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Oriented distance point of view on random sets with application to shape optimization

M. Dambrine and B. Puig

Abstract: Motivated by free boundary problems under uncertainties, we consider the oriented distance function as a way to define the expectation for a random compact or open set. In order to provide a large numbers law and a central limit theorem for this notion of expectation, we also address the question of the convergence of the level sets of f_n to the level sets of f when (f_n) is a sequence of functions uniformly converging to f . We provide error estimates in terms of Hausdorff convergence. We illustrate our result on a free boundary problem.

Key words: Random sets, continuity of level sets, oriented distance functions, large numbers law, central limit theorem, free boundary problem.

1 Introduction

This work is motivated by the consideration of uncertainties in free boundary problem and in shape optimization problem. Roughly speaking, we consider situations where a shape is the output of a mathematical problem like the minimization of a functional or an over determined boundary value problem and the situation depends of some parameters that are not exactly known but only through some statistical information over its distribution. A typical problem we have in mind is the optimal design of a bridge for when the applied loading (typically the weight of the people on the bridge) is random.

For a fixed value p of the parameter, the problem is a classical shape optimization problem associated to an objective $D \mapsto J(D, p)$ where D stands for a open subset of \mathbb{R}^d in a given class of domain. This question has been widely studied. However, its solution is a shape in $Argmin J(., p)$ and depends on p . When p is random, the object we are interested in is a random set. After the first pioneering works of Matheron and Kendall (see [20],[19], [21]), the study of random sets is receiving growing attention in the statistical and probabilistic literature. Let us first precise some general notions about random sets and what will be our point of view on this work.

We fix a dimension d . Let B_∞ be a given open ball in \mathbb{R}^d of radius R_∞ , this ball will play the role of a box and we will consider only subsets of B_∞ . Let \mathcal{K} be the set of all compact sets contained in B_∞ . We also denote the probability space by $(\Omega, \mathcal{A}, \mathbb{P})$. The space \mathcal{K} endowed with the Hausdorff metric (see section 2 for precise definition and properties) is a complete separable metric space and is then endowed with the Borel σ - field associated with this metric. A random compact set in B_∞ is a measurable function from Ω to \mathcal{K} . As well-known, \mathcal{K} is not a vector space but just a metric space: therefore the notion of expectation of a random compact set is difficult, there is no canonical definition.

The most usual choice is the so-called Aumann mean since it corresponds to the limit of the natural Minkowski empirical means. Central Limit Theorems for this case have already been derived (see [4]). This choice is not satisfactory for our purpose since the Aumann mean of a random set is convex. If this notion is perfectly suited for situation where the minimizers are known to be convex, it seems unadapted to application to usual mechanical devices. In [11], we have explored two possibilities in the context of free boundary problem: a notion based on a parametrization of the sets and the Vorob'ev mean based on the quantiles of the coverage function of the random set (that is the probability that a given point belongs to the random set). We would like to choose a notion of mean set adapted to the shape optimization context, leading to computations that remains realistic and sufficiently rich to provide a good description of the object.

In this work, we are interested by random sets that arising in shape optimization. We will make the assumption pertinent in many application to deal in a bounded situation: we shall consider subset of a fixed ball B_∞ that plays the role of a box. In that context, the compact assumption made in the probabilistic literature is unsuited: the theory of partial differential equations requires to work with domains that are open sets. Moreover, the usual existence results for optimal shapes are stated in the class of quasi open set. However, the optimality conditions usually allow to prove more regularity of the set: typically, one can expect to deal with open sets with a piecewise smooth boundary. Therefore, an alternative appears: either, one deals with random compact sets that are closure of open sets, either one deals directly with random open sets defined as follows. The class \mathcal{O} of open subsets of B_∞ is a separable metric space endowed with the Hausdorff distance for open sets (see Section 2) and a random open set is a measurable function from Ω to \mathcal{O} endowed with the Borel σ - field associated with the Hausdorff distance.

There are two important ways to parametrize domains. The first one is the Hadamard point of view leading to the shape calculus and the notion of shape derivative: the idea is to parametrize a domain as the image of a reference domain by a diffeomorphism and to use the Banach structure of diffeomorphisms to define derivative. The second one is to use a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ to implicitly define the domain under study D as $\{x \mid f(x) < 0\}$ that will be denoted in the sequel as $[f < 0]$. We will denote level sets defined by inequalities or equalities as $[f < 0]$. This parametrization is the ground of the level set method which coincides in that context to the shape gradient flow. The leading idea of this work is to use such an implicit parametrization of shapes to define the expectation. Therefore, we are interested in basing the notion of mean on the level set functions.

We then have to face a difficulty: the choice of f is not unique. Therefore, in order to define an expectation, for each compact set $K \in \mathcal{K}$, we have to choose a specific function f such that $K = [f \leq 0]$. The natural choice is the oriented distance function to K . Since some decades, the oriented distance function to a compact set K has become a used way to describe the property of the set. It has been first used to study the motion by mean curvature on a theoretical level. Then, Osher and Sethian [22] introduced the level set method that has become the reference encoding of evolving domains. In shape optimization, it is used to implement the gradient method [2] or even the Newton method [1] and also to obtained compactness result on class of domain [12] by exploiting the deep connection between the geometric properties of the boundary of K and the oriented distance function to K . Notice that few theoretical convergence results have been obtained for this level method in the context of shape optimization even at the continuous level [8, 9].

Of course, the oriented distance function is a parametrization and we have to face the

difficulties of parametrization based approaches to expectation. In particular, it is not intrinsic but depends on the choice of parametrization. In addition, the natural procedure to define the expectation of the random domain as the domain whose distance function is the expectation of the distance functions to the random set makes no sense: the set of distance functions is not convex: a mean of oriented distance functions is not the oriented distance to some set (see section 2.2) and we have to consider some relaxation of the definition.

This article is organized as follows. In section 2, we recall the definitions of the Hausdorff distance and oriented distance functions and provide useful result on their use to implicitly represent domains. In section 3, we study approximation properties. The underlying question is: how far is $[f_n \leq 0]$ from $[f \leq 0]$ when (f_n) is a sequence converging to f ? We provide condition to prove convergence of the level sets and also convergence rates. Then, in section 4, we define a notion of expectation of random compact set through oriented distance functions. It is therefore natural to consider an estimator of this expectation defined like a level set of the empirical mean of this oriented distance function. Some results of convergences are given. In the last part, an application of these tools to the Bernoulli exterior free boundary problem is presented.

Notice that many parts of this work can be found spread in the literature on shape optimization, convex analysis and statistical estimation. In particular, we discovered during this work that the notion of oriented distance based expectation was introduced by Jankowski and Stanberry in [18]. However, we provide many improvements in the results and we believe to have simplified many proofs in a self contained work.

2 Implicit representation of sequence of sets by distance functions

In this section, we consider various implicit representations of compact sets and provide technical results that will be used in the sequel.

2.1 Distance functions and Hausdorff convergence.

Let us recall the definition of the Hausdorff distance on \mathcal{K} . We first consider compact sets. Let K_1 and K_2 in \mathcal{K} , then the excess of K_1 with respect to K_2 is

$$\rho(K_1, K_2) = \sup_{x \in K_1} d(x, K_2).$$

The Hausdorff distance is then $d_{\mathcal{H}}(K_1, K_2) = \max(\rho(K_1, K_2), \rho(K_2, K_1))$. As a consequence, a sequence (K_n) in \mathcal{K} Hausdorff converges to K if $d_{\mathcal{H}}(K_n, K) \rightarrow 0$. This will be denoted by $K_n \xrightarrow{\mathcal{H}} K$.

An important property is the characterization of the Hausdorff distance by the distance functions. For $K \in \mathcal{K}$, we set $d_K : x \mapsto d(x, K)$ and then it holds

$$d_{\mathcal{H}}(K_1, K_2) = \|d_{K_1} - d_{K_2}\|_{\infty}.$$

Since we will only consider continuous functions defined on the whole \mathbb{R}^d , the uniform convergence on \bar{B}_{∞} provides the notion of uniform convergence on B_{∞} .

