheses (H1)-(H2)-(H3) are not all satisfied (what is considered in this section), over-yielding cannot be guaranteed for any value of T as in the previous section. However, Theorem 4.1 provides optimal periodic solutions that present over-yielding.

5. Periodic versus constant strategies in a single population model. We consider an exploited stock of a renewable resource (fish, forest..) represented by its density $x(t)$ which follows a dynamics

$$\dot{x} = f_0(x) - E(t)x, \quad (36)$$

where the growth function $f_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$ is of class C^1 and satisfies $f_0(0) = 0$. The harvesting effort E , which is considered as a measurable control, takes values within an interval $[0, E_{max}]$ (with $E_{max} > 0$). Such models have been extensively studied in the bio-economics literature (see for instance [10] and the references cited herein). Typically an optimal steady state \bar{x} associated with a constant control \bar{E} is determined as maximizing a bio-economic profit of the harvesting over a discounted infinite horizon. However, it is not always possible or desirable to apply the theoretical value \bar{E} of the harvesting effort in a constant manner (because of labor laws, seasonality...), but its average value is usually guaranteed on a period T . In this context, our objective is to study the impacts on the stock of applying a periodic harvesting effort instead of a constant one. We study conditions on the growth function for periodic harvesting effort having negative impact or not. In the case of negative impact, we then consider the worst scenarios to estimate the maximal loss that could be expected. There are several ways of measuring the impacts on a stock, in terms of a function $\ell(x)$ which measures the *well-being* of the stock or its *utility* (such as recreative activities). In the simplest case, $\ell(x)$ is just equal to the stock density x but more generally one can consider that $\ell : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a C^1 concave increasing function.

Given a constant control $\bar{E} \in (0, E_{max})$, we then consider an associated steady-state \bar{x} of (36) such that

$$h(\bar{x}) = \bar{E}, \quad (37)$$

where $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the function defined as

$$h(x) := \begin{cases} \frac{f_0(x)}{x} & \text{if } x > 0, \\ f'_0(x) & \text{if } x = 0. \end{cases}$$

Note that equation (37) may have several solutions. We consider one of them which leads to a stable equilibrium (one can easily check that this amounts to have $\bar{E} > f'_0(\bar{x})$). Our aim is to study if the average criterion

$$J_T(E) := \frac{1}{T} \int_0^T \ell(x(t)) dt, \quad (38)$$

can be improved by considering T -periodic inputs $E(\cdot)$ satisfying

$$\frac{1}{T} \int_0^T E(t) dt = \bar{E}, \quad (39)$$

and T -periodic solutions of (36) associated with $E(\cdot)$ with

$$x(0) = x(T) = \bar{x}. \quad (40)$$

In order to use the previous setting, we consider the following change of variables:

$$u := 1 - \frac{2E}{E_{max}}; \quad f(x) := f_0(x) - \frac{E_{max}}{2}x; \quad g(x) := \frac{E_{max}}{2}x,$$

and the function ψ becomes

$$\psi(x) = -\frac{f(x)}{g(x)} = 1 - \frac{2}{E_{max}}h(x).$$

So, (36) has exactly the form (1) with $u \in [-1, 1]$. Let $\bar{u} \in (-1, 1)$ be the constant control associated with \bar{E} such that $\bar{u} = \psi(\bar{x})$. We now study the effects of T -periodic inputs for two growth functions f_0 : the classical logistic function, and the modified one with a depensation term (that will highlight Section 4).

5.1. The logistic growth. We recall the classical expression of this model

$$f_0(x) := rx \left(1 - \frac{x}{K}\right),$$

where $r > 0$ and $K > 0$. One can easily check that there exists a positive equilibrium \bar{x} of (36) satisfying (37) as soon as $\bar{E} < r$. Moreover, \bar{x} is a stable equilibrium (see [10]). We assume hereafter that one has $\bar{E} < r$. Since one has

$$(f - g)(x) = x \left(r - E_{max} - \frac{r}{K}x\right) \quad ; \quad (f + g)(x) = rx \left(1 - \frac{x}{K}\right),$$

Hypotheses (H1)-(H2) are satisfied for the interval $I := (\lambda(E_{max}), K)$ where

$$\lambda(E_{max}) = \begin{cases} 0 & \text{if } E_{max} > r, \\ h^{-1}(E_{max}) & \text{if } E_{max} < r. \end{cases}$$

Note that ψ is an affine function: $\psi(x) = c_1x + c_0$ with $c_0 = 1 - \frac{2r}{E_{max}}$, $c_1 = \frac{2r}{KE_{max}}$. When ℓ is strictly concave, the function $\gamma = \psi \circ \ell^{-1}$ is strictly convex (and increasing). Hypothesis (H3) is thus satisfied. According to Proposition 2.1, there is a systematic over-yielding whatever is $T > 0$, *i.e.*, the average criterion J_T is always below $\ell(\bar{x})$. Its lowest value is given by the two strategies $B_+B_-B_+$ or $B_-B_+B_-$ (see Theorem 3.6). Note that when $\ell(x) = x$, the function γ is affine and consequently the criterion J_T is always equal to \bar{x} , *i.e.*, the average of the stock is always equal to \bar{x} .