Lemma 2.1. *Let K_n be a sequence of compact subsets of B_{∞} and let K be a compact subset of B_{∞} . The sequence (d_{K_n}) converges uniformly to d_K if and only if $K_n \xrightarrow{\mathcal{H}} K$.*

In particular, this lemma means that the Hausdorff distance does not see to interior of the compact set and is not suited to study properties of boundaries and that natural geometric quantities like volume and perimeter are not continuous with respect to this topology as emphasized as the example 2.6. However, the Hausdorff convergence enjoys nice properties for monotone sequences of sets ([15, Section 2.2.3.2, page 32]).

Lemma 2.2. *A decreasing sequence of nonempty compact sets Hausdorff converges to their intersection. An increasing sequence of nonempty compact sets converges to the closure of their union.*

We shall also use the following lemma.

Lemma 2.3. *Let A_1, A_2, B_1 and B_2 be compact sets such that $A_1 \subset B_1 \subset A_2$. Then it holds $d_{\mathcal{H}}(B_1, B_2) \leq d_{\mathcal{H}}(A_1, A_2)$.*

Proof of Lemma 2.3. Notice that if K_1 and K_2 are two compact sets with $K_1 \subset K_2$ then $d_{K_2} \leq d_{K_1}$. Hence, $d_{A_2} \leq d_{B_i} \leq d_{A_1}$ for $i = 1, 2$ and therefore $|d_{B_1}(x) - d_{B_2}(x)| \leq |d_{A_1}(x) - d_{A_2}(x)|$ for all x and passing to the supremum we obtain $\|d_{B_1} - d_{B_2}\|_{\infty} \leq \|d_{A_1} - d_{A_2}\|_{\infty}$. \square

The Hausdorff distance is extended to open subsets of B_{∞} by the definition:

$$d_{\mathcal{H}}(\Omega_1, \Omega_2) = d_{\mathcal{H}}(\bar{B}_{\infty} \setminus \Omega_1, \bar{B}_{\infty} \setminus \Omega_2),$$

when Ω_1, Ω_2 are open subsets of B_{∞} . It is sometimes called the Hausdorff complementary distance.

2.2 Oriented distance functions.

Let us recall the main properties of oriented distance functions and their connection with the Hausdorff distance between compact sets. We first give definition and fix notations. For any subset A of B_{∞} , the oriented distance function to A is the function b_A defined as

$$b_K(A) = d(x, A) - d(x, A^c).$$

Notice $d_A = (b_A)_+$ and $d_{A^c} = (b_A)_-$ where $(t)_+ = \max(0, t)$ is the positive part and where $(t)_- = \max(0, -t)$ is the negative part.

The oriented distance function b_K provides a nice implicit representation of open and compact sets.

Lemma 2.4 (Implicit representation of domains by oriented distance function). *Let A denote a non empty subset of \mathbb{R}^d .*

1. *Its closure is given by $\bar{A} = [b_A \leq 0]$. In particular, if $K \in \mathcal{K}$ is a compact set then*

$$K = [b_K \leq 0] \text{ and } \partial K = [b_K = 0].$$

2. *If A satisfies $\bar{\bar{A}} = \bar{A}$ then $\mathring{A} = [b_A < 0]$.*

3. *for any real λ and any set A , $\overline{[b_A < \lambda]} \subset [b_A \leq \lambda]$. If $\lambda > 0$ and K is compact, then the equality holds $\overline{[b_K < \lambda]} = [b_K \leq \lambda]$.*

Proof of Lemma 2.4.

Step 1: Proof of the first point. The inclusion $\bar{A} \subset [b_A \leq 0]$ is clear since A is contained in the closed set $[b_A \leq 0]$. Conversely, if x is a point where $b_A(x) \leq 0$, then $d_A(x) = 0$ and, by definition of the distance, there is a sequence (y_n) of points in A such that $y_n \rightarrow x$. The characterization of the boundary is stated in [12, Theorem 2-1,(iii),p 338]).

Step 2: case of open sets. We now prove that if A is open then $A = [b_A < 0]$. Indeed, on the one hand, if $b_A(x) < 0$, then $d_A(x) = 0$ and $d_{A^c}(x) > 0$ hence $x \notin A^c$ that is to say $x \in A$. On the other hand if $x \in A$ then first $d_A(x) = 0$ and second there is a open ball centered in x and contained in A so that $d_{A^c}(x) > 0$.

Step 3: Proof of the second point. The interior $\overset{\circ}{A}$ is open and $\overset{\circ}{A} = [b_{\overset{\circ}{A}} < 0]$ by the second step. To conclude, it suffices to prove that $b_{\overset{\circ}{A}} = b_A$. If $\bar{\overset{\circ}{A}} = \bar{A}$, one also has $\partial \overset{\circ}{A} = \bar{\overset{\circ}{A}} \setminus \overset{\circ}{A} = \bar{A} \setminus \overset{\circ}{A} = \partial A$. Now, we use the equivalence

$$b_{B_1} = b_{B_2} \Leftrightarrow \bar{B}_1 = \bar{B}_2 \text{ and } \partial B_1 = \partial B_2$$

proved in [12, Theorem 2-1,(ii),p 338]) to deduce that $b_{\overset{\circ}{A}} = b_A$.

Step 4: Proof of the third point. The inclusion $\overline{[b_A < \lambda]} \subset [b_A \leq \lambda]$ is clear: any point $x \in \overline{[b_A < \lambda]}$ is the limit of a sequence (x_n) such that $b_A(x_n) < \lambda$ then by continuity of b_A , $b_A(x) \leq \lambda$.

We now assume $\lambda > 0$ and K compact and prove $[b_K \leq \lambda] \subset \overline{[b_K < \lambda]}$. Set $x \in [b_K \leq \lambda]$. If $x \in K$, then $B(x, \lambda) \subset [b_K < \lambda]$ and $x \in \overline{[b_K < \lambda]}$. If $x \notin K$, let $y \in K$ such that $\|x - y\| = d_K(x) = b_K(x) \leq \lambda$. Then, for any $t \in (0, 1)$, the point $x(t) = y + t(x - y)$ of the segment $[y, x]$ satisfies $\|y - x(t)\| = t\|y - x\| < \|x - y\| \leq \lambda$ and $x(t) \in [b_K < \lambda]$. Since $x(t)$ converges to x when $t \rightarrow 1$, $x \in \overline{[b_K < \lambda]}$. \square

We state useful properties of the parametrization by oriented distance function.

Lemma 2.5. *Let K_1 and K_2 be two compact subsets of B_∞ .*

1. $b_{K_2} \leq b_{K_1}$ if and only if $K_1 \subset K_2$ and $\overline{(K_1^c)} \subset \overline{(K_2^c)}$
2. *The Hausdorff distance is dominated by the gap of oriented distance function:*

$$d_{\mathcal{H}}(K_1, K_2) = \|d_{K_1} - d_{K_2}\|_\infty \leq \|b_{K_1} - b_{K_2}\|_\infty.$$

Proof of Lemma 2.5. The first property is stated in [12, Theorem 2-1,(ii),p 338]. To prove the second one, it suffices to check that $d_K = (b_k)_+$ and to notice that the map $t \mapsto (t)_+$ is 1 Lipschitz on \mathbb{R} . Hence, for any $x \in B_\infty$, we get

$$|d_{K_1}(x) - d_{K_2}(x)| \leq |b_{K_1}(x) - b_{K_2}(x)|.$$

The conclusion follows by taking the supremum to the right hand side then to the left hand side. Notice that a converse inequality is not possible if $\overbrace{K_1 \cap K_2}^{\neq \emptyset} \neq \emptyset$ and $K_1 \neq K_2$. \square

As a consequence of the second point, the convergence of oriented distance functions to compact subsets implies the Hausdorff convergence. However, the equivalence stated in Lemma 2.1 is lost when one replaces distance functions by oriented distance function as shown by the following example on \mathbb{R} .

Example 2.6. Let (x_n) be a dense sequence in $[0, 1]$. Set $K_n = \{x_0, \dots, x_n\}$, then $K_n \xrightarrow{\mathcal{H}} [0, 1]$ while $b_{K_n} \rightarrow d_{[0,1]} \neq b_{[0,1]}$.

The reason is that oriented distance function contains also information on the interior of the set as stated in the following result.

Proposition 2.7 (Convergence of oriented distance functions to compact sets). *Let K_n be a sequence of compact subsets of B_∞ and let K be a compact subset of B_∞ . The statements*

- (i) (b_{K_n}) uniformly converges to b_K ,
- (ii) $K_n \xrightarrow{\mathcal{H}} K$ and $\overset{\circ}{K}_n \xrightarrow{\mathcal{H}} \overset{\circ}{K}$;

are equivalent.

Proof of Proposition 2.7.

Step 1: (i) \implies (ii). Since the positive $(\)_+$ part is 1 Lipschitz, the sequence (d_{K_n}) uniformly converges to d_K implying $K_n \xrightarrow{\mathcal{H}} K$. In the same manner, the negative $(\)_-$ part is 1 Lipschitz and the sequence $(d_{K_n^c})$ uniformly converges to d_{K^c} . Since $d_A = d_{\bar{A}}$ for any $A \subset B_\infty$, the sequence $(d_{\overline{(K_n^c)}})$ uniformly converges to $d_{\overline{(K^c)}}$ and the compact sets $\overline{(K_n^c)} \xrightarrow{\mathcal{H}} \overline{(K^c)}$ that is $\overset{\circ}{K}_n \xrightarrow{\mathcal{H}} \overset{\circ}{K}$ since for any $A \subset B_\infty$, $\overline{A^c} = (\overset{\circ}{A})^c$.

Step 2: (ii) \implies (i). If $K_n \xrightarrow{\mathcal{H}} K$, then (d_{K_n}) uniformly converges to d_K . If $\overset{\circ}{K}_n \xrightarrow{\mathcal{H}} \overset{\circ}{K}$ that is $\overline{(K_n^c)} \xrightarrow{\mathcal{H}} \overline{(K^c)}$, then $(d_{\overline{(K_n^c)}})$ uniformly converges to $d_{\overline{(K^c)}}$ that is $(d_{K_n^c})$ uniformly converges to d_{K^c} .

□

As a clear consequence, we get a similar statement for open sets.

Proposition 2.8. *Let (Ω_n) be a sequence of non empty open sets in B_∞ and let $\Omega \subset B_\infty$ be open. The statements*

- (i) (b_{Ω_n}) uniformly converges to b_Ω ,
- (ii) $\Omega_n \xrightarrow{\mathcal{H}} \Omega$ and $\bar{\Omega}_n \xrightarrow{\mathcal{H}} \bar{\Omega}$;

are equivalent.

Let us denote by \mathcal{D} the set of oriented distance functions to compact sets that is $f \in \mathcal{D}$ if and only if there is a compact K in B_∞ such that $f = b_K$. The main properties of functions of \mathcal{D} are:

Lemma 2.9 (Properties of \mathcal{D}). *Ones has :*

1. $\mathcal{D} \subset Lip(B_\infty, 1)$ the space of 1-Lipschitz functions on B_∞ ,
2. \mathcal{D} is not convex.

The first point is [12, Theorem 2-1,(vi),p 338]. The second point is just a calculus on \mathbb{R} : the function $(b_{\{0\}} + b_{\{1\}})/2$ is not a oriented distance function.

Remark 2.10. *It would be more natural to work in the sequel within the convex hull of \mathcal{D} than in the space $Lip(B_\infty, 1)$. However, we did not manage to characterize of this convex hull. Is it the whole $Lip(B_\infty, 1)$?*

We shall work within the space $Lip(B_\infty, 1)$. Let us recall its main properties that we shall use.

Lemma 2.11. *The space $Lip(B_\infty, 1)$ is endowed with the norm*

$$\|f\|_\infty = \sup_{x \in \bar{B}_\infty} |f(x)|.$$

It is a convex closed set. Moreover, if (f_n) is a pointwise convergent sequence of functions in $Lip(B_\infty, 1)$, then the convergence is uniform.

3 Implicit representation of sequence of sets by continuous functions.

We say that a continuous function f is a parametrization of a compact K if K is the 0 sublevel set of $\{x \mid f(x) \leq 0\}$ that we will denote by $[f \leq 0]$. Of course, such a parametrization is not unique since for any non decreasing function ϕ with $\phi(0) = 0$ one has $[f \leq 0] = [\phi \circ f \leq 0]$.

3.1 Convergence of approximated level set.

We first discuss the following general question posed here in a rough way: *if the sequence of functions (f_n) converges to some function f , does the sequence of level-sets $[f_n(x) \leq 0]$ converges in the Hausdorff sense to the level-set $[f(x) \leq 0]$?* In general, the answer is negative as shown by the following examples in dimension one.

Example 3.1. *The sequence of functions $f_n(x) = \inf(b_{[0,2]}, d_{\{3\}} + 1/n)$ converges to $f(x) = \inf(b_{[0,2]}, d_{\{3\}})$. One has $\|f_n - f\|_\infty = 1/n \rightarrow 0$ while*

$$K_n = [0, 2] \xrightarrow{\mathcal{H}} [0, 2] \neq K = [0, 2] \cup \{3\}.$$

Example 3.2. *Set f be the piecewise linear function such that $f(-1) = 1 = f(3)$, $f(0) = f(1) = f(2) = 0$ and $f(1.5) = -0.5$. Set $f_n = f + \phi/n$ where ϕ is a continuous function supported in $(0, 1)$ with $\phi(x) > 0$ on $(0, 1)$. Then $\|f_n - f\|_\infty = \|\phi\|_\infty/n \rightarrow 0$ while*

$$K_n = \{0\} \cup [1, 2] \xrightarrow{\mathcal{H}} \{0\} \cup [1, 2] \neq K = [0, 2].$$

We nevertheless obtain such a result under an additional assumption of topological nature of the limit set. Notice that we then obtain also convergence of the 0-level-set.

Theorem 3.3. *If the sequence of continuous functions (f_n) uniformly converges to f in B_∞ such that*

$$[f \leq 0] \neq \emptyset.$$

If the following regularity from the inside condition

$$(\mathbf{A}_<) \quad \overline{[f < 0]} = [f \leq 0]$$

holds then the compact level sets converge

$$[f_n \leq 0] \xrightarrow{\mathcal{H}} [f \leq 0], \tag{1}$$

If the following regularity from the outside condition

$$(\mathbf{A}_{>}) \overbrace{[f \leq 0]}^{\circ} = [f < 0]$$

holds, then the open level sets converge

$$[f_n < 0] \xrightarrow{\mathcal{H}} [f(x) < 0], \quad (2)$$

If both conditions $(\mathbf{A}_{<})$ and $(\mathbf{A}_{>})$ hold, then the boundaries converge

$$[f_n = 0] \xrightarrow{\mathcal{H}} [f = 0]. \quad (3)$$

We propose an elementary proof of this theorem. It relies on a pinching lemma for the Hausdorff convergence.

Lemma 3.4. *Let (A_n) , (B_n) and (C_n) be three sequences of compact sets (resp. open sets) such that for all n , $A_n \subset B_n \subset C_n$. Let K be a compact set (resp. open set). If (A_n) and (C_n) Hausdorff converges to K then (B_n) Hausdorff converges to K .*

Proof of Lemma 3.4. It suffices to prove the lemma for compact sets. Indeed the case of open sets is obtained by passing to the complement. One checks that

$$\rho(B_n, K) = \sup_{x \in B_n} d(x, K) \leq \sup_{x \in C_n} d(x, K) = \rho(C_n, K) \leq d_{\mathcal{H}}(C_n, K),$$

and

$$\rho(K, B_n) = \sup_{x \in K} d(x, B_n) \leq \sup_{x \in K} d(x, A_n) = \rho(K, A_n) \leq d_{\mathcal{H}}(A_n, K).$$

Then, $0 \leq d_{\mathcal{H}}(B_n, K) \leq \sup(d_{\mathcal{H}}(A_n, K), d_{\mathcal{H}}(C_n, K)) \rightarrow 0$.