We now illustrate the over-yielding with the function

$$\ell(x) := \frac{4x}{1+x},$$

which is concave increasing. Numerical simulations have been conducted with the parameters values $r = 3$, $K = 7$, $\bar{x} = 3.5$, $E_{max} = 2.5$ and $\bar{E} = 1.5$. Results are depicted on Fig. 4.

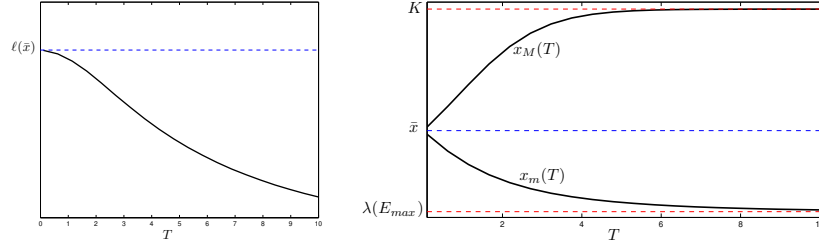


FIGURE 4. Optimal criterion $J_T(\hat{u}_T)$ (left) and x_m, x_M (right) as functions of the period T for the logistic growth.

5.2. The logistic with depensation. Some populations are known to present a *depensation* in the first part of their growth function [10], which is also called a *weak Allee effect*. This is represented by the following modification of the logistic function

$$f_0(x) := rx^\alpha \left(1 - \frac{x}{K}\right),$$

with $\alpha > 2$. For this function, one has

$$h(x) = rx^{\alpha-1} \left(1 - \frac{x}{K}\right),$$

which is increasing on $[0, x^*)$ and decreasing on $(x^*, K]$ with

$$x^* := \frac{\alpha - 1}{\alpha} K,$$

(see Fig. 5). In presence of depensation in the model, one can also easily check that the function ψ is concave decreasing on $[0, x_c)$, convex decreasing on (x_c, x^*) , and convex increasing on $(x^*, K]$ with

$$x_c := \frac{\alpha - 2}{\alpha} K < x^*,$$

(see Fig. 5).

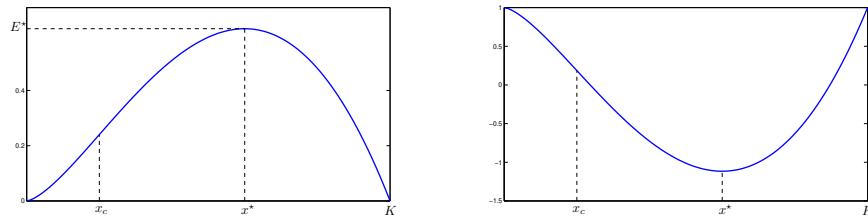


FIGURE 5. Graphs of the functions h (left) and ψ (right) for $r = 0.3$, $K = 5$ and $\alpha = 2.5$.

We shall consider here the function $\ell(x) = x$ (*i.e.*, the criterion is simply the level of the stock x). Let us define

$$E^* := h(x^*).$$

We distinguish now two cases depending if E_{max} is below or above E^* .

5.2.1. *Case 1: $E_{max} < E^*$.* Note first that there are two solutions $\lambda_1(E_{max})$ and $\lambda_2(E_{max})$ on the interval $(0, K)$ of the equation $h(x) = E_{max}$ such that $\lambda_1(E_{max}) < x^* < \lambda_2(E_{max})$. One can then check that Hypotheses (H1)-(H2)-(H3) are fulfilled on the interval $I := (\lambda_2(E_{max}), K)$. For any $\bar{E} \in (0, E_{max})$, one can also show, as in the logistic model, that there exists a unique solution $\bar{x} \in I$ of (37) which is moreover a stable steady-state of (36) (see [10]). Proposition 2.1 guarantees then an over-yielding whatever is $T > 0$.

Fig. 6 depicts the optimal cost value $J_T(\hat{u}_T)$ for the following parameter values: $r = 0.3$, $K = 5$, $a = 2.5$, $\bar{x} = 4$, $\bar{E} = 0.48$, and $E_{max} = 0.5893$.