□

Proof of Theorem 3.3.

Step 1: construction of barrier domains. Set $\epsilon_n = \sup_{k \geq n} \|f - f_k\|_{\infty}$. Since the sequence (f_n) uniformly converges to f , the sequence ϵ_n is decreasing and converges to 0. For each integer n , we define compact sets by

$$A_n = [f \leq -\epsilon_n], B_n = [f_n(x) \leq 0] \text{ and } C_n = [f \leq \epsilon_n].$$

The triangle inequality

$$f(x) - \epsilon_n \leq f(x) - \|f_n - f\|_{\infty} \leq f_n(x) \leq f(x) + \|f_n - f\|_{\infty} \leq f(x) + \epsilon_n,$$

valid for each $x \in B_{\infty}$ translates to inclusions for level sets

$$\forall n, \quad A_n \subset B_n \subset C_n.$$

Notice that the sets A_n is not empty for n large enough since $[f < 0] \neq \emptyset$. In a similar way, set

$$\mathcal{A}_n = [f < -\epsilon_n], \mathcal{B}_n = [f_n < 0] \text{ and } \mathcal{C}_n = [f < \epsilon_n]$$

so that $\mathcal{A}_n \subset \mathcal{B}_n \subset \mathcal{C}_n$. Notice that the monotony of the sequence (ϵ_n) implies the monotony of the sequences (A_n) , (C_n) , (\mathcal{A}_n) and (\mathcal{C}_n) .

Step 2: proof of (1). Now, we study the limits of (A_n) and (C_n) . The sequence (C_n) is a decreasing sequence of non empty compact sets, hence it converges in the sense of Hausdorff to its intersection by Lemma 2.2

$$C_n \xrightarrow{\mathcal{H}} \bigcap_{n \geq 0} C_n = \bigcap_{n \geq 0} [f \leq \epsilon_n] = [f \leq 0].$$

The sequence (A_n) is an increasing sequence of compact sets, hence it converges in the sense of Hausdorff to the closure of its union by Lemma 2.2:

$$A_n \xrightarrow{\mathcal{H}} \overline{\bigcup_{n \geq 0} A_n} = \overline{\bigcup_{n \geq 0} [f \leq -\epsilon_n]} = \overline{[f < 0]}.$$

We conclude thanks to the comparison principle (Lemma 3.4).

Step 3: proof of (2). It is a straightforward adaptation of the previous step: (\mathcal{A}_n) is an increasing sequence of open sets, then it converges in the sense of Hausdorff to its union that is $[f < 0]$. Then, (\mathcal{C}_n) is a decreasing sequence of open sets. Therefore it converges to the interior of the intersection of all the open sets: namely the interior of $[f \leq 0]$ which is exactly $[f < 0]$ since (\mathbf{A}'') holds.

Step 4: proof of (3). Set $\Gamma = [f = 0]$ and $\Gamma_n = [f_n(x) = 0]$. These are compact sets.

We first prove that $\rho(\Gamma_n, \Gamma)$ the excess from Γ_n to Γ tends to 0. For all n , there exists $x_n \in \Gamma_n$ such that $d(x_n, \Gamma) = \rho(\Gamma_n, \Gamma)$. Since the sequence (x_n) stays in the compact $\overline{B_\infty}$, there are accumulation points. Let $\bar{x} = \lim x_{n_k}$ be such an accumulation point, then $f(\bar{x}) = \lim f_{n_k}(x_{n_k}) = 0$ since (f_n) converges uniformly to f and $\bar{x} \in \Gamma$. Then $d(x_{n_k}, \Gamma) \leq \|x_{n_k} - \bar{x}\| \rightarrow 0$.

We then prove that $\rho(\Gamma, \Gamma_n)$ the excess from Γ to Γ_n tends to 0. For all n , there exists $x_n \in \Gamma$ such that $d(x_n, \Gamma_n) = \rho(\Gamma, \Gamma_n)$. By *reductio ad absurdum*, assume that there exists $\eta > 0$ such that $d(x_n, \Gamma_n) \geq \eta$.

Since the sequence (x_n) stays in the compact Γ , there are accumulation points. Let $\bar{x} = \lim x_{n_k}$ be such an accumulation point in Γ . Hence there is a rank k_0 such that for $k \geq k_0$, $\|\bar{x} - x_{n_k}\| \leq \eta/2$ then $d(\bar{x}, \Gamma_{n_k}) \geq \eta/2$.

Now, we use $(\mathbf{A}_<)$ since $\bar{x} \in \partial[f < 0]$, there exists a point y with $\|y - \bar{x}\| \leq \eta/4$ such that $f(y) < 0$. Since $f_n(y) \rightarrow f(y)$, there exists a rank $k_1 > k_0$ such that for $k \geq k_1$, $f_{n_k}(y) < 0$. Using $(\mathbf{A}_>)$, $\bar{x} \in \partial[f > 0]$, there exists a point z with $\|z - \bar{x}\| \leq \eta/4$ such that $f(z) > 0$. Since $f_n(z) \rightarrow f(z)$, there exists a rank $k_2 > k_1$ such that for $k \geq k_2$, $f_{n_k}(z) > 0$.

Since the ball $B(\bar{x}, \eta/4)$ is convex, it contains the whole segment $[y, z]$. For $k \geq k_2$, f_{n_k} is continuous on $[y, z]$ with $f_{n_k}(y)f_{n_k}(z) < 0$ then by the intermediate value theorem, there is a point $t \in [y, z] \subset B(\bar{x}, \eta/4)$ with $f_{n_k}(t) = 0$. Then, $d(\bar{x}, \Gamma_{n_k}) \leq \|\bar{x} - t\| \leq \eta/4$ that negates $d(\bar{x}, \Gamma_{n_k}) \geq \eta/2$

□

Remark 3.5. Notice that the implication,

$$\text{If } \sup_{B_\infty} |f_n(x) - f(x)| \rightarrow 0 \text{ and } \overline{[f < 0]} = [f \leq 0], \text{ then } [f_n \leq 0] \xrightarrow{\mathcal{H}} [f \leq 0]$$

remains true under the weaker assumption that the functions f and f_n are only lower semi-continuous. In the same spirit,

$$\text{If } \sup_{B_\infty} |f_n(x) - f(x)| \rightarrow 0 \text{ and } \overbrace{[f \leq 0]}^\circ = [f < 0], \text{ then } [f_n < 0] \xrightarrow{\mathcal{H}} [f < 0],$$

when the functions are upper semi-continuous.

Let us comment the meaning of the topological assumption $(\mathbf{A}) = (\mathbf{A}_<)$ and $(\mathbf{A}_>)$. On the one hand, it prohibits the 0 level-set $[f = 0]$ to be fat. In general, the two boundaries $\partial[f < 0]$ and $\partial[f > 0]$ do not coincide to the 0 level-set that is the space delimited by these boundaries. But it also prohibits isolated points and more generally parts of the boundary of codimension bigger than two.

The topological assumption (\mathbf{A}) is sharp as stated by the following proposition.

Proposition 3.6. *Let f be a continuous function such that*

$$[f = 0] \not\subset \overline{[f < 0]}.$$

There exists a sequence of continuous functions (f_n) that converges uniformly to f and such that $[f_n \leq 0]$ Hausdorff converges to a compact set that is not $[f \leq 0]$.

Proof of Proposition 3.6. Since

$$[f = 0] \not\subset \overline{[f < 0]},$$

there exists a with $f(a) = 0$ and a real $\delta > 0$ such that $d(x, \overline{[f < 0]}) \geq \eta$ that is to say $f(x) \geq 0$ on $B(a, \delta)$. Let χ be a continuous function on $[0, +\infty)$ with $\chi(t) > 0$ on $[0, 1)$ and $\chi(x) = 0$ for $t > 1$. Set

$$f_n(x) = f(x) + \frac{1}{n} \chi\left(\frac{n\|x - a\|}{\delta}\right).$$

By construction, $[f_n \leq 0] = [f \leq 0] \setminus B(a, \delta/n)$ is an increasing sequence of compact sets then it converges to its union that is

$$[f_n \leq 0] \xrightarrow{\mathcal{H}} [f \leq 0] \setminus \{a\},$$

while $\|f_n - f\|_\infty \leq \|\chi\|_\infty/n \rightarrow 0$.

□

3.2 Convergence rate.

A natural question is to evaluate the rate of convergence of $[f_n \leq 0]$ to $[f \leq 0]$ stated in Theorem 3.3. A first elementary remark is that this problem reduces to the convergence rate of the sublevel sets of f reducing the problem to the study of the mapping:

$$\Psi : \lambda \mapsto [f \leq \lambda].$$

Notice f is continuous from \mathbb{R}^d with values in a compact set and hence Ψ maps a real number to a compact set of \mathbb{R}^d .

Proposition 3.7. *One has :*

$$d_{\mathcal{H}}([f \leq 0], [f_n \leq 0]) \leq d_{\mathcal{H}}([f < -\|f - f_n\|_{\infty}], [f \leq 0]) + d_{\mathcal{H}}([f \leq 0], [f \leq \|f - f_n\|_{\infty}])$$

Proof of Proposition 3.7. The basic estimation

$$f(x) - \|f - f_n\|_{\infty} \leq f_n(x) \leq f(x) + \|f - f_n\|_{\infty},$$

translates into the pinching of $[f \leq 0]$:

$$[f \leq -\|f - f_n\|_{\infty}] \subset [f_n \leq 0] \subset [f \leq \|f - f_n\|_{\infty}].$$

Since one of course also has

$$[f \leq -\|f - f_n\|_{\infty}] \subset [f \leq 0] \subset [f \leq \|f - f_n\|_{\infty}],$$

then by Lemma 2.3:

$$d_{\mathcal{H}}([f \leq 0], [f_n \leq 0]) \leq d_{\mathcal{H}}([f \leq -\|f - f_n\|_{\infty}], [f \leq \|f - f_n\|_{\infty}]).$$

One concludes by the triangle inequality and by Proposition 3.8. \square

We now prove that Ψ is a Lipschitz map. To that end, we require stronger regularity. A sufficient condition for $(\mathbf{A}_{>})$ and $(\mathbf{A}_{>})$ is that the 0 level set is a (at least Lipschitz) manifold of codimension 1. By the implicit function theorem, this is true if the strong regularity assumption S is made

(S) there are constants η and c that f is continuously differentiable on Γ_{η} and

$$\|\nabla f(x)\| \geq c \text{ for } x \in \Gamma_{\eta}.$$

where $\Gamma_{\eta} = [d_{\Gamma} \leq \eta]$ is the η tubular neighborhood of Γ .

Proposition 3.8. *Assume that (S) holds, let (ϵ_n) be a sequence of non negative real converging to 0 then there exist a non negative real c depending only of f and a rank n_0 depending on (f_n) such that for $n \geq n_0$*

$$d_{\mathcal{H}}([f \leq 0], [f \leq \pm\epsilon_n]) \leq c\epsilon_n.$$

Proof of Proposition 3.8.

Step 1: preliminary remarks. Under (S), using the continuity of the gradient ∇f on the compact set Γ_{η} , there exists δ such that

$$\|x - y\| \leq \delta \implies \nabla f(x) \cdot \nabla f(y) \geq \frac{1}{2}|\nabla f(y)|^2.$$

Set $\bar{\epsilon} = \inf(\eta, \delta)$.

Since the sequence of functions $g_n^{\pm} = f \pm \epsilon_n$ uniformly converges to f , Theorem 3.3 insures that the level sets $[f \leq \pm\epsilon_n] = [g_n^{\pm} \leq 0]$ converges in the Hausdorff sense to $[f \leq 0]$ and then there is a rank n_0 such that $[f \leq \pm\epsilon_n] \subset \Gamma_{\bar{\epsilon}}$ for all $n \geq n_0$. In particular, by the local inverse theorem, the level sets $[f = 0] = \partial[f \leq 0]$ and $[f = \epsilon_n] = \partial[f_n \leq \epsilon_n]$ are smooth for $n \geq n_0$. In the sequel, we assume $n \geq n_0$

Step 2: an upper bound to $d_{\mathcal{H}}([f \leq 0]), [f \leq \epsilon_n]$. Since the level sets of f are nested: $[f \leq 0] \subset [f \leq \epsilon_n]$, it suffices to dominate the excess of $[f \leq \epsilon_n]$ to $[f \leq 0]$:

$$d_{\mathcal{H}}([f \leq \epsilon_n], [f \leq 0]) = \rho([f \leq \epsilon_n], [f \leq 0]).$$

Let $y \in [f \leq \epsilon_n]$ such that $d_{[f \leq 0]}(y) = \rho([f \leq \epsilon_n], [f \leq 0]) > 0$. In particular, $y \notin [f \leq 0]$ and $d_{[f \leq 0]}(y) = d_{\Gamma}(y)$, we therefore get

$$d_{\mathcal{H}}([f \leq \epsilon_n], [f \leq 0]) = d_{\Gamma}(y).$$

Consider the function

$$\varphi(x) = \left| \frac{f(x)}{b_{\Gamma}(x)} \right| = \frac{|f(x)|}{d_{\Gamma}(x)}$$

is continuous on $\bar{B}_{\infty} \setminus \Gamma$ and can be extended by continuity on the whole \bar{B}_{∞} with $\varphi(x) = \partial_n f(x)$ for any $x \in \Gamma$. By construction, this function takes positive values. In particular, by compactness of \bar{B}_{∞} there are a constant $c_2 > 0$ such that for all $x \in B_{\infty}$

$$c_2 d_{\Gamma}(x) \leq |f(x)|.$$

We conclude

$$d_{\Gamma}(y) \leq \frac{|f(y)|}{c} \leq \frac{\epsilon_n}{c}.$$

Step 3: an upper bound to $d_{\mathcal{H}}([f \leq 0], [f \leq -\epsilon_n])$. We deduce from the inclusion $[f \leq -\epsilon_n] \subset [f \leq 0]$ that it holds:

$$d_{\mathcal{H}}([f \leq -\epsilon_n], [f \leq 0]) = \rho([f \leq 0], [f \leq -\epsilon_n]).$$

Let $x \in [f \leq 0]$ and $y \in [f \leq -\epsilon_n]$ such that

$$\|x - y\| = d(x, [f \leq -\epsilon_n]) = \rho([f \leq 0], [f \leq -\epsilon_n]).$$

By construction, $x \in [f = 0]$ and $y \in [f = -\epsilon_n]$ and these level sets are smooth. As y minimizes the distance to x over $[f \leq -\epsilon_n]$, the Euler-Lagrange equation implies that

$$x - y = \frac{\|x - y\|}{\|\nabla f(y)\|} \nabla f(y),$$

and we get

$$\begin{aligned} \epsilon_n &= f(x) - f(y) = \int_0^1 \nabla f(x + t(y - x)) \cdot (y - x) dt \\ &= \int_0^1 \frac{\|x - y\|}{\|\nabla f(y)\|} \nabla f(x + t(y - x)) \cdot \nabla f(y) dt \\ &\geq \frac{1}{2} \int_0^1 \|\nabla f(y)\| \|x - y\| dt \\ &\geq \frac{c_2}{2} \|x - y\| = \frac{c_2}{2} d_{\mathcal{H}}([f \leq -\epsilon_n], [f \leq 0]). \end{aligned}$$

□

Let us notice that **(S)** is not mandatory, one can get convergence when it is not satisfied. The idea is to straighten the surface $y = f(x)$ in order to reach a unit slope. This corresponds to the redistancing procedure that appears in the numerical implementation of the level set method.