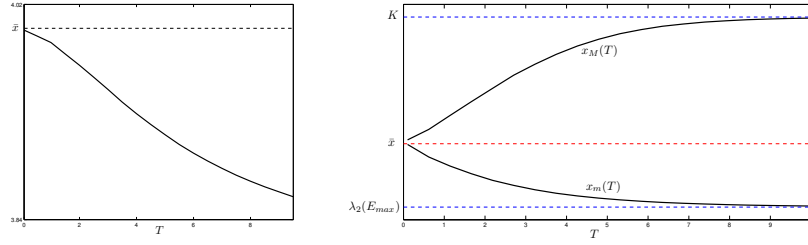


FIGURE 6. Optimal criterion $J_T(\hat{u}_T)$ (left) and x_m , x_M (right) as functions of the period T for the depensation model (case 1).

Finally, in presence of depensation in the model with a maximal harvesting effort $E_{max} < E^*$, our analysis shows that periodic solutions cause a systematic decrease of the mean value of the stock (compared to constant harvesting).

5.2.2. *Case 2: $E_{max} > E^*$.* One can easily check that Hypotheses (H1)-(H2) are fulfilled on the interval $(0, K)$, but not Hypothesis (H3). Since \bar{x} is a stable steady-state of the dynamics, the point \bar{x} belongs to the interval (x^*, K) (see [10]). Note also that ψ is increasing in a neighborhood of \bar{x} . Proposition 4.1 guarantees then the existence of the T -periodic trajectory $B_+B_-B_+$ (or $B_-B_+B_-$) that satisfies the integral constraint, for T not too large. Moreover for T small enough, the function ψ is strictly convex on $[x_m(T), x_M(T)]$, and we can conclude about the optimality of these trajectories according to Theorem 4.1.

Using the same parameter values except $E_{max} = 0.8235$, the function F defined in (22) is depicted on Fig. 7 (left) for different values of T . We recall (see the proof of Proposition 3.2) that the existence of x_m , x_M is equivalent to the existence of a zero of F . One can see on this figure that for $T > 6$, the equation $F(\alpha) = 0$ has no roots, and therefore x_m and x_M no longer exist for these values of the parameter T . This means that the $B_+B_-B_+$ (or $B_-B_+B_-$) strategy cannot satisfy the periodicity and the input constraint for $T > 6$. On the contrary, for $T < 6$ (in this example $T_{max} = 6$ where T_{max} is given by Proposition 4.1), the $B_+B_-B_+$ and $B_-B_+B_-$ strategies are admissible, and x_m , x_M , x_T^- , x_T^+ are plotted as function of T on Fig. 7 (right). Remark that property (34) is fulfilled, for all $T < 6$. Note that

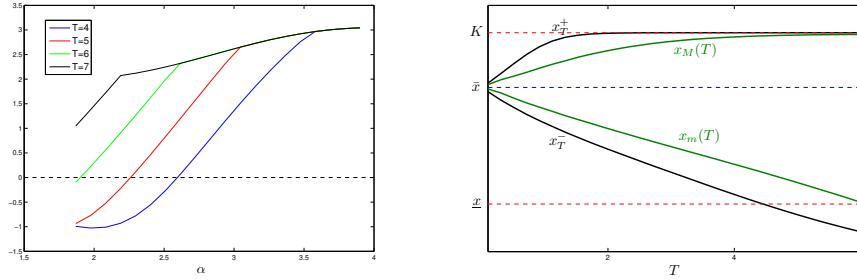


FIGURE 7. Plot of the function F defined by (22) (left), and x_m , x_M , x_T^- , x_T^+ (right) as functions of the period T ($T < 6$) for the depensation model (case 2).

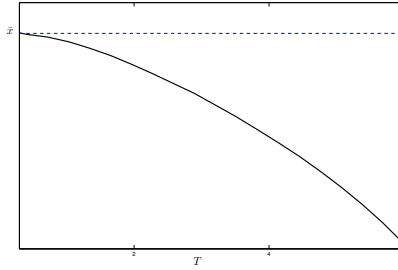


FIGURE 8. Optimal criterion $J_T(\hat{u}_T)$ for the depensation model (case 2)

equation $h(x) = \bar{E}$ has two solutions $\underline{x} < \bar{x}$ (such that $\psi(\underline{x}) = \psi(\bar{x})$). Finally, on Fig. 8, we present the cost of the $B_+B_-B_+$ (or $B_-B_+B_-$) strategy as a function of T (for $T < 6$).

6. Conclusion. In this work, we have shown that under concavity assumptions, the optimal trajectory is the steady-state solution, that is, no over-yielding is possible.

On the contrary, under convexity assumptions, we have proved that there is exactly one optimal trajectory (up to a time translation) which is bang-bang with two switches on a period. This optimality result is global and valid for any period T . We have also relaxed the hypotheses to prove the same optimality result globally, but for a limited range of values of the period T , when only local convexity is fulfilled.