For a given continuous function φ on \mathbb{R} , \mathcal{C}^1 on \mathbb{R}^* with $\varphi'(t) > 0$ for $t \neq 0$. We set the assumption

(S $_{\varphi}$) there are constants η and c that $\varphi \circ f$ is continuously differentiable on Γ_{η} and

$$\|\nabla(\varphi \circ f)(x)\| \geq c \text{ for } x \in \Gamma_{\eta}.$$

It is a generalization of **(S)** since the first condition corresponds to the case $\varphi_0(t) = t$ i.e. **(S)** = **(S $_{\varphi_0}$)**. Notice that φ is strictly increasing and $[f \leq \pm\epsilon_n] = [\varphi \circ f \leq \pm\varphi(\epsilon_n)]$, then one deduces from Proposition 3.8 that

Theorem 3.9. *Assume that **(S $_{\varphi}$)** holds, let (ϵ_n) be a sequence of non negative real converging to 0 then there exist a non negative real c depending only of f and a rank n_0 depending on (f_n) such that for $n \geq n_0$*

$$d_{\mathcal{H}}([f \leq 0], [f \leq \pm\epsilon_n]) \leq c\varphi(\epsilon_n).$$

Remark 3.10. *Let us mention that condition **(S $_{\varphi}$)** is also known as the Kurdyka-Lojasiewicz inequality. A nice reference on the topic is [7] see in particular Theorem 2 and its corollaries for statements similar to Proposition 3.9 where it is shown that the Lipschitz property of Ψ from \mathbb{R} endowed with the metric $d(s, t) = |\phi(t) - \phi(s)|$ is equivalent to **(S $_{\varphi}$)**.*

4 Oriented distance Expectation of random sets

4.1 Definitions

We follow the usual strategy of a parametrization based expectation as done in [17, 18].

Definition 4.1 (Oriented distance based expectation of a random compact set).

Let \mathbf{K} be a random compact set. Let $b_{\mathbf{K}}$ be the process of its oriented distance function. If this process has an expectation $\mathbb{E}[b_{\mathbf{K}}]$, then the oriented distance expectation of \mathbf{K} is defined as the compact set $\mathbb{E}[\mathbf{K}] = [\mathbb{E}[b_{\mathbf{K}}] \leq 0]$. The oriented distance expectation of the boundary of \mathbf{K} is defined as the compact set $\mathbb{E}[\partial\mathbf{K}] = [\mathbb{E}[b_{\mathbf{K}}] = 0]$.

In order to validate this definition, one has to observe that each realization of $b_{\mathbf{K}}$ is a 1-Lipschitz function. As the set $Lip(B_{\infty}, 1)$ of 1-Lipschitz functions is a closed convex set, its expectation $\mathbb{E}[b_{\mathbf{K}}]$ is also a 1-Lipschitz function. Of course, in order to obtain a computable notion we need to use samples of the random compact set and define an oriented distance empirical mean as follow.

Definition 4.2 (Oriented distance empirical mean of a random compact set).

If $\mathbf{K}_1, \dots, \mathbf{K}_n$ are n random compact sets independent identically distributed. Then the empirical mean of the oriented distance mean is defined as

$$\hat{b}_n = \frac{1}{n} \sum_{i=1}^n b_{\mathbf{K}_i}.$$

The oriented distance empirical mean $\hat{\mathbf{K}}_n$ is defined as the compact set $[\hat{b}_n \leq 0]$. The oriented distance empirical mean of the boundary $\partial\hat{\mathbf{K}}_n$ is defined as the compact set $[\hat{b}_n = 0]$.

This definition corresponds to a plug-in estimator of the random sets based on the oriented distance parametrization.

In the very same way, let \mathbf{O} be a random open set and $b_{\mathbf{O}}$ be the process of its oriented distance function. If this process has an expectation $\mathbb{E}[b_{\mathbf{O}}]$, we define the oriented distance expectation of \mathbf{O} as $\mathbb{E}[\mathbf{O}] = [\mathbb{E}[b_{\mathbf{O}}] < 0]$ and the oriented distance expectation of the boundary as $\mathbb{E}[\partial\mathbf{O}] = [\mathbb{E}[b_{\mathbf{O}}] = 0]$. We also define the oriented distance empirical means as $\hat{\mathbf{O}}_n = [\hat{b}_n < 0]$ and $\partial\hat{\mathbf{O}}_n = [\hat{b}_n = 0]$ where \hat{b}_n is the empirical mean of the process $b_{\mathbf{O}}$.

4.2 Limits of the oriented distance empirical mean and application for the random compact sets

Large Number law. We first establish the consistency of the oriented distance empirical mean estimators. This result is crucial as it provides a way to numerically access to the object we just defined.

Proposition 4.3 (Consistency of the empirical estimators). *Let \mathbf{K} be a random compact set. It holds :*

$$\lim_{n \rightarrow +\infty} \|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty} = 0 \text{ almost surely.}$$

Then, if moreover the function $\mathbb{E}[b_{\mathbf{K}}]$ has the property (A),

$$\hat{\mathbf{K}}_n \xrightarrow{\mathcal{H}} \mathbb{E}[\mathbf{K}] \text{ and } \partial\hat{\mathbf{K}}_n \xrightarrow{\mathcal{H}} \mathbb{E}[\partial\mathbf{K}] \text{ almost surely.}$$

The same properties hold also for random open sets.

Proof of Theorem 4.3. The usual law of large numbers implies pointwise convergence. As all the functions \hat{b}_n and $\mathbb{E}[b_{\mathbf{K}}]$ are 1-Lipschitz functions, we also obtain uniform convergence over compact sets. Finally, the Hausdorff convergences are direct applications of Theorem 3.3. \square

Central limit theorem. We now quote the central limit theorem obtained by Jankowski and Stanberry in [17, Theorem 2-6 and Proposition 2-7].

Proposition 4.4 (Central limit theorem for the ODF). *There is a centered Gaussian random field Z with covariance*

$$\text{cov}[Z(x), Z(y)] = \mathbb{E}[b_{\mathbf{K}}(x)b_{\mathbf{K}}(y)] - \mathbb{E}[b_{\mathbf{K}}](x) \mathbb{E}[b_{\mathbf{K}}](y),$$

for any $x, y \in B_{\infty}$ such that

$$Z_n = \sqrt{n} \left(\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}] \right) \Rightarrow Z,$$

in the space $\mathcal{C}(B_{\infty}, \mathbb{R})$ of continuous functions on \bar{B}_{∞} endowed with the uniform convergence topology. Moreover,

$$\text{var}[Z(x) - Z(y)] \leq |x - y|^2,$$

and the sample paths of the process Z are α -Hölder for any $\alpha \in (0, 1)$.

It is a direct application of the following general central limit theorem of uniformly Lipschitz processes that relies on two classical arguments: first, an estimate of the moment based on a combinatorial argument and second the Garsia, Rademich and Rumsay inequality [10, Theorem B.1.5] applied on B_∞ . We check that B_∞ is regular in the sense that there is a sequence of open set O_n in B_∞ and a nonnegative real $\kappa > 0$ such that

$$\cup_{x \in O_n} B(x, 1/n) \subset B_\infty \text{ and } \mathcal{L}_d(O_n \cap B(x, 1/n)) \geq \frac{\kappa}{n^d}, \forall x \in O_n,$$

where \mathcal{L}_d stands for the usual Lebesgue measure on \mathbb{R}^d .

Theorem 4.5. *Let L and M be two non negative reals and d an integer. Let K be a regular compact set $K \subset \mathbb{R}^d$. Let f be a random process defined on K such that for all $x, y \in K$*

$$|f(x) - f(y)| \leq L|x - y| \text{ a.s. and } |f(x)| \leq M \text{ a.s..}$$

And let f_1, \dots, f_n be n independent and identically distributed random processes defined on K with the same distribution as f .

Set $\hat{f}_n = (f_1 + \dots + f_n)/n$ the empirical mean. Then, there exists a centered Gaussian random field Z with covariance

$$\text{cov}[Z(x), Z(y)] = \mathbb{E}[f(x)f(y)] - \mathbb{E}[f](x) \mathbb{E}[f](y),$$

for any $x, y \in K$ such that

$$Z_n = \sqrt{n} \left(\hat{f}_n - \mathbb{E}[f] \right) \xrightarrow{\mathcal{D}} Z,$$

in the space $\mathcal{C}(K, \mathbb{R})$ of continuous functions on K endowed with the uniform convergence topology.