The determination of the optimal solution for large values of T when neither convexity nor concavity assumptions are fulfilled appears to be much more complex, as the bang-bang solution is no longer admissible.

This analysis was illustrated in the context of a population model subject to a harvesting effort. Depending on the growth model and the criterion, we are able to predict the effect of a periodic harvesting efforts (with the same given mean value) compared to the constant value at steady-state. Such analysis in this context is new to our best knowledge.

Some of the techniques we have proposed here to cope with the integral constraint on the control variable, which is the main characteristic of the problem we have considered, could be deployed for systems in higher dimensions, and will be the matter of a future work.

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REFERENCES

- [1] E.M. Abulesz and G. Lyberatos, *Periodic Impulse Forcing of Nonlinear Systems: a New Method*, International Journal of Control, vol. 48, pp. 469–480, 1988.
- [2] E.M. Abulesz and G. Lyberatos, *Periodic Optimization of Microbial Growth Processes*, Biotechnology & Bioengineering, vol. 29, pp. 1059–1067, 1987.
- [3] E.M. Abulesz and G. Lyberatos, *Periodic Operation of a Continuous Culture of Baker's Yeast*, Biotechnology & Bioengineering, vol. 34, pp. 741–749, 1989.
- [4] A. O. Belyakov and V.M. Veliov, *Constant versus periodic fishing: age structured optimal control approach* Math. Model. Nat. Phenom., vol. 9, pp. 20–37, 2014.
- [5] D.S. Bernstein and E.G. Gilbert, *Optimal periodic control: The π test revisited*, IEEE Transactions on Automatic Control, vol. 25, pp. 673–684, 1980.
- [6] S. Bittanti, G. Fronza and G. Guardabassi, *Periodic control: A frequency domain approach*, IEEE Transactions on Automatic Control, vol. 18, pp. 33–38, 1973.
- [7] S. Bittanti, A. Locatelli and C. Maffezzoni, *Second-variation methods in periodic optimization*, J. Optimization Theory and Appl., vol. 14, pp. 31–49, 1974.
- [8] S. Bittanti, A. Locatelli and S. Rinaldi, *Status of periodic optimization of dynamical systems*, J. Optimization Theory and Appl., vol. 14, pp. 1–20, 1974.
- [9] L. Cesari, *Optimization-Theory and Applications. Problems with ordinary differential equations*, Springer, 1983.
- [10] C.W. Clark *Mathematical Bioeconomics: The Mathematics of Conservation* Wiley, 2010.
- [11] R.T. Evans, J.L. Speyer and C.H. Chuang, *Solution of a periodic optimal control problem by asymptotic series*, J. Optimization Theory and Appl., vol. 52, pp. 343–364, 1987.
- [12] E.G. Gilbert, *Optimal periodic control: A general theory of necessary conditions*, SIAM J. Control Optim., vol. 15, pp. 717–746, 1987.
- [13] E.G. Gilbert, Q. Wang, Q. and J.L. Speyer, *Necessary and Sufficient Conditions for Local Optimality of a Periodic Process*, SIAM J. Control Optim., vol. 28, pp. 482–497, 1990.
- [14] J. Harmand, C. Lobry, A. Rapaport and T. Sari, *The Chemostat: mathematical theory of microorganisms cultures*, ISTE Wiley, 2017.
- [15] V. Hatzimanikatis, G. Lyberatos, S. Pavlou and S.A. Svoronos, *A Method for Pulsed Periodic Optimization of Chemical Reaction Systems*, Chemical Engineering Science, vol. 48, n. 4, pp. 789–797, 1993.
- [16] L.V. Idels, *Stability analysis of periodic Fox production models*, Can. Appl. Math. Q. 14, no. 3, pp. 331–341, 2006.
- [17] L.V. Idels, M. Wang *Harvesting Strategies with Modified Effort Function*, Intern. J. of Modelling, Identification and Control, Special issue Modeling Complex Systems (IJMIC) vol. 3, n. 1, pp. 83–87, 2008.
- [18] C. Maffezzoni, *Hamilton-Jacobi theory for periodic control problems*, J. Optimization Theory and Appl., vol. 14, pp. 21–29, 1974.
- [19] L. Perko, *Differential equations and dynamical systems*, Springer Sciences, 2013.
- [20] L.S. Pontryagin, V.G. Boltyanskiy, R.V. Gamkrelidze and E.F. Mishchenko, *Mathematical theory of optimal processes*, The Macmillan Company, 1964.
- [21] J.L. Speyer and R.T. Evans, *A second variation theory for optimal periodic processes*, IEEE Transactions on Automatic Control, vol. 29, pp. 138–148, 1984.

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