Proof of Theorem 4.5. We first prove a moment estimate, then derive tightness and finally prove the convergence of the finite dimensional laws. Let us introduce some notations: we define independent centered processes on K^2 by

$$g_i(x, y) = (f_i(x) - \mathbb{E}[f](x)) - (f_i(y) - \mathbb{E}[f](y)).$$

One checks that $|g_i(x, y)| \leq 2L|x - y|$ a.s. and therefore for any k

$$\mathbb{E}[g_i^k](x, y) \leq (2L)^k |x - y|^k.$$

Let $N \geq 2$ be an integer and \mathfrak{C}_N be the set

$$\mathfrak{C}_N = \{\mathbf{c} \in (\mathbb{N} \setminus \{1\})^N \mid \mathbf{c}_1 \geq \mathbf{c}_2 \geq \dots \geq \mathbf{c}_N \text{ and } \mathbf{c}_1 + \mathbf{c}_2 + \dots + \mathbf{c}_N = N\},$$

that is the set of the ways to write N as a sum of integers distinct from 1. For $\mathbf{c} \in \mathfrak{C}$, let $l(\mathbf{c}) = \max\{k \mid \mathbf{c}_k \neq 0\}$ be the number of non-zero terms of the sum. Since $\mathbf{c}_i \neq 0$ implies $\mathbf{c}_i \geq 2$, it holds $l(\mathbf{c}) \leq N/2$.

Step 1: estimating the moments. We compute

$$\begin{aligned} \mathbb{E}[|Z_n(x) - Z_n(y)|^N] &= \frac{1}{n^{N/2}} \sum_{i_1, \dots, i_k=1}^n \mathbb{E}[g_{i_1} \dots g_{i_k}](x, y), \\ &= \frac{1}{n^{N/2}} \sum_{\mathbf{c} \in \mathfrak{C}_N} \sum_{\substack{i_1, \dots, i_{l(\mathbf{c})}=1 \\ i_j \neq i_k \text{ if } j \neq k}} \prod_{j=1}^{l(\mathbf{c})} \mathbb{E}[g_{i_j}^{\mathbf{c}_j}], \end{aligned}$$

since the processes g_i are independent and centered. We check

$$\alpha_{\mathbf{c},n} = n^{-N/2} \sum_{\substack{i_1, \dots, i_{l_{\mathbf{c}}}=1 \\ i_j \neq i_k \text{ if } j \neq k}}^n 1 = n^{-N/2} \prod_{i=0}^{l(\mathbf{c})-1} (n-i) \underset{n \rightarrow +\infty}{\sim} n^{l(\mathbf{c})-N/2},$$

by a basic combinatorial argument. Since for any $\mathbf{c} \in \mathfrak{C}$ one has

$$\prod_{j=1}^{l_{\mathbf{c}}} \mathbb{E}[g_{i_j}^{\xi_j}] \leq (2L)^N |x - y|^N,$$

we deduce

$$\mathbb{E}[|Z_n(x) - Z_n(y)|^N] \leq \sum_{\mathbf{c} \in \mathfrak{C}_N} \alpha_{\mathbf{c},n} (2L)^N |x - y|^N.$$

Since $l(\mathbf{c}) - N/2 \leq 0$, all the sequences $\alpha_{\mathbf{c},n}$ are bounded and there exists a constant (depending on N) such that :

$$\mathbb{E}[|Z_n(x) - Z_n(y)|^N] \leq C(N) |x - y|^N.$$

Step 2: obtaining tightness. In order to derive the tightness, we precise the properties of the map $\omega \mapsto C(\omega)$. To that end, we use Garsia, Rademich and Rumsay inequality. Since Z_n is continuous, there exists a universal constant $c > 0$ such that

$$|Z_n(\omega, x) - Z_n(\omega, y)| \leq c |x - y|^{(\beta-2d)/\delta} \left(\int_{B_{\infty}^2} \frac{|Z_n(\omega, \eta) - Z_n(\omega, \xi)|^{\delta}}{|\eta - \xi|^{\beta}} \right)^{1/\delta}.$$

We set

$$C(\omega) = c^{\delta} \int_{B_{\infty}^2} \frac{|Z_n(\eta) - Z_n(\xi)|^{\delta}}{|\eta - \xi|^{\beta}}.$$

Since by Fubini's theorem for positive functions,

$$\mathbb{E}[C] = \int_{B_{\infty}^2} \frac{\mathbb{E}[|Z_n(\eta) - Z_n(\xi)|^{\delta}]}{|\eta - \xi|^{\beta}} \leq C(\delta) \int_{B_{\infty}^2} |\eta - \xi|^{\delta-\beta},$$

for any integer $\delta > d$. Therefore, now if $\delta > \beta$ then the right hand side is finite and therefore the random variable $C(\omega)$ has a finite expectation $\mathbb{E}[C] < +\infty$. We fix $\varepsilon > 0$. For $\delta > \beta > 2d$, Markov inequality implies that

$$\mathbb{P} \left[C > \frac{\mathbb{E}[C]}{\varepsilon} \right] \leq \varepsilon.$$

Now we consider the set K_{ε} of continuous functions φ such that

$$|\varphi(x) - \varphi(y)| \leq \left(\frac{\mathbb{E}[C]}{\varepsilon} \right)^{1/\delta} |x - y|^{(\beta-2d)/\delta} \text{ and } |\varphi(x)| \leq M.$$

Since $(\beta-2d)/\delta > 0$, the set K_{ε} is compact in $\mathcal{C}(K, \mathbb{R})$ by Ascoli's theorem. Since $Z_n(\omega, \cdot) \in K_{\varepsilon}$ if $C(\omega) \leq \mathbb{E}[C]/\varepsilon$ and owing to the Markov inequality, we have proven that the sequence (Z_n) is tight in $\mathcal{C}(K, \mathbb{R})$.

Step 3: Conclusion. The convergence of the finite-dimensional distributions is guaranteed by the classical central limit theorem. As (Z_n) is tight in $\mathcal{C}(K, \mathbb{R})$, we deduce that Z_n converges in distribution to Z in the space $\mathcal{C}(K, \mathbb{R})$ (see for instance [6]).

□

4.3 On confidence neighborhood

In practical computations, the limit sets $\mathbb{E}[\mathbf{K}]$ and $\mathbb{E}[\partial\mathbf{K}]$ are approximated through a numerical realization of a term of the sequences $\hat{\mathbf{K}}_n$ and $\partial\hat{\mathbf{K}}_n$ obtained by a sampling method like Monte Carlo. Two distinct sources of errors then appear: the first one is a deterministic one connected to the computation of each of the realizations of the random sets \mathbf{K}_i , the second one is a stochastic one due to the Monte-Carlo method. Since there are usually no convergence estimates for shape optimization problem, we shall only consider the second source of error and assume for a while that the sets are exactly known. Given a threshold $p_{min} \in (0, 1)$, we aim at compute sets V such that $\mathbb{P}[\mathbb{E}[\mathbf{K}] \subset V] \geq p_{min}$.

A basic error estimate is given by the following remark: the triangle inequality

$$\hat{b}_n(x) - \|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty} \leq \mathbb{E}[b_{\mathbf{K}}](x) \leq \hat{b}_n + \|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty},$$

valid for each $x \in B_{\infty}$ translates to inclusions for level sets

$$\forall n, \quad [\hat{b}_n(x) \leq -\|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty}] \subset \mathbb{E}[\mathbf{K}] \subset [\hat{b}_n(x) \leq \|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty}].$$

which pinch the target set $\mathbb{E}[\mathbf{K}]$ between two sublevel sets $[\hat{b}_n \pm \|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty}]$ where the function \hat{b}_n is computable. Assuming that $E_{\mathbf{K}}$ satisfies (\mathbf{S}_{φ}) holds for some function φ , the inclusions

$$\begin{aligned} [\mathbb{E}[b_{\mathbf{K}}](x) \leq -2\|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty}] &\subset [\hat{b}_n(x) \leq -\|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty}] \\ [\hat{b}_n(x) \leq \|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty}] &\subset [\mathbb{E}[b_{\mathbf{K}}](x) \leq 2\|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty}] \end{aligned}$$

provide, by the pinching Lemma 2.3 and Theorem 3.9, the estimate

$$d_{\mathcal{H}}(\hat{\mathbf{K}}_n, \mathbb{E}[\mathbf{K}]) \leq C \varphi(\|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty}).$$

This allows to translates an information of the residual $\|\hat{b}_n - \mathbb{E}[b_{\mathbf{K}}]\|_{\infty}$ into an information of the set. Of course, we have no direct access to this quantity but the central limit theorem provides the asymptotic approximation Z/\sqrt{n} where Z is a zero mean Gaussian field of variance $var(b_{\mathbf{K}})$. In particular, Z is a continuous process by Proposition 4.4, hence the random variable $SZ = \sup_{x \in B_{\infty}} |Z(x)|$ is well defined.

Proposition 4.6 (Convergence rate for the plug-in estimator). *If $E_{\mathbf{K}}$ satisfies (\mathbf{S}_{φ}) holds for some function φ and $\mathbb{P}(SZ \leq M) \geq \alpha$ then*

$$P \left(d_{\mathcal{H}}(\hat{\mathbf{K}}_n, \mathbb{E}[\mathbf{K}]) \leq C \varphi\left(\frac{M}{\sqrt{n}}\right) \right) \geq \alpha.$$

In [17], the authors define a confidence neighborhood based on the law of sup. However, it is very rare to be able to analytically determine this law and extremely difficulty to numerically determine its quantiles.

5 Application to the Bernoulli free boundary problem.

As a first application of the tools build in the previous section, we consider the exterior Bernoulli free boundary problem.

Presentation of the Bernoulli free boundary problem. Let K be a compact regular domain in \mathbb{R}^N and let λ a positive real number. The problem of finding a domain D and a function u such that the capacitary potential of K in D defined ad the solution of

$$\begin{cases} -\Delta u = 0 & \text{in } D \setminus K \\ u = 1 & \text{on } K \\ u = 0 & \text{on } \partial D \end{cases} \quad (4)$$

satisfies the overdetermination condition

$$|\nabla u| = \lambda \text{ on } \partial D, \quad (5)$$

is called the exterior Bernoulli free boundary problem. Notice that this overdetermination condition means that for any $x \in \partial D$

$$\lim_{y \in \Omega, y \rightarrow x} |\nabla u(y)| = \lambda.$$

Existence of a solution has been established by Beurling [5] more than fifty years ago. Contemporary existence proofs are based on variational methods in the sense of shape optimization, the regularity of the free boundary has been studied by Alt and Caffarelli in [3]. If the inner boundary is convex, uniqueness of the solution has been shown by Tepper [23]. For general compact sets K , there may exist more than one solution to the free boundary value problem. But, there are classes of compact sets K such that the free boundary problem has a unique solution. The largest one is the star shaped domains. Tepper [23, 24] has also shown that if the inner boundary is starlike, then so is the outer boundary. Another interesting class is the one of convex domains. in [16][Theorem 2-1], Henrot and Shahgholian proved that, if K is convex, the free boundary problem admits exactly one solution (Ω, u) and moreover this Ω is convex. A nice introduction lecture is [13].

Properties of the exterior Bernoulli free boundary problem. Consider \mathcal{S}_c the space of compact domains in \mathbb{R}^N that are star-shaped with respect to the origin and \mathcal{S}_o the space of open domains that are star-shaped with respect to the origin. We define the map $\mathcal{B} : \mathcal{S}_c \times (0, +\infty) \rightarrow \mathcal{S}_o$ that maps a convex compact set K to the closure of the unique domain Ω solution of the exterior Bernoulli free boundary problem (4). We shall describe now the properties of the application \mathcal{B} gathering results of [14, Theorem 2-2, Theorem 3-1].

Proposition 5.1 (Properties of the map). *The map \mathcal{B} is continuous and moreover:*

1. *increasing with respect to inclusion: let K_1 and K_2 be two compact sets with $K_1 \subset K_2$ then for any $Q > 0$, $\mathcal{B}(K_1, Q) \subset \mathcal{B}(K_2, Q)$.*
2. *non increasing with respect to the constant: let Q_1 and Q_2 be two non negative real numbers with $Q_1 \leq Q_2$, then for any K , $\mathcal{B}(K, Q_2) \subset \mathcal{B}(K, Q_1)$ with $\mathcal{B}(K, Q_2) \neq \mathcal{B}(K, Q_1)$.*

As a consequence, the map \mathcal{B} is measurable and given a random compact set K , one defines a random open set as $D = \mathcal{B}(K)$. We have already considered notion of expectation based on parametrization and the Vorobe'v expectation for this free boundary problem in [11].

An semi-analytic case. Let us consider the case where K is a random disk of radius r centered at the origin. Then, the domain D is also a random annulus centered at the origin and of outer radius $f_d(r)$ where the function f_d is defined as:

$$f_2(r) = \frac{1}{\lambda W(\lambda r)} \text{ while } f_3(r) = \frac{\lambda r + \sqrt{\lambda^2 r^2 + 4\lambda r}}{2\lambda},$$

where W is the inverse of $x \mapsto xe^x$. The oriented distance function to D is then explicitly known:

$$b_D(x) = |x| - f_d(r) \text{ if } |x| > (r + f_d(r))/2 \text{ and } r - |x| \text{ else.}$$

If one can write an analytic expression of the expectation of the distance function, it is not very explicit. In order to go further, we have to do simulations even for such a simple case. Figure 1 present the simulation obtained with 10 000 samples with the inner radius $r = 1 + 0.3\alpha$ where α is random variable following the uniform law on $[-1/2, 1/2]$ and a centered Gaussian law of variance 0.3. In red, we present the characteristic function of the empirical mean of the domain and in blue the empirical mean of the distance function.

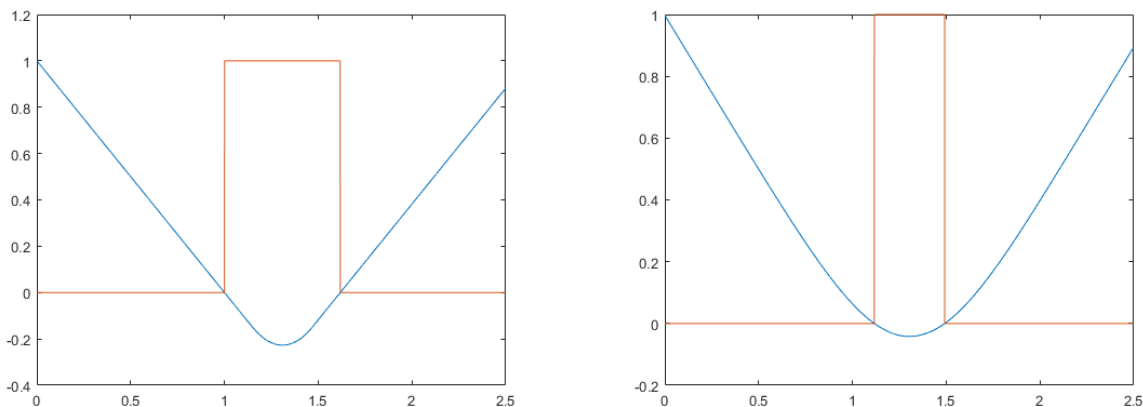


Figure 1: The expected distance function and expected domain: centered uniform (left), centered Gaussian of variance 0.3 (right).

Conclusion. For more general inner boundary, one has no analytic expression for the distance function and numerical scheme are mandatory. This requires precise analysis and is out of scope of this paper, it will be presented in a work in preparation.

